INFINITE CARTESIAN PRODUCTS OF DIFFERENTIAL AND FRÖLICHER LIE GROUPS

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ABSTRACT

Given a set, the Sikorski differential space structure is determined on it by a collection of real-valued functions, while the Frölicher (smooth) structure is defined by a pair of paths into along with real-valued functions which fulfil specified sets of axioms. According to A. Batubege and P. Ntumba, when these structures are provided with an additional group operation that is compatible with the smooth structure, they are then called differential groups or Frölicher Lie groups, respectively. The infinite Cartesian product of differential groups was investigated by W. Sasin, and since a Frölicher space is a differential space in the sense of Sikorski, it turns out that a Frölicher Lie group is a differential group. Now, the differential structure on the product of differential groups is the product of structures of the factors. On the product of Frölicher Lie groups as for general Frölicher spaces, it is rather the set of structure curves that has this property, and not the set of structure functions. In this study we use a class of Frölicher spaces free of this defect, in order for the resulting Frölicher Lie groups to satisfy the property similar to that of smooth functions on the product of differential groups. To this end, we consider a class of differential groups made of differential spaces whose set of structure functions is reflexive in the sense that it generates Frölicher curves from which the generated Frölicher functions are exactly the Sikorski functions which induced the smooth structure. Such differential spaces, so-called pre-Frölicher spaces by A. Batubege, induce a class of Frölicher spaces (Frölicher Lie groups) so-called $\mathbb{DF}$-spaces (groups) on which, unlike differential groups, the set of smooth functions is the product of sets of structure functions from the factors. This induces the results similar to the study by W. Sasin.

DECLARATION

I, Stanley Mukuka Chewe, declare that this dissertation titled **INFINITE CARTESIAN PRODUCTS OF DIFFERENTIAL AND FRÖLICHER LIE GROUPS**, and the work presented in it are my own. I confirm that:

- Where any part of this dissertation has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.

- Where I have consulted the published work of others, this is always clearly attributed.

- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this dissertation is entirely my own work.

- I have acknowledged all main sources of help.

- Where the dissertation is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed:

Stanley Mukuka Chewe (Student)

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Prof. Batubenge A. (Supervisor)

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DEDICATION

To my children, my wife Ivy Chimfwembe and my parents, this is as a result of your sacrifice and I dedicate this work to you.

ACKNOWLEDGEMENTS

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APPROVAL

This dissertation of STANLEY MUKUKA CHEWE is a fulfilling requirement of the award of the degree of master of science in mathematics of the University of Zambia.

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Contents

INTRODUCTION

In this dissertation, we consider differential and Frölicher Lie groups. Since Lie groups are both groups and manifolds (see [35], [18]) it will be necessary to understand the elementary properties of Lie groups and their connection to differential spaces and Frölicher spaces. This fact allows us to use concepts from algebra and analysis to study differential and Frölicher Lie groups (see [27]).
Lie groups are differentiable manifolds which are also groups and in which the group operation is smooth. The study of the classical concept of Lie groups has been extended to modern smooth spaces such as diffeological spaces, differential spaces and Frölicher spaces. When these smooth spaces are provided with an algebraic group structure that is compatible with the differential one, one speaks of diffeological groups, differential groups and more recently, of Frölicher Lie groups. We are more interested in the latter two classes of smooth spaces.

In the first chapter of this dissertation we shall look at the geometry and topology of differential spaces and differential groups. Here we show the relation between differential spaces and differential groups. Henceforth, the dissertation will discuss some properties of differential groups in the sense of Sikorski, looking at the standard facts concerning left invariant vector fields and left invariant forms on a differential group. Furthermore, we will introduce the main concepts on differential groups as they were investigated by W. Sasin (see [30]), P. Multarzynski (see [24]), and Z. Pasternack-Winiarski (see [25], [26]). To this end, this part will contain the definitions and facts from the theory of Sikorski differential spaces, differential subgroups, Hausdorff differential group, tangent vector to a differential space as well as some important results associated with differential spaces and groups.

Chapter two of the dissertation is devoted to the study of smooth structures which are generated on a set M by a collection of maps from $\mathbb{R}$ into M, called contours (which will become smooth curves) and the set of maps from M into $\mathbb{R}$, the scalar functions which will be called smooth functions. We point out that the smooth structure on M is obtained without requiring that M be a linear or Banachable space. These spaces were introduced by Alfred Frölicher who called them 'smooth spaces', which were later named after him by P. Cherenack (see [10]), P. Michor and A. Kriegl (see [16]). The link between Frölicher spaces and Frölicher Lie groups will be discussed. We mention here that the concept of Frölicher Lie groups was introduced by A. Batubenge, P. Ntumba and M. Laubinger to cite but very few (see [33], [1], [10], [29]). The topics to be considered here include definitions, diffeomorphisms of Frölicher spaces, the notion of bundles on Frölicher spaces and so forth. Lie groups in the setting of Frölicher spaces are called Frölicher Lie groups or $\mathbb{F}$-Lie groups. A similar concept of Lie groups exists in the category of Sikorski differential spaces, the so-called differential groups.

In chapter three we investigate the infinite Cartesian products of smooth spaces. Here we investigate and describe the topologies underlying these spaces, the Cartesian products and the infinite Cartesian product of differential groups (see [30]). In particular we will show that the infinite Cartesian product of Lie groups may be viewed as a differential group. Similarly we examine if the infinite Cartesian product of Frölicher Lie groups is a Frölicher Lie group. We will show that the set of structure functions on the Cartesian product of Frölicher Lie groups is not the product of sets of structure functions in general. Therefore, we will introduce a class of spaces which allows resemblance of product structures for both types of spaces. A way out for this will be that of considering a class of differential groups made of differential spaces whose set of structure functions is reflexive (see [37], [38]) $\mathcal{F} = \Phi\Gamma\mathcal{F}$, the so-called pre-Frölicher spaces ([6]), on which the process of yielding a Frölicher structure on the same set is smooth function preserving. The resulting Frölicher spaces we shall call $\mathbb{DF}$-spaces, the smooth groups of which are the spaces of high interest for this research. They are called $\mathbb{DF}$-Lie groups.

The last chapter which is the conclusion will highlight on the geometry and the topology of infinite products of the $\mathbb{DF}$-spaces. Then we will construct infinite Cartesian products on $\mathbb{DF}$-Lie groups.

1. DIFFERENTIAL SPACES

1.1 Preliminary Definitions

In this section we recall the basics on differentiable manifolds since it is known that they are the building blocks of Lie groups. In simpler terms a manifold is a Hausdorff second countable topological space, which locally (i.e in a close-up view) resembles the spaces described by Euclidean geometry but which globally (i.e, when viewed as a whole) may have a complicated structure. e.g the surface of the earth is a manifold, locally it seems to be flat, but viewed as a whole from the outer (globally) it is round. We begin by defining a topological space as this will help to understand the differential space and subsequently the differential group.

Definition 1.1.1. *([36]) A topological space is a non-empty set X equipped with a distinguishable family of subsets, called the open sets which forms a topology τ such that:*

1. *the empty set and the set X are both open,*

2. *the intersection of any finite collection of open sets is again open,*

3. *the union of any collection (finite or infinite) of open sets is again open.*

Example 1.1.2. *For any X, $\mathscr{P}(X) = U$ is a topology called the discrete topology on X.*

Example 1.1.3. *Take $U = \{\phi, X\}$. This is a topology called indiscrete topology on X.*

Definition 1.1.4. *([35]) A locally Euclidean space M of dimension d is a Hausdorff topological space M for which each point has a neighbourhood homeomorphic to an open subset of Euclidean space $\mathbb{R}^d$.*

Definition 1.1.5 (Induced topology). *([36]) For any non-empty subset A of a topological space (X, τ), the induced (or relative) topology, τ_A, on A is defined to be that given by the collection $A \cap \tau = \{A \cap U : U \in \tau\}$ of subsets of A.*

Definition 1.1.6 (Hausdorff topological space). *([36]) A Topological space (X, τ) is said to be Hausdorff if for any pair of distinct points $x, y \in X$, $(x \neq y)$, there exist sets $U, V \in \tau$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$.*

Definition 1.1.7 (Smooth manifold). *([33], [35]) Given any integers N, M with $N \geq M \geq 1$, an m-dimensional smooth manifold in $\mathbb{R}^N$, for short, a manifold, is a nonempty subset S of $\mathbb{R}^N$ such that for every point $p \in S$ there are two open subsets $\Omega \subseteq \mathbb{R}^M$ and $U \subseteq S$ and a smooth function $\varphi : \Omega \to \mathbb{R}^N$ such that φ is a homeomorphism between Ω and $U = \varphi(\Omega)$ and $\varphi'(t_o)$ is injective, where $t_o = \varphi^{-1}(p)$.*

Definition 1.1.8. *([18]) Let $U \subset \mathbb{R}^d$ be open, and let $f : U \to \mathbb{R}$. We say that f is* differentiable of class C^k *(or simply that f is C^k), for k a non-negative integer, if all the partial derivatives $\partial^\alpha f/\partial r^a$ exist and are continuous on U for $[\alpha] \leq k$. If $f : U \to \mathbb{R}^n$, then f is* differentiable of class C^k *if each of the component functions $f_i = r_i o f$ is C^k, where f_i is called the $\imath$th component function of f and $[\alpha] = \sum \alpha_i$ for $\alpha = (\alpha_1, ..., \alpha_n)$ d-tuples.*

Definition 1.1.9. *([18]) A differentiable structure $\mathcal{F}$ of class $C^k(1 \leq k \leq \infty)$ on a locally Euclidean space M is a collection of coordinate systems $\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ satisfying the following properties*

(a) $\bigcup_{\alpha \in \mathbf{A}} U_\alpha = M$

(b) $\varphi_\alpha o \varphi_\beta^{-1}$ *is C^k for all $\alpha, \beta \in A$*

(c) *The collection $\mathcal{F}$ is maximal with respect to (b); that is , if (U, φ) is a coordinate system such that $\varphi o \varphi_\alpha^{-1}$ and $\varphi_\alpha o \varphi^{-1}$ are C^k for all $\alpha \in A$, then $(U, \varphi) \in \mathcal{F}$.*

We note that a *d-dimensional differentiable manifold of class C^k* (similarly C^ω or complete analytic) is a pair $(M, \mathcal{F})$ consisting of a d-dimensional, second countable, locally Euclidean space M together with a differentiable structure $\mathcal{F}$ of class C^k. We denote the differentiable manifold $(M, \mathcal{F})$ simply by M, with the understanding that when we speak of the "differentiable manifold M" we are considering the locally Euclidean space M with some given structure $\mathcal{F}$. Our attention will be restricted solely to the case of class C^∞, so by *differentiable* we will always mean *differentiable of class C^∞*. We also use the terminology *smooth* to indicate differentiability of class C^∞. Thus we shall always refer to differentiable manifolds simply as *manifolds*, with differentiability of class C^∞ always implicitly assumed (see [35]).

A differential space or d-space is one of the ways of generalising the classical concept of a smooth manifold. We recall that a smooth manifold is a Hausdorff second countable topological space which is locally homeomorphic to a Euclidean space. When looking at a differential space, the starting point is to properly choose the set together with a topology in it and some family of continuous real functions on this set. Then one determines the differential structure on the considered set if these functions are required to satisfy specific conditions. We mention that every differentiable manifold is a differential space, but not every differential space is a differentiable manifold. The details are given in the literature below. It is assumed that $\mathbb{N} = \{1, 2, ...\}$.

Let X be a nonempty set, $\mathcal{F}$ an $\mathbb{R}$-algebra of real functions on X with the usual operations of pointwise addition and multiplication. We consider the weakest topology $\tau_{\mathcal{F}}$ on X in which functions of $\mathcal{F}$ are continuous (see [11],[12]).

Definition 1.1.10. *A function $f : X \to \mathbb{R}$ is called a local $\mathcal{F}$-function on X if for every $p \in X$ there is a neighborhood V of p and $\alpha \in \mathcal{F}$ such that $f|V = \alpha|V$. The set of all local $\mathcal{F}$-functions on X shall be denoted by $\mathcal{F}_X$.*

Remark 1.1.1. *Note that any function $f \in \mathcal{F}_X$ is continuous with respect to the topology $\tau_{\mathcal{F}}$. Then $\tau_{\mathcal{F}_X} = \tau_{\mathcal{F}}$.*

Definition 1.1.11. *A function $f : X \to \mathbb{R}$ is called $\mathcal{F}$-smooth function on X if there exist $n \in \mathbb{N},\ \omega \in C^\infty(\mathbb{R}^n)$ and $\alpha_1, ..., \alpha_n \in \mathcal{F}$ such that*

$$f = \omega \circ (\alpha_1, ..., \alpha_n)$$

.

The set of all $\mathcal{F}$-smooth functions on X will be denoted by $sc\mathcal{F}$.

Since $\mathcal{F} \subset sc\mathcal{F}$ and any composition $\omega \circ (\alpha_1, ..., \alpha_n)$ is continuous with respect to $\tau_{\mathcal{F}}$, that is $(\alpha_1, ..., \alpha_n) : X \to \mathbb{R}^n$, we obtain $\tau_{sc\mathcal{F}} = \tau_{\mathcal{F}}$ (see [12], [23]).

Lemma 1.1.1. *If $A \subset B \subset X$, then $(\mathcal{F}_B)_A = \mathcal{F}_A$. In particular, $(\mathcal{F}_A)_A = \mathcal{F}_A$*

Proof. Let us take $f \in (\mathcal{F}_{\mathcal{B}})_A$, where $(\mathcal{F}_{\mathcal{B}})_A$ is the function on B to the subset A. For any point $p \in A$, there is a neighbourhood U of $p \in A$ such that $f|U = g|U$ for $g \in \mathcal{F}_B$. Since $g \in \mathcal{F}_B$, there exist a neighbourhood V of p in B and a function $h \in \mathcal{F}$ such that $g|V = h|V$. But $W := U \cap V$ is a neighbourhood of p in A and $f|W = h|W$, which means that $f \in \mathcal{F}_A$. Therefore $\mathcal{F}_B \subset \mathcal{F}_A$.

Now let us show that $\mathcal{F}_A \subset (\mathcal{F}_B)_A$. Given $f \in \mathcal{F}_A$ and a point p in A, there exist a neighbourhood U of p in A and a function $g \in \mathcal{F}$ such that $f|U = g|U$. But since $U \subset A \subset B$, and $h := g|_B \in \mathcal{F}|_B$ is such that $f|U = h|U$, it follows that $f \in (\mathcal{F}_B)_A$.

Proposition 1.1.1. *If $\mathcal{V}$ is an open covering of X, f a function defined on X, and $f|V \in \mathcal{F}_V$ for every $V \in \mathcal{V}$, then $f \in \mathcal{F}_X$.*

Proof. Let p be a point in X. There exists a neighbourhood $V \in \mathcal{V}$ such that $p \in V$. Since $f|V \in \mathcal{F}_V$, it follows that there is a neighbourhood U of p in V such that $f|U = g|U$, for some $g \in \mathcal{F}_M$.

Definition 1.1.12 (Differential space). *([39], p.12) Let X be a nonempty set. A differential structure, sometimes called a Sikorski structure on X, is a nonempty family $\mathcal{F}$ of functions into $\mathbb{R}$, along with the functional topology τ, which is the weakest topology on X for which every element of $\mathcal{F}$ is continuous, satisfying:*

1. *(Smooth Compatibility) For any positive integer k, functions $f_1, f_2, ..., f_k \in \mathcal{F}$, and $F \in C^\infty(\mathbb{R}^k)$, the composition $F(f_1, f_2, ..., f_k)$ is contained in $\mathcal{F}$.*

2. *(Locality) Let $f : X \to \mathbb{R}$ be a function $g \in \mathcal{F}$ satisfying $f|U = g|U$, $U \in \tau$. Then $f \in \mathcal{F}$.*

Definition 1.1.13. *A set X equipped with a differential structure $\mathcal{F}$ is called a differential space, or a Sikorski space, and shall be denoted by $(X, \mathcal{F})$. The functions $f_1, ..., f_k$ are called* generators, *and $\mathcal{F}_0 = \{f_1, f_2, ..., f_k\}$ the generating set for the structure $\mathcal{F}$.*

Equivalently we have that;

Definition 1.1.14. *([29]) A differential space (sometimes called a Sikorski space) X is a topological space equipped with a differential structure. A differential structure is a family of functions, denoted $C^\infty(X)$, that satisfies the following:*

1. *The set $\{f^{-1}(a, b) \subseteq X \mid f \in C^\infty(X), (a, b) \subseteq \mathbb{R}\}$ is a sub-basis of the topology of X.*

2. *if $f_1, f_2, .., f_n \in C^\infty(X)$ and $F \in C^\infty(\mathbb{R}^N)$ then $F(f_1, f_2, ..f_n) \in C^\infty(X)$.*

3. *If $f : X \to \mathbb{R}$ is such that for any $x \in X$ there exists a neighbourhood $U \subseteq X$ of x such that $x \in X$ and a function $f_x \in C^\infty(X)$ so that $f\mid_U = f_x\mid_U$, then $f \in C^\infty(X)$.*

Remark 1.1.2. 1. *Let X be a set and $\mathcal{F}$ a family of real-valued functions on X. The weakest topology on X such that $\mathcal{F}$ is a set of continuous functions will be called the topology induced or generated by $\mathcal{F}$, and will be denoted by τ_F (see definition 1.1.5). A sub-basis for this topology is given by*

$$\{f^{-1}(I) | f \in \mathcal{F},\ I \textit{ is an open interval in } \mathbb{R}\}.$$

2. *The smooth compatibility of a differential structure guarantees that $\mathcal{F}$ is a commutative*

$\mathbb{R}$ algebra under pointwise addition and multiplication.

Example 1.1.15. *([19]) Consider $X = \mathbb{R}^n$ for some $n \in \mathbb{N}$ and $\mathcal{F} = C^\infty(\mathbb{R}^n)$. Then $(X, \mathcal{F})$ is a smooth Euclidean n-dimensional differential space. Note that $\mathcal{F}$ is generated by projections $\pi_1, ..., \pi_n$, where $\pi_i(x_1, x_2, .., x_n) = x_i, i = 1, ..., n, (x_1, x_2, ..., x_n) \in \mathbb{R}^n$.*

Example 1.1.16. *The pair $\mathbb{R}_d := (\mathbb{R}, C^\infty(\mathbb{R}, \mathbb{R}))$ is a differential space, and is called the canonical Euclidean differential space.*

Example 1.1.17. *Consider some classical smooth manifold X and all smooth (in the classical sense) real functions on it, $C^\infty(X)$. Then $(X, C^\infty(X))$ is a differential space.*

Example 1.1.18. *Consider $M = \{(x, y) \in \mathbb{R}^2 | xy = 0\}$. From classical point of view (0,0) is a singular point and M cannot be equipped with a classically smooth differential structure. It is an example of a differential space, which is not a manifold. This differential structure consists of restrictions of smooth functions from $\mathbb{R}^2$, i.e. if $\mathcal{F} = \{f|_M;\ f \in C^\infty(\mathbb{R}^2)\}$, then $(M, \mathcal{F})$ is a differential space.*

Now we define induced and coinduced differential structures (see [24]).

Definition 1.1.19. *([10], [24]) Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be differential spaces. A differential structure $\mathcal{F}$ is said to be induced on X from a family $\{X_i, \mathcal{F}_i \mid i \in I\}$ of differential spaces by a family of mappings $\{f_i : X \to X_i, i \in I\}$ if $\mathcal{F}$ is the weakest differential structure on X with respect to which all mappings $f_i, i \in I$, are smooth (see also 1.1.5). Analogously, a differential structure $\mathbb{G}$ is said to be coinduced on Y from a family $\{Y_i, \mathcal{G}_i \mid i \in I\}$ of differential spaces by a family of mappings $f_i : Y \to Y_i, i \in I$ if $\mathcal{G}$ is the strongest differential structure on Y with respect to which all mappings f_i, $(i \in I)$ are smooth.*

Furthermore, we have that

Proposition 1.1.2. *([24]) Let $\{f_i : X_i \to X\}_{i\in I}$ be a collection of set maps where the $X_{i's}$ are differential spaces with structure functions $\mathcal{F}_i$, correspondingly. Then the pair $(X, \mathcal{F})$ is the coinduced differential space corresponding to the family of set maps $\{f_i : X_i \to X\}_{i\in I}$ if*

1. *the maps $f_i : (X_i, \mathcal{F}_i) \to (X, \mathcal{F})$ are smooth, and*

2. *for any other differential space $(X, \mathcal{G})$ in the diagram*

$$\begin{tikzcd} (X_i, \mathcal{F}_i) \arrow[r, "f_i"] \arrow[dr, "f_i"'] & (X, \mathcal{G}) \\ & (X, \mathcal{F}) \arrow[u, "I"] \end{tikzcd}$$

with the f_i smooth in both cases, the identity map I in the given direction is smooth. Thus,

$\mathcal{F}$ is the greatest differential structure on X with respect to which each $f_i, i \in I$, is a smooth mapping.

Proposition 1.1.3. *The differential space $(X, \mathcal{F})$ has the induced structure arising from the family $\{f_i : X \to X_i\}_{i \in I}$ of maps.*

Proof (see [24], p. 4)

Next we have the coinduiced structure

Proposition 1.1.4. *Given a family of set maps $\{f_i : X_i \to X\}_{i \in I}$ where the $X_{i's}$ are the differential spaces, the set*

$$\mathcal{F} := \cap_{i \in I} (f_{*i})^{-1}(\mathcal{F}_i)$$

where

$$(f_{*i})^{-1}(\mathcal{F}_i) = \{f : X \to \mathbb{R} \mid f \circ f_i \in \mathcal{F}_i\}$$

for all $i \in I$, defines the coinduced differential structure on X corresponding to the collection $\{f_i : X_i \to X\}_{i \in I}$ of set maps.

Proof (see[24], p. 5)

Definition 1.1.20. *([37], p. 16) Having fixed some differential space $(X, \mathcal{F})$, any function from $\mathcal{F}$ is called smooth in the sense of Sikorski.*

In his thesis, Jordan W.(see [37], p. 16), also (see[19]) has shown the following result.

Proposition 1.1.5. *([38], [37]) $(X, \mathcal{F})$ is a differential space.*

Proof. Let X be a set, and let $\mathcal{Q}$ be a family of real valued functions on X. Equip X with the topology induced by $\mathcal{Q}$.

First, we show smooth compatibility. Let $f_1, ..., f_k \in \mathcal{F}$ and $F \in C^\infty(\mathbb{R}^k)$. Then, we want to show $F(f_1, ..., f_k) \in \mathcal{F}$. Fix $x \in X$. Then for each $i = 1, ..., k$, there exist an open neighborhood U_i of x, $q_i^1, ..., q_i^{mi} \in \mathcal{Q}$ and $F_i \in C^\infty(\mathbb{R}^{mi})$ such that

$$f_i \mid U_i = F_i(q_i^1, ..., q_i^{mi}) \mid U_i.$$

Let U be the intersection of the neighbourhoods of U_i, which itself is an open neighbourhood of X. Then

$$F(f_1, ..., f_k)|U = F(F_1(q_i^1, ..., q_i^{mi}), ..., F_k(q_k^1, ..., q_k^{mk})|U$$

Let $N := m_1 + ... + m_k$. Define $\tilde{F} \in C^\infty(\mathbb{R}^N)$ by

$$\tilde{F}(x^1, ..., x^N) = F(F_1(x^1, ..., x^{m1}), F_2(x^{m_1+1}, ..., x^{m_1+m_2}), ..., F_k(x^{m_1+...+m_{k-1}+1}, ..., x^N)),$$

then

$$F(f_1, ..., f_k)|u = \tilde{F}(q_1^1, ..., q_1^{m1}, q_2^1, ..., q_2^{m2}, q_k^1, ..., q_k^{mk})|U$$

By definition of $\mathcal{F}$, we have $F(f_1, ..., f_k) \in \mathcal{F}$.

Next, we show locality. Let $f : X \to \mathbb{R}$ be a function with property that for every $x \in X$ there is an open neighbourhood U of x, and a function $g \in \mathcal{F}$ such that $g|U = f|U$. Fix x, and let U and g satisfy this property. Shrinking U if necessary, there exist $q_1, ..., q_k \in \mathcal{Q}$ and $F \in C^\infty(\mathbb{R}^k)$ such that

$$g \mid U = F(q_1, ..., q_k)|U.$$

Hence,

$$f \mid U = F(q_1, ..., q_k) \mid U.$$

Since this is true at each $x \in X$, by definition, $f \in \mathcal{F}$. This completes the proof.

Definition 1.1.21. *A differential space $(X, \mathcal{F})$ is said to be Hausdorff if the induced topology τ is Hausdorff (see definition 1.1.6).*

Example 1.1.22. *(see [29]) The differential space $(\mathbb{R}^n, \varepsilon_n)$ is Hausdorff.*

Example 1.1.23. *Let $X = \mathbb{R}$ and $p, q \in M$ $(\mathrm{M} \subseteq \mathrm{X}),\ \mathrm{p} \neq \mathrm{q}$. Let $\mathcal{F} = \{f \in C^\infty(\mathbb{R}) | f(p) = f(q)\}$. It follows that $(X, \mathcal{F})$ is a differential space, where M is a subspace of X.*

Remark 1.1.3. *Note that $(X, \mathcal{F})$ is Hausdorff if and only if, for any two distinct points $x, y \in X$, there is a function $f \in \mathcal{F}$ such that $f(x) \neq f(y)$.*

Definition 1.1.24. *Given a differential space $(X, \mathcal{F})$, we say that a set F_o of functions generates the differential structure $\mathcal{F}$ if and only if, given any point $p \in X$, there are functions $f_1, f_2, ..., f_n \in F_0$, $\omega \in C^\infty(\mathbb{R}^n, \mathbb{R})$ and a neighborhood $U \in \tau$ (where τ is the topology induced by F_0) such that*

$$f|U = \omega \circ (f_1, f_2, ..., f_n)|U$$

The differential structure $\mathcal{F}$ is the smallest structure that contains F_0.

Definition 1.1.25 (Subset differential structure). *([37], [38]). Let $(X, \mathcal{F})$ be a differential space, and let $Y \subseteq X$ be any subset. Then Y acquires a differential structure $\mathcal{F}_Y$ generated by restrictions to Y of functions in $\mathcal{F}$. That is, $f \in \mathcal{F}_Y$ if and only if for every*

$x \in Y$ there is an open neighbourhood $U \subseteq X$ and a function $\tilde{f} \in \mathcal{F}$ such that

$$f|U \cap Y = \tilde{f}|U \cap Y$$

We call $(Y, \mathcal{F}_Y)$ a differential subspace of X.

Equivalently we have;

Definition 1.1.26. *([25]) Let (M, C) be a differential space and $A \subset M$, $A \neq \emptyset$, then C_A is a differential structure on A and a differential space (A, C_A) is called a differential subspace of (M, C). We have that $C_A = (i^*C)_A$, where i is the inclusion mapping of A in M.*

It follows, therefore, that every subset of a Euclidean space is a differential space. This property of Euclidean spaces is enough to show one the scope of the differential space concept. Indeed, differential spaces are a generalisation of differentiable manifolds since manifolds are locally smooth spaces.

Example 1.1.27. *The graph of the function $|x| : [-1, 1] \to \mathbb{R}$ is not a smooth manifold, but it is a differential space. For, let $X = \{(x, |x|) \subset \mathbb{R}^n | x \in [-1, 1]\}$, then $(X, C^\infty(\mathbb{R}^2)|_X)$ is a differential space.*

In his works W. Jordan, (see[22]) had also shown the following result;

Lemma 1.1.2. *([37], [38]) Let $(X, \mathcal{F})$ be a differential space. Then for any subset $Y \subseteq X$ the subspace topology on Y is the weakest topology for which the restrictions of $\mathcal{F}$ to Y are continuous.*

Proof. We first set some notation. Let τ_Y be the subspace topology on Y and let $\mathcal{G}$ be all the restrictions of functions in $\mathcal{F}$ to Y.

Fix $U \in \tau_Y$ and $x \in U$. We will show that there exists a basic open set $W \in \tau_{\mathcal{G}}$ such that $x \in W \subseteq U$. By definition of the subspace topology on Y, there exists an open set $V \in \tau_F$ such that

$$U = V \cap Y$$

There exists $f_1, ..., f_k \in \mathcal{F}$ such that

$$\tilde{W} := \bigcap_{i=1}^{k} f_i^{-1}((0, 1)).$$

is basic open set of X containing x and contained in V. Define

$$W := \tilde{W} \cap Y.$$

Then

$$\begin{aligned} W &= \bigcap_{i=1}^{k} f_i^{-1}((0,1)) \cap Y \\ &= \bigcap_{i=1}^{k} (f_i|Y)^{-1}((0,1)). \end{aligned}$$

But $f_i|Y \in \mathcal{G}$, and so W is a basic open set in $\tau_{\mathcal{G}}$ that contains x and is contained in U.

Next we show that for $U \in \tau_{\mathcal{G}}$, U is in fact open in the subspace topology. It is sufficient to show this for any basic open set U, in the basis generated by $\mathcal{G}$. To this end, fix a basic open set $U \in \tau_{\mathcal{G}}$ and $x \in U$. There exist $g_1, ..., g_k \in \mathcal{G}$ such that

$$U = \bigcap_{i=1}^{k} g_i^{-1}((0,1))$$

But then there exist $f_1, ..., f_k \in \mathcal{F}$ such that for $i = 1, ..., k$ we have

$$g_i = f_i \mid Y.$$

Then,

$$U = \bigcap_{i=1}^{k} f_i^{-1}((0,1)) \cap Y.$$

Since $\bigcap_i^k f_i^{-1}((0,1))$ is open in X, we have that U is open in the subspace topology on Y.

We have shown that the subspace topology on Y and the topology generated by restrictions of functions $\mathcal{F}$ to Y are one and the same.

Proposition 1.1.6. *([37],[38]) The intersection of any family of differential structures defined on a set $X \neq \emptyset$ is a differential structure on X.*

Proof. Let $\{\mathcal{F}_i\}_{i\in I}$ be a family of differential structures defined on a set X and let

$$\mathcal{F} = \bigcap_{i\in I} \mathcal{F}_i.$$

Then $\mathcal{F}$ is nonempty family of real-valued functions on X (it contains all constant functions).

If $n \in \mathbb{N}, \omega \in \mathcal{F}^\infty(\mathbb{R}^n$ and $\alpha_1, ..., \alpha_n \in \mathcal{F})$, then for any $i \in I, \alpha_1, .., \alpha_n \in \mathcal{F}_i$ and consequently $\omega \circ (\alpha_1, ..., \alpha_n) \in \mathcal{F}_i$. Hence $\omega \circ (\alpha_1, ..., \alpha_n) \in \mathcal{F}$ which means $sc\mathcal{F} = \mathcal{F}$.

Since $\mathcal{F} \subset \mathcal{F}_i$ for any $i \in I$ we have $\tau_\mathcal{F} \subset \tau_{\mathcal{F}i}$. It means that any subset of X open with respect to $\tau_\mathcal{F}$ is open with respect to $\tau_{\mathcal{F}_i}$, for $i \in I$.

Let $\beta \in \mathcal{F}_m$. Choose for any $p \in X$ a set $U_p \in \tau_\mathcal{F}$ and a function $\alpha_p \in \mathcal{F}$ such that $p \in U_p$ and $\beta U_p = \alpha | U_p$. Since $\alpha_p \in \mathcal{F}_i$ and $U_p \in \tau_{\mathcal{F}i}$ we obtain $\beta \in (\mathcal{F}_i)_m = \mathcal{F}_i$ for any $i \in I$. Then $\beta \in \mathcal{F}$ and consequently $\mathcal{F}_m = \mathcal{F}$. Equalities $\mathcal{F}_m = \mathcal{F} = sc\mathcal{F}$ means that $\mathcal{F}$ is a differential structure on X.

As it is a well known fact from the theory of manifolds, subsets of differentiable manifolds are not, generally speaking, differentiable manifolds. But in the differential spaces context, differential structures can be induced from a base space to a subset. Thus we have;

Proposition 1.1.7. *([38]) Given a differential space X and a subset $Y \subseteq X$, Y is a differential space called a differential subspace of X (see definitions 1.1.25 and 1.1.2).*

Proof (see[38], p. 14)

1.2 Differential basis on differential spaces

The notion of differential basis on differential spaces is important when one is dealing with the dimensionality of the differential space.

Let $(M, \mathcal{F})$ be a differential space.

Definition 1.2.1. *[28]* A function $f \in \mathcal{F}$ is said to be differentially dependent (briefly, d-dependent) on functions $g_1, g_2, ..., g_n \in \mathcal{F}$ at a point $p \in M$ if there exist a neighbourhood $U \in \tau_\mathcal{F}$ of the point p and a function $\omega \in \varepsilon_n$ such that

$$f|U = \omega \circ (g_1, g_2, ...g_n)|U$$

Example 1.2.2. *Any function $f \in \varepsilon_n$ differentially depends on projections $\pi_1, \pi_2, ..., \pi_n \in \varepsilon_n$ at any point $p \in \mathbb{R}^n$.*

Proof. Since $\varepsilon_n = Gen\{\pi_1, \pi_2, ..., \pi_n\}$, $f \in \varepsilon_n$ if and only if for any point $p \in \mathbb{R}^n$ there exists a neighbourhood $U \in \tau_{\varepsilon_n}$ and a smooth function $\omega : \mathbb{R}^n \to \mathbb{R}$ such that

$$f|_U = \omega \circ (\pi_1, ..., \pi_n)|_U$$

Definition 1.2.3. *(see [28])* A set $\{f_1, f_2..., f_n\} \subset \mathcal{F}$ is said to be differentially independent at a point $p \in M$ if no function $f_i, \imath = 1, ...n$ differentially depends on other functions of this set at p. Any set $\mathcal{F}_0 \subset \mathcal{F}$ is said to be differentially independent at $p \in M$ if every finite subset of $\mathcal{F}_0$ is differentially independent at p.

Example 1.2.4. *The set $\{\pi_1, \pi_2, ..., \pi_n\} \subset \varepsilon_n$ is differentially independent at any point $p \in \mathbb{R}^n$.*

Evidently, from Definitions 1.2.1 and 1.2.3 it follows that both d-dependence and d-independence of a set $\mathcal{F}_0 \subset \mathcal{F}$ are local properties of $\mathcal{F}_0$.

Definition 1.2.5. *The tangent space T_pM to M at a point $p \in M$ is the set of all tangent vectors at the point p. That is, the set of all mappings $X_p : C^\infty(p) \to \mathbb{R}$ satisfying for all $\alpha, \beta \in \mathbb{R}$ and $f, g \in C^\infty(p)$ the two conditions*

(i) $X_p(\alpha f + \beta g) = \alpha(X_p f) + \beta(X_p(g))$ *(linearity)*

(ii) $X_p(fg) = (X_p f)g(p) + f(p)(X_p g)$ *(Leibniz rule)*

Proposition 1.2.1. *([28]) Let $M \subset \mathbb{R}^n$ be a non-empty subset and $\mathcal{F} = (\varepsilon_n)_M$. The set of the projections $\{\pi_1|_M, ..., \pi_n|M\}$ is d- independent at $p \in M$ if and only if $dimT_pM = n$, where T_pM is the tangent space on M at $p \in M$.*

More generally we also prove

Proposition 1.2.2. *Let $(M, \mathcal{D})$ be a d-space with $\mathcal{F} = Gen\{f_1, f_2, ..., f_n\}$, where the set $\{f_1, f_2, ..., f_n\}$ is d-independent at any point $p \in M$. Then, $dimT_pM = n$ for any $p \in M$.*

Proof. Since $\mathcal{F} = Gen\{f_1, f_2, ..., f_n\}$ and $\{f_1, f_2..., f_n\}$ is d-independent, the function $\varphi := (f_1, f_2, ..., f_n)$ is a diffeomorphism from $(M, \mathcal{F})$ onto $(\varphi(M), (\varepsilon_n)_{\varphi(M)})$. It is easy to see that φ is one-to-one and onto. On the other hand, φ is smooth since $\omega \circ (\pi_1|_{\varphi(M)}, ..., \pi_n|_{\varphi(M)}) \circ (f_1, f_2, ..., f_n) = \omega(f_1, f_2, ..., f_n) \in \mathcal{F}$, for any $\omega \in \varphi_n$. φ^{-1} is also smooth; indeed, for any $\sigma \in \varepsilon_n$, $\sigma \circ (f_1, f_2, ..., f_n) \circ \varepsilon^{-1} = \sigma$ Hence, for any $p \in (M, \mathcal{F})$, $dimT_pM = dimT_x\varphi(M) = n$, where $x = (f_1(p), f_2(p), ...f_n(p))$.

Corollary 1.2.1. *.Let $(M, \mathcal{F})$ be a differential space finitely generated by $\mathcal{F}_0 := \{f_1, ..., f_n\}$ and $p \in M$. The following conditions are equivalent:*

(i) *$\mathcal{F}_0$ is differentially independent at p*

(ii) *$dimT_pM = n$*

The immediate proof to this corollary is omitted.

Another useful characterisation of the d-independence of a set of real-valued functions belonging to $\mathcal{F}$ is given by the following:

Theorem 1.2.0.1. A subset $\{f_1, f_2, ..., f_n\} \subset \mathcal{F}$ is differentially independent at $p \in M$ if and only if for any functions $\omega \in \varepsilon_n$ and any neighbourhood $U \in \tau_{\mathcal{F}}$ of p, the following condition is satisfied

$$\omega \circ (f_1, f_2, ..., f_n) = 0 \Rightarrow \quad \text{for all } 1 \leq i \leq n, \partial_i(f_1(p), ..., f_n(p)) = 0.$$

Proof. The implication($\Rightarrow$) is immediate.
($\Leftarrow$) Let us assume that for any function $\omega \in \varepsilon_n$ and any neighbourhood $U \in \tau_{\mathcal{F}}$ of 1.2.3 is true. Let us suppose that one of the $f_i's$ differentially depends on the other functions of the set,that is on $f_i, ..., f_{i-1}, f_{i+1,...,f_n}$. Without loss of generality, suppose that there exists a function $\sigma \in \varepsilon_{n-1}$ such that $f_1 = \sigma \circ (f_2, ..., f_n)$ on some neighbourhood of p, i.e, f differentially depends on $f_1, f_2, ..., f_n$. Let ω be a function in ε_n, such that $\omega \circ (f_1, f_2, ..., f_n) = \sigma \circ (f_1, f_2, ..., f_n) = 0$, and then $\partial_1 \omega = 1$. This contradict 1.2.3. Thus the set $(f_1, ..., f_n)$ is d-independent.

To further deepen our understanding of a local structure of a differential space we introduce the following:

Definition 1.2.6. A subset $\mathbb{G} \subset \mathcal{F}$ reproduces $\mathcal{F}$ at $p \in M$ if, for any functions $f \in \mathcal{F}$, there exists a neighbourhood $U \in \tau_{\mathcal{F}}$ of p and functions $g_1, ..., g_n \in \mathbb{G}$, $\omega \in \varepsilon_n$ such that $f|_U = \omega \circ (g_1, ..., g_n)|_U$.

Remark 1.2.7. *Let $M, \mathcal{F}$ be a locally finitely generated differential space.One can easily see that a subset $\mathcal{G} \subset \mathcal{F}$ reproduces $\mathcal{F}$ at $p \in M$ if and only if $\mathcal{G}$ locally generates the structure $\mathcal{F}$ in a certain neighbourhood of the point p.*

Definition 1.2.8. A set $\mathcal{G} \subset \mathcal{F}$ is a differential basis of the differential structure $\mathcal{F}$ at $p \in M$ if $\mathcal{G}$ is differentially independent at p and $\mathcal{G}$ reproduces $\mathcal{F}$ at p.

1.3 Functionally smooth maps

In differential geometry (see [35], [18] and [38]), the role of a structure-preserving map is played by C^∞-maps between two manifolds, which is defined as follows.

Definition 1.3.1. *([18]) The* local representative *of a function f (from a manifold M to a manifold N) with respect to the coordinate charts (U, ϕ) and (V, ψ) on M and N respectively, is the map*

$$\psi \circ f \circ \phi^{-1} : \phi(U) \subset \mathbb{R}^m \to \mathbb{R}^n.$$

This leads us to the following:

Let $\mathcal{M}$ and $\mathcal{N}$ be C^∞ -manifolds of dimensions m and n, respectively.

Definition 1.3.2. *([18]) A map of sets $f : \mathcal{M} \to \mathcal{N}$ is a C^∞- map of manifolds if , for all atlases of $\mathcal{M}$ and $\mathcal{N}$, the local representatives are C^∞ functions as defined in the standard real analysis of functions between the topological vector spaces $\mathbb{R}^m$ and $\mathbb{R}^n$, as in 1.1.7*

Definition 1.3.3. *[35] Let $U \subset M$ be open. We say that $f : U \to \mathbb{R}$ is a C^∞ function on U(denoted $f \in C^\infty(U)$) if $f \circ \varphi^{-1}$ is C^∞ for each coordinate map φ on M. A continuous map $\psi : M \to N$ is said to be differentiable of class C^∞ (denoted $\psi \in C^\infty(M, N)$ or simply $\psi \in C^\infty$) if $g \circ \psi$ is a C^∞ function on ψ^{-1}(domain of g) for all C^∞ functions g defined domain of charts in N. Equivalently, the continuous map ψ is C^∞ if and only if $\varphi \circ \psi \circ \tau^{-1}$ is C^∞ for each coordinate map τ on M and φ on N. More generally, we have*

Definition 1.3.4. *([18]) Let M, N be smooth manifolds. A continuous map $f : M \to N$ is called smooth (C^∞) if for each $p \in M$, for some (hence for every) charts φ and ψ, of M and N respectively, with p in the domain of φ and $f(p)$ in the domain of ψ, the composition $\psi \circ f \circ \varphi^{-1}$ (which is a map between open sets in $\mathbb{R}^n$, $\mathbb{R}^k$, where $n = dimM$, $k = dimN$) is smooth on its domain of definition.*

From the definition above, we can say that the composition of two differentiable maps is again differentiable. Furthermore, we see that the mapping $\psi : M \to N$ is C^∞ if and only if for each $m \in M$ there exists an open neighbourhood U of m such that $\psi|U$ is C^∞.

In particular, a *differentiable function is defined to be a C^1 function.* A function that is C^∞ is also said to be smooth.

1.4 Examples of differential maps

Recall the following notions from the theory of smooth manifolds:

1. A continuous map $\psi : M \to N$ is said to be $C^\infty(\psi \in C^\infty(M,N))$ or $\psi \in C^\infty$ if $g \circ \psi$ is a C^∞ function for all C^∞ functions g defined on open sets in N.

2. Equivalently, the continuous map ψ is C^∞ if and only if $\varphi \circ \psi \circ \tau^{-1}$ is C^∞ for each map τ on M and φ on N. We denote by $\hat{\psi} = \varphi \circ \psi \circ \tau^{-1}$ this Euclidean function and we call it the local representative of the map ψ. The rank of ψ is understood as the rank of its associated local representation.

For differential spaces (see [15], [10], [11],[28],[12], [25] and [29]), we rather have what follows. Let (M,C) and (N,D) be differential spaces, then we have;

Definition 1.4.1. *A map $F : M \to N$ is said to be functionally or Sikorski smooth if any $\beta \in D$, one has $\beta \circ F \in C$.*

Equivalently, we note that;

Definition 1.4.2. Diffeomorphism *([35]) Let $\psi : M \to N$ be C^∞, then ψ is a diffeomorphism if ψ is one-to-one onto $N$and ψ^{-1} is C^∞.*

In addition to the above we have;

Definition 1.4.3. *Two differential spaces (M, C) and (N, D) are said to be diffeomorphic if there exists a smooth bijective map $M \to N$ having smooth inverse. Informally, diffeomorphic differential spaces can be thought of as 'the same'.*

Proposition 1.4.1. *([23]) Let (M, C) and (N, D) be differential spaces and let $\mathcal{D}_0$ generate D. A mapping $\alpha : M \to N$ is smooth if for each $f \in \mathcal{D}_0$ we have $f \circ \alpha \in C$.*

Further more a smooth map $F : M \to N$ is called a C^∞ diffeomorphism if there is a map $G : N \to M$ such that $G \circ F$ is the identity on M.

Remark 1.4.1. *A functionally smooth map is continuous with respect to the topologies induced by the differential structure.*

1.5 Tangent vectors and spaces

A smooth manifold is a generalisation to higher dimensions of a smooth surface immersed in $\mathbb{R}^3$. That is, it can be linearized at each point. The resulting linear space is called a tangent space (recall from definition 1.2.5). Between tangent spaces one can define a linear map associated to the differentiable map between manifolds. The idea of a tangent space to a manifold, and to a smooth space in general, is very important in differential geometry. This is based in part on the intuitive geometric idea of a tangent plane to a surface in $\mathbb{R}^3$, such as 2-sphere, a cylinder, a-2-torus and many more (see [18],[35],[14],[28], [15], [11], [12], [24] and [29]).

Definition 1.5.1. *(see [18], p. 73)*

1. *A curve on a manifold M is a smooth (i.e.,C^∞) map σ from some interval $(-\epsilon, \epsilon)$ of the real line into M. Note that the 'curve' is defined to be the map itself.*

2. *Two curves σ_1 and σ_2 are tangent at a point p in M if*

   (a) $\sigma_1(0) = \sigma_2(0) = p$;

   (b) *in some local coordinate chart $(x^1, x^2, ..., x^m)$ around the point, the curves are 'tangent' in the usual sense as curves in $\mathbb{R}^m$;*

$$\frac{dx^i}{dt}(\sigma_1(t))\Big|_{t=0} = \frac{dx^i}{dt}(\sigma_2(t))\Big|_{t=0}$$

   for i= 1,2,...,m.

   Note that if σ_1 and σ_2 are tangent in one coordinate chart, then they are tangent in any other coordinate chart that covers the point $p \in M$. Thus this definition is independent of coordinate charts.

3. *A tangent vector at $p \in M$ is an equivalence class of curves in M where the equivalence relation between two curves is that they are tangent at the point p.*

Definition 1.5.2. *([18]) The tangent bundle TM is given by $TM := \cup_{p\in M} T_P M$.*
Theorem 1.5.0.2. *([35]) The tangent space T_pM is a vector space of dimension n if $dim M = n$.*

Equivalently (see [10], [27], [29], [28] and [30]), we have ;
Definition 1.5.3. *([10])*

1. *Let $(X, \mathcal{F})$ be a differential space. A tangent vector V at $p \in X$ is a derivation $V : \mathcal{F} \to \mathbb{R}$ at P, i.e. a linear map such that*

$$V(fg) = f(p)V(g) + g(p)V(f).$$

   The set of such vectors form the tangent space T_pX or TX_p of X at p.

2. *Let $(X, \mathcal{F})$ be a differential space and $p \in X$. Let $c :]-\epsilon, \ \epsilon[\to X$ be a differentiable curve on X such that $c(a) = p$. Let $f \in \mathcal{F}$. Suppose that V_c is the derivation defined by setting*

$$V_c(f) = \lim_{t\to a} \frac{f\circ c(t) - f\circ c(a)}{t-a}$$

   Set $TCX_p = \{V_c | c(a) = p\}$. We see that $TCX_p \subset TX_p$. TXC_p is the tangent cone to X at p.

Now for any $\alpha \in \mathcal{F}$, we have the differential of α at $p \in X$ as a linear mapping $T_PX \to \mathbb{R}$ given by the formula $d_p\alpha(v) = v(\alpha)$, where $v \in T_pX$.

We denote by TX the disjoint sum of all tangent spaces to $(X, \mathcal{F})$, that is

$$TX := \bigcup_{p\in X} T_pX.$$

Definition 1.5.4. *The mapping $d\alpha : TX \to \mathbb{R}$, for a function $\alpha \in \mathcal{F}$, satisfying the condition $d\alpha|_{T_pX} = d_p\alpha$ is called the tangent mapping associated to the differential of a smooth function α.*

Definition 1.5.5. *The differential structure on TX will be denoted by $T\mathcal{F}$, and is generated by the set $\{\alpha \circ \pi : \alpha \in \mathcal{F}\} \cup \{d\alpha : \alpha \in \mathcal{F}\}$. One has:*

$$T\mathcal{F} = sc(\{\alpha \circ \pi : \alpha \in \mathcal{F}\} \cup \{d\alpha : \alpha \in \mathcal{F}\})_{TX},$$

where $\pi : TX \to X$ is the natural projection, satisfying $\pi(v) = p$, for any $v \in T_pX$. The triple $((TX, T\mathcal{F}), \pi, (X, \mathcal{F}))$ is called the tangent bundle of a d-space $(X, \mathcal{F})$.

Definition 1.5.6. *[10] A tangent vector field on a d-space $(X, \mathcal{F})$ is any mapping V which associates with every point $p \in X$ a tangent vector $V(p) \in T_pX$. A tangent vector field V on $(X, \mathcal{F})$ is said to be smooth if and only if $V : (X, \mathcal{F}) \to (TX, T\mathcal{F})$. In addition we can also say that a (smooth) vector field is a (smooth) section of the tangent bundle.*

Definition 1.5.7. *([19]) A mapping $X : M \to TM$, $X : p \mapsto X_p$ is called a tangent vector field to $(X, \mathcal{F})$. It is called smooth, if for all $f \in \mathcal{F}$, $X(f)$ belongs to $\mathcal{F}$. The set of all vector fields tangent to $(X, \mathcal{F})$ is denoted by $\mathfrak{X}(X)$.*

A vector field X on $(M, \mathcal{F})$ induces a linear mapping also denoted by X, $X : \mathcal{F} \to \mathcal{F}$ satisfying the Lebniz rule, that is a derivation on the algebra $\mathcal{F}$ of smooth functions on X.

Proposition 1.5.1. *([28]) Let $(M, \mathcal{F})$ be a differential space, $f \in \mathcal{F}$ and $f|_A = 0$ for a neighbourhood A of a point $p \in M$. Then $\partial_v f = 0$ for every $v \in T_pM$. Consequently, if functions $f, g \in \mathcal{F}$ are equal on a neighbourhood A of a point $p \in M$, then $\partial_v f = \partial_v g$ for every $v \in T_pM$.*

Proof. Let $(A, \mathcal{F}_A)$ be a differential subspace of a differential space $(M, \mathcal{F})$ and let $p \in A$. If $v \in T_pA$, i.e, if v is a vector tangent to A at p, then the formula

$$\bar{v}(f.g) = v(f|_A)$$

for all $f, g \in \mathcal{F}$, defines a vector $v \in T_pM$. Indeed, $\bar{v}$ is linear and

$$\bar{v}(f.g) = v(f|_A)g(p) + f(p)v(g|_A)$$

$$= \bar{v}(f)g(p) + f(p)\bar{v}(g)$$

for all $f, g \in \mathcal{F}$ and $p \in A$. Clearly the map $T_pA \to T_pM$ which assigns $\bar{v} \in T_pM$ to $v \in T_pA$ is a linear monomorphism. We shall identify v with $\bar{v}$.

Proposition 1.5.2. The tangent space T_pA at $p \in A$ to a subspace $(A, \mathcal{F}_A)$ of a differential space $(M, \mathcal{F})$ is a linear subspace of T_pM. If A is an open subset of M, then $T_pA = T_pM$ for every $p \in A$.

Proof. It is easy to see that T_pA is a linear subspace of T_pM by virtue of 1.5.1. Now, let us assume that A is open in M. If $f, g \in \mathcal{F}_A$, there exists a $\tau_{\mathcal{F}}$-open subset U in A such that

$$f|U = h|U, h \in \mathcal{F}$$

$$g|U = k|U, k \in \mathcal{F}$$

Therefore, for all $\in T_pM$, we have, using 1.5.1

$$v(f.g) = v(h.k) = v(h)k(p) + h(p)v(k) = v(f)g(p) + f(p)v(g),$$

which proves that $v \in T_pA$.

1.6 Cartesian product of differential spaces

We know from set theory that if A and B are sets, $A \times B$ denotes the Cartesian products of A with B. This is defined by the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$. Warner F. W and Isham J. C. (see [35],[18]) in their studies on differential geometry have observed the following on Cartesian products of smooth manifolds

Theorem 1.6.0.3. *[18] If M_1 and M_2 are two differentiable manifolds then the Cartesian product $M_1 \times M_2$ can be given a manifold structure in a natural way.*

Proof. Let $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ be differentiable manifolds of dimensional d_1 and d_2 respectively. Then $M_1 \times M_2$ becomes a differentiable manifold of dimension $d_1 + d_2$ (*Since* $\mathrm{U}_\alpha, \rightarrow \mathbb{R}^{\mathrm{d}_1}, \mathrm{V}_\beta \rightarrow \mathbb{R}^{\mathrm{d}_2} \implies \mathrm{U}_\alpha \times \mathrm{V}_\beta \rightarrow \mathbb{R}^{\mathrm{d}_1} \times \mathbb{R}^{\mathrm{d}_2} = \mathbb{R}^{\mathrm{d}_1+\mathrm{d}_2}$), with differentiable structure $\mathcal{F}$ the maximal collection containing

$$\{(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta : U_\alpha \times V_\beta \rightarrow \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}) : (U_\alpha, \varphi_\alpha) \in \mathcal{F}_1, (V_\beta, \psi_\beta) \in \mathcal{F}_2\}.$$

Thus this example shows that the Cartesian product of manifolds is also a manifold. The dimensions of the product manifold is the sum of the dimensions of factors. Its topology is given by the collection of the product of all charts on factors. Thus, an atlas for the product manifold can be constructed using atlases for its factors.

Similarly as a follow up to the above definition, we have the following for differential spaces which were studied by Sasin W. (see [30]) and by Ntumba P. (see [28]).

Definition 1.6.1. *([28]) Let* $(M, \mathcal{F})$ *and* $(N, \mathcal{G})$ *be non-empty d-spaces. Let* $\mathcal{F} \times \mathcal{G}$ *be the differential structure on the Cartesian product* $M \times N$*, generated by the set of real-valued functions*

$$\{f \circ \pi_1 : f \in \mathcal{F}\} \cup \{g \circ \pi_2 : g \in \mathcal{G}\},$$

where $\pi_1(p, q) = p$*,* $\pi_2(p, q) = q$ *for all* $(p, q) \in M \times N$*. The d-space* $(M \times N, \mathcal{F} \times \mathcal{G})$ *is called the Cartesian product of the d- spaces* $(M, \mathcal{F})$ *and* $(N, \mathcal{G})$*.*

Proposition 1.6.1. *The natural projections*

$$\pi_1 : (M \times N, \mathcal{F} \times \mathcal{G}) \to (M, \mathcal{F})$$

and

$$\pi_2 : (M \times N, \mathcal{F} \times \mathcal{G}) \to (M, \mathcal{G})$$

are smooth.

Proof. . Clearly, for every $f \in \mathcal{F}$, $f \circ \pi_1 \in \mathcal{F} \times \mathcal{G}$ and, for every $g \in \mathcal{G}$, $g \circ \pi_2 \in \mathcal{F} \times \mathcal{G}$.

Let $(M \times N, \mathcal{F} \times \mathcal{G})$ be the Cartesian product of differential spaces $(M, \mathcal{F})$ and $(N, \mathcal{G})$. For an arbitrary point $p \in M$, let $j_p : M \times N$ be the embedding defined by

$$j_p(q) = (p, q) \quad \text{for } q \in N.$$

In the same way, let $j_q : M \times N$, $q \in N$, the embedding defined by

$$j_q(p) = (p, q) \quad \text{for } p \in M.$$

Now we see that

$$\pi_1 \circ j_q = id_M$$

$$\pi_2 \circ j_p = id_N$$

for $p \in M$ and $q \in N$.

In addition Ntumba (see [28]) has also observed the following;

Proposition 1.6.2. *[28] Let us consider the Cartesian product* $(M \times N, \mathcal{F} \times \mathcal{G})$ *of d-spaces* $(M, \mathcal{F})$ *and* $(N, \mathcal{G})$*. For any tangent vector* $\omega \in T_{(p,q)}(M \times N)$*, let us put*

$$\omega_M = (j_q \circ \pi_1) * (p, q)^{\omega}$$

$$\omega_N = (j_p \circ \pi_2) * (p, q)^{\omega}$$

Then, we have the following:

$$\omega = \omega_M + \omega_N$$

(1)

$$\omega_M(g \circ \pi_2) = 0 \quad \textit{for any } g \in \mathcal{G}$$

(2)

$$\omega_N(g \circ \pi_1) = 0 \quad \textit{for any } g \in \mathcal{F}$$

Proof. For any $u \in \mathcal{F} \times \mathcal{G}$, we have

$$\omega_M(u) + \omega_N(u) = \omega(u \circ j_q \circ \pi_1(p, q)) + \omega(u \circ j_p \circ \pi_2(p, q)) = \omega(u)$$

Moreover, for any $g \in G$, we have

$$\omega_M(g \circ \pi_2)(p, q) = \omega(g(q)) = 0,$$

since $g(q)$ is constant which proves (2) and (3).
More generally, we have
Proposition 1.6.3. *[28]) Let* $U \in \chi(M)$ *and* $V \in \chi(N)$ *be vector fields tangent to the differential spaces* $(M, \mathcal{F})$ *and* $(N, \mathcal{G})$ *respectively. Then,we have*

1. $(j_q)_{*p} U_p(g \circ \pi_2) = 0$

2. $(j_p)_{*q} V_q(f \circ \pi_1) = 0$

3. $T_{(p,q)}(M \times N) = (j_q)_{*p}(T_p M) \oplus (j_p)_{*q}(T_q N)$.

For $(p, q) \in M \times N$*,* $f \in \mathcal{F}$*,* $g \in \mathcal{G}$*. Note that we have used, in the equations (1) and (2), the identifications* $U(p) = U_p$ *and* $V(p) = V_p$ *.*

Proof (see [28])

Definition 1.6.2. *([30]) Let* $(M, \mathcal{F})$ *and* $(N, \mathcal{G})$ *be two differential spaces. A vector* $w \in T_{(p,q)}(M \times N)$ *is said to be parallel to* $(M, \mathcal{F})$ *if* $w(g \circ \pi_2) = 0$ *for any* $g \in \mathcal{G}$*. A vector* $w \in T_{(p,q)}(M \times N)$ *is said to be parallel to* $(N, \mathcal{G})$ *if* $w(f \circ \pi_1) = 0$ *for any* $f \in \mathcal{F}$*.*
Definition 1.6.3. *[30]) A vector field* $Z \in \chi(M \times N)$ *is said to be parallel to* $(M, \mathcal{F})$ *if* $(\pi_2) * Z(p, q) = 0$ *for every* $(p, q) \in M \times N$*.*
A vector field $Z \in \chi(M \times N)$ *is said to be parallel to* $(N, \mathcal{G})$ *if* $(\pi_1) * Z(p, q) = 0$ *for every* $(p, q) \in M \times N$*.*
Definition 1.6.4 (Infinite Cartesian product). *([25]) Let* $\{(M_i, C_i)\}_{i \in I}$ *be an indexed family of differential spaces. Then the differential structure* $\Pi_{i \in I} C_i$ *generated on the Cartesian product* $\Pi_{i \in I} M_i$ *by the family* $\{f_i \circ pr_i : i \in I, f_i \in C_i\}$ *(*pr_j is the natural projection of $\Pi_{\mathrm{i} \in \mathrm{I}} \mathrm{M_i}$) *onto* M_j*) is said to be the Cartesian product of the family* $\{(M_i, C_i)\}_{i \in I}$*.*
Remark 1.6.1. *For a finite set of indices* $I = \{i_1, ..., i_k\}$*, we write* $C_{i_1} \times \cdots \times C_{i_k}$ *instead of* $\Pi_{i \in I} C_i$*. The topology* $\tau_{\Pi_{i \in I} C_i}$ *coincides with the standard topology of the Cartesian product of topological spaces.*

1.7 Differential groups

In order for us to define the differential group, we first briefly discuss Lie group as this is foundation of differential groups. The key idea of a Lie group is that it is a group in the usual algebraic sense, but with the additional property that it is also a differentiable manifold, and in such a way that the group operation and the inversion map are smooth with respect to this structure.
Definition 1.7.1. *([18]) A real Lie group, or briefly Lie group* G *is a set that is*

(a) *a group in the usual algebraic sense;*

(b) *a differentiable manifold with the properties that taking the product of two group elements, and taking the inverse of a group element, are smooth operations. Specifically, the maps*

$$\begin{aligned} \mu : G \times G &\rightarrow G \\ (g_1, g_2) &\mapsto g_1 g_2 \end{aligned}$$

and

$$\begin{aligned} i : G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

are both C^∞.

We can as well simply say that a Lie group is a differentiable manifold which is also endowed with a group structure such that the map $G \times G \to G$ *defined by* $(\delta, \tau) \mapsto \delta\tau^{-1}$ *is* C^∞.

For a complex Lie group G, one requires that G is equipped with a complex analytic structure and that multiplication and inversion are holomorphic. The following definition is equivalent to 1.7.1

Definition 1.7.2. *([35]) A Lie group* G *is a differentiable manifold which is also endowed with a group structure such that the map* $G \times G \to G$ *defined by* $(\sigma, \tau) \mapsto \sigma\tau^{-1}$ *is* C^∞.

Remark 1.7.1.

- *If we let* G *to be a Lie group then, the map* $\tau \mapsto \tau^{-1}$ *is* C^∞ *since it is the composition* $\tau \to (e, \tau) \mapsto \tau^{-1}$ *of* C^∞ *maps. Also the map* $(\delta, \tau) \mapsto \delta\tau$ *of* $G \times G \to G$ *is* C^∞ *since it is the composition* $(\delta, \tau) \mapsto (\delta, \tau^{-1}) \mapsto \delta\tau$ *of* C^∞ *maps.*

- *The identity component of a Lie group is itself a Lie group; and the components of a Lie group are mutually diffeomorphic.*

- *If* G *and* H *are Lie groups (both real or complex), then a Lie group homomorphism* $f : G \to H$ *is a group homomorphism which is a smooth map; so a continuous map.*

Example 1.7.3. (a) *The Euclidean space* $\mathbb{R}^n$ *is a Lie group under vector addition.*

(b) *The product* $G \times H$ *of two Lie groups is itself a Lie group with the product manifold structure and the componentwise product group structure; that is,* $(\delta_1, \tau_1)(\delta_2, \tau_2) = (\delta_1\delta_2, \tau_1\tau_2)$.

Accordingly, in a similar manner, we define a differential group;

Definition 1.7.4 (Differential group). *([25]) A pair* $(G, \mathcal{G})$ *is said to be differential group if and only if*

- G *is a group*

- $(|G|, \mathcal{G})$ *is a differential space, where* $|G|$ *denotes the set of elements of* G;

- *a map* $\delta : (G \times G, \mathcal{G} \times \mathcal{G}) \to (G, \mathcal{G})$, *defined by the formula*

$$\delta(g, h) := gh^{-1},$$

is a smooth map.

Equivalently, Batubenge A. and Ntumba P. (see[29], p. 74) have also studied and investigated differential groups were they gave the following;

Definition 1.7.5. *A differential group is a differential space G together with two d-smooth maps $\mu : G \times G \to G$ and $\nu : G \to G$ such that G is a group with multiplication μ and inversion ν.*

Definition 1.7.6. *([27]) A group homomorphism $G \to H$ of differential groups is called a map of differential groups provided it is a d-smooth map of the underlying differential spaces G and H. Differential groups and maps between them constitute a category $\mathbb{DIFFG}$ (see1.4.2 for more on smooth maps).*

Remark 1.7.2. *We see that a differential group is automatically a topological group (with the topology $\tau_{\mathcal{G}}$ in G) because it is a differential space.*

Example 1.7.7. *Let G be an arbitrary group. If $\mathcal{G}_o$ denotes the differential structure of all constant functions on G then $(G, \mathcal{G}_o)$ is a differential group. Similarly, if $\mathbb{R}^G$ is the differential structure of all real-valued functions on G then $(G, \mathbb{R}^G)$ is also a differential group. In the last case the topology $\tau_{\mathbb{R}^G}$ is the discrete topology on G.*

Proposition 1.7.1. *([25],[27])Let H be a group,$(G, \mathbb{G})$ - a differential group and ϕ - (an algebraic) homorphism on H into G. Then $(H, \phi^*(\mathbb{G})_H)$ is a differential group and $\phi : (H, \phi^*(\mathbb{G})_H) \to (G, \mathbb{G})$ is a smooth map.*

Proof. The smoothness of ϕ follows directly from the definition of the differential structure $\phi^*(\mathbb{G})_H$. This implies that

$$H * H \ni (g, h) \to \eta(g, h) := \phi(g)\phi(h^{-1}) \in G$$

is smooth with respect to the differential structure $\phi^*(\mathbb{G})_H \times \phi^*(\mathbb{G})_H$ and $\mathbb{G}$, respectively.

Example 1.7.8. *(see [25]) If H is a Lie group then $(H, C^\infty(H))$ is a differential group. If G is an arbitrary subgroup of H, ϕ is the natural embedding of G into H and $\mathbb{G} := \phi^*((C^\infty(H))_G = C^\infty(H)_G$ then $(G, \mathbb{G})$ is a differential group.*

Example 1.7.9. *Let $\theta : G \to Gl(n, \mathbb{R})$ be an n-dimensional matrix representation of a group $G, n \in \mathbb{N}$. By 1.7.1 the pair $(G, \theta^*(C^\infty(Gl(n, \mathbb{R})))_G)$ is a differential group. We see that the differential structure $\theta^*(C^\infty(Gl(n, \mathbb{R})))_G$ is generated by the family $\{\theta_{ij}\}_{1 \le i,j \le n}$ of all matrix* *elements of the representation θ.*

Example 1.7.10. *Let G be a locally compact, connected topological group. Let U be an arbitrary neighbourhood of the identity element of G. Then there exists a normal subgroup N of G such that $N \subset U$ and G/N is a Lie group.*

Example 1.7.11. *Let $\theta : G \to Gl(n, R)$ be an n-dimensional matrix of a group G, $n \in \mathbb{N}$, the pair $(G, \theta^*(C^\infty(Gl(n, \mathbb{R}))_G)$ is a differential group.*

Denote by ϕ the canonical map on G onto G/N. From proposition 3.1 it follows that $(G, \phi^*(C^\infty(G/N)_G)$ is a differential group.

Theorem 1.7.0.4. *([25]) Let $\mathcal{F}$ be a family of real-valued functions defined on a group G. Let $\mathcal{G} := sc\mathcal{F}_G$ be a differential structure generated by $\mathcal{F}$ on G. The pair $(G, \mathcal{G})$ is a differential group if and only if the following condition is satisfied:*
For any $f \in \mathcal{F}$ and any $(g, h) \in G \times G$, there exists a neighbourhood $U \in \tau_{\mathcal{F}}$ of g, a neighbourhood $V \in \tau_{\mathcal{F}}$ of h, mapping $\sigma \in \mathcal{F}^r, \beta \in \mathcal{F}^s$ and a function $\omega \in C^\infty(\mathbb{R}^{r+s})$ such that for each $(g', h') \in U \times V$,

$$f(g'h'^{-1}) = \omega(\sigma(g'), \beta(h')).$$

Proof. Suppose that $(G, \mathcal{G})$ is a differential group. Since the map σ is smooth we obtain that, for any $f \in \mathcal{F} \subset \mathcal{G}$, the map $f \circ \sigma \in \mathcal{G} \times \mathcal{G}$. This follows directly from the definition of $\mathcal{G}$ and $\mathcal{G} \times \mathcal{G}$,i.e, $(G \times G, \mathcal{G} \times \mathcal{G}) \xrightarrow{\sigma} (G, \mathcal{G})$.

Suppose now that $\mathcal{F}$ satisfies the condition in the definition of the differential structure. Since any function G is locally a function from $sc\mathcal{F}$, we obtain that $\mathcal{G}$ also fulfils the definition. Hence σ is a smooth map, and $(G, \mathcal{G})$ is a differential group.

1.8 Differential Subgroup

Definition 1.8.1. *([25]) Let $(G, \mathcal{G})$ be a differential group and H be any subgroup of G. The differential structure $\mathcal{G}$ satisfies the conditions in 1.7.0.4. This then means that the family $\mathcal{F} := f_{|H} : f \in \mathcal{G}$ also satisfies the condition in the theorem. Consequently the group H with the differential structure $\mathcal{H} = \mathcal{F}_H$ generated by $\mathcal{F}$ on H is a differential subgroup of $(G, \mathcal{G})$.*

Equivalently, we have;

Definition 1.8.2. *([17]) If (G, C) is a differential group with the differential structure C on G then, for any subgroup G_0 of G, the pair (G_0, C_{G_0}) is a differential group, called a differential subgroup of the differential group (G, C).*

1.9 Cartesian products of differential groups

Definition 1.9.1. *Let $(G_i, \mathcal{G}_i)_{i\in I}$ be a family of differential groups, where I is an arbitrary set of indices. Let also for any $j \in I$, $pr_j : \Pi_{i\in I}G_i \to G_j$ be the natural projections of the Cartesian product $\Pi_{i\in I}G_i$ onto G_j. For any $j \in I$, $\mathcal{G}$ the family $\mathcal{F} = \{f_j \circ pr_j : f_j \in \mathcal{G}_j, j \in I\}$ of functions on the product $\Pi_{i\in I}G_i$ satisfies the condition $(\Pi_{i\in I}G_i, \Pi_{i\in I}\mathcal{G}_i)$ is a differential group which is a direct product of the family of differential groups $(G_i, \mathcal{G}_i)_{i\in I}$.*

Example 1.9.2. *Let G be a group and $\{\theta^i\}_{i\in I}$ be an arbitrary family of matrix representations of G. For any $i \in I$, the map $\theta^i : G \to Gl(n_i, \mathbb{R})$ is a homomorphism of groups. Define the map $\theta : G \to \Pi_{i\in I}Gl(n_i, \mathbb{R})$ by the following way*

$$\theta(g) := (\theta^i(g))_{i\in I} \in \Pi_{i\in I}Gl(n_i, \mathbb{R}), g \in G.$$

It is obvious that θ is a homomorphism of G into the direct product $\Pi_{i\in I}Gl(n_i, \mathbb{R})$. By proposition 1.7.1 the pair $(G, \mathcal{G})$ where $\mathcal{G} = \theta^*[\Pi_{i\in I}C^\infty(Gl(n_i, \mathbb{R}))]_G$, is a differential group. The differential structure $\mathcal{G}$ is generated by the family $\{\theta^i_k\}_{i\in I}, i \le k, 1 \le n_i$.

1.10 The tangent space of a differential group

Definition 1.10.1. *For any $g \in G$, by the symbols L_g, R_g we shall denote the* left *and* right *multiplication in the group G, which are defined as mappings of on G such that*

$$L_g(h) := gh, \quad (1.1)$$
$$R_g(h) = hg \quad (1.2)$$

and the automorphism $ad_g(h) := ghg^{-1}$. It is obvious that

$$ad_g \equiv L_g \circ R_{g^{-1}}$$

Proposition 1.10.1. *([7], [26]) If $(G, \mathcal{G})$ is a differential group then, for any $g \in G$, the translations L_g, R_g and the automorphism ad_g are diffeomorphisms.*

Corollary 1.10.1. *Let $(G, \mathcal{G})$ be a differential group. Then, for any $g \in G$,*

$$dimT_gG = dimT_eG,$$

where e is the identity element of G.

Proof. Since L_g and R_g are diffeomorphisms, the tangent mapping $d_eL_g : T_eG \to T_gG$ (differential) proves to be an isomorphism of linear spaces.
Thus, the vector space T_gG and T_eG have same dimension.
Theorem 1.10.0.5. *If (G, C) is a differential group, then the symmetry*

$$inv(g) = g^{-1},$$

a right multiplication $R_a(g) = ga$ and a left translation $L_a(g) = ag$ are diffeomorphisms of the differential space (G, C). Moreover the group operation

$$G \times G \ni (g, h) \to A(A, h) = gh \in G$$

is a smooth mapping of the differential space $(G \times G, C \times C)$ onto the differential space (G, C).

Proof. The following mappings

1. $G \ni g \mapsto i_a(g) = (a, g) \in G \times G$

2. $G \ni g \mapsto j_a(g) = (g, a) \in G \times G$

are smooth with respect to the differential structures C and $C \times C$, respectively. Taking

$$a = e$$

we obtain that $inv = Q \circ i_e$ is a smooth mapping on (G, C). On the other hand $inv = inv^{-1}$ and this implies that the symmetry is a diffeomorphism on (G, C). Hence the mapping

$$G \times G \ni (g, h) \mapsto B(g, h) = (g, h^{-1}) \in G \times G$$

is smooth on $(G \times G, C \times C)$ and we conclude that the group operation $A = Q \circ B$ is smooth as a composition of smooth mappings. The smoothness of right and left translations follows now from equalities

$$R_g = A \circ j_a \quad \text{and } L_g = A \circ i_a.$$

Taking into account that $R_g^{-1} = R_{g^{-1}}$ and $L_g^{-1} = L_{g^{-1}}$ we obtain that right and left translations are diffeomorphisms of (G, C).

1.11 Lie Algebra of a differential group

In this section it is important before we look at the Lie algebra of a differential group to look back at the Lie algebra of a differentiable manifold as given by Warner F.W. (see [35], p. 84).

Definition 1.11.1. *Let $(G, \mathcal{G})$ be a differential group. The vector space $\mathcal{L}(G)$, over $\mathbb{R}$, of all left-invariant and smooth vector fields on G, together with the Lie multiplication $[.,.]$ is said to be the Lie algebra of $(G, \mathcal{G})$, is the pair $(\mathcal{L}(G), [.,.])$, where $[V, W] = VW - WV$ for $V, W \in \mathcal{L}(G)$.*

Proposition 1.11.1. *For any differential group $(G, \mathcal{G})$, the linear space T_eG and $\mathcal{L}(G)$ are isomorphic.*

Proof. (see [25])

Proposition 1.11.2. *Let $(G, \mathcal{G})$ and $(H, \mathcal{H})$ be differential groups. For any smooth homomorphism $f : G \to H$ (f is said to be a homomorphism of differential groups), the mapping $\mathcal{L}(f) : \mathcal{L}(G) \to \mathcal{L}(H)$, defined by*

$$\mathcal{L}(f) : \jmath_H \circ d_e f \circ \jmath_G^{-1}$$

is a homomorphism of Lie algebras. Moreover, for any smooth homomorphism $f : G \to H$ and $g : H \to Z$

$$\mathcal{L}(f) : \mathcal{L}(g) \circ \mathcal{L}(f)$$

$$\mathcal{L}(id_G) = id_{\mathcal{L}(G)}.$$

2. FRÖLICHER SPACES

Frölicher spaces were first studied by A. Frölicher who referred to them as 'smooth spaces'. Later on they were for the first time called Frölicher spaces by Cherenack P. (see[16], [9]). Let $C = C^\infty(\mathbb{R}, \mathbb{R})$ be the set of smooth maps from $\mathbb{R}$ to $\mathbb{R}$, X be a nonempty set and C_X be a subset of the set $Map(\mathbb{R}, X)$, a collection of curves c from the real line $\mathbb{R}$ to X and $\mathcal{F}_X$ be a subset $Map(X, \mathbb{R})$, a collection of real-valued scalar functions from X to $\mathbb{R}$. Now, consider the following diagram

$$\begin{tikzcd} & X \arrow[d, "f"] \\ \mathbb{R} \arrow[ur, "c"] \arrow[r, "f\circ c"'] & \mathbb{R} \end{tikzcd}$$

Definition 2.0.2. *The pair $(C_X, \mathcal{F}_X)$ is called a smooth structure or a Frölicher structure on X if the following compatibility condition is satisfied,*

$$\Gamma\mathcal{F}_X = C_X \text{ and} \tag{2.1}$$

$$\Phi C_X = \mathcal{F}_X, \tag{2.2}$$

where we denote by $\Gamma\mathcal{F}_X$ the set of all paths (contours) $(c : \mathbb{R} \to X)$, the composition of which with every $f \in \mathcal{F}_X$ is a C^∞ real function. Similarly, ΦC_X is a set of all maps $(f : X \to \mathbb{R})$ such that $f \circ c$ is a C^∞ real function for all choices of paths (contours) in C_X. A mapping $c \in C_X$ is called a structure curve, and $f \in \mathcal{F}_X$ is called a structure function.

Definition 2.0.3. *([33],[2],[9]) A Frölicher space is a triple $(X, C_X, \mathcal{F}_X)$, where $(C_X, \mathcal{F}_X)$ is a Frölicher structure and X is the underlying set.*

Equivalently, (see [29]), we have that;

Definition 2.0.4. *A Frölicher (smooth) space is a set M together with a set C_M of curves $c : \mathbb{R} \to M$ $(C_M \subseteq M^{\mathbb{R}})$ and a set $\mathcal{F}_M$ of real valued functions $f : M \to \mathbb{R}(\mathcal{F} \subseteq \mathbb{R}^M)$ such that*

- *for any $c \in C_M$ and any $f \in \mathcal{F}_M$ we have $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$.*

- *Note that the curves and functions determine each other in the following sense:*
  If $c \in M^{\mathbb{R}}$ is such that $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for any $f \in \mathbb{R}^M$ then $c \in C_M$, and if $f \in \mathbb{R}^{\mathbb{M}}$

is such that $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ *for any* $c \in C_M$ *then* $f \in \mathcal{F}_M$.

In this work, when there is no confusion, we shall mention $(M, C, \mathcal{F})$ or simply M as a Frölicher space instead of $(M, C_M, \mathcal{F}_M)$ with subscript M.

Definition 2.0.5. *Given Frölicher structures* $(C, \mathcal{F})$ *and* $(C', \mathcal{F}')$ *on a set* X*, we say that* $(C, \mathcal{F})$ *is finer than* $(C', \mathcal{F}')$ *if* $C \subset C'$ *or equivalently if* $\mathcal{F}' \subset \mathcal{F}$*. Similarly,* $(C, \mathcal{F})$ *is coarser than* $(C', \mathcal{F}')$ *if* $C' \subset C$.

In his work, Laubinger M.(see [21]) has given the following results:

Definition 2.0.6. *Let* X *be a set,* $\{(X_i, C_i, \mathcal{F}_i)_{i \in I}\}$ *be a collection of Frölicher spaces, and* $g_i : X_i \to X$ *and* $f_i : X \to X_i$ *set maps. The initial Frölicher struture with respect to the maps* f_i *is the Frölicher structure generated by all* $f \circ f_i$ *with* $i \in I$ *and* $f \in \mathcal{F}_i$*. Similarly, the final structure with respect to the maps* g_i *is the Frölicher structure generated by all* $g_i \circ c$ *with* $i \in I$ *and* $c \in C$.

The definition above is similar to 1.1.19 for differential spaces. In particular, if X is a Frölicher space and $i : A \to X$ the inclusion of a subset and $\pi : X \to B$ the projection onto a quotient, then the subset structure on A is the initial structure with respect to i, and the quotient structure on B is the final structure with respect to π.

Example 2.0.7. *The finite-dimensional smooth manifolds where if* X *is such a manifold, then* $C_X = \{c : \mathbb{R} \to X \mid \text{c is smooth}\}$ *and* $\mathcal{F}_X = \{f : X \to \mathbb{R} \mid \text{ f is smooth}\}$ *are examples of Frölicher spaces.*

Example 2.0.8. *Let* $(\mathbb{R}, \mathcal{C}, \mathcal{F})$*, where both* $\mathcal{C}$ *and* $\mathcal{F}$ *are the set* $C^\infty(\mathbb{R}, \mathbb{R})$*. The pair* $(\mathcal{C}, \mathcal{F})$ *is a smooth structure called the standard Frölicher structure on the real line, on which all smooth* (C^∞) *usual functions are smooth curves and functions in the Frölicher sense. Then,* $(\mathbb{R}, C, \mathcal{F})$ *is the standard (canonical) Frölicher space.*

Example 2.0.9. *If* $(M, C_M, \mathcal{F}_M)$ *is a Frölicher space, and* $A \subset M$ *a subset of* M*. Then* A *is a Frölicher subspace of* M.

Example 2.0.10. $M = \mathbb{R}^n$, $(M, C, \mathcal{F})$*, where* $C = C^\infty(\mathbb{R}, \mathbb{R}^n)$ *and* $\mathcal{F} = C^\infty(\mathbb{R}^n, \mathbb{R})$*,* M *is a smooth space as a smooth Manifold.*

Theorem 2.0.0.6 (Boman's Theorem). *([20]) Let* $f \in \mathrm{map}(\mathbb{R}^n, \mathbb{R})$ *be such that* $f \circ c$ *is* C^∞ *whenever* $c : \mathbb{R} \to \mathbb{R}^n$ *is* C^∞*, then* f *is* C^∞.

Example 2.0.11. *We show that the canonical structure of* $\mathbb{R}$ *is generated by* $\mathcal{F}_0 = \{id_{i\mathbb{R}}\}$

$$\mathcal{F}_0 \xrightarrow{\Gamma} \Gamma\mathcal{F}_0 \xrightarrow{\Phi} \Phi\Gamma\mathcal{F}_0$$

$$\begin{aligned} \Gamma\mathcal{F}_, &= \{c : \mathbb{R} \to \mathbb{R} : f \circ c \in C^\infty \quad \text{for all} \quad f \in \mathcal{F}_0\} \\ &= \{c : \mathbb{R} \to \mathbb{R} : c \in C^\infty(\mathbb{R}, \mathbb{R})\} \\ \Phi\Gamma\mathcal{F}_0 &= \{f : \mathbb{R} \to \mathbb{R} : f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \quad \text{for all} \quad c \in C^\infty(\mathbb{R}, \mathbb{R})\} \\ &= C^\infty(\mathbb{R}, \mathbb{R}) \end{aligned}$$

For, if $f \notin C^\infty(\mathbb{R}, \mathbb{R})$*, then particular choice* $c = id_{\mathbb{R}}$ *will yield a contradiction.*

This first example above appears then as an immediate consequence of Boman's theorem. That is:

Corollary 2.0.1. *If* M *is a smooth finite-demensional manifold, then*

$$(C^\infty(\mathbb{R}, M), C^\infty(M, \mathbb{R}))$$

is a Frölicher structure on M*.*

It is important to mention here that there are some Frölicher spaces that are not smooth manifolds, see[29]. The following are some of the examples of Frölicher spaces;

Example 2.0.12. $\mathbb{K} = \{(x, y) \in \mathbb{R}^2 | xy = 0\} = \{(x, 0) : x \neq 0\} \cup \{(0, y) : y \neq 0\}$ *is a Frölicher space as a subset of the Frölicher space* $\mathbb{R}^2$*. It is not a smooth manifold.*

Example 2.0.13. $G = \{(x, |x|), x \in \mathbb{R}\}$ *is the graph of the absolute value function in* $\mathbb{R}$ *such that* $x \mapsto |x|$ *is a Frölicher subspace of* $\mathbb{R}^2$ *which is not a smooth manifold.*

2.1 Frölicher subspace

Definition 2.1.1. *(see [37], [38]) Let* $(X, C_X, \mathcal{F}_X)$ *be a Frölicher space and* A *a subset of* X*. Then, the inclusion* $i_A : A \to X$ *places an initial structure on* A*, where the resulting Frölicher space is* $(A, C_A, \mathcal{F}_A)$ *with*

- $C_A = \{c : \mathbb{R} \to A | i_A \circ c \in C_X\}$

- $\mathcal{F}_A = \Phi\Gamma\{f \circ i_A | f \in \mathcal{F}_X\}$.

With this structure A is called a Frölicher subspace of X.

Example 2.1.2. *Let $X = \mathbb{R}$ and $A = \mathbb{Q}$ denote the rationals. Then, C_A consists of the constant maps and $\mathcal{F}_A$ thus consists of all functions, and then has the discrete topology. We call it a discrete Frölicher space.*

2.2 Frölicher smooth maps

Definition 2.2.1. *([1], [29] and [34]) A map $\varphi : (M, \mathcal{C}_M, \mathcal{F}_M) \longrightarrow (N, \mathcal{C}_N, \mathcal{F}_N)$ between Frölicher spaces is termed smooth if it satisfies one of the following equivalent conditions:*

(1) $\varphi \circ c \in \mathcal{C}_N$ for all $c \in \mathcal{C}_M$

(2) $\varphi \circ f \in \mathcal{F}_M$ for all $f \in \mathcal{F}_N$

(3) $h \circ \varphi \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $h \in \mathcal{F}_N$

Equivalently (see [37]) we have that;

Definition 2.2.2. *Let $(X, C_X, \mathcal{F}_X)$ and $(Y, C_Y, \mathcal{F}_Y)$ be Frölicher spaces. Let $F : X \to Y$ be a map. Then F is Frölicher smooth if for every $f \in \mathcal{F}_Y$, $f \circ F \in \mathcal{F}_X$.*

Proposition 2.2.1. *The composite of two Frölicher smooth maps is also a Frölicher smooth map.*

Proof. If M, N and P are Frölicher spaces and $\varphi_1 : M \to N$ and $\varphi_2 : N \to P$ are Frölicher smooth maps then for some $f \in \mathcal{F}_P$ and for some $c \in C_M$, the composite $f \circ (\varphi_2 \circ \varphi_1) \circ c = (f \circ \varphi_2) \circ (\varphi_1 \circ c)$ is in $C^\infty(\mathbb{R}, \mathbb{R})$ since $(f \circ \varphi_2) \in \mathcal{F}_N$ and $(\varphi_1 \circ c) \in C_N$. That is, $(\varphi_2 \circ \varphi_1) : M \to P$ is a Frölicher smooth map.

Proposition 2.2.2. *Let M be a Frölicher space. The structure curves in M and the structure functions on M are Frölicher smooth.*

Proof. Let us consider the canonical Frölicher structure on $\mathbb{R}$. Each structure curve in M is a map $c : \mathbb{R} \to M$ such that for all $f \in \mathcal{F}_M$, $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) = \mathcal{F}_\mathbb{R}$. And each structure function on M is a map $f : M \to \mathbb{R}$ such that for all $c \in C_M$, $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) = C_\mathbb{R}$. Hence, according to definition 2.2.1 the structure function f is Frölicher smooth. So is each structure curve c.

Remark 2.2.1. *Using the same notation as in the definition above, note that for any $c \in C_X$, we have $F \circ c \in C_Y$. Indeed, for every $f \in \mathcal{F}_Y$, we have $f \circ F \in \mathcal{F}_X$. Hence $F \circ c \in \Gamma\mathcal{F}_Y = C_Y$. Moreover, $F : X \to Y$ is Frölicher smooth if and only if for any $C \in C_X$, we have $F \circ c \in C_Y$.*

Frölicher spaces and maps between them form a category, which we denote $\mathbb{FRL}$, which has the following properties as shown by the works done by Frölicher and Kriegl[16] and Cherenack[10], these further discussed by Batubenge and Tshilombo (see [5]):

1. It is complete, that is, arbitrary limits exist. The underlying set is formed as in the category of sets as a subset of the Cartesian product, and the smooth Frölicher structure is generated by smooth functions on the factors.

2. It cocomplete, that is, arbitrary colimits exist. The underlying set is formed as in the category of sets, that is, as a certain quotient of the disjoint union, and the smooth functions are exactly those which induce smooth functions on the factors.

3. It is Cartesian closed. That is, for any Frölicher spaces X, Y, and Z , the set $C^\infty(Y, Z)$ of all $\mathbb{FRL}$ - morphism from Y to Z carries a canonical smooth Frölicher structure following an exponential law:

   $$C^\infty(X \times Y, Z) \cong C^\infty(X, C^\infty(Y, Z)).$$

   Now if $X = \mathbb{R}$ in this formula,we construct the set $C_{Y,Z}$ of curves $C : \mathbb{R} \to C^\infty(Y, Z)$ by requiring that the map $\tilde{c} : \mathbb{R} \times Y \longrightarrow Z$ where $\tilde{c}(t, y) := c(t)(y)$, is smooth. Then using the functors Φ and Γ , we shall generate a Frölicher structure $C^\infty(Y, Z)$, the proof for this was done by Laubinger M (see[20]).

4. It is topological over **Set**. (see[10],[21]). That is, the category $\mathbb{FRL}$ behaves like the category of topological spaces . One defines induced $\mathbb{F}$-structures on new sets constructed on **Set** and induces topologies. So, quotients, subsets, products and co-products exist in $\mathbb{FRL}$ as limits or colimits lifted from the category of sets.

Proposition 2.2.3. *$\varphi : M \to N$ is said to be a smooth map of Frölicher spaces ($\mathbb{F}$-smooth) if $\varphi \circ c \in C_N$ for all $c \in C_M$.*

Proposition 2.2.4. *[34] Let M be a Frölicher space. The identity map id_M on M is Frölicher smooth map.*

Proof. One considers the fact that $\varphi = id_M$ and $N = M$ in definition 2.2.1. Hence the proposition holds.

Definition 2.2.3. *An $\mathbb{F}$-diffeomorphism or diffeomorphism between Frölicher spaces is that smooth map which has a smooth inverse.*

Proposition 2.2.5. *([2]) Let $(M, C_M, \mathcal{F}_M)$ be a Frölicher space. Consider a set N and assume that $F : (M, C_M, \mathcal{F}_M) \to N$ is an injective mapping. Then there exists on the image $F(M) \subseteq N$ a Frölicher structure making F an $\mathbb{F}$ diffeomorphism of M onto $F(M)$.*

A. Batubenge and Ntumba P. (see [29]) have shown the following two results

Proposition 2.2.6. *Let $(X, C_X, \mathcal{F}_X)$ be a Frölicher space, and let Y be a set, and let $S = \{f_i : X \to Y, i \in I\}$ be a family of set map $\varphi : X \to Y^I$ by setting*

$$\varphi(x) = (f_i(x))_I.$$

If φ is one-to-one, then $(X, C_X, \mathcal{F}_X)$ is diffeomorphic to the subspace $\varphi(X)$ of the Frölicher space Y^I ($\mathrm{Y^I} = \prod_\mathrm{I} \mathrm{Y}$, where Y is the Frölicher space whose structure is the structure coinduced by the family S).

Proof. First note that the structure on Y is generated by the family

$$F_0 = \{f : Y \to \mathbb{R} | f \circ f_i \in \mathcal{F}_X \quad \text{for all } i \in I\},$$

and the structure on Y^I has as generating set the family

$$\{g \circ \pi_i : g \in F_0, i \in I\}.$$

Since $g \circ \pi_i \circ \varphi(x) = g \circ \pi_i((f_i(x)) = g \circ f_i(x)$; then φ is smooth.

Now consider $\varphi^{-1} : \varphi(X) \to X$. Curves on $\varphi(X)$ have the form $c(t) = (f_i \circ \tilde{c}(t))_I$, where $\tilde{c} : \mathbb{R} \to X$ is a structure of X. It follows clearly that

$$\varphi^{-1}((f_i \circ (\tilde{t})) = \tilde{c}.$$

that is φ^{-1} is smooth. As a straightforward consequence (see [29]), to Proposition 3.0.6, one has

Corollary 2.2.1. *Let X and Z be Frölicher spaces, and let Y be a set, and let* $\mathbb{S} = \{f_1, f_2, ..., f_n : X \to Y\}$ *and* $\mathbb{S}' = \{g_1, ..., g_m : Z \to Y\}$ *be families of set maps. Suppose that* $\varphi := (f_1, f_2, ..., f_n)$ *and* $\psi := (g_1, ..., g_m)$ *are one-to-one maps* $X \to Y^n$ *and* $Z \to Y^m$ *respectively. Then, the map* $\alpha : X \times Z \to Y^{m+n}$ *denoted by*

$$\alpha = (f_1 \circ \pi_1, ..., f_n \circ \pi_1, g_1 \circ \pi_2, ..., g_m \circ \pi_2)$$

is one-to-one, and the product space $X \times Z$ *is diffeomorphic to the subspace* $\alpha(X \times Z)$ *of the Frölicher space* Y^{m+n} (every Y in the product Y^{m+n} is a Frölicher space whose structure is coinduced by the family $\{f_1 \circ \pi_1, ..., f_n \circ \pi_1, g_1 \circ \pi_2, ..., g_m \circ \pi_2\}$).

In (see [29] and [7]),we have the following results;

Theorem 2.2.0.7. *Let Y be a Frölicher space, and the pair* $(C_X, \mathcal{F})$ *the Frölicher structure induced on the set X via maps* $f_i : X \to Y, i \in I$. *Assume that the map* $\varphi : X \to Y^I$, *given by* $\varphi(x) = (f_i(x))_I$, *is one-to-one. Then* φ *is a diffeomorphism onto the subspace* $\varphi(X)$ *of* Y^I.

Proof. . Let $c : \mathbb{R} \to X$ be a curve on X. Then

$$\varphi \circ c(t) = (f_i \circ c(t))_I$$

for all $t \in \mathbb{R}$. Since the structure Y^I is generated by the family $\{g \circ \pi_i : g \in \mathcal{F}_Y, i \in I\}$, it follows that $\varphi \circ c : \mathbb{R} \to \varphi(X)$ is a smooth curve on $\varphi(X)$. Hence φ is smooth.

Now, Let $(x_i)_I \in \varphi(X)$. It is clear that

$$g \circ f_i \circ \varphi^{-1}((x_i)_I) = g \circ \pi_i \circ \varphi \circ \varphi^{-1}((x_i)_I) = g \circ \pi_i((x_i))_I),$$

$\pi \circ f = f_i$ by assumption. It follows that φ^{-1} is smooth, and the proof is finished.

Corollary 2.2.2. *Let M be a set, and let* $f_1, f_2, ..., f_n : X \to \mathbb{R}$ *be real-valued functions on M such that the map* $\varphi : M \to \mathbb{R}^n$, $\varphi(x) = (f_1(x), f_2(x), ..., f_n(x))$, *is one-to-one. If* $(C_M, \mathcal{F}_M)$ *is a Frölicher structure generated by the* $\{f_1, f_2, ..., f_n\}$ *then* φ *is a diffeomorphism onto the subspace* $\varphi(M)$ *of* $\mathbb{R}^n$.

2.3 Smooth structure generated by a set

Recall from definition 2.0.2 the operators Γ and Φ. We have that the set F_0 is called a generating set of functions for the Frölicher space $(X, \Gamma F_0, \Phi\Gamma F_0)$. Analogously a set C_0 of maps $\mathbb{R} \to X$ on a set X generates a Frölicher space $(X, C_X, \mathcal{F}_X)$, where

- $\mathcal{F}_X := \Phi C_0 = \{f : X \to \mathbb{R} | f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \quad \text{for all } c \in C_0\}$
- $C_X = \Gamma \mathcal{F}_X = \Gamma\Phi C_0$

Note that for any family of functions $\mathcal{F}_0$ from X to $\mathbb{R}$ and family of functions C_0 from $\mathbb{R}$ to X, we have

$$C_0 \subseteq \Gamma\Phi C_0, \quad \text{and } \mathcal{F}_0 \subseteq \Gamma\mathcal{F}_0$$

These facts imply that

$$\Phi\Gamma C_0 = \Phi C_0 \quad \text{and that } \Gamma\Phi\Gamma\mathcal{F}_0 = \Gamma\mathcal{F}_0.$$

Definition 2.3.1. *([1],[29],[5],[34]) Let $\mathcal{F}_M$ be the set of all $f \in \mathbb{R}^M$ such that $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $f \in \mathcal{F}_M$. Then $(C_M, \mathcal{F}_M)$ is a Frölicher structure on M generated by the family C of curves.*
Dually we define the smooth structure on M generated by a set of functions $\mathcal{F} \subseteq \mathbb{R}^M$.

For the two sets $\mathscr{P}(M^{\mathbb{R}})$ and $\mathscr{P}(\mathbb{R}^M)$, we have that;

- if $C \in \mathscr{P}(M^{\mathbb{R}})$, then $\Phi C = \{f : M \to \mathbb{R}/f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } c \in C\}$
- if $\mathcal{F} \in \mathscr{P}(\mathbb{R}^M)$, then $\Gamma\mathcal{F} = \{c : \mathbb{R} \to M/f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all } \ f \in \mathcal{F}\}$

Therefore,

1. $\Gamma\Phi C = \{c : \mathbb{R} \to M/f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}), \text{ for all f} \in \Phi\text{C}\}$
2. $\Phi\Gamma F = \{f : M \to \mathbb{R}/f \circ c \in C^\infty(\mathbb{R}, \mathbb{R}), \text{ for all c} \in \Gamma\mathcal{F}\}.$

Definition 2.3.2. *The pair $(\Gamma\mathcal{F}, \Phi\Gamma\mathcal{F})$ is called the Frölicher structure generated by $\mathcal{F}$ and the pair $(\Gamma\Phi C, \Phi C)$ is the Frölicher structure generated by C.*

The following lemma states that the operations Γ and Φ are inclusion-reversing.

Lemma 2.3.3. *([34]) Let C_1, C_2 be subsets of $M^{\mathbb{R}}$ and $\mathcal{F}_1$, $\mathcal{F}_2$ subsets of $\mathbb{R}^M$. We have*

1. *If* $C_1 \subseteq C_2$ then $\Phi C_2 \subseteq \Phi C_1$,

2. If $\mathcal{F}_1 \subseteq \mathcal{F}_2$ then $\Gamma\mathcal{F}_2 \subseteq \Gamma\mathcal{F}_1$.

Proof.

1. Let $f \in \Phi C_2$. We have by definition 2.0.4 that $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $c \in C_2$. In particular, $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $c \in C_1$ since $C_1 \subseteq C_2$. Thus, $f \in \Phi C_1$.

2. Let $c \in \Gamma\mathcal{F}_2$. Then by definition 2.0.4, $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $f \in \mathcal{F}_2$. Since $\mathcal{F}_1 \subseteq \mathcal{F}_2$, we have particularly $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $f \in \mathcal{F}_1$. That is $c \in \Gamma\mathcal{F}_1$.

Remark 2.3.1. *From the Lemma 2.3, one can observe that the functors Γ and Φ are order reversing . As a consequence a small set generates a richer Frölicher structure.*

Proposition 2.3.1. *([34]) Let M be a set. Let $\mathcal{F}_0$ be a subset of $\mathbb{R}^M$ and C_0 a subset of $M^{\mathbb{R}}$. Then*

1. $\mathcal{F}_0 \subseteq \Phi\Gamma\mathcal{F}_0$

2. $C_0 \subseteq \Gamma\Phi C_0$

Proof. (see [34])

Proposition 2.3.2. *The following identities hold for the functors Φ and Γ :*

1. $\Gamma\Phi\Gamma = \Gamma$

2. $\Phi\Gamma\Phi = \Phi$

Proof.

1. Let $\mathcal{F}_0$ be a subset of $\mathbb{R}^M$. From Proposition 2.3.1 (1), it yields $\mathcal{F}_0 \subseteq \Phi\Gamma\mathcal{F}_0$. Applying Lemma 2.3.3 to $\mathcal{F}_0 \subseteq \Phi\Gamma\mathcal{F}_0$, we obtain $\Gamma\Phi\Gamma\mathcal{F}_0 \subseteq \Phi\Gamma\mathcal{F}_0$. Since $\Gamma\mathcal{F}_0$ is a subset of $M^{\mathbb{R}}$, proposition 2.3.3 (2) gives $\Gamma\mathcal{F}_0 \subseteq \Gamma\Phi\Gamma\mathcal{F}_0$. From the two inclusions $\Gamma\Phi\Gamma\mathcal{F}_0 \subseteq \Gamma\mathcal{F}_0$ and $\Gamma\mathcal{F}_0 \subseteq \Gamma\Phi\Gamma\mathcal{F}_0$, it follows that $\Gamma\Phi\Gamma\mathcal{F}_0 = \Gamma\mathcal{F}_0$ for any subset $\mathcal{F}_0$ of $\mathbb{R}^M$. Thus, $\Gamma\Phi\Gamma = \Gamma$.

2. Similarly, let C_0 be a subset of $M^{\mathbb{R}}$. From proposition 2.3.1 (2), we obtain $C_0 \subseteq \Gamma\Phi C_0$. By lemma 2.3.3 (1), it follows from $C_0 \subseteq \Gamma\Phi C_0$ that $\Phi\Gamma\Phi C_0 \subseteq \Phi C_0$. By definition 2.0.4 , ΦC_0 is a subset of $\mathbb{R}^M$. Then proposition 2.3.1 (1) gives $\Phi C_0 \subseteq \Phi\Gamma\Phi C_0$. The two inclusions $\Phi\Gamma\Phi C_0 \subseteq \Phi C_0$ and $\Phi C_0 \subseteq \Phi\Gamma\Phi C_0$ imply that $\Phi\Gamma\Phi C_0 = \Phi C_0$ for any subset C_0 of $M^{\mathbb{R}}$. Thus, $\Phi\Gamma\Phi = \Phi$.

Remark 2.3.2. *We can interpret proposition 2.3 as that, one is able to obtain more than one Frölicher structure on a set. But the Frölicher structure generated by a fixed set $\mathcal{F}_0$ or C_0 is unique.*

2.4 Topological properties of Frölicher spaces

For some reasons as from its structure which is a pair of input and output mappings, a Frölicher space $(M, \mathcal{C}_M, \mathcal{F}_M)$ carries two natural topologies induced by functions and curves as follows:

Definition 2.4.1. *1. $\tau_{\mathcal{F}} = \{O \subset M : O = \cup_{f\in\mathcal{F}} f^{-1}(I); I \in \tau_{\mathbb{R}}\}$ which is a functional topology or initial topology on M. More generally (see [5]) it is the topology induced by functions which is the collection of all subsets O that are pre-images $f^{-1}(V)$, for $f \in \mathcal{F}_M$, of open sets V of the standard topology $\Gamma_{\mathbb{R}}$ of $\mathbb{R}$.*

2. $\tau_{\mathcal{C}} = \{U \subseteq M : c^{-1}(U) \in \tau_{\mathbb{R}}; c \in \mathcal{C}\}$ which is a curvaceous topology or final topology on M. Equivalently (see [5])

We note that since the composite of each function with each curve is a C^∞ real function, it is noticed that the functional topology is the weakest one in which all maps are continuous. Furthermore, smooth maps in general , smooth curves and functions in particular are continuous irrespective of topologies.

Proposition 2.4.1. *([4]) $\tau_{\mathcal{F}} \subseteq \tau_{\mathcal{C}}$ is the weakest topology on M such that all functions and curves are continuous.*

Proof. Fix O in $\tau_{\mathcal{F}}$, that is $O = \cup_{f\in\mathcal{F}} f^{-1}(I)$ such that $I \in \tau_{\mathbb{R}}$.

$$\begin{aligned} \text{Now } \mathrm{c}^{-1}(\mathrm{O}) &= c^{-1}(\cup_{f\in\mathcal{F}} f^{-1}) \\ &= \cup_{f\in\mathcal{F}}(c^{-1}(f^{-1}(I)) \\ &= \cup_{f\in\mathcal{F}}(f \circ c)^{-1}(I). \end{aligned}$$

Therefore $O \in \tau_{\mathcal{C}}$ and thus, $\tau_{\mathcal{F}} \subseteq \tau_{\mathcal{C}}$.

But $f \circ c$ is C^∞, so continuous. I is open in $\mathbb{R}$, and $I \in \tau_{\mathbb{R}}$.

Lemma 2.4.1. *If $\varphi : M \to N$ is $\mathbb{F}$ -smooth, then it is continuous in both $\tau_{\mathcal{F}}$ and $\tau_{\mathcal{C}}$.*

2.5 Tangent spaces of Frölicher spaces

We recall that given a Frölicher space $(X, C_X, \mathcal{F}_X)$, $\mathcal{F}_X$ consists of all smooth maps $X \to \mathbb{R}$, where $\mathbb{R}$ is the canonical Euclidean Frölicher space. Since $\mathbb{FRL}$ is Cartesian closed (see [10], [14]), the collection $\mathcal{F}_X$ of structure functions of the Frölicher space X can be made into a Frölicher space, in a way that if $C_{X,\mathbb{R}}$ is the set of structure curves : $\mathbb{R} \to \mathcal{F}_X$, then $c \in C_{X,\mathbb{R}}$ provided that there exists a smooth map $\tilde{c} : \mathbb{R} \times X \to \mathbb{R}$, given by $\tilde{c}(t, x) = c(t)(x)$. Now let D_X denote the set of all smooth maps $v : \mathcal{F}_X \to X$ with properties that v is linear and

$$v(f.g) = f(p)v(g) + g(p)v(f).$$

i.e. v is a derivation at $p \in X$, then regarding D_X as a Frölicher subspace of $\mathbb{FRL}(\mathcal{F}_X, \mathbb{R})$, we have that;

Definition 2.5.1. *(see [9], [10], [29])*

1. *The tangent bundle TX on X is the Frölicher subspace of $X \times D_X$ consisting of all pairs (p, v) such that v is a derivation at p.*

2. *The set T_pX of all derivations to the space X at p is linear over $\mathbb{R}$ and is called the tangent space to X at p.*

3. *Fix $c \in C_X$, and suppose that $c(t_0) = p$ for some $t_0 \in \mathbb{R}$ and we let a map $v_c : \mathcal{F}_X \to \mathbb{R}$ be a derivation defined by*

$$v_c(f) = \lim_{t \to t_0} \frac{(f \circ c)(t) - (f \circ c)(t_0)}{t - t_0}$$

   If we set $TC_pX = \{v_c : c \in C_X : c(t_0) = p\}$, then TC_pX is called the tangent cone to X at p.

4. *The tangent cone bundle TCX on the Frölicher space X is the Frölicher subspace of $X \times \mathbb{FRL}(\mathcal{F}_X, \mathbb{R})$ consisting of all pairs (p, v_c) such that there exists a $t_0 \in \mathbb{R}$ with $c(t_0) = p$ and v_c is the derivation given in the equation above.*

Equivalently (see [33], [10], [5], [21]) we have that;

Definition 2.5.2. *The tangent bundle TX (* tangent cone bundle TCX*) on X is the Frölicher subspace of $X \times \mathcal{D}_X$ (resp., $X \times \mathcal{D}_{X,c}$) consisting of all (p, D) such that D is a derivation at p. The projection map $\pi : TX \to X$ ($\prod : TXC \to X$) is the smooth map sending (p,D) to p. A vector field on X is (most properly) a section of $\prod$ or (more generally) π.*

Lemma 2.5.1. *Let $\varphi : X \to Y$ be a map of Frölicher spaces X and Y. Then the following canonical mappings are smooth:*

1. $\tilde{\varphi} : \mathcal{F}_Y \to \mathcal{F}_X$, $\tilde{\varphi}(\beta) = \beta \circ \varphi$,
2. $\chi : \mathbb{FRL}(X, Y) \to \mathbb{FRL}(\mathcal{F}_Y, \mathcal{F}_X), \chi(f) = \tilde{f},\ \tilde{f}(\beta) = \beta \circ f.$

Proof (see [14])

Lemma 2.5.2. *Let $\varphi : X \to Y$ be a smooth map of Frölicher smooth spaces X and Y. Then for all $x \in X$, the associated tangent mapping $\varphi * x : T_{\varphi(*)}Y$ is smooth.*

Proof (see [14])

Example 2.5.3. *The rationals as Frölicher subspace of $\mathbb{R}$ have trivial tangent spaces to their tangent cones: since contours must have constant values, the tangent cones must be trivial. Let $q \in \mathbb{Q}$. The function $f : \mathbb{Q} \to \mathbb{R}$ such that $f(q) = 1$ and f(r) = 0 if $r \neq q$ belongs to $\mathcal{F}_{\mathbb{Q}}$. Since $f^2 = f$, one can show that, for any derivation D at q, D(f) = 0. Let $g \in \mathcal{F}_{\mathbb{Q}}$ and g' = fg. Then, D(g')= f(g)D(g). Since $g(q)g' = (g')^2$, D(g') = 0 and thus D(g) = 0. Hence, the tangent space at q is trivial.*

Example 2.5.4. *Except at (0,0) the Frölicher curve c above has a one-dimensional tangent space (= tangent cone). At (0, 0) the tangent is trivial (as in 2.0.0.6).*

Example 2.5.5. *Let B be the Frölicher subspace of $\mathbb{R}^2$ defined by xy = 0. The tangent cone to B agrees with the tangent space and is one-dimensional except at (0, 0) where the tangent cone is B and the space is $\mathbb{R}^2$. A scalar on B is a function $f : B \to \mathbb{R}$ such that f is smooth on the x- and y-axes, respectively.*

Example 2.5.6. *$\mathbb{Q}$ as a differential subspace of $\mathbb{R}$ has scalars $f : \mathbb{Q} \to \mathbb{R}$ which are locally in the usual topology the restrictions of locally smooth functions on $\mathbb{R}$. Thus, the tangent space to a point $q \in \mathbb{Q}$ is the same as the tangent space when q is regarded as a point of $\mathbb{R}$ and one-dimensional.*

2.6 Frölicher Lie groups

In their works 'On the Way to Frölicher Lie Groups', Ntumba P. and Batubenge A. (see[29], p. 81-90) have investigated and shown what follows.

Definition 2.6.1. *Assume that G is a group with identity element e. A triple $(G, C, \mathcal{F})$ is called a Frölicher Lie group if:*

- *$(G, C, \mathcal{F})$ is a Frölicher space*
- *the mapping $\sigma : G \times G \to G$ given by*

$$\sigma(x, y) = xy^{-1}$$

is smooth.

In this mapping σ, we are assuming that the space $G \times G$ is equipped with the product structure.

Lemma 2.6.1. *The condition that the map $\sigma : G \times G \to G, \sigma(x, y) = xy^{-1}$, be smooth is equivalent to requiring that the product map $\mu : G \times G \to G$ and the inversion map $i : G \to G$, given respectively by*

$$\mu(x, y) = xy, \quad i(x) = x^{-1},$$

be smooth maps.

Proof. . Clearly, $i = \sigma(e, -)$, where e is the identity element of G. Let $c \in C$, and $f \in \mathcal{F}$; then for $t \in \mathbb{R}$, one has

$$f \circ i \circ c(t) = f \circ \sigma(e, c(t))$$

Since the map $e : \mathbb{R} \to G$, e(t) = e for all $t \in \mathbb{R}$, is a curve into G, it follows that $f \circ \sigma(e, -) : \mathbb{R} \to \mathbb{R}$ is smooth. Therefore, i is smooth.

Now, consider the map $Id \times i : G \times G \to G \times G$, where $Id : G \to G$ is the identity map. Since $\mu = \sigma Id \times i$, it follows that μ is smooth.

Conversely, assume that μ and i are smooth. Since $\sigma = \mu \circ Id \times i$, it follows that σ is smooth.

Now if G and H are Frölicher Lie groups, an $\mathbb{F}$-map $\varphi : G \to H$ is a smooth map of Frölicher

Lie group provided φ is a homomorphism of groups.

Proposition 2.6.1. *The category* **FrLiG** *of Frölicher Lie groups has initial and final structures to the forgetful functor* U : **FrLiG** $\to$ **Grp**.

Proof (see [29])

Example 2.6.2. *Let G be an arbitrary group, and let F_0 denote the collection of all constant real-valued functions on G. Since F_0 is contained in $\mathcal{F}$ for every Frölicher structure $(C, \mathcal{F})$ on G, it follows that the set ΓF_0 of curves consists of all functions $\mathbb{R} \to G$, and $\Phi\Gamma F_0 = F_0$. Hence the triple $(G, \Gamma F_0, \Phi\Gamma F_0)$ is a Frölicher Lie group.*

Example 2.6.3. *Let $F_0 = \mathbb{R}^G$, where G is a group. It is clear that the Frölicher structure generated by F_0 on G is the pair $(C, \mathcal{F})$, where $\mathcal{F} = F_0$ and C consists of all constant maps $\mathbb{R} \to G$. It is equally evident that the collection of structure functions on $G \times G$ is the set of all real-valued functions on $G \times G$. Since any curve $c : \mathbb{R} \to G \times G$ is of the form $c = (c_1, c_2)$, where c_1 and c_2 are curves on G, it follows that $\sigma \circ c : \mathbb{R} \to G$ is smooth. Consequently σ is smooth, and thus the Frölicher space G is a Frölicher Lie group.*

Example 2.6.4. *Finite dimensional smooth manifolds form an important class of Frölicher spaces where if X is such a manifold, then C_X is the set of all smooth maps $\mathbb{R} \to X$ and $\mathcal{F}_X$ consists of all smooth maps $X \to \mathbb{R}$. Moreover, if a smooth manifold X is a Lie group, then the triple $(X, C_X, \mathcal{F}_X)$ is a Frölicher Lie group.*

2.7 Left invariant vector fields on Frölicher Lie groups

The purpose of this section is to reformulate results presented in Lie groups and differential spaces and associate them with Frölicher spaces. This endeavor emanates from the close relationship that links differential spaces (in the sense of Sikorski) to Frölicher spaces.

Definition 2.7.1. *([28], [5]) For any $g \in G$, notations L_g and R_g are defined as mappings $G \to G$ such that*

- $L_g(h) = gh$,
- $R_g(h) = hg$,

for $h \in G$, and are called left and right translations respectively.

Lemma 2.7.1. *Let $\eta \in T_eG$ and let α be a smooth function on G. Then the function $f : G \to \mathbb{R}$, given by*

$$f(g) = \eta(\alpha \circ L_g),$$

is a smooth function.

Proof Let $\lambda : G \to C^\infty(G, \mathbb{R})$ be a map given by

$$\lambda(g) = \alpha \circ L_g,$$

for all $g \in G$. If $c : \mathbb{R} \to G$ is a curve into G, then

$$(\lambda \circ c)(t)(g) = \alpha \circ \mu(c(t), g).$$

Next define the map $\hat{c} : \mathbb{R} \times G \to \mathbb{R}$ by setting

$$\hat{c}(t, g) = \alpha \circ \mu(c(t), g).$$

It is clear that $\hat{c}$ is smooth as a curve on $C^\infty(G, \mathbb{R})$. Hence λ is smooth. Since tangent vectors are smooth mappings, it follows that the function f is smooth.

3. INFINITE CARTESIAN PRODUCTS OF SMOOTH SPACES

3.1 Products of differential spaces and of Frölicher spaces

In this discussion we will make reference to the discussion of Cartesian products of differential spaces and differential groups in sections 1.6 and 1.9.

Definition 3.1.1. *([25], [27],[30]) Let (M, C) and (N, D) be differential spaces. By $C \times D$ we denote the differential structure on $M \times N$ generated by the set*

$$\{\alpha \circ pr_M : \alpha \in C\} \cup \{\beta \circ pr_N : \beta \in D\},$$

where $pr_M : M \times N \longrightarrow M$ and $pr_N : M \times N \longrightarrow N$ are canonical projections on M and N, respectively. The pair $(M \times N, C \times D)$ is called the Cartesian product of differential spaces (M, C) and (N, D).

Now for differential groups in particular, an arbitrary Cartesian product $G = \times_{i \in I} G_i$ has differential structure denoted by $F(G)$ generated by the set $\cup_{i \in I} pr_i^*(F(G_i))$ where $pr_i^*(F(G_i))$ is the differential structure generated on each differential group G_i, for an indexed family $(G_i)_{i \in I}$ of differential groups. Then the infinite product of differential groups is written as $(G, F(G))$, where $F(G) = \times_i F_i$.

For Frölicher spaces, the smooth structure consists of a pair of structure curves and structure functions, and the above translates as follows.

Let $(M, \mathcal{C}_M, \mathcal{F}_M)$ and $(N, \mathcal{C}_N, \mathcal{F}_N)$ be Frölicher spaces such that the structure $\mathcal{C}_M$ is generated by a set $\mathcal{F}_0$ of functions $\alpha : M \longrightarrow \mathbb{R}$ and $\mathcal{C}_N$ is generated by a set $\mathcal{G}_0$ of functions $\beta : N \longrightarrow \mathbb{R}$. To get a Frölicher structure on the Cartesian product $M \times N$ we

1. First form the generating set, that is,

$$F_0 = \{\alpha \circ pr_M; \alpha \in \mathcal{F}_M\} \cup \{\beta \circ pr_N; \beta \in \mathcal{F}_N\}.$$

2. Next, generate structure curves for the product. That is,

$$\Gamma F_0 = \{c : \mathbb{R} \longrightarrow M \times N;\ f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})\},$$

   with f being of the form $f = \alpha \circ pr_M$ or $f = \beta \circ pr_N$. Since we required $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$, and a path into $M \times N$ is $c = (c_1, c_2)$, where $c_1 : \mathbb{R} \longrightarrow M$ and $c_2 : \mathbb{R} \longrightarrow N$, then the requirement $f \circ c \in C^\infty(\mathbb{R}, \mathbb{R})$ implies that both components $f \circ c_1$ and $f \circ c_2$ be $C^\infty(\mathbb{R}, \mathbb{R})$ which forces $c_1 \in \mathcal{C}_M$ and $c_2 \in \mathcal{C}_N$. One concludes that

$$\mathcal{C}_{M \times N} = C_M \times C_N.$$

3. Form the set of structure functions by setting

$$\Phi\Gamma F_0 = \{h : M \times N \longrightarrow \mathbb{R};\ h \circ c \in C^\infty(\mathbb{R}, \mathbb{R}) \text{ for all c} \in \mathrm{C_{M\times N}}\}.$$

   Since a function $h : M \times N \longrightarrow \mathbb{R}$ need not be **of the form** (h_1, h_2), there is no possible characterization of the set of structure functions on $M \times N$ considered as Frölicher spaces, and particularly for Frölicher Lie groups.

Following this difference between Frölicher structure $(\mathcal{C}_M, \mathcal{F}_M)$ and Sikorski differential structure $\mathcal{F}$, the (infinite) product of Frölicher Lie groups will not be dealt with as the one on differential groups. A way out for a similar study (similar results) will be that of considering a class of differential groups made of differential spaces whose set of structure functions is reflexive. That is, $\Phi\Gamma\mathcal{F} = \mathcal{F}$, where $\mathcal{F}$ is generated by a set $\mathcal{F}_0$ of some real-valued functions. We shall refer to these spaces as pre-Frölicher spaces in the sense of A. Batubenge (see [6]).

Suppose that one is given a collection $\{(X_i, \mathcal{F}_i)\}_{i\in I}$ of differential spaces or a collection $\{(X_i, C_i, \mathcal{F}_i)\}_{i\in I}$ of Frölicher spaces. Let $\prod_{i\in I} X_i$ be the set product of the sets $\{X_i\}_{i\in I}$ and $\pi_j : \prod_{i\in I} X_i \to X_j$ for $j \in I$ denote the projection map. The initial structure on $\mathrm{P} = \prod_{i\in I} X_i$

in both the sense of Sikorski and Frölicher is generated by the set

$$\mathcal{F}_{\mathrm{P}}^{*} = \bigcup_{i \in I} \{ f_i \circ \pi_i | f_i \in \mathcal{F}_i, i \in I \}.$$

In the Sikorski sense one obtains $\sum \mathcal{F}_{\mathrm{P}}^{*}$ as the initial structure on P, whereas in the Frölicher sense one obtains

$$C_{\mathrm{P}} = \{ c : \mathbb{R} \to \mathrm{P} | \quad \text{if } f_i \in \mathcal{F}_i, i \in I \},$$

$$\mathcal{F}_{\mathrm{P}} = \Phi \Gamma \mathcal{F}_{\mathrm{P}}^{*}.$$

Here,the requirement that each component of C is a smooth map is most useful.

Example 3.1.2. *Let $\mathbb{R}^N_{\oplus}$ denote the Frölicher space and thus differential space whose underlying set is $\mathbb{R}^N$ and whose Frölicher space structure is generated by the set C_0 of curves which is equal to*

$$\{ (x_i(t))_{i \in N} \in C_{\mathbb{R}^N} | \quad \textit{except for finitely many } i, x_i(t) \textit{ is identically 0} \}.$$

Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\alpha(t) = t$ if $-\frac{3}{4} \leq t \leq \frac{3}{4}$ and $|\alpha(t)| \leq 1$ otherwise. It is clear that the function $l : \mathbb{R}^N_{\oplus} \to \mathbb{R}$ defined by setting

$$l((x_i)_{i \in N}) = \sum_{i=1}^{\infty} \frac{1}{2i} \alpha(x_i)$$

is a scalar on $\mathbb{R}^N_{\oplus}$.

3.2 Infinite Cartesian product of differential groups

We now define the Cartesian product of differential groups which is one of the main objectives of this research.

Definition 3.2.1. *[30] Let $(G_i)_{i \in I}$ be an indexed family of differential groups. Let $G = \times_{i \in I} G_i$ be their Cartesian product. For $j \in I$, we denote by $pr_j : \times_{i \in I} G_i \to G_j$ the natural projection.*

Let $F(G)$ be the differential structure on G generated by the set $\bigcup_{i \in I} pr_i^*(F(G_i))$. One can easily prove that $(G, F(G))$ is a differential group. $(G, F(G))$ is called the Cartesian product of differential groups $(G_i, F(G_i)), i \in I$.

For $j \in I$ we put $G(\tilde{j}) = \underset{i \in I - \{j\}}{\times G_i}$. For $q \in G(_j)$. Let $J_q : G_j \to G$ be the imbedding defined by

$$J_q(s) = (q, s) \quad \text{for } s \in G_j,$$

where

$$(q, s) = \begin{cases} q_i & \text{for } i \in I \\ s & \text{for } i = j \end{cases}$$

We see that

- $pr_j \circ J_q = id_G$,
- $(pr_i \circ J_q)(s) = q_i \quad \text{for any } s \in G_j \quad \text{if } i \neq j$

It follows therefore, that J_q is a smooth mapping.

Proposition 3.2.1. *For any $g \in G$ the mapping $E : T_g G_i \to \times_{i \in I} T_g i G_i$ defined by*

$$E(w) = (pr_{i * g^w})_{i \in I} \quad \textit{for } w \in T_g G$$

is an isomorphism.

Now for a vector $w \in T_g G$ put

$$w_i = (j_{g(i)} \circ pr_i)_* g^w \quad \text{for } i \in I,$$

where $g(i) \in G(i)$ is defined by $g(i) = g|(I - \{i\})$. Of course $w_i \in T_g G$ for $i \in I$ and

$$w_i(\alpha \circ pr_j) = 0 \quad \text{for } \alpha \in F(G_j) \text{ and } j \neq i.$$

where w_i is called the *i-th component of* w.

Now, a vector $v \in T_g G$ is said to be *parallel* to $(G_j, F(G_j))$ if $v(\alpha \circ pr_j) = 0$ for $\alpha \in F(G_i)$ and $i \neq j$. Clearly the *i-th component* w_i of any $w \in T_g G$ is parallel to $(G_i, F(G_i))$. It is easy to see that $J_{g(\hat{j}) * gj}$ is the subspace of all vectors in $T_g G$ parallel to $(G_j, F(G_j))$. The mapping $J_{g(\hat{j}) * gj} : T_{gj} G_{gj} \to T_g G$ is an isomorphism onto its image.

Definition 3.2.2. *[30] Let H be the set of all vector fields on the product group G denoted by$\mathcal{H}(\mathcal{G})$. We say that a vector field $Z \in \mathcal{H}(\mathcal{G})$ is said to be parallel to $(G_j, F(G_j))$ if $Z(g)$ is parallel to $(G_j, F(G_j))$ for every $g \in G$.*

4. INFINITE PRODUCTS OF $\mathbb{DF}$-FRÖLICHER LIE GROUPS

As stated earlier, we recall that it is easier to work on products of differential spaces than on products of Frölicher spaces, because the smooth functions on these products in $\mathbb{DSP}$ are the coordinate functions on the factors. In this section we introduce a class of Sikorski spaces, called pre-Frölicher spaces, on which the process of yielding a Frölicher structure on the same set is smooth functions preserving. If we let M to be a non empty set and $\mathcal{D}$ to be the differential structure on M, then the differential structure $\mathcal{D}$ induces a Frölicher structure. In addition we also point out that a differential space can be made into a Frölicher space by performing the operations $\Gamma\mathcal{D}$ followed by by $\Phi\Gamma\mathcal{D}$. It is noted that the structure functions in $\Phi\Gamma\mathcal{D}$ are not always the same. Special attention will be given to the generating set $\mathcal{G}$ for $\mathcal{D}$ and the resulting Frölicher one. Now if the same generators produce a differential structure $\mathcal{D}$ that coincides with the set of Frölicher functions $\mathcal{F}$ in the smooth structure $(C, \mathcal{F})$, then $(M, \mathcal{D})$ will be called a pre-Frölicher space. Thus given a Frölicher space, there is a natural way to construct a differential space out of it. The work on pre-Frölicher spaces was done by Batubenge A (see [2], also [22]). This subcategory turns out to be isomorphic to the full subcategory of Frölicher spaces.

4.1 Frölicher space and reflexivity

Definition 4.1.1. *Let $\mathcal{F}$ be differential structure on the set X. We say that $\mathcal{F}$ is reflexive if $\Phi\Gamma\mathcal{F} = \mathcal{F}$.*

This section is important in that it will help us understand the definition of $\mathbb{DF}$-spaces. The works in this section were a joint project of Batubenge A., Patrick Iglesias Z. and Yael k. (see [3], [37], [38], [39]).
Theorem 4.1.0.8 (Reflexive Theorem). *([3]) There is a natural isomorphism of categories of Frölicher spaces to reflexive differential spaces.*

Furthermore (see [37]) if we Let Ξ to be the forgetful functor from Frölicher spaces to differential spaces : $\Xi(X, C, \mathcal{F}) = (X, \mathcal{F})$, and Ξ takes maps to themselves.

We now state the following theorem:

Proposition 4.1.1 (Frölicher Stability). *Let X be a set, and let $\mathcal{F}_0$ be a family of functions on X, C_0 be a family of curves into X.*

1. *Let $C = \Gamma\mathcal{F}_0$ and $\mathcal{F} = \Phi\Gamma\mathcal{F}_0$. Then X equipped with C and $\mathcal{F}$ is a Frölicher space.*

2. *$\mathcal{F} = \Phi C_0$ and $C = \Gamma\Phi C_0$ $\dot{T}$hen X equipped with C and $\mathcal{F}$ is a Frölicher space.*

Proof. (see [37], pages 26-27).

Proposition 4.1.2. *([24]) The forgetful functor $\Lambda : \mathbb{FRL} \to \mathbb{DSP}$ sending $(X, C, \mathcal{F}) \to (X, \mathcal{F})$ preserves final structures.*

In his works "Frölicher versus differential spaces: a prelude to cosmolgy", Cherenack P. (see [10]) has pointed out that given a Frölicher space $(M, C, \mathcal{F})$, then $(M, \mathcal{F})$ is a differential space. This differential structre $\mathcal{F}$ is Sikorski. Conversely given differential space $(M, \mathcal{D})$, the differential structure $\mathcal{D}$ induces a Frölicher structure. However, it is a property of Frölicher spaces that this Sikorski structure $\mathcal{D}$ considered as generating set will be a subset of the set of all Frölicher structure functions. Thus we have the inclusion $\mathcal{D} \subseteq \Phi\Gamma\mathcal{D}$. In this dissertation we show the case when the differential struture $\mathcal{D}$ equals the set of structure functions for the Frölicher structure generated on M by $\mathcal{D}$. We mention here again that the result comes about due to reflexivity. We recall the theorem 4'.0.1.1 by Cherenack P. (see [10], p 393) which was his first comparison study on Frölicher and differential spaces. It is observed that in this study Cherenack did not take into account the *generating process* for the structures obtained on the underlying set.

Lemma 4.1.1. *([2] and [10]) The category $\mathbb{FRL}$ is a full subcategory of $\mathbb{DSP}$.*

Using a differential structure $\mathcal{D}$ of a $\mathfrak{D}$-object one can generate a unique $\mathcal{M}$-structure, from which the inclusion (1) above becomes $\mathcal{D} \subseteq \Phi\Gamma\mathcal{D}$. We employ the following notation. We use $\widehat{M}$ to denote the $\mathfrak{D}$-object $(M, \mathcal{D}_M)$, $\Upsilon\widehat{M}$ the resulting $\mathcal{M}$-object $(M, \Gamma\mathcal{D}_M, \Phi\Gamma\mathcal{D}_M)$, $\Gamma\mathcal{D}_i := \Upsilon\mathcal{C}_i$ the set of smooth curves generated by the differential structure $\mathcal{D}$, and $\Phi\Gamma\mathcal{D}_i := \Upsilon\mathcal{F}_i$ the set of Frölicher smooth functions obtained from $\mathcal{D}$.

Lemma 4.1.2. *The association of a Frölicher space $\Upsilon\widehat{M}$ to a differential space $\widehat{M}$ induces a functor $\Upsilon : \mathbb{DSP} \longrightarrow \mathbb{FRL}$.*

Proof. (see [10], p. 402)

Furthermore, Batubenge A.(see [6]) in his works 'A Survey On Frölicher Spaces'has shown the following results:
Lemma 4.1.3. *Let $\varphi : (M_1, \mathcal{D}_{M_1}) \longrightarrow (M_2, \mathcal{D}_{M_2})$ be a $\mathfrak{D}$-morphism, then φ is an $\mathcal{M}$-morphism of $\Upsilon\widehat{M_1}$ into $\Upsilon\widehat{M_2}$.*

Proof. Observe first that $\mathcal{D}_{M_1} \subseteq \Upsilon\mathcal{F}_1$ and $\mathcal{D}_{M_2} \subseteq \Upsilon\mathcal{F}_2$ by Equation of Lemma 4.1. Now, since φ is $\mathfrak{D}$-smooth, then for all $g \in \mathcal{D}_{M_2}$ one has $g \circ \varphi \in \mathcal{D}_{M_1}$. It follows that $g \circ \varphi \in \Upsilon\mathcal{F}_1$. That is, φ is an $\mathcal{M}$-morphism.
Theorem 4.1.0.9. *Let $\varphi : (M, \mathcal{D}_M) \longrightarrow (N, \mathcal{D}_N)$ be a $\mathfrak{D}$-isomorphism, then φ is an $\mathcal{M}$-isomorphism of $\Upsilon\widehat{M}$ onto $\Upsilon\widehat{N}$.*

Proof. From Lemma 4.1.3 above, φ is smooth in both categories. Hence, in $\mathbb{DSP}$ we have $\varphi^*\mathcal{D}_N = \mathcal{D}_M$. Now, φ is a diffeomorphism as a map of differential spaces by assumption, then the inverse φ^{-1} exists. It remains to be shown that it is both an $\mathcal{M}$-morphism and a $\mathcal{D}$-morphism. It is enough to show this on the associated $\mathcal{M}$-objects, i.e. either φ^{-1} maps $\Upsilon\mathcal{C}_2$ into $\Upsilon\mathcal{C}_1$ or, equivalently, it pulls back $\Upsilon\mathcal{F}_1$ into $\Upsilon\mathcal{F}_2$.
Let $f \in (\varphi^{-1})^*\Upsilon\mathcal{F}_1$. That is, $f = h \circ \varphi^{-1}$ for some $h \in \Upsilon\mathcal{F}_1$. Now let $c \in \Upsilon(\varphi_*\mathcal{C}_1)$ so that $\varphi \circ c = \gamma$, where $\gamma \in \Upsilon\mathcal{C}_2$. Hence,

$$h \circ \varphi^{-1} \circ \gamma = h \circ \varphi^{-1} \circ \varphi \circ c = h \circ c \in C^\infty(\mathbb{R}, \mathbb{R}).$$

Since $\gamma \in \Upsilon\mathcal{C}_2$, then $h \circ \varphi^{-1} = f \in \Upsilon\mathcal{F}_2$. Hence $(\varphi^{-1})^*\Upsilon\mathcal{F}_1 \subseteq \Upsilon\mathcal{F}_2$, which shows the smoothness of φ^{-1}. Thus, φ is an $\mathcal{M}$-isomorphism, i.e. a diffeomorphism of Frölicher space. Now we are able to prove the reverse inclusion $\Upsilon\mathcal{F}_2 \subseteq (\varphi^{-1})^*\Upsilon\mathcal{F}_1$. Let $f \in \Upsilon\mathcal{F}_2$. By the smoothness of φ, one has $f \circ \varphi \in \Upsilon\mathcal{F}_1$. Since φ^{-1} is smooth, one has $(f \circ \varphi) \circ \varphi^{-1} \in (\varphi^{-1})^*\Upsilon\mathcal{F}_1$. That is, $f \in (\varphi^{-1})^*\Upsilon\mathcal{F}_1$. Thus, $(\varphi^{-1})^*\Upsilon\mathcal{F}_2 = \Upsilon\mathcal{F}_1$.

4.2 Pre-Frölicher and $\mathbb{DF}$-spaces

From differential geometry view point, differential spaces and Frölicher spaces generalise the calculus on a smooth manifold. We focus on two important aspects.

Firstly we state that all smooth manifolds are both Frölicher spaces and Sikorski differential spaces. Furthermore, their smooth maps are both Frölicher and Sikorski smooth maps. We state, therefore, the following important theorem by Paul Cherenack, (see [10]):

Theorem 4.2.0.10. *Let $(X, \mathcal{C}, \mathcal{F})$ be a Frölicher space. Then $(X, \mathcal{F})$ is a differential space.*

Proof. We use the notion in the definition of differential space. Let $c : \mathbb{R} \to X$ be a contour. As $h_i \circ c$ is smooth, so that $g|U_i \circ c|c^{-1}(u_i)$. Since the sets $c^{-1}(U_i)$ for $i \in K$ cover $\mathbb{R}$, $g \circ c$ must be smooth. But, since $(X, \mathcal{C}, \mathcal{F})$ is a Frölicher space, then $g \in \mathcal{F}$.

Suppose that c is again a contour. Since $f_i \circ c, i = 1, ..., n,$ is smooth, the composite $(f_1, f_2, ...f_n) \circ c$ is smooth. But then $g \circ (f_1, f_2..., f_n) \circ c$ is smooth. This implies that $g \circ (f_1, f_2..., f_n)$ belongs to $\mathcal{F}$.

It is noted by Batubenge (see [1], [29] , [6]) that the definition of a Frölicher smooth structure need not refer to any topology, contrary to the definition of a Sikorski differential strucuture. The topologies naturally induced (by smooth curves and smooth functions) depend on the structure, the inconvenience of this dependence being that one ends up having discrete structures and topologies on dense subspaces. On the other hand, unlike smooth manifolds that are modeled on Euclidean spaces, the topology is part of the defining axioms for a Sikorski structure, which probably implies good behavior on dense subsets. A typical example is provided by the Sikorski smooth space of rationals.
A further advantage is that it is easier to work on products of differential spaces than on the product of Frölicher spaces, since smooth functions on these products differential spaces are the coordinate functions on the factors.
It must be noted that the comparison between differential spaces and Frölicher spaces raises a two-part key question, which we consider in these works.

1. First, consider that we are given a Frölicher space $(M, \mathcal{C}, \mathcal{F})$. If we 'forget' the smooth curves in the structure $(\Gamma\mathcal{F}, \Phi\Gamma\mathcal{F})$ (of course, we do not do this in reality since there would not be a Frölicher structure without these curves), then $\mathcal{F}$ is a Sikorski differential structure. The first question is, therefore: Is it true that $\Phi\Gamma\mathcal{F}$ is the original set $\mathcal{F}$? Of course this is clearly true by the compatibility condition satisfied by $\mathcal{C}$ and $\mathcal{F}$ on M.

2. Now, let $(M, \mathcal{D})$ be a differential space. The differential structure $\mathcal{D}$ induces a Frölicher structure $(\Gamma\mathcal{F}, \Phi\Gamma\mathcal{F})$. However, it is a property of Frölicher spaces that this Sikorski structure $\mathcal{D}$ considered as generating set will be a subset of the set of all Frölicher structure functions. That is, we have the inclusion $\mathcal{D} \subseteq \Phi\Gamma\mathcal{D}$. The second question, therefore, is: In which case does the differential structure $\mathcal{D}$ equal the set of struc-

ture functions for the Frölicher structure generated on M by $\mathcal{D}$? Of course the strict inclusion yields undesirable situations, where a well behaved Sikorski structure can generate a discrete Frölicher structure. Thus, the topology and geometry that follow will be different.

Proposition 4.2.1. *([6]) Let $(M, \mathcal{D})$ be a differential space, and $\Upsilon\widehat{M}$ its associated Frölicher space with structure function given by $\mathcal{D} = \Phi\Gamma\mathcal{D}$. Then $\Gamma\mathcal{D}$ is not a collection of constant curves.*

Proof. First, note that in this case the generating set $\mathcal{D}$ is a differential structure, which is in turn generated by an arbitrary collection of real-valued functions $\mathcal{F}_0 = \{\alpha_0, ..., \alpha_k\}$ on M. So the condition

$$\mathcal{D} = \Phi\Gamma\mathcal{D}$$

reads

$$Gen\{\alpha_0, ..., \alpha_k\} = \Phi\Gamma Gen\{\alpha_0, ..., \alpha_k\}.$$

In other words we consider the differential and Frölicher structures induced by the same generators. Hence, the fact that smooth functions induced by the function Φ are not exactly those in $\mathcal{D}$ depends on the set of curves $\Gamma\mathcal{F}$. From the above argument, if we assume that $\Gamma\mathcal{D}$ is a collection of constant curves, then the associated Frölicher structure is discrete (without loss of generality, refer to the case of $\mathbb{Q}$), which implies that $\mathcal{D} \neq \Phi\Gamma\mathcal{D}$.

Definition 4.2.1. *[6] A pre- Frölicher space is a differential space $(M, \mathcal{D})$ with structure $\mathcal{D}$ such that $\mathcal{D} = \Phi\Gamma\mathcal{F}_0$, where $(M, \Gamma\mathcal{F}_0, \Phi\Gamma\mathcal{F}_0)$ is the associated Frölicher space so-called $\mathbb{DF}$-space and $\mathcal{F}_0$ a generating set.*

The diagram below explains that such a class of differential spaces exists, and serves to illustrate the definition above.

$$\begin{tikzcd}
 & & (M, \Gamma\mathcal{F}_0, \Phi\Gamma\mathcal{F}_0) \arrow[drr, shift left, "U"] & & \\
(M, \Phi\Gamma\mathcal{F}_0) \arrow[urr, "\Upsilon"] & & = & & (M, \mathcal{D}) \arrow[ull, shift left, "\Upsilon"] \\
\Gamma\mathcal{F}_0 \arrow[u, "\Phi"] & & & & \mathcal{D} \arrow[u] \\
 & & \mathcal{F}_0 \arrow[ull, "\Gamma"] \arrow[urr, "sc+loc"'] & &
\end{tikzcd}$$

We refer again to the case $M = \mathbb{Q}$, and clearly show that it is not a pre-Frölicher space. For, let $\mathcal{F}_0 = \{id_{\mathbb{Q}}\}$ be the generating set. A $\mathfrak{D}$-structure is given by taking

$$sc\{id_{\mathbb{Q}}\} = \{\omega|\mathbb{Q};\ \omega \in C^{\infty}(\mathbb{R}, \mathbb{R})\}.$$

If we further examine the locality property, it turns out that the structure $\mathcal{D}$, say, is given by all smooth real-valued functions defined on $\mathbb{Q}$. That is, $sc\{id_{\mathbb{Q}}\} = C^{\infty}(\mathbb{Q}, \mathbb{R})$. However, the Frölicher structure generated by $\{id_{\mathbb{Q}}\}$ on $\mathbb{Q}$ is the $(\mathcal{C}, \mathcal{F})$, where $\mathcal{C}$ is a set of constant functions from $\mathbb{R}$ to $\mathbb{Q}$ and $\mathcal{F}$ is simply the set of all functions $f : \mathbb{Q} \longrightarrow \mathbb{R}$. Clearly $\mathcal{D}$ in $\mathbb{DSP}$ is a subset of $\mathcal{F}$ in $\mathbb{FRL}$ for the same generating set considered on the same underlying set $\mathbb{Q}$. So, on the $\mathfrak{M}$-space $\mathbb{Q}$ generated by $\{id\}$ the forgetful functor $U : \mathbb{FRL} \longrightarrow \mathbb{DSP}$ gives a trivial differential space, which is not the one generated by the same set $\{id\}$ on $\mathbb{Q}$.

Lemma 4.2.1. *([6],[1]) Let $(M_1, \mathcal{D}_1)$ and $(M_2, \mathcal{D}_2)$ be differential spaces. If $(M_1, \mathcal{D}_1)$ is a pre- Frölicher space and $\varphi : (M_1, \mathcal{D}_1) \longrightarrow (M_2, \mathcal{D}_2)$ is a diffeomorphism of differential spaces, then $(M_2, \mathcal{D}_2)$ is a pre-Frölicher space.*

Proof. Let us denote the associate Frölicher structures as $(\Upsilon\mathcal{C}_i, \Upsilon\mathcal{F}_i)$ $(i = 1, 2)$. From theorem 4.1.0.9 above and using the subsequent notations, the diffeomorphism

$$\varphi : (M_1, \mathcal{D}_1) \longrightarrow (M_2, \mathcal{D}_2)$$

between differential spaces is a diffeomorphism of the associated Frölicher spaces $(M_1, \Upsilon\mathcal{C}_1, \Upsilon\mathcal{F}_1)$ and $(M_2, \Upsilon\mathcal{C}_2, \Upsilon\mathcal{F}_2)$. Assume that $(M_1, \mathcal{D}_1)$ is a pre-Frölicher space. That is, $\Upsilon\mathcal{F}_1 = \mathcal{D}_1$. We need to show that $\Upsilon\mathcal{F}_2 = \mathcal{D}_2$. From the assumption, we have

$$(\varphi^{-1})^*\Upsilon\mathcal{F}_1 = (\varphi^{-1})^*\mathcal{D}_1.$$

Then $(\varphi^{-1})^*(\Upsilon\mathcal{F}_1) = \mathcal{D}_2$ according to the first equality in 4.1.0.9. Also, $(\varphi^{-1})^*(\Upsilon\mathcal{F}_1) = \Upsilon\mathcal{F}_2$ as shown in the last identity of 4.1.0.9. Hence, $\Upsilon\mathcal{F}_2 = \mathcal{D}_2$ as required.

Proposition 4.2.2. *([1],[6]) Let M be a set and N be a pre-Frölicher space. Let $(\mathcal{C}_N, \mathcal{F}_N)$ be the Frölicher structure induced on M by means of maps $f_i : M \longrightarrow N,\ i \in I$ where I is a set of indices. Assume that the map $\varphi : M \longrightarrow N^I$, given by $\varphi(x) = (f_i(x))_I$, is one-to-one. Then φ is a Frölicher diffeomorphism onto the subspace $\varphi(M)$ of N^I.*

Proof. First, note that for φ to be one-to-one it is enough that one of the functions f_i separates points on M. Since φ is surjective onto $\varphi(M)$, then it is bijective. Now, let $c : \mathbb{R} \longrightarrow M$ be a curve on M. Then $\varphi \circ c(t) = (f_i \circ c(t))_I$ for all $t \in \mathbb{R}$. Since the structure on N^I is generated by the family $\{g \circ \pi_i : g \in \mathcal{F}_N,\ i \in I\}$, it follows that $\varphi \circ c : \mathbb{R} \longrightarrow \varphi(M)$ is a smooth curve on $\varphi(M)$. Hence φ is smooth.

Next, let $(x_i)_I \in \varphi(M)$. It is clear that

$$g \circ f_i \circ \varphi^{-1}(x_i)_I) = g \circ \pi_i \circ \varphi \circ \varphi^{-1}((x_i)_I) = g \circ \pi_i((x_i)_I).$$

Thus, φ^{-1} is smooth, which ends the proof.

As a corollary we shall state under the same assumptions that if $N = \mathbb{R}$, then we have a Frölicher diffeomorphism of M onto $\mathbb{R}^n$. Notice that the diffeomorphism obtained under this construction is not necessarily a homeomorphism since, as shown in [5.8], the topology on the subset $\varphi(M) \subset \mathbb{R}^n$ can be different from the relative topology. However, since M is a pre-Frölicher space, $\varphi(M)$ is not dense in $\mathbb{R}^n$ and therefore, it follows from the argument that φ provides a homeomorphism onto $\varphi(M)$. Also, we may consider a pre-Frölicher space that is *locally* diffeomorphic to $\mathbb{R}^n$, then revert back and transfer necessary topological properties inherited from $\mathbb{R}^n$ to an open set in $\tau_{\mathcal{F}_M}$.

Example 4.2.2. *Let $X = [0, \pi)$. Consider the map given on X by setting $\varphi(x) = (-cos\ x, -1)$, for all $x \in X$. The function $-cos\ x$ is point-separating in X so that the map φ is an $\mathbb{F}$-diffeomorphism of $[0, \pi)$ onto the interval $J = \varphi[X]$ in $\mathbb{R}^2$, which is neither open, nor closed. More generally, one can consider $Y = [0, \infty)$ and the map defined on Y by setting $\varphi(x) = (id(x), \theta(x))$ for all $x \in Y$, where $\theta(x) = 0$.*

Example 4.2.3. *Let us denote by $G(f)$ the graph of f, then the set $G(f) = \{(x, |x|) : x \in \mathbb{R}\} \subset \mathbb{R}^2$ is locally $\mathbb{F}$-diffeomorphic to $\mathbb{R}^2$ at $x = 0$ (open discs $B(r)$ are neighborhoods of zero) and to $\mathbb{R}$ at every point away from the origin.*

Definition 4.2.4. *[6] A $\mathbb{DF}$-space is a differential space $(M, \mathcal{D})$ with the structure $\mathcal{D}$ such that $\mathcal{D} = \Phi\Gamma\mathcal{F}_0$, where $(M, \Gamma\mathcal{F}_0, \Phi\Gamma\mathcal{F}_0)$ is the associated Frölicher space and $\mathcal{F}_0$ a generating set.*

The lemma below, the proof of which is due to A. Cap (see [8]) in the setting of Frölicher spaces, plays a key role.

Lemma 4.2.2. *Let M, N be $\mathbb{DF}$-spaces. Let $U \subset M$ be a $\tau_{\mathcal{C}_M}$-open subset, $f : M \longrightarrow N$ a function. Then the following conditions are equivalent:*

(1) For any $c \in \mathcal{C}_M$ with $c(\mathbb{R}) \subset U$ the curve $f \circ c$ is smooth.

(2) For any $c \in \mathcal{C}_M$ the curve $f \circ c : c^{-1}(U) \longrightarrow N$ is smooth.

Proof. (see [6])

4.3 Product of $\mathbb{DF}$-spaces

Unlike the class of $\mathbb{DF}$-spaces is to the class of pre-Frölicher spaces, the class of $\mathbb{DF}$-Frölicher Lie groups is isomorphic to that of pre-Frölicher d-spaces. Therefore, they have same behaviour regarding their sets of smooth functions.

Definition 4.3.1. *([6]) A $\mathbb{DF}$-space is said to be of constant dimension if, for a fixed positive integer n, it is locally diffeomorphic to $\mathbb{R}^n$ at each point.*

Definition 4.3.2. *A pre-Frölicher space that is also a differential group is called pre-Frölicher differential group.*

Definition 4.3.3. *A Frölicher space $(G, \mathcal{C}_G, \mathcal{F}_G)$ which is both a $\mathbb{DF}$-space and a Frölicher Lie group is called a $\mathbb{DF}$-Lie group.*

Lemma 4.3.1. *Let $(G_i, \mathcal{C}_i, \mathcal{F}_i)_i$ be an infinite collection of Frölicher Lie groups. Then the infinite Cartesian product $\Pi_I G_I$ is a Frölicher Lie group with the smooth structure $(\bar{\mathcal{C}}, \bar{\mathcal{F}})$, where $\bar{c} \in \bar{\mathcal{C}}$ is of the form $\bar{c} = (c_i)_{i\in I}$, with $c_i \in \mathcal{C}_i$.*

The proof is a straightforward consequence of the property on the set of structure curves on the product of general Frölicher spaces (see section 3.1). We have not been able to prove the same result for the set of structure functions on the product of Frölicher spaces, although it is true on the product of differential spaces. Nevertheless, it holds true for the product of $\mathbb{DF}$-spaces, and in particular, for $\mathbb{DF}$-Lie groups.

Proof. (see section 3.1)

Theorem 4.3.0.11. *Let $(G_I, \mathcal{C}_I, \mathcal{F}_I)_I$ be an infinite collection of $\mathbb{DF}$-Lie groups. Then the infinite Cartesian product $\Pi_I G_I$ is a Frölicher Lie group with the smooth structure $(\bar{\mathcal{C}}, \bar{\mathcal{F}})$, where $\bar{\mathcal{C}} = \Pi_I \mathcal{C}_I$ and $\bar{\mathcal{F}} = \Pi_I \mathcal{F}_I$.*

Proof. Again, the proof is a straightforward consequence of the fact that the class of $\mathbb{DF}$-spaces (Lie groups) is isomorphic to the class of those differential spaces (groups) with underlying smooth space being pre-Frölicher.

Recall that every Frölicher space is a differential space (see Cherenack [10]). Moreover a $\mathbb{DF}$ space is a differential space whose underlying space is a pre-Frölicher space. Since pre-Frölicher spaces are differential spaces, then the product of G_I has the product structure $\Pi \mathcal{F}_I$. Therefore we conclude from lemma 4.3.1 above that the structure on the product $(G_I, \mathcal{C}_I, \mathcal{F}_I)$ is the product $(\Pi \mathcal{C}_I, \Pi \mathcal{F}_I)$, which ends the proof.

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