THE STRUCTURE OF EXTENDED REAL-VALUED QUASI-METRIC SPACES

By

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ABSTRACT

An extended quasi-metric q on a nonempty set X without any assumed structure is a distance functional that satisfies the usual properties of a quasi-metric except that it can assume values of infinity, in addition to non-negative real values. Given a quasi-metrizable space X we exhibit a universal space for all extended quasi-metric spaces compatible with the asymmetric topologies of X. Defining a set in an extended quasi-metric space (X, q) to be bounded if it is contained in an intersection of the left-q and right-q open (or closed)-balls, we characterize these kinds of bornologies on X and, obtain necessary and sufficient conditions in order for the same bornologies to be realized as those for quasi-metrically bounded sets. We also consider in this setting a second possible definition of bounded sets involving quasi-components.

Keywords: Quasi-metric, Extended real-valued quasi-metric, uniform equivalent quasi-metrics, Bounded set, Partial function, Bornology, Quasi-metric bornology, Quasi isometry, Free union bitopology, Generalized Hus Theorem.

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DECLARATION

The work described in this Master of Science (MSc) dissertation was carried out under the supervision of Dr. Isaac Daniel Tembo, Department of Mathematics and statistics, University of Zambia, Lusaka.

The MSc dissertation represents original work by the author and has not otherwise been submitted in any form for any degree or diploma to any other University. Where use has been made of the work of others it is duly acknowledged in the text.

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APPROVAL

This dissertation of Levy K. Matindih has been approved as fulfilling the requirements or partial fulfillment of the requirements for the award of Master of Science in Mathematics by the University of Zambia.

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DEDICATION

I dedicate this work to my parents Jones M. Matindih and Emeldah K. Makayi and, my siblings Mercy, Percy, Zia, Early, Jones, Lucky, Buto, Friday, Dezzy, Samba, Gabriel and Ruth, for the support and words of courage they have rendered.

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Below is a list of symbols that will be frequently used and a brief indication of their meaning.

$B(x,r)$	Open ball of radius r centred at x
$C(x,r)$	Closed ball of radius r centred at x
$cl(A)$	closure of set A
$int(A)$	interior of set A
$\mathscr{B}(X)$	Bornology (or space of bounded subsets) on X
$\mathscr{B}_q(X)$	A quasi-metric bornology (or space of quasi-metrically bounded subsets) on X.
$\mathbb{N}$	the set of natural numbers
$\mathcal{FP}(X)$	Space of all paired functions $f = (f_1, f_2)$ on X
$C[a,b]$	Space of continuous functions on [a, b]
$C(X)$	Space of continuous functions on X
$C^b(X)$	Space of bounded continuous functions on X
$q(x,y)$	A quasi-metric distance from x to y
$\dim X$	Dimension of a space X
inf	Infimum (greatest lower bound)
max($\vee$)	Maximum
min($\wedge$)	Minimum
$x \dot{-} y$	$\max\{x-y, 0\}$
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
sup	Supremum (least upper bound)
$\mathscr{P}(X)$,	the set of all subsets of X (Power set of X)
$\mathscr{P}_0(X)$	the set of all non-empty subsets of X
$\mathscr{F}(X)$	the set of all finite subsets of X
$\mathscr{F}_o(X)$	the set of all non-empty finite subsets of X
$\mathscr{K}(X)$	the set of all compact subsets of X
$\mathscr{K}_0(X)$	the set of all non-empty compact subsets of X
$\mathscr{C}(X)$	the set of all closed subsets of X
$\mathscr{C}_0(X)$	the set of all non-empty closed subsets of X

CHAPTER 1 : INTRODUCTION

1.1. Background

The concept of universal spaces and their groups of isometries first constructed by Urysohn [37] in the 1920s have received considerable attention by many scholars in different perspectives [7], [14], [19,20], [36], [39] etc. For instance, Dugundji [14, Theorem 5.2, p. 286] proved that when X is metrizable for each compatible metric d, then (X, d) can be isometrically embedded into the metric space $(C(X), d_e^+)$ using the map $x \longrightarrow d(x, \cdot) - d(x_0, \cdot)$ where x_0 is a fixed point of X and d_e^+ is the supremum metric. Recently, Beer [7] has investigated the results of Urysohn [37] and of Dugundji [14, Theorem 5.2, p. 286] and concluded that the space of bounded continuous real-valued functions on X equipped with uniform distance is universal for the family of metric spaces based on X. He has then extended these results to produced a universal space for the family of extended metric spaces based on X. His universal space was the space of partial functions defined for any Hausdorff space. In addition, Stojmirović in his Phd thesis [36] generalized the results of Urysohn [37] to quasi-metric spaces.

The notion of bounded sets has also been very important in the theory of metric spaces, quasi-metric spaces and normed vector spaces. For that reason this concept has been extended to some different settings, like (bi)-topological vector spaces, uniform spaces or even in the general context of (bi)-topological spaces. In this way one can see that there exists new notions of bounded sets. In spite of the fact that there are different generalizations of the notion of bounded sets, we can observe that all of them have a common property of forming a bornology. By a bornology for any space X we mean a family $\mathscr{B}$ of bounded subsets of X which: covers X, is stable under set inclusions and closed under finite unions [16]. The pair $(X, \mathscr{B})$ consisting of any (bi)-topological space X and a bornology $\mathscr{B}$ on X, is called a bornological (bi)-universe [18], [29].

The systematic study of bornologies in topological spaces starts with a paper by Hu [18] some 68 years ago. The main result of that paper was to characterize bornologies that are metric bornologies. He studied and solved a general problem of metrizability of bornologies, that is, he investigated bornologies defined on a metric space that corresponds to the bornologies of bounded sets. Hu's results were an extension of boundedness in the realm of general topology introduced in 1939 by Alexander [2].

Garrido and Meroño in [15] applied the theorem of uniform metrization to a metric space with a bornology of bounded sets in the sense of Bourbaki, that is, the subsets of X which are bounded for every continuous function. This was done by use of a characteristic function. They also applied this theorem to a bornology of compact subsets of a topological space X and, this was achieved by using metrics on X with the Heine-Borel property. Their results were generalizations of works by Hu [18].

Recently, Beer [7] has extended Hu's metrization theorem and, Garrido and Meroño's uniform metrization theorem for a bornology to subsets equipped with an extended real-valued metric. He has proved that given a quasi-metrizable space X with a bornology $\mathscr{B}$, the bornology $\mathscr{B} = \mathscr{B}_d(X)$ for some compatible extended metric d on X. He further showed that, given an extended real valued metric d on X, there exists a metric $d' = \{1, d\}$ which is uniformly equivalent to d on X such that $\mathscr{B}_d(X) = \mathscr{B}_{d'}(X)$.

More recently, Piękosz and Wajch [29] have shown that, Hu's [18] metrization theorem for bornological universes can be applied in ZF-(*Zermelo-Fraenkel*) spaces and adopted it to a quasi-metrization theorem for bornologies in bitopological spaces. They further investigated the problem of uniform quasi-metrization of bornological biuniverse using a characteristic function of which was a generalization of results by Garrido and Meroño [15].

Since a quasi-metric q defined on a set without any assumed structure X can either be real-valued provided $q : X \longrightarrow \mathbb{R}$ or complex valued provided $q : X \longrightarrow \mathbb{C}$, our main goal in this dissertation is to study the structure of sets equipped with an extended real-valued quasi metric, with an emphasis on large structure, that is, our quasi-metric q will be defined from X onto $\mathbb{R}_+ \cup \{+\infty\} = [0, +\infty]$ unless stated. We are devoted to resolve the following fundamental questions:

(i) Given a quasi-metrizable space X, when can I exhibit up to isomorphism an extended real-valued quasi-metric space (W, Q) which is universal for all extended real-valued quasi-metric spaces that can be built on X. This means that whenever q is an extended real-valued quasi-metric on X, we can find an isometry $\phi : (X, q) \to (W, Q)$, i.e. $Q(\phi(x), \phi(y)) = q(x, y)$, $\forall x, y \in X$.

(ii) Given a quasi-metrizable space X with a bornology $\mathscr{B}$, when does there exist an extended real-valued quasi-metric q such that $\mathscr{B} = \mathscr{B}_q(X)$? If that is the case, we say that $\mathscr{B}$ is quasi-metrizable with respect to q.

(iii) Given an extended real-valued quasi-metric q on X with a bornology $\mathscr{B}$, when does there exist a real-valued quasi-metric q' on X, uniformly equivalent to q, such that the bornology of q'-bounded sets coincides with $\mathscr{B}$, that is to say, $\mathscr{B}_q(X) = \mathscr{B}_{q'}(X)$? If that happens, we say that $\mathscr{B}$ is uniformly quasi-metrizable. q' is some times referred to as a bona-fide quasi-metric.

We will characterize these kinds of bornologies on X, and obtain necessary and sufficient conditions in order for the same bornologies to be realised as bornologies for (uniformly

-) quasi-metrically bounded sets. We will also consider bornologies generated by pairwise compact subsets of X.

1.2. Organisation of the dissertation

This dissertation is organized as described below.

Chapter 1. This chapter presents a background on universal spaces and bonorlogies in topological spaces, metric spaces and quasi-metric spaces as investigated by different scholars.

Chapter 2. In this chapter, we recall some of the important definitions to be used throughout the dissertation. The first section introduces quasi-metric spaces, thereafter, we present a summary of notions related to asymmetric topologies. The second section briefly discusses the concept of completeness and compactness in relation to quasi-metric spaces. Section three briefly discusses the concept of a Hausdorff quasi-metric.

Chapter 3. In this chapter we present the results of Beer [7]. The first section gives the definition of an extended-real valued metric, thereafter, we briefly discuss some results related to metric components on the structure of sets equipped with extended real-valued metrics. In the second section, we give some of Beer's construction of a universal metric space of some class of sets equipped with an extended real valued metric with emphasis on large structure. The third section outlines the concept of a bornology induced by an extended metric, which is an extension of a metrizable bornology constructed by Hu [**?**] and uniform metrization of a bornology by Garrido and Meroño [15].

Chapter 4. In this chapter, we start our own investigations. We generalize Beer's [7] work to extended real-valued quasi-metric spaces. In the first section, we give the concepts of extended-real valued quasi-metrics with some examples, after which, we present a brief summary of results related to quasi-components on sets that assume the value infinite. The second section is devoted to constructing a universal quasi-metric space on sets equipped with an extended real-valued quasi metric with emphasis on large structure of which, is a generalization of Beer's [7] extension of the Urysohn universal metric space and also an extension of the universal quasi-metric space constructed by Stojmirović in [36]. In the third section, we discuss the notion of a bornology of quasi-metrically bounded subsets, which is a generalisation of Beer's [7] results and an extension of works by Piękosz and Wajch [29]. We also construct some results in relation to bornologies generated by pairwise compact subsets.

Chapter 5. In this chapter, we summarize our investigations and present some open problems to be studied in future.

CHAPTER 2 : PRELIMINARIES

In this chapter, we recall the basic concepts from the theory of quasi metric spaces to be used throughout the Msc dissertation and give some of the examples related to their asymmetric topologies. For more details, we refer the reader to [9, 10], [21], [22] [25] [31, 33], [34], [27], [35], [36]. Most known applications of quasi-metrics come from theoretical computer science.

2.1. Basic concepts and definitions in quasi-metric spaces

Definition 2.1.1. Let X be a nonempty set and consider a function $q : X \times X \longrightarrow [0, \infty)$. Then, q is called a quasi-pseudo-metric on X if

(i) $q(x, x) = 0$ whenever $x \in X$,

(ii) $q(x, z) \leq q(x, y) + q(y, z)$ whenever $x, y, z \in X$.

In addition, we shall say that q is a T_0-quasi-metric provided that q also satisfies the following T_0-condition: For each $x, y \in X$,

(iii) $q(x, y) = 0 = q(y, x)$ implies that $x = y$.

A set X endowed with a quasi-pseudo-metric q is called a quasi-pseudo-metric space denoted by (X, q). In particular, if q satisfies the symmetric condition, then (X, q) is the well-known metric space. So, metric spaces are a particular case of quasi-pseudo-metric spaces.

Remark 2.1.2. If we replace $[0, \infty)$ by $[0, \infty]$ where for a q attaining the value ∞ the triangle inequality is interpreted in the obvious way, then in such a case we shall speak of an extended quasi-pseudo-metric. If X is a nonempty set and q is an extended quasi-pseudo-metric on X, then the pair (X, q) is called an extended quasi-pseudo-metric space. Moreover, if q is a T_0-extended quasi metric on X, the pair (X, q) is called an extended quasi-metric space.

Note that, in a T_0-quasi-metric space, we can have $q(x, y) = 0$ with $x \neq y$. For example if $x \in A$ and $y \in cl(A)$ the $q(x, y) = 0$ but $x \neq y$.

The next definition is an obvious consequence of the lack of symmetry of the distance from a point to another.

Definition 2.1.3. Let q be a T_0-quasi metric on X. Then a mapping $q^{-1} : X \times X \longrightarrow [0, \infty)$ defined by $q^{-1}(x, y) = q(y, x)$ whenever $x, y \in X$ is also a quasi metric on X, called the conjugate (or dual) quasi metric of q.

Definition 2.1.4. Let q be a T_0-quasi metric on X. Then, a map $q^s : X \times X \longrightarrow [0, \infty)$ defined by

$$q^s(x, y) = \max\{q(x, y), q^{-1}(x, y)\}$$

whenever $x, y \in X$ is called the associated metric of q.

The associated metric q^s is the smallest metric majorising q and q^{-1}. That is for q^s on X, the following inequalities hold for all $x, y \in X$:

$$q(x, y) \leq q^s(x, y) \quad and \quad q^{-1}(x, y) \leq q^s(x, y).$$

A quasi metric q on X such that $q = q^{-1}$ is called a metric.

The following describes some concepts related to asymmetric topologies of a quasi-metric space.

The topology $\mathscr{T}_q$ of a quasi metric space (X, q) can be defined starting from the family $\mathcal{V}_q(x)$ of neighborhoods of an arbitrary point $x \in X$: for any $V \subseteq X$, we have $V \in \mathcal{V}_q(x)$ if and only if there exists $\delta > 0$ such that $B_q(x, \delta) = \{y \in X : q(x, y) < \delta\} \subset V$, if and only if there exists $\epsilon > 0$ such that $C_q(x, \epsilon) = \{y \in X : q(x, y) \leq \epsilon\} \subseteq V$. A set $U \subseteq X$ is $\mathscr{T}_q$-open if and only if for every $x \in U$ there exists $\delta = \delta_x > 0$ such that $B_q(x, \delta) \subseteq U$. We shall say that U is a q-neighborhood of x or that the set U is q-open.

However, taking into consideration the lack of symmetry, a quasi metric q generates three different topologies (see [10]), that we recall next.

Definition 2.1.5. Let (X, q) be a quasi metric space. The topologies $\mathscr{T}_q$ and $\mathscr{T}_{q^{-1}}$ are generated by the quasi metrics q and q^{-1} respectively, where the open balls are described as follows: given $x \in X$ and $\delta > 0$, we have, $B_q(x, \epsilon), B_{q^{-1}}(x, \epsilon) \subseteq X$, where

$$B_q(x, \delta) = \{y \in X : q(x, y) < \delta\}, \quad and \quad B_{q^{-1}}(x, \delta) := \{y \in X : q(y, x) < \delta\}$$

and the closed balls are described as follows: given $x \in X$ and $\delta > 0$, we have, $C_q(x, \delta), C_{q^{-1}}(x, \delta) \subseteq X$, where

$$C_q(x, \delta) = \{y \in X : q(x, y) \leq \delta\} \quad and \quad C_{q^{-1}}(x, \delta) := \{y \in X : q(y, x) \leq \delta\}.$$

The balls with respect to q are called forward (or left) balls and the topology $\mathscr{T}_q$ is called the forward topology and the balls with respect to q^{-1} are called backward (or right) balls and the topology $\mathscr{T}_{q^{-1}}$ is called the backward topology.

Remark 2.1.6. Note that, $\{C_q(x,\delta) : \delta > 0\}$ is a neighborhood base of the point x formed of $\mathscr{T}_{q^{-1}}$-closed sets: this set is $\mathscr{T}_{q^{-1}}$ closed but not $\mathscr{T}_q$ closed because $C_q(x,\delta)$ contains the closure of the open balls $B_q(x,\delta)$ and $\overline{B_q(x,\delta)} = cl_{\mathscr{T}_{q^{-1}}} B_q(x,\delta)$. Similarly, $\{C_{q^{-1}}(x,\delta) : \delta > 0\}$ is a neighborhood base of the point x formed of $\mathscr{T}_q$-closed sets. The proof to this remark will be shown in the later stage of this section.

Definition 2.1.7. Let (X,q) be a quasi metric space. The topology $\mathscr{T}_{q^s} = \max\{\mathscr{T}_q, \mathscr{T}_{q^{-1}}\}$ is generated by the associated metric q^s, where the open balls are described as follows: given $x \in X$ and $\delta > 0$, we have, $B_{q^s}(x,\delta) \subset X$, where

$$B_{q^s}(x,\delta) = \{y \in X : q^s(x,y) < \delta\}.$$

Remark 2.1.8. Note that if q is a T_0-quasi-metric, then $B_{q^s}(x,\delta) \subseteq B_{q^{-1}}(x,\delta)$ and $B_{q^s}(x,\delta) \subseteq B_q(x,\delta)$, whenever $x \in X$ and $\delta > 0$. This also holds true for closed balls. The proof of this remark will be shown in the later stage of this section

Definition 2.1.9. Let (X,q) be a quasi-metric space. Given $A \in \mathscr{P}_0(X)$ and $x \in X$, we will set $q(x,A) = \inf\{q(x,a) : a \in A\}$ and $q(A,x) = \inf\{q(a,x) : a \in A\}$ for the left and right distance from x to A respectively.

Definition 2.1.10. Let (X,q) be a quasi-metric space, $x \in X$ and $\delta > 0$. For $A \subset X$, we define the δ-neighborhood (or enlargement) of A with respect to q by $[A]_q^\delta$ where,

$$[A]_q^\delta = B_q(A,\delta) = \{x \in X : q(A,x) < \delta\} = \{x \in X : \inf\{q(x,a) : a \in A\} < \delta\} = \bigcup_{a \in A} B_q(a,\delta).$$

Similarly,

$$[A]_q^\delta = C_q(A,\delta) = \{x \in X : q(A,x) \le \delta\} = \{x \in X : \inf\{q(x,a) : a \in A\} \le \delta\} = \bigcup_{a \in A} C_q(a,\delta).$$

Definition 2.1.11. Let (X,q) be a quasi-metric space. Then a pair $(B_q(x,\epsilon); B_{q^{-1}}(x,\delta))$ with $x \in X$ and $\epsilon, \delta \in [0,\infty)$ is called a double ball at $x \in X$. Generally,
$[(B_q(x_i,\epsilon_i))_{i\in I}; (B_{q^{-1}}(x_i,\delta_i))_{i\in I}]$ with $x_i \in X$, $\epsilon_i, \delta_i \in [0,\infty)$ is a family of double balls.

$[(B_q(x_i,\epsilon_i))_{i\in I};(B_{q^{-1}}(x_i,\delta_i))_{i\in I}]$ with $x_i \in X$ and $\epsilon_i,\delta_i \in [0,\infty)$ whenever $i\in I$ is said to have the mixed binary intersection property if for all indices $i,j\in I$,

$$B_q(x_i,\epsilon_i)\cap B_{q^{-1}}(x_j,\delta_j)\neq\emptyset.$$

Definition 2.1.12. Let (X,q) be a quasi-metric space. For $A\in\mathscr{P}_0(X)$, we will set $\text{diam}(A)=\sup\{q(x,y):x,y\in A\}$ for the diameter of set A.

Definition 2.1.13. Let (X,q) be a quasi-metric space. A subset A of X will be called q-bounded provided that there is a real constant M such that $q(x,y)<M$ whenever $x,y\in A$. Equivalently, an arbitrary subset A of X is said to be q-bounded if there exists $x\in X$ and $\epsilon,\delta>0$ such that $A\subseteq B_q(x,\epsilon)\cap B_{q^{-1}}(x,\delta)$. Note that one can replace $B_q(x,\epsilon)\cap B_{q^{-1}}(x,\delta)$ by $C_q(x,\epsilon)\cap C_{q^{-1}}(x,\delta)$.

It is obvious that if a set A is q-bounded and q^{-1} bounded, then A is q^s-bounded.

Remark 2.1.14. Note that in a quasi-metric space (X,q), a set $A\subseteq X$ can be simultaneously q bounded and q^{-1} unbounded as the following example below indicates.

Example 2.1.15. (cf. Example 1.6 of [29]) For $x,y\in X=\mathbb{N}$, let $q(x,y)=0$ if $x=y$ and $q(x,y)=2^x$ if $x\neq y$. Then $X=B_q(0,2)=\{x\in X:q(0,x)<2\}$, so X is q-bounded. However, for arbitrary $x\in X$ and $\delta\in(0,+\infty)$, if $y\in X$ is such that $2^y>\delta$, then $y\notin B_{q^{-1}}(x,\delta)$. Therefore, X is q^{-1} unbounded.

Definition 2.1.16. [27] Let X be a nonempty set without any assumed structure and $\mathcal{B}=\{B_i\}_{i\in I}$ be a collection of subsets of X. We say that $\{B_i\}_{i\in I}$ is a basis for a topology $\mathscr{T}$ of X if the following conditions hold:

(i) For each $x\in X$, there exists B_i such that $x\in B_i$.

(ii) For every $B_i,B_j\in\mathcal{B}$, if $x\in B_i\cap B_j$, then there exists $B_k\in\mathcal{B}$ such that $x\in B_k\subset B_i\cap B_j$.

The topology generated by $\mathcal{B}$ is defined as follows: a set $U\subseteq X$ is open if for each $x\in U$, there is a $B\in\mathcal{B}$ such that $x\in B\subset U$ and thus, is given by $\mathscr{T}_{\mathcal{B}}=\bigcup\limits_{B\in\mathcal{B}}B$.

As a space with two topologies, a quasi-metric space can be viewed as a bitopological space in the sense of Kelly [21] and so, all the results valid for bitopological spaces apply to quasi-metric spaces. A bitopological space is simply a set X endowed with two (distinct) topologies $\mathscr{T}_1$, $\mathscr{T}_2$ and is denoted by $(X,\mathscr{T}_1,\mathscr{T}_2)$. For the real line $\mathbb{R}$, the topology $\mathscr{T}_l=\{\emptyset,\mathbb{R}\}\cup\{(x,\infty):$

$x \in \mathbb{R}\}$ is called the left topology on $\mathbb{R}$, while the topology $\mathscr{T}_u = \{\emptyset, \mathbb{R}\} \cup \{(-\infty, x) : x \in \mathbb{R}\}$ is called the right topology on $\mathbb{R}$ and so, $(\mathbb{R}, \mathscr{T}_l, \mathscr{T}_u)$ is a bitopological space [35, 36]. The relationship between quasi-metric and bitopological spaces are well researched in [25].

Definition 2.1.17. A bitopological space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is said to be quasi-metrizable if there exists a quasi-metric q on X such that $\mathscr{T}_1 = \mathscr{T}_q$ and $\mathscr{T}_2 = \mathscr{T}_{q^{-1}}$ ([21], [39])

The following describe some properties of maps between two quasi metric spaces.

Definition 2.1.18. Let (X, q_X) and (Y, q_Y) be quasi-metric spaces. A map $f : (X, q_X) \longrightarrow (Y, q_Y)$ is said to be:

(i) (q_X, q_Y)-uniformly continuous provided that for each $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$, $q_Y(f(x), f(y)) < \epsilon$ whenever $q_X(x, y) < \delta$;

(ii) a homeomorphism if it is a continuous bijective mapping and its inverse is continuous.

(iii) an isometry provided that $q_Y(f(x), f(y)) = q_X(x, y)$ whenever $x, y \in X$, that is, f is distance preserving. Two quasi metric spaces (X, q_X) and (Y, q_Y) will be called isometric provided that there exists a bijective isometry $f : (X, q_X) \longrightarrow (Y, q_Y)$ between them. It is clear that an isometry is always injective, uniformly continuous, and is a homeomorphism of X onto $f(X)$;

(iv) nonexpansive provided that $q_Y(f(x), f(y)) \leq q_X(x, y)$ whenever $x, y \in X$.

Definition 2.1.19. Let $(X, \mathscr{T}_q, \mathscr{T}_{q^{-1}})$ be a bitopological space, with $x \in X$. Then;

(i) X is said to be countable if there exist an onto function from $\mathbb{N}$ to X.

(ii) X is said to be first-countable if it has a countable basis at each of its points $x \in X$. By a countable basis at $x \in X$, we mean a sequence $\{U_n\}_{n \in \mathbb{N}}$ of neighborhoods of x such that if U is a neighborhood of x, then $x \in U_n \subset U$ for some n. The sets U_n need not be distinct.

(iii) X is said to be second-countable if it has a countable basis for its asymmetric topologies.

(v) A point $x \in X$ is an accumulation point (or limit point) of $A \subseteq X$ if there exists $a \in A - \{x\}$ such that $a \in B_{q^s}(x, \delta)$, $\forall \delta > 0$, that is, every $\mathscr{T}_{q^s}$ ball centered at x contains at least one point of A different from x.

(vi) $A \subseteq X$ is an (i,j)-dense subset in X if $cl_{\mathscr{T}_i}(cl_{\mathscr{T}_j}A) = X$, where $i,j = 1,2$. Basically a subset A is called pairwise-dense in $(X, \mathscr{T}_q, \mathscr{T}_{q^{-1}})$, if $V \cap A \neq \emptyset$ for every pairwise open set V. On the other hand, for any random point x in a large space X, one can draw a circle around x using a random $a \in A$ as a radius and some element of the circle will be in A.

If two elements x, y of a quasi-metric space are not equal, it is possible to put these elements in two separate open sets. This is not always possible in a general asymmetric topological space unless the asymmetric topological space is quasi-metrizable. We introduce, following Kelly [21], some useful separation properties specific to a bitopological space.

Definition 2.1.20. A space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is said to be pairwise Hausdorff (T_2) if, for each two distinct points x and y, there is a $\mathscr{T}_1$-neighborhood U of x and $\mathscr{T}_2$-neighborhood V of y such that $U \cap V = \emptyset$.

Definition 2.1.21. In a space $(X, \mathscr{T}_1, \mathscr{T}_2)$, $\mathscr{T}_1$ is said to be regular with respect to $\mathscr{T}_2$ if, for each point x in X ,there is a $\mathscr{T}_1$-neighborhood base of $\mathscr{T}_2$-closed sets ,or, as is easily seen to be equivalent, if, for each point x in X and each $\mathscr{T}_1$-closed set F such that $x \notin F$, there is a $\mathscr{T}_1$-open set U and a $\mathscr{T}_2$-open set V such that

$$x \in U,\ F \subset V, \text{ and } U \cap V = \emptyset.$$

$(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise regular if $\mathscr{T}_1$ is regular with respect to $\mathscr{T}_2$ and vice-versa.

Definition 2.1.22. A space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is said to be pairwise normal if, given $\mathscr{T}_1$-closed set A and $\mathscr{T}_2$-closed set B with $A \cap B = \emptyset$, there exist a $\mathscr{T}_2$-open set U and a $\mathscr{T}_1$ -open set V such that $A \subset U$,$B \subset V$, and $U \cap V = \emptyset$.

Equivalently, $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise normal if, given a $\mathscr{T}_2$ -closed set C and a $\mathscr{T}_1$ open set D such that $C \subseteq D$, there are a $\mathscr{T}_1$-open set G and a $\mathscr{T}_2$-closed set F such that

$$C \subseteq G \subseteq F \subset D.$$

Definition 2.1.23. Let $(X, \mathscr{T}_q, \mathscr{T}_{q^{-1}})$ be a bitopological space. Two quasi-metrics on X will be called bitopologically equivalent if they define the same asymmetric topologies on X. Generally, quasi-metrics q_0, q_1 on a set X are called uniformly equivalent if for every $\epsilon \in (0, \infty)$, there exists a $\delta_0, \delta_1 \in (0, \infty)$ such that $\forall x, y \in X$, $\forall i \in \{0, 1\}$ the following condition holds:

$$q_i(x,y) < \delta_i \Longrightarrow q_{1-i}(x,y) < \epsilon.$$

Proposition 2.1.24. *Let q and ρ be quasi-metrics on X inducing the asymmetric topologies $\mathscr{T}_q$, $\mathscr{T}_{q^{-1}}$ and $\mathscr{T}_\rho$, $\mathscr{T}_{\rho^{-1}}$ respectively. Then, $\mathscr{T}_{q^s}$ is finer than $\mathscr{T}_{\rho^s}$ (or $\mathscr{T}_{\rho^s}$ is a coarser topology than $\mathscr{T}_{q^s}$) if and only if for all $x \in X$ and $\epsilon > 0$, there is a $\delta > 0$ such that $B_{q^s}(x, \delta) \subseteq B_{\rho^s}(x, \epsilon)$.*

Proof. Suppose first that $\mathscr{T}_{\rho^s} \subset \mathscr{T}_{q^s}$. Let $x \in X$ and $\epsilon > 0$. $B_{\rho^s}(x, \epsilon)$ is open in $\mathscr{T}_{\rho^s}$, so it's open in $\mathscr{T}_{q^s}$. Since the q^s-open balls form a basis for $\mathscr{T}_{q^s}$, then $\forall x \in X$, there is a $\delta > 0$ such that

$$x \in B_{q^s}(x, \delta) \subseteq B_{\rho^s}(x, \epsilon)$$

by the definition of a basis.

Conversely, suppose that for all $x \in X$ and $\epsilon > 0$, there is a $\delta > 0$ such that $B_{q^s}(x, \delta) \subseteq B_{\rho^s}(x, \epsilon)$. We need to show that $\mathscr{T}_{\rho^s} \subset \mathscr{T}_{q^s}$. Let U be open in $\mathscr{T}_{\rho^s}$, we must show that it is open in $\mathscr{T}_{q^s}$. Let $x \in U$. Since the ρ^s open balls forms a basis for $\mathscr{T}_{\rho^s}$, there is an $\epsilon > 0$ such that

$$x \in B_{\rho^s}(x, \epsilon) \subset U.$$

By assumption, there is a $\delta > 0$ such that $x \in B_{q^s}(x, \delta) \subseteq B_{\rho^s}(x, \epsilon)$. Thus, $x \in B_{q^s}(x, \delta) \subset U$. Since $x \in U$ was arbitrary, U is open in $\mathscr{T}_{q^s}$. Therefore, $\mathscr{T}_{\rho^s} \subset \mathscr{T}_{q^s}$. □

The following results show the inversion of Remarks 2.1.6 and 2.1.8.

Lemma 2.1.25. *([10]). Let (X, q) be a quasi-metric space, then whenever $x \in X$ and $\delta > 0$,*

(i) $B_q(x, \delta)$ is $\mathscr{T}_q$ open, $B_{q^{-1}}(x, \delta)$ is $\mathscr{T}_{q^{-1}}$ open and $C_q(x, \delta)$ is $\mathscr{T}_{q^{-1}}$ closed. The ball $C_q(x, \delta)$ need not be $\mathscr{T}_q$ closed.

(ii) $B_{q^s}(x, \delta) \subseteq B_{q^{-1}}(x, \delta)$ and $B_{q^s}(x, \delta) \subseteq B_q(x, \delta)$. Similarly, for $\delta \geq 0$, $C_{q^s}(x, \delta) \subseteq C_{q^{-1}}(x, \delta)$ and $C_{q^s}(x, \delta) \subseteq C_q(x, \delta)$.

(iii) the topologies $\mathscr{T}_q$ and $\mathscr{T}_{q^{-1}}$ are T_0; but not necessarily T_1 (and so nor T_2, in contrast to the case of metric spaces). The topology $\mathscr{T}_q$ is T_1 if and only if $q(x, y) > 0$ whenever $x \neq y$. In this case, $\mathscr{T}_{q^{-1}}$ is also T_1 and, as a bitopological space, X is pairwise Hausdorff.

Proof. The proof is taken from [10].

(i) For $y \in X$ such that $q(x,y) < \delta$, let $\epsilon = \delta - q(x,y) > 0$. If $z \in X$ is such that $q(y,z) < \epsilon$, then $q(x,z) \leq q(x,y) + q(y,z) < q(x,y) + \epsilon = \delta$ showing that $B_q(x,\epsilon) = \{z \in X : q(x,z) < \epsilon\} \subset \{y \in X : q(x,y) < \delta\} = B_q(x,\delta)$. Hence $B_q(x,\delta)$ is $\mathscr{T}_q$ open.

Similarly, for $y \in X$ with $q(x,y) > \delta$ take $\epsilon = q(x,y) - \delta > 0$. If $z \in X$ is such that $q(z,y) = q^{-1}(y,z) < \epsilon$, then $q(x,y) \leq q(x,z) + q(z,y) < q(x,z) + \epsilon$, so that $q(x,z) > q(x,y) - \epsilon = \delta$. Consequently, $B_{q^{-1}}(x,\epsilon) = \{z \in X : q(x,z) > \epsilon\} \subset \{y \in X : q(x,y) > \delta\} = B_{q^{-1}}(x,\delta)$. So, $B_{q^{-1}}(x,\delta)$ is $\mathscr{T}_{q^{-1}}$-open. This equivalently shows that, the mapping $y \longrightarrow q(x,y)$ (i. e., $q(x,\cdot)$) is $\mathscr{T}_q$ upper continuous and $\mathscr{T}_{q^{-1}}$ lower continuous. The $\mathscr{T}_q$ lower and $\mathscr{T}_{q^{-1}}$ upper continuity of $x \longrightarrow q(x,y)$ also follows easily.

Next, to prove that $C_q(x,\delta)$ is $\mathscr{T}_{q^{-1}}$-closed, let $y \in X$ be such that $y \in X \setminus C_q(x,\delta)$ with $q(x,y) > \delta$. Put $\epsilon = q(x,y) - \delta > 0$. Then $B_{q^{-1}}(x,\epsilon) \cap C_q(x,\delta) = \emptyset$, or equivalently, $B_{q^{-1}}(x,\epsilon) \subseteq X \setminus C_q(x,\delta)$. Indeed, if there exists an element $z \in B_{q^{-1}}(x,\epsilon) \cap C_q(x,\delta)$, with $q(z,y) = q^{-1}(y,z) < \epsilon$, then

$$q(x,y) \leq q(x,z) + q(z,y) \leq \delta + q(z,y) < \delta + \epsilon = q(x,y),$$

a contradiction. Consequently, $X \setminus C_q(x,\delta)$ is $\mathscr{T}_{q^{-1}}$ open and so, $C_q(x,\delta)$ is $\mathscr{T}_{q^{-1}}$ closed.

(ii) The inclusions follow from the inequalities $q(x,y) \leq q^s(x,y)$ and $q^{-1}(x,y) \leq q^s(x,y)$. Indeed, for if $q^{-1}(x,y) \leq q^s(x,y)$ from $q^s(x,y) = \max\{q(x,y), q^{-1}(x,y)\}$, $G \in \mathscr{T}_{q^s}$ is equivalent to the fact that for every $x \in G$ there exists a compatible quasi-metric q and $\delta > 0$ such that $B_{q^s}(x,\delta) \subset G$. Because $q(x,y) < \delta$ implies that $q^s(x,y) \leq q(x,y) < \delta$, we have $B_q(x,\delta) \subseteq B_{q^s}(x,\delta) \subset G$, so that $G \in \mathscr{T}_q$. As a result, $B_{q^s}(x,\delta) \subseteq B_q(x,\delta)$. Similarly, if $q(x,y) \leq q^s(x,y)$, then $G \in \mathscr{T}_{q^s}$ with $x \in G$ implies, there exist a compatible quasi-metric q and $\delta > 0$ such that $B_{q^s}(x,\delta) \subset G$. As $q(y,x) < \delta$ implies $q^s(x,y) \leq q(y,x) < \delta$, we have $B_{q^{-1}}(x,\delta) \subseteq B_{q^s}(x,\delta) \subset G$, so that $G \in \mathscr{T}_{q^{-1}}$. Hence, $B_{q^s}(x,\delta) \subseteq B_{q^{-1}}(x,\delta)$. Thus, $B_{q^s}(x,\delta) = B_q(x,\delta) \cap B_{q^{-1}}(x,\delta)$ and also $C_{q^s}(x,\delta) = C_q(x,\delta) \cap C_{q^{-1}}(x,\delta)$. We infer that the topology $\mathscr{T}_{q^s}$ is finer than the asymmetric topologies $\mathscr{T}_q$ and $\mathscr{T}_{q^{-1}}$.

(iii) If x, y are distinct points in the quasi-metric space (X,q) then $\max\{q(x,y), q(y,x)\} > 0$. If $q(x,y) > 0$, then $y \notin B_q(x,\epsilon)$ where $\epsilon = q(x,y)$. Similarly, if $q(y,x) > 0$, then $x \notin B_q(y,\delta)$, where $\delta = q(y,x)$. Consequently, $\mathscr{T}_q$ is T_0 and $\mathscr{T}_{q^{-1}}$ as well.

Next, suppose that $q(x,y) > 0$ for every $x \neq y$. Then $y \notin B_q(x, q(x,y))$. Since $q(y,x) > 0$ too, $x \notin B_q(y, q(y,x))$, showing that the topology $\mathscr{T}_q$ is T_1. Similarly $\mathscr{T}_{q^{-1}}$ is T_1. Conversely suppose that $\mathscr{T}_q$ is T_1 and let $x, y \in X$, $x \neq y$. Then, there exists a quasi-metric q and $\delta > 0$ such that $x \notin B_q(y,\delta)$, implying $q(x,y) \geq \delta$.

Also, $B_q(x,\delta) \cap B_{q^{-1}}(y,\delta) = \emptyset$ where $\delta > 0$ is given by $2\delta = q(x,y) > 0$. Indeed, if $z \in B_q(x,\delta) \cap B_{q^{-1}}(y,\delta)$, then

$$q(x,y) \leq q(x,z) + q(z,y) < \delta + \delta = q(x,y)$$

a contradiction which shows that $(X, \mathscr{T}_q, \mathscr{T}_{q^{-1}})$ is pairwise Hausdorff.

□

Any unbounded quasi-metric can be converted to a bounded quasi-metric while preserving the asymmetric topologies in the following way:

Definition 2.1.26. Let (X,q) be a quasi-metric space. If q is unbounded, it can be converted to a bounded quasi-metric in the form $q'(x,y) = \min\{1, q(x,y)\}$ for each $x, y \in X$ while preserving the asymmetric topologies, $\mathscr{T}_{q'} = \mathscr{T}_q$ and $\mathscr{T}_{(q')^{-1}} = \mathscr{T}_{q^{-1}}$. q' is called the truncation or bona-fade of q and, q and q' are bitopologically equivalent.

Proposition 2.1.27. *Let (X,q) be a quasi-metric space. Define $q' : X \times X \longrightarrow \mathbb{R}$ by $q'(x,y) = \min\{1, q(x,y)\}$ for all $x, y \in X$. Then,*

(i) q' is a quasi-metric.

(ii) q and q' induce the same asymmetric topologies on X.

Proof. (i) Let $x, y \in X$. Since $q(x,y) \geq 0$, $q'(x,y) = \min\{1, q(x,y)\} \geq 0$, and $q'(x,x) = \min\{1, q(x,x)\} = \min\{1, 0\} = 0$.

If $q'(x,y) = \min\{1, q(x,y)\} = 0 = \min\{1, q(y,x)\} = q'(y,x)$, then $q(x,y) = 0 = q(y,x)$, so $x = y$. Thus, the T_0 condition holds.

To verify the triangle inequality, let $x, y, z \in X$. Now, if either $q(x,y) \geq 1$ or $q(y,z) \geq 1$, then $q'(x,y) = 1$ or $q'(y,z) = 1$. Thus, $q'(x,y) + q'(y,z) = 2 \geq 1 = q'(x,z)$.

Assume that $q(x,y) < 1$ and $q(y,z) < 1$ then

$$q'(x,y) + q'(y,z) = q(x,y) + q(y,z) \geq q(x,z) \geq q'(x,z).$$

Hence, q' is a quasi-metric and if defined on $\mathbb{R} \cup \{\infty\}$, it becomes an extended quasi-metric.

(ii) Let $x \in X$. Then, by applying Proposition 2.1.24 and shrinking the balls if necessary to make their radii less than 1 (i.e., $0 < \delta < 1$), we have that

$$x \in B_{q^s}(x, \delta) = B_{(q^s)'}(x, \delta).$$

If $\delta \geq 1$ then $x \in B_{q^s}\left(x, \frac{\delta}{2}\right) \subseteq B_{(q^s)'}(x, \delta)$.

Therefore, the q^s topology is finer than the $(q^s)'$ topology. The other inclusion follows by swapping the q^s and $(q^s)'$.

□

2.2. Some concepts on completeness and compactness in quasi-metric spaces

The lack of symmetry in the definition of a quasi-metric causes a lot of troubles, mainly concerning completeness, compactness and total boundedness in such spaces. There are a lot of completeness notions in quasi-metric spaces, all agreeing with the usual notion of completeness in the case of metric spaces, each of them having its advantages and weaknesses, see for example [10], [21], [25], [32,33]. For a quasi-metric space (X, q), we shall present briefly following [33] some of these notions of completeness.

Definition 2.2.1. Let (X, q) be a quasi-metric space and (x_n) be a sequence in X.

(i) The convergence of a sequence (x_n) to x with respect to $\mathscr{T}_q$ (respectively $\mathscr{T}_{q^{-1}}$), called left(respectively right) q-convergence and denoted by, $x_n \xrightarrow{q} x$ (respectively $x_n \xrightarrow{q^{-1}} x$) as $n \to \infty$, is defined in the following way: $\lim\limits_{n \xrightarrow{q} \infty} x_n = x \iff \lim\limits_{n \longrightarrow \infty} q(x, x_n) = 0$, (respectively $\lim\limits_{n \xrightarrow{q^{-1}} \infty} x_n = x \iff \lim\limits_{n \longrightarrow \infty} q(x_n, x) = 0$).

(ii) A sequence (x_n) q^s-converges to x if it is both left q and right q-convergent to x. Hence $x_n \xrightarrow{q^s} x \iff x_n \xrightarrow{q} x$ and $x_n \xrightarrow{q^{-1}} x$.

Definition 2.2.2. Let q be a quasi-metric on X. A sequence (x_n) in (X, q) is said to be

(i) left q-Cauchy if for each $\epsilon > 0$, there exists $x \in X$ and $n_\epsilon \in \mathbb{N}$ such that $\forall n \geq n_\epsilon$, $q(x, x_n) < \epsilon$. Equivalently it implies, $\forall \epsilon > 0$, $\exists n_\epsilon \in \mathbb{N}$ such that $x_n \in B_q(x, \epsilon)$, $\forall n > n_\epsilon$;

(ii) right q-Cauchy if for each $\epsilon > 0$, there exists $x \in X$ and $n_\epsilon \in \mathbb{N}$ such that $\forall n \geq n_\epsilon$, $q(x_n, x) < \epsilon$. Equivalently, $\forall \epsilon > 0$, $\exists n_\epsilon \in \mathbb{N}$ such that $x_n \in B_{q^{-1}}(x, \epsilon)$, $\forall n > n_\epsilon$;

(ii) q^s-Cauchy if for each $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $\forall k, n \geq n_\epsilon$, $q(x_n, x_k) < \epsilon$;

(iv) left K-Cauchy if for each $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $\forall k, n : n \geq k \geq n_\epsilon$, $q(x_k, x_n) < \epsilon$;

(v) right K-Cauchy if for each $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $\forall k, n : n \geq k \geq n_\epsilon$, $q(x_n, x_k) < \epsilon$;

(vi) weakly left (right) K-Cauchy if for each $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $\forall n : n_\epsilon \leq n, q(x_{n_\epsilon}, x_n) < \epsilon$, (respectively $q(x_n, x_{n_\epsilon}) < \epsilon$).

The following results concerning sequences in quasi-metric spaces are true.

Proposition 2.2.3. *Let (x_n) be a sequence in a quasi-metric space (X, q).*

(i) If (x_n) is $\mathscr{T}_q$-convergent to x and $\mathscr{T}_{q^{-1}}$-convergent to y, then $q(x, y) = 0$.

(ii) If (x_n) is $\mathscr{T}_q$-convergent to x and $q(y, x) = 0$, then (x_n) is also $\mathscr{T}_q$-convergent to y.

(iii) If (x_n) is left K-Cauchy and has a subsequence which is $\mathscr{T}_q$-convergent to x, then (x_n) is $\mathscr{T}_q$-convergent to x.

(iv) If (x_n) is left K-Cauchy and has a subsequence which is $\mathscr{T}_{q^{-1}}$-convergent to x, then (x_n) is $\mathscr{T}_{q^{-1}}$-convergent to x.

Proof. This proof comes from Cobzas [10]

(i) Fix $\delta > 0$. By assumption, $x_n \xrightarrow{q} x$ so there exists $N_1 \in \mathbb{N}$ such that $q(x, x_n) < \delta$ for all $n \geq N_1$. Also, $x_n \xrightarrow{q^{-1}} y$, so there exists $N_2 \in \mathbb{N}$ such that $q(x_n, y) < \delta$ for all $n \geq N_2$. Then for all $n \geq N := \max\{N_1, N_2\}$, $q(x, y) \leq q(x, x_n) + q(x_n, y) < \delta$. As δ was arbitrary, we deduce that $q(x, y) = 0$, which implies $x = y$. Equivalently, letting $n \longrightarrow \infty$ in the inequality $q(x, y) \leq q(x, x_n) + q(x_n, y)$, one obtains $q(x, y) = 0$.

(ii) Follows from the relations $q(y, x_n) \leq q(y, x) + q(x, x_n) = 0 + q(x, x_n) = q(x, x_n) \longrightarrow 0$ as $n \longrightarrow \infty$.

(iii) Suppose that (x_n) is left K-Cauchy and (x_{n_k}) is a subsequence of (x_n) such that $\lim_{k \longrightarrow \infty} q(x, x_{n_k}) = 0$. For $\delta > 0$ choose n_0 such that $n_0 \leq m \leq n$ implies $q(x_m, x_n) < \delta$, and let $k_0 \in \mathbb{N}$ be such that $n_{k_0} \geq n_0$ and $q(x, x_{n_k}) < \delta$ for all $k \geq k_0$. Then, for $n \geq n_{k_0}$, $q(x, x_n) \leq q(x, x_{n_{k_0}}) + q(x_{n_{k_0}}, x_n) < 2\delta$.

(iv) Suppose that (x_n) is left K-Cauchy such that there exists a subsequence (x_{n_k}) which is $\mathscr{T}_{q^{-1}}$ -convergent to some $x \in X$: For $\delta > 0$ let $k_0 \in \mathbb{N}$ be such that for all $k \geq k_0$, $q(x_{n_k}, x) < \delta$, and let $n_0 \in \mathbb{N}$ be such that for all $m, n \in \mathbb{N}$, $n_0 \leq m < n$ implies $q(x_m, x_n) < \delta$. For $n \geq n_{k_0}$ let $k > k_0$ be such that $n_k \leq n$, $k \in \mathbb{N}$. Then $q(x_n, x) \leq q(x_n, x_{n_k}) + q(x_{n_k}, x) < 2\delta$.

□

Remark 2.2.4. Obviously, if q and q^{-1} are a pair of conjugate quasi-metrics on X, then a sequence is left Cauchy (in some sense) with respect to q if and only if it is right Cauchy (in the same sense) with respect to q^{-1}. And also, a sequence is q^s-Cauchy if and only if it is both left and right K-Cauchy if and only if it is both left and right q-Cauchy.

Definition 2.2.5. [33] Let (X, q) be a quasi-metric space. We say that (X, q) is

(i) left K-complete provided that any left K-Cauchy sequence in X is q-convergent.

(ii) right K-complete provided that any right K-Cauchy sequence in X is q^{-1}-convergent.

Definition 2.2.6. Let (X, q) be a T_0-quasi-metric space. Then (X, q) is called bicomplete provided that the associated metric space (X, q^s) is complete.

At this point, we present some results on compactness specific to quasi-metric spaces following [1], [9], [21], [32].

Definition 2.2.7. In a bitopological space $(X, \mathscr{T}_1, \mathscr{T}_2)$ the topology $\mathscr{T}_i$ is compact with respect to the topology $\mathscr{T}_j$ if for every i-open cover $\mathcal{U}$ of X and for each point $x \in X$ there is a j-neighborhood of x covered by a finite subfamily of $\mathcal{U}$, i.e., every family $\mathcal{U} = \{U_s\}_{s \in S}$ such that $\mathcal{U} \subset \mathscr{T}_1 \cup \mathscr{T}_2$, $X = \bigcup\limits_{s \in S} U_s$ and $\mathcal{U} \cap \mathscr{T}_1$ contains a nonempty set, has a finite subfamily.

Definition 2.2.8. [1] Let (X, q) be a quasi metric space. We say that a subset A of X is

(i) q-precompact if for all $\epsilon > 0$ we can find a finite set of points $\{a_1, ..., a_n\}$ in A such that $A \subset \bigcup_{i=1}^{n} B_q(a_i, \epsilon)$ for all $i \in \{1, 2, ..., n\}$.

(i) Outside q-precompact if for each $\epsilon > 0$ there is a finite set $\{x_1, ..., x_n\}$ in X such that $A \subset \bigcup_{i=1}^{n} B_q(x_i, \epsilon)$.

The same notions can be obtained if one works with closed balls. Obviously if a set A is q-precompact, then it is outside q-precompact; the converse is not true in general, a q-convergent sequence is outside q-precompact but it is not necessarily q-precompact.

Definition 2.2.9. [30, 32] Let $(X, \mathscr{T}_1, \mathscr{T}_2)$ be a bitopological space. $(X, \mathscr{T}_1, \mathscr{T}_2)$ is (i, j)-locally quasi-compact if each point $x \in X$ has an i-neighborhood $U_{(x)}$ such that $cl_{\mathscr{T}_j} U_{(x)}$ is quasi-compact. Equivalently, we say that $\mathscr{T}_1$ is locally compact with respect to $\mathscr{T}_2$ if each point of X has a $\mathscr{T}_1$ open neighbourhood whose $\mathscr{T}_2$ closure is pairwise compact.

It is clear that, pairwise compactness implies pairwise locally compactness. Note that, a pairwise locally compact bitopological space is not a pair of locally compact topological spaces as the example below shows.

Example 2.2.10. [32] Let X be the set of real numbers, $\mathscr{T}_1$ be the left hand topology with base the family $\{(-\infty, a) : a \in X\}$ and $\mathscr{T}_2$ the right hand topology. Then $(X, \mathscr{T}_1)$ is not locally compact, since for each $\mathscr{T}_1$ open set U we have $cl_{\mathscr{T}_1} U = X$, and $(X, \mathscr{T}_1)$ is not compact. Similarly, $(X, \mathscr{T}_2)$ is not locally compact. However, $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise locally compact because it is pairwise compact.

Lemma 2.2.11. *(Compare [14, Theorem 6.2]) Let $((X, q)$ be a pairwise Hausdorff space. Then the following are equivalent*

(i) X is pairwise locally compact.

(ii) For each $x \in X$ and each $\mathscr{T}_1$ open neighborhood $U_{(x)}$ of x, there is a pairwise relatively compact $\mathscr{T}_1$ open set V with $x \in V \subseteq cl_{\mathscr{T}_2} V \subseteq U$ and $cl_{\mathscr{T}_2} V$ is pairwise compact.

(iii) For each pairwise compact set K and $\mathscr{T}_1$ open set $U \supset K$, there is a pairwise relatively compact $\mathscr{T}_1$ open set V with $K \subseteq V \subseteq cl_{\mathscr{T}_2} V \subseteq U$.

(iv) X has a basis consisting of pairwise relatively compact open sets.

Proof. The proof comes from [32].

(i) $\Rightarrow$ (ii) Let $x \in X$ and $U_{(x)}$ be the $\mathscr{T}_1$ neighborhood of x. There is some $\mathscr{T}_1$ open set W in X with $x \in W \subset cl_{\mathscr{T}_2} W$ and $cl_{\mathscr{T}_2} W$ is pairwise compact. Since the space $cl_{\mathscr{T}_2} W$ is pairwise compact and pairwise Hausdorff, it is hence pairwise regular and $cl_{\mathscr{T}_2} W \cap U$ is a neighborhood of x in $cl_{\mathscr{T}_2} W$. Thus, there is a $\mathscr{T}_1$ open set G in $cl_{\mathscr{T}_2} W$ such that $x \in G \subseteq cl_{\mathscr{T}_2} G_{(cl_{\mathscr{T}_2} W)} \subset cl_{\mathscr{T}_2} W \subset cl_{\mathscr{T}_2} W \cap U$. Now, $G = E \cap cl_{\mathscr{T}_2} W$ for some $\mathscr{T}_1$ open subset E of X. If $V = E \cap W$, then V is a $\mathscr{T}_1$ set containing x, and

$$cl_{\mathscr{T}_2} V = cl_{\mathscr{T}_2} V \cap cl_{\mathscr{T}_2} W = cl_{\mathscr{T}_2} V_{(cl_{\mathscr{T}_2} W)}$$

so that, $cl_{\mathscr{T}_2} V$ is pairwise compact. Moreover, $x \in V \subset cl_{\mathscr{T}_2} V \subset cl_{\mathscr{T}_2} G_{(cl_{\mathscr{T}_2} W)} \subset U$.

(ii) ⇒ (iii) For each $c \in K$, there is a relatively compact $\mathscr{T}_1$ neighborhood $V_{(c)}$ with $cl_{\mathscr{T}_2}V_{(c)} \subset U$. Since K is pairwise compact, finitely many of these $\mathscr{T}_1$ neighborhoods cover K and this union has $\mathscr{T}_2$ compact closures which is pairwise compact, as it is a finite union of pairwise compact subsets of a pairwise Hausdorff space [1].

(iii) ⇒ (iv) Let $\mathscr{B}_o$ be a family of all pairwise relatively compact subsets of X. Since X is pairwise Hausdorff and locally compact, for each $x \in X$, every $\mathscr{T}_1$ open set in $\mathscr{B}_0$ is $\mathscr{T}_2$ closed and bounded, and (iii) asserts that $\mathscr{B}_0$ is a basis.

(iv) ⇒ (i) Let $x \in X$ and $\mathscr{B}_0$ be a base of X consisting of pairwise relatively compact open subsets. Since each $\mathscr{T}_1$ member of $\mathscr{B}_0$ has $\mathscr{T}_2$ compact closures and x lies at least in one of these members, finitely many of these $\mathscr{T}_1$ neighborhoods cover X and the union has $\mathscr{T}_2$ compact closures. Hence, X is pairwise locally compact as it is a union of pairwise relatively compact sets. □

Lemma 2.2.12. *(Compare [14, Theorem 6.3]) Let (X, q) be a second countable quasi-metric space. If X is pairwise-locally compact, it has a countable basis of pairwise-relatively compact open sets.*

Proof. Let X be a pairwise locally compact space satisfying the second axiom of countability and $\{U_n : n \in \mathbb{N}\}$ be a basis formed by $\mathscr{T}_1$ open sets in X. Since X is pairwise locally compact, for each fixed n and $x \in U_n$ there is a family $\{V_{(x)} : x \in U_n\}$ of $\mathscr{T}_1$ open sets covering U_n such that $x \in V \subset cl_{\mathscr{T}_2}V_{(x)} \subset U_n$ [1,21]. Since a subspace of a second countable space is second countable, we can extract countable sub-coverings $\{V_{n,i} : i \in \mathbb{N}\}$ of U_n. Repeating for each n, the family of pairwise relatively compact sets $\{V_{n,i} : (n, i) \in \mathbb{N} \times \mathbb{N}\}$ forms a countable basis of X. □

2.3. Some concepts on a Hausdorff quasi-metric

Asymmetric variants of the Hausdorff metric provide further examples of quasi-metrics. For a quasi-metric space (X, q), we shall present briefly some notions of a Hausdorff-quasi-metric following [36].

Definition 2.3.1. Let A and B be nonempty subsets of a metric space (X, d). Then, we define the gap D_d and the excess e_d of A over B with respect to the metric d by

$$D_d(A, B) = \inf\{d(a, b) : a \in A, b \in B\} = \inf\{d(a, B) : a \in A\} = \inf_{a \in A} d(a, B)$$

and

$$e_d(A,B) = \sup\{d(a,b) : a \in A, b \in B\} = \sup_{a\in A} d(a,B) = \sup_{a\in A}\{\inf\{q(a,b) : b \in B\}\}.$$

Gap is symmetric in A and B while excess is not; additionally, excess can assume values of infinity, and this will occur if A is unbounded and B is bounded (see [3, 39]).

Definition 2.3.2. Let (X,d) be a metric space. For any two nonempty subsets A, B of X, a map $d_H : \mathscr{P}_0(X) \times \mathscr{P}_0(X) \longrightarrow \mathbb{R}_+$ define by

$$d_H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\} = \inf\{\delta > 0 : A, B \in [B]_d^\delta\}$$

is called the Hausdorff metric between A and B. Observe that: $D_d(A,B) \leq d_H(A,B)$. The quantity $e_d(A,B) = \sup_{a\in A} d(a,B)$ is sometimes termed as the semi-(Hausdorff) distance from the set A to the set B.

Clearly, if $\{a,b\} \subseteq X$, then $d_H(\{a\},\{b\}) = d(a,b)$ so that distance so defined is an extension of ordinary d-distance. When the metric d is unbounded, we can obtain an infinite Hausdorff metric, for whenever $x \in X$, we get $d_H(\{x\}, X) = \infty$. Notice that the Hausdorff distance is invariant under replacing sets by their closures, and in particular, the Hausdorff distance between a set A and its closure must be zero. Further, d_H as defined also makes sense for arbitrary closed sets, and if restricted to the family $\mathscr{C}_0(X)$ of nonempty closed subsets of X, one yields an infinite valued metric which is not a metric unless the space (X,d) is bounded. Hausdorff distance on a variety of set structures has been extremely well studied by various scholars e.g., see [3, 7], [36], [39], etc.

We omit the proof of a Hausdorff metric being a metric as it follows from the properties of the Hausdorff quasi-metric given below.

Definition 2.3.3. Let (X,q) be a T_0-quasi-metric space. Denote by q_H^+, q_H^- and q_H the maps $\mathscr{P}_0(X) \times \mathscr{P}_0(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ where for all $A, B \in \mathscr{P}_0(X)$,

$$q_H^+(A,B) = \sup_{a\in A} q(a,B), \quad q_H^-(A,B) = \sup_{b\in B} q(A,b)$$

and

$$q_H(A,B) = \max\{q_H^+(A,B), q_H^-(A,B)\}.$$

Lemma 2.3.4. *Let (X, q) be a T_0-quasi-metric space. Then q_H^+, q_H^-, and q_H as defined are extended quasi-metrics.*

Proof. The proof is taken from [36]. For any $A \in \mathscr{P}_0(X)$ it is obvious that,

$$q_H^+(A, A) = q_H^-(A, A) = q_H(A, A) = 0$$

as q is a T_0-quasi-metric. To prove the triangle inequality let $A, B, C \in \mathscr{P}_0(X)$. Take any $a \in A$, $b \in B$, then we have $q(a, C) \leq q(a, b) + q(b, C) \leq q(a, b) + q_H^+(B, C)$, by the definition of q_H^+. Now, taking infimum over $b \in B$ on both sides gives $q(a, C) \leq q(a, B) + q_H^+(B, C)$. Taking supremum over $a \in A$ on both sides we get

$$q_H^+(A, C) \leq q_H^+(A, B) + q_H^+(B, C).$$

Similarly, by noting that $q_H^-(A, B) = \sup_{b \in B} q(A, b) = \sup_{b \in B} q^{-1}(b, A)$, we have $q^{-1}(b, C) = q(C, b) \leq q(C, a) + q(a, b) \leq q_H^-(C, A) + q(a, b)$ by the definition of $q_H^-(A, B)$. Hence, $q(C, b) \leq q_H^-(C, A) + q(A, b)$. Taking supremum over $b \in B$ on both sides we get

$$q_H^-(C, B) \leq q_H^-(C, A) + q_H^-(A, B).$$

It is obvious that if both q_H^+ and q_H^- satisfy the triangle inequality, so does q_H. □

Lemma 2.3.5. *Let (X, q) be a quasi-metric space with $d = q^s$, the associated metric. Then for any $A, B \in \mathscr{P}_0(X)$,*

$$d_H^+(A, B) = \max\{q_H^+(A, B), q_H^-(B, A)\} \text{ and } d_H^-(A, B) = \max\{q_H^-(A, B), q_H^+(B, A)\}.$$

Proof. The proof is taken from [36]. By the definition and positivity of any distance $q(a, B)$, we have

$$\begin{aligned} \max\{q_H^+(A, B), q_H^-(B, A)\} = \max\left\{\sup_{a \in A} q(a, B),\ \sup_{b \in B} q(b, A)\right\} &= \sup_{a \in A} \max\{q(a, B), q(B, a)\} \\ &= \sup_{a \in A} d(a, B) \\ &= d_H^+(A, B). \end{aligned}$$

Similarly,

$$\begin{aligned}\max\{q_H^-(A,B), q_H^+(B,A)\} = \max\left\{\sup_{a\in A} q(a,B),\ \sup_{b\in B} q(b,A)\right\} &= \sup_{b\in B}\max\{q(A,b), q(b,A)\}\\ &= \sup_{b\in B} d(A,b)\\ &= d_H^-(A,B).\end{aligned}$$

□

Lemma 2.3.6. *[36] Let (X, q) be a quasi-metric space. Then q_H restricted to $\mathscr{C}_0(X, q)$ satisfying $\mathscr{C}_0(X, q) \supset A = cl_{\mathscr{T}_q} A \cap cl_{\mathscr{T}_{q^{-1}}} A$ whenever $A \subset X$ is an extended (Hausdorff) quasi-metric and restricted to $\mathscr{K}_0(X, q)$ is a (Hausdorff) quasi-metric.*

Proof. To show that q_H is an extended quasi-metric, only the separation axiom needs to be provided as the rest follows by the Lemma 2.3.4.

Suppose $A, B \in \mathscr{C}_0(X, q)$ and $q_H(A, B) = 0 = q_H(B, A)$. Let $d = q^s$. By the Lemma 2.3.5, we have $d_H^+(A, B) = 0 = d_H^-(A, B)$. Now, if $d_H^+(A, B) = 0$, then for all $a \in A$ there exists $b \in B$ such that $d(a, b) = 0$ as B is closed, implying $a = b$ since d is a metric. Hence, $d^+H(A, B) = 0 \Longrightarrow A \subseteq B$. Similarly, if $d_H^-(A, B) = 0 \Longrightarrow B \subseteq A$ as $d_H^-(A, B) = q_H^+(B, A)$. Therefore, $q_H(A, B) = 0 = q_H(B, A)$ implies $A = B$.

If $A, B \in \mathscr{K}_0(X, q)$, for any $a \in A$, the function $a \longmapsto q(a, B)$ is continuous and bounded since A is compact ([36]). Hence $q_H(A, B) < \infty$ and thus q_H is a quasi-metric. □

CHAPTER 3 : THE STRUCTURE OF EXTENDED REAL-VALUED METRIC SPACES

This chapter is a literature review of the works done by Beer [7]. We investigate his resolutions to the fundamental questions he outlined as below:

(i) Given a metrizable space X, when can we exhibit or display upto isomorphsim, an extended real-valued metric space (W, D) which is universal for all extended real-valued metric spaces that can be built on X. This means that when ever d is an extended real-valued metric on X, we can find an isometry $\phi : (X, d) \longrightarrow (W, D)$.

(ii) Given a metrizable space X and a bornology $\mathscr{B}$ on X, when does there exist an extended real-valued metric d such that $\mathscr{B} = \mathscr{B}_d(X)$?

(iii) Given an extended real-valued metric d on X, when does there exist a bona fide metric d' on X such that $\mathscr{B}_{d'}(X) = \mathscr{B}_d(X)$?

For the sake of understanding by the reader and in preparation for generalization in chapter 3, we give proofs to most of the results.

3.1. Definitions and basic properties

Definition 3.1.1. ([7]) Let X be a set without any assumed structure. An extended real-valued metric defined on a set X, is a distance function $d : X \times X \longrightarrow [0, \infty]$ satisfying the conditions below $\forall x, y, z \in X$:

(i) $d(x, y) = 0$ if $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, z) \leq d(x, y) + d(y, z)$.

A set X endowed with an extended real-valued metric d is called an extended real-valued metric space denoted by (X, d).

Throughout this chapter, we shall call an extended real-valued metric d on a metrizable space X simply an extended metric, and a set equipped with such a metric (X, d) an extended metric space.

Next are some concepts in relation to metric components.

Definition 3.1.2. (Compare [18, Definition 10.3]) Let $((X, \mathscr{T}), d)$ be an extended metric space. Two extended metrics on X will be called topologically equivalent if they define the same topology on X. They will be called topologically and bornologically equivalent if they determine the same open sets and the same collection of bounded sets [17].

Definition 3.1.3. Let $((X, \mathscr{T}), d)$ be an extended metric space. Then for each $x, y \in X$, we define a natural relation $\mathrm{R_d}$ (or $\sim$) on X by $x \ \mathrm{R_d} \ \mathrm{y}$ (or $x \sim y$) provided $d(x, y) < \infty$.

Lemma 3.1.4. *[7] If (X, d) is an extended metric space, then the natural relation $\mathrm{R_d}$ is an equivalence relation on X.*

Proof. Let (X, d) be an extended metric space on which is defined a relation $\mathrm{R_d}$. Then, for each $x \in X$, $\infty > 0 = d(x, x)$ and so $x \ \mathrm{R_d} \ \mathrm{x}$ implying reflexivity. For each $x, y \in X$ let $x \ \mathrm{R_d} \ \mathrm{y}$. By the symmetrization of d, we have that $\infty > d(x, y) = d(y, x)$ implying $y \ \mathrm{R_d} \ \mathrm{x}$. Thus, $x \ \mathrm{R_d} \ \mathrm{y}$ if and only if $y \ \mathrm{R_d} \ \mathrm{x}$. Finally, for every $x, y, z \in X$ lets assume that $x \ \mathrm{R_d} \ \mathrm{y}$ and $y \ \mathrm{R_d} \ \mathrm{z}$ holds. We aim to show that $x \ \mathrm{R_d} \ \mathrm{z}$. By the triangle inequality, $d(x, z) \leq d(x, y) + d(y, z) < \infty + \infty = \infty$ which implies $x \ \mathrm{R_d} \ \mathrm{z}$ and so, transitivity holds. Thus, $\mathrm{R_d}$ is an equivalence relation. $\square$

Definition 3.1.5. ([11]) Let $((X, \mathscr{T}), d)$ be an extended metric space.

(i) We say that a subset A of X is clopen if A is both open and closed in X.

(ii) $(X, \mathscr{T})$ is said to be connected if X could not be represented as the union of disjoint sets A and B in $\mathscr{T}$.

(iii) A connected component of a topological space X is a maximal connected subset $Y \subseteq X$, i.e. if $Z \subseteq X$ is connected and $Z \supseteq Y$ then $Z = Y$. Two connected components of X can be equal or disjoint.

(iv) The metric-component $C_d(x)$ of $x \in X$ is the union of all clopen subsets of X containing x .i.e, $C_d(x) = \bigcup\{U_{(x)} : x \in U_{(x)}$ clopen subset of $X\}$. We will denote the metric-component of $x \in X$ by $mc_d(x)$.

Remark 3.1.6. As $\mathrm{R_d}$ is a natural equivalence relation on the extended metric space X, it provides a partition of X into equivalence classes. These equivalence classes of $\mathrm{R_d}$ are clopen subsets of X which are called metric components of X.

Observe that, if a metric component happens to be connected, it is already a connected component of X. Each point $x \in X$ is contained in a unique connected component of X, namely, $\bigcup\{Z \subset X : Z \text{ is connected and } x \in Z\}$. However, if we equip each metric component with the relative topology, the topology of X then is the free union topology, equivalently, X is the direct sum of (or partitioned by) these subspaces (see, e.g., ([41, p. 65])).

Lemma 3.1.7. *[7] Let $(X, \mathscr{T})$ be a (Hausdorff)-topological space. If X can be partitioned into nonempty clopen sets $\{X_i : i \in I\}$ and each X_i is metrizable, then choosing a compatible metric d_i for X_i, the functional d on $X \times X$ defined by*

$$d(x,y) = \begin{cases} d_i(x,y) \quad if \;\; \exists i \;\; with \;\; \{x,y\} \subseteq X_i \\ \\ \infty \;\; otherwise \end{cases}$$

is an extended metric compatible with the topology of X and has $\{X_i : i \in I\}$ as metric components.

***Proof*.** Let $x, y, z \in X$. If $\{x, y\} \subseteq X_i$, then $d(x, y) = d_i(x, y) \geq 0$ and $d(x, y) = d_i(x, y) = 0$ if and only if $x = y$. Also, $d(x, y) = d_i(x, y) = d_i(y, x) = d(y, x)$. If $\{x, y, z\} \subseteq X_i$ we clearly have that $d(x, z) = d_i(x, z) \leq d_i(x, y) + d_i(y, z) = d(x, y) + d(y, z)$.

Next, if $\{x\} \subseteq X_i$, $\{y\} \subseteq X_j$ for all $i \neq j$, then either $\{z\} \subseteq X_j$, $\{x\} \subseteq X_i$ or $\{z\} \subseteq X_i$, $\{y\} \subseteq X_j$:

If $\forall i \neq j$ $\{z\} \subseteq X_j$, $\{x\} \subseteq X_i$, then $\infty = d(x, y) + d(y, z) \geq d(x, z)$.

If $\forall i \neq j$ $\{z\} \subseteq X_i$, $\{y\} \subseteq X_j$, then $\infty = d(x, y) + d(y, z) \geq d(x, z)$. Thus, d as defined is an extended metric.

Finally, since d_i is a restriction of d on $X_i \times X_i$ and the family of nonempty clopen sets $\{X_i : i \in I\}$ partitions X, we infer that d as defined has $\{X_i : i \in I\}$ as the metric components. □

These concepts immediately lead to the following.

Definition 3.1.8. Let $(X, \mathscr{T}_q)$ be a topological space. Two subsets A, B of X are separated if the closure of each of them does not meet the other (this is equivalent to say that A and B are clopen in $A \cup B$). X is said to be connected if it cannot be partitioned in two separated sets. A subset S of X separates the nonempty sets A and B if the complement of S can be partitioned in two separated sets, one of which containing A, the other containing B.

Proposition 3.1.9. *([7]) A metrizable space X is connected if and only if each extended metric d on X is a metric.*

Proof. For sufficiency, suppose X is not connected, that is X admits a partition into two nonempty clopen sets X_1 and X_2. Then the construction

$$d(x_1, x_2) = \begin{cases} d_i(x_1, x_2) \quad if \ \ \exists i = 1, 2 \ \ with \ \ \{x_1, x_2\} \subseteq X_i \\ \\ \infty \ \ otherwise \end{cases}$$

produces an extended metric on X that assumes values of infinity since $x_1 \in X_1$, $x_2 \in X_2$ implies $d(x_1, x_2) = \infty$. Let $x \in X$ be arbitrary. Let $mc_d(x)$ be the metric component of x with respect to d. We claim that $\{mc_d(x)\}$ and $\{X \setminus mc_d(x)\}$ are separated. Now, $mc_d(x)$ is closed implies that $\overline{(mc_d(x))} = mc_d(x)$, and $mc_d(x)$ is open implies $X \setminus mc_d(x)$ is closed so that $\overline{(X \setminus mc_d(x))} = X \setminus mc_d(x)$ as well. By separability, $\emptyset = \overline{(mc_d(x))} \cap (X \setminus mc_d(x)) = mc_d(x) \cap (X \setminus mc_d(x)) = mc_d(x) \cap \overline{(X \setminus mc_d(x))}$. Since $X = (mc_d(x)) \cup (X \setminus mc_d(x))$, we see that $\{mc_d(x), X \setminus mc_d(x)\}$ is a partition of X into two nonempty clopen sets, that is, X has been written as a union of two nonempty separated sets, and so X is not connected. □

3.2. A universal space for extended metric spaces

In this section, we present works among others done by Beer [7] involving completeness and universality of extended-metric spaces. The universal space will be a space of partial functions and can be defined for any Hausdorff space X, that is, there is no need to assume metrizability.

We first present in the sense of Beer [7] an extended metric that will be useful throughout this chapter.

Definition 3.2.1. [23, 24] A partial function from a nonempty set X to a nonempty set Y denoted $f : X \rightharpoonup Y$, is a function with domain X_o and codomain Y, where X_o is some subset of X.

Definition 3.2.2. Let $((X, \mathscr{T}), d)$ be a (Hausdorff) metric space and $A \subseteq X$. We denote by $C(X)$ the set of continuous real-valued functions on X and within $C(X)$, we denote the set of bounded continuous real-valued functions by $C^b(X)$. If we let $X = \mathbb{R}$ and $C(A)$ set of continuous functions $f : A \longrightarrow \mathbb{R}$, then we can define a supremum metric on the set $A \times A$ by

$$d(f, g) = \sup_{x \in A} \mid f(x) - g(x) \mid .$$

Definition 3.2.3. [7] Let X be a metrizable space and $\Delta_X = \{(f, A) : f \in C(A)$ and $A \in \mathscr{C}_o(X)\}$. Then, for any compatible metric d on X, the map D on Δ_X defined by

$$D((f,A),(g,B)) = \begin{cases} \infty \quad if \quad A \neq B \\ \\ \sup_{a\in A} \mid f(a) - g(a) \mid \quad A = B \end{cases}$$

is an extended metric on Δ_X.

For completeness purpose, we first recall the following standard result.

Proposition 3.2.4. *Let X be a metrizable space. Then $(C(X), d)$ the space of all real-valued continuous functions on X, with metric*

$$d(f,g) = \sup_{x\in X} \mid f(x) - g(x) \mid$$

is complete.

Proposition 3.2.5. *[7] Let X be a Hausdorff space. Then (Δ_X, D) is a complete extended metric space.*

Proof. Let (f_n, A_n) be a Cauchy sequence in the space Δ_X. Lets assume without loss of generality that the partial functions all have a common domain A. Then, for each $t \in A$, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$D((f_n, A_n), (f_j, A_j)) = \sup_{t\in A} \mid f_n(t) - f_j(t) \mid < \epsilon, \quad A_n = A_j, \quad \forall j, n > N$$

where $A \in \mathscr{C}_o(X)$, $f_n, f_j \in C(A)$. Hence, for any fixed $t = t_o \in A$ we have

$$\mid f_n(t_o) - f_j(t_o) \mid \leq \epsilon \quad \forall j, n > N.$$

This shows that, $(f_n(t_o), A_n)_{n\in\mathbb{N}}$ is a Cauchy sequence of real numbers and so converges. In other words, this means that the function sequence (f_n, A_n) converges point-wise to (f, A).

We now show that, this point-wise convergence is actually uniform in $t \in A \in \mathscr{C}_o(X)$, that is, $\forall \epsilon \in (0, \infty), \exists N \in \mathbb{N}$ such that

$$\sup_{t\in A} \mid f_n(t) - f(t) \mid < \epsilon \quad \forall n > N.$$

Given $\epsilon \in (0, \infty)$ choose N such that $D((f_n, A_n), (f_j, A_j)) < \frac{\epsilon}{2}$, $\forall n, j > N$. Then for $n > N$,

$$\begin{aligned} \mid f_n(t) - f(t) \mid \leq \mid f_n(t) - f_j(t) \mid + \mid f_j(t) - f(t) \mid &\leq \sup_{t \in A} \{\mid f_n(t) - f_j(t) \mid + \mid f_j(t) - f(t) \mid\} \\ &= \sup_{t \in A} \mid f_n(t) - f_j(t) \mid + \mid f_j(t) - f(t) \mid \\ &= D((f_n, A_n), (f_j, A_j)) + \mid f_j(t) - f(t) \mid . \end{aligned}$$

By choosing j sufficiently large (j may depend on t), each term on the right-hand side can be made less than $\frac{\epsilon}{2}$ so that $\sup_{t \in A} \mid f_n(t) - f(t) \mid < \epsilon \ \forall n > N$. So, the sequence is uniformly Cauchy on A and thus uniformly convergent to some f. Since the f_n's are continuous on A and the convergence is uniform, it follows that, $f(t)$ is continuous on A. Hence, $\lim_{n \to \infty} D((f_n, A_n), (f, A)) = 0$. Since $D((f, A), (g, B)) = \infty \ \forall A \neq B$, (Δ_X, D) is a complete extended metric space. □

For the universality of an extended metric space we consider the following characterizations.

Definition 3.2.6. (cf. S. D. Iliadis [19,20]). A space X is said to be (topologically) universal in a class say $\mathbb{P}$ of spaces if:

(i) X is an element of $\mathbb{P}$, and

(ii) for every element Y of $\mathbb{P}$ there exists a topological embedding e^Y of Y into X.

If only condition (ii) is satisfied, then X is said to be (topologically) containing for the class $\mathbb{P}$. If, moreover, the space X and the elements of $\mathbb{P}$ are metric spaces and e^Y is considered to be an isometry, then X is said to be isometrically universal in $\mathbb{P}$ or isometrically containing for $\mathbb{P}$, respectively.

Definition 3.2.7. (cf. S. D. Iliadis- [19, 20]). Assume any arbitrary considered mapping to be continuous. For every mapping f we denote by D_f and R_f the domain and range of f, respectively.

(i) Let f and F be two mappings. A pair (ϕ, ψ), where ϕ is a topological embedding of D_f into D_F and ψ is a topological embedding of R_f into R_F such that $\psi \circ f = F \circ \phi$, is said to be a (topological) embedding of f into F. If, moreover, the spaces D_f, R_f, D_F, and R_F are metric and the embeddings ϕ and ψ are isometries, then the pair (ϕ, ψ) is called an isometric embedding of f into F.

(ii) A mapping F is said to be (topologically) universal in a class $\mathbb{F}$ of mappings if:

(a) F is an element of $\mathbb{F}$ and

(b) for every element f of $\mathbb{F}$ there exists a topological embedding $(\phi f, \psi f)$ of f into F.

If only condition (ii) is satisfied, then F is said to be (topologically) containing for the class $\mathbb{F}$. If, moreover, the domain and range of F, as well as, the domains and ranges of all elements of $\mathbb{F}$ are metric spaces and the embedding $(\phi f, \psi f)$ is considered to be isometric, then F is called isometrically universal in $\mathbb{F}$ or isometrically containing for $\mathbb{F}$, respectively, [also see [14], p. 286].

Generally, the concepts above imply the following definition:

Definition 3.2.8. (Compare [36, Definition 2.8.2]) Let $\mathscr{C}$ be a class of extended metric spaces. An extended metric space $\mathbb{V} = (\mathbb{U}, d_{\mathbb{U}})$ of class $\mathscr{C}$ is called universal or Urysohn if it satisfies the following properties:

i) For every extended metric space $X = (X, d_X)$ of class $\mathscr{C}$ there exists an isometric embedding $X \hookrightarrow \mathbb{U}$; (Universality).

(ii) For every two isometric finite extended metric subspaces F_1, F_2 of $\mathbb{U}$, the isometry $F_1 \leftrightarrow F_2$ extends to a global isometry $\mathbb{U} \leftrightarrow \mathbb{U}$; (Ultrahomogeneity).

Theorem 3.2.9. [7] Let X be a metrizable space. Then for each compatible extended metric d on X, $\phi : (X, d) \longrightarrow (\Delta_X, D)$ defined by

$$\phi(x) = (d(x, \cdot)|_{mc_d(x)}, mc_d(x))$$

where $x \in X$, is an isometry.

Proof. Choose fixed distinct points x_1 and x_2 in X. Let $\phi : X \longrightarrow \Delta_X$ be the map $x_1 \longmapsto f_{x_1}$ given by

$$f_{x_1}(x) = d(x_1, x) - d(x_2, x).$$

where we assume x to be from a common metric component.

We verify that $f_{x_1} \in C^b(X)$. By the triangle inequality, and each x in the common metric component say $mc_d(x_1)$ we have the inequality

$$| \, d(x_1, x) - d(x_2, x) \, | \leq d(x_1, x_2).$$

Hence $| \, f_{x_1}(x) \, | \leq d(x_1, x_2)$ and since $d(x_1, x_2)$ is a constant independent of x we infer that $f_{x_1} \in C^b(X)$.

Next, we prove that ϕ is an isometry, that is $D(\phi(x_1), \phi(x_2)) = d(x_1, x_2)$ by showing that

$$\sup_{x \in mc_d(x_1)} | \, f_{x_1}(x) - f_{x_2}(x) \, | = d(x_1, x_2) \qquad \forall x_1, x_2 \in X.$$

There are two cases to consider

Case 1 If $d(x_1, x_2) = \infty$, then $mc_d(x_1) \neq mc_d(x_2)$, and so by the definition of D,

$$D((d(x_1, x)|_{mc_d(x_1)}, mc_d(x_1)), (d(x_2, x)|_{mc_d(x_2)}, mc_d(x_2))) = \infty.$$

Case 2. If $d(x_1, x_2) < \infty$, then $mc_d(x_1) = mc_d(x_2)$, and the definition of D implies

$$D((d(x_1, x)|_{mc_d(x_1)}, mc_d(x_1)), (d(x_2, x)|_{mc_d(x_2)}, mc_d(x_2))) < \infty.$$

Now, since $| \, f_{x_1}(x) - f_{x_2}(x) \, | \equiv | \, d(x_1, x) - d(x, x_2) \, | \leq d(x_1, x_2)$, then for each x in the common domain, the supremum taken over x can not exceed $d(x_1, x_2)$, that is

$$\sup_{x \in mc_d(x_1)} | \, f_{x_1}(x) - f_{x_2}(x) \, | \leq d(x_1, x_2).$$

On the other hand, selecting $x = x_1$ and/or $x = x_2$, we get that

$$\sup_{x \in mc_d(x_1)} | \, d(x_1, x) - d(x_2, x) \, | \geq d(x_1, x_2).$$

Hence,

$$\sup_{x \in mc_d(x_1)} | \, f_{x_1}(x) - f_{x_2}(x) \, | = d(x_1, x_2),$$

implying that ϕ as defined is an isometry. $\square$

In the following result, we aim to show that a generalization of the kind of functions used in Theorem 3.2.9 from singletons to nonempty subsets of X leads to the extended Hausdorff metric. Indeed this fact motivates the definition of the distance D on Δ_X above.

Proposition 3.2.10. *[14] Let X be a metrizable space. Then for each compatible extended metric d on X and $A \in \mathscr{P}_0(X)$, the map $f : X \longrightarrow (\Delta_X, D)$ defined by $x \longmapsto d(x, A)$ is continuous.*

Proof. Let x, y be any two elements of X and define $f : X \longrightarrow \Delta_X$ by $f(z) = d(x, z) - d(y, z)$ for some fixed $z \in X$. Then, for each $a \in A$, we have

$$d(x, a) \leq d(x, y) + d(y, a).$$

Taking infimum over $a \in A$ gives

$$d(x, A) = \inf_{a \in A} d(x, a) \leq d(x, y) + \inf_{a \in A} d(y, a) = d(x, y) + d(y, A),$$

which implies $d(x, A) - d(y, A) \leq d(x, y)$. Similarly, $d(y, a) \leq d(y, x) + d(x, a)$, so that

$$d(y, A) = \inf_{a \in A} d(x, a) \leq d(x, y) + \inf_{a \in A} d(x, A) = d(x, y) + d(x, A),$$

implying $d(y, A) - d(x, A) \leq d(x, y)$.

Together, we obtain $\mid d(x, A) - d(y, A) \mid \leq d(x, y)$ and $d(x, y)$ is independent of $a \in A$, which clearly gives the continuity of f. $\square$

The next result shows that the Hausdorff distances between two nonempty subsets is equivalent to the uniform distance between distance functionals.

Definition 3.2.11. [3](Pompeiu-Hausdorff distance) Let (X, d) be a metric space, $A, B \in \mathscr{P}_0(X)$. Then the Pompeiu-Hausdorff distance between A and B is given by

$$\mathbb{D}_\infty(A, B) = \sup\{\mid d(x, A) - d(x, B) \mid : x \in X\} = \sup_{x \in X} \mid d(x, A) - d(x, B) \mid .$$

Note that, if either $A = \emptyset$ or $B = \emptyset$, then $\mathbb{D}_\infty$ is undefined.

Proposition 3.2.12. *[3] Let X be a metrizable space. Then for each compatible metric d on X and $A, B \in \mathscr{C}_0(X)$, $d_H(A, B) = \sup_{x \in X} \mid d(x, A) - d(x, B) \mid$.*

Proof. $\Rightarrow$ Let $A, B \in \mathscr{C}_0(X)$, $a \in A$ and $b \in B$. For a fixed $x \in X$, define $f_x : X \longrightarrow [0, \infty)$ by $x \longmapsto d(x, a) - d(x, b)$. By the triangle inequality we have

$$d(x, A) \leq d(x, b) + d(b, A).$$

Since $b \in B$ is arbitrary, this implies that

$$d(x, A) \leq \inf_{b \in B} d(x, b) + d(b, A) \leq \inf_{b \in B} d(x, b) + \sup_{b \in B} d(b, A) \leq d(x, B) + e_d(B, A).$$

So that $d(x, A) - d(x, B) \leq e_d(B, A)$.

Similarly, we have $d(x, B) - d(x, A) \leq e_d(A, B)$. Hence,

$$\max\{d(x, A) - d(x, B), d(x, B) - d(x, A)\} \leq \max\{e_d(B, A), e_d(A, B)\},$$

implying $\mid d(x, A) - d(x, B) \mid \leq d_H(A, B)$.

Taking supremum over $x \in X$ on both sides gives $\mathbb{D}_\infty(A, B) \leq d_H(A, B)$ and $d(A, B)$ is independent of x, implying that $\mathbb{D}_\infty$ is continuous.

$\Leftarrow$ For any $x \in X$ and arbitrary $b \in B$

$$\begin{aligned} e_d(B, A) = \sup_{b \in B} d(b, A) = \sup_{b \in B}\{d(b, A) - d(b, B)\} &\leq \sup_{x \in X}\{d(x, A) - d(x, B)\} \\ &\leq \sup_{x \in X} \mid d(x, A) - d(x, B) \mid . \end{aligned}$$

Analogously, $e_d(A, B) \leq \sup_{x \in X} \mid d(A, x) - d(B, x) \mid = \sup_{x \in X} \mid d(x, A) - d(x, B) \mid$. Consequently, we have

$$\max\{e_d(B, A), e_d(A, B) \leq \sup_{x \in X} \mid d(x, A) - d(x, B) \mid$$

implying $d_H(A, B) \leq \mathbb{D}_\infty(A, B)$, which completes the proof.

$\square$

Remark 3.2.13. From Proposition 3.2.12 it is clear that the Hausdorff metric convergence of a sequence of closed sets $\{A_n\}$ to A amounts to the uniform convergence of $\{d(., A_n)\}$ to $d(., A)$, that is to say Hausdorff distance is just a uniform distance between distance functionals.

Proposition 3.2.14. *[7] Let X be a metrizable space. Then for each compatible metric d on X, $(\mathscr{C}_0(X), d_H)$ can be isometrically embedded in (Δ_X, D).*

Proof. Let d be a compatible metric on X; if A and B are contained in $\mathscr{C}_0(X)$, then $d_H(A, B) = \sup_{x \in X} \mid d(x, A) - d(x, B) \mid$ by Proposition 3.2.12. Choose a fixed point $x \in X$ and let $\phi : A \longrightarrow \Delta_X$ be the map defined by

$$f_A(x) = d(x, A) - d(x, B).$$

We need to verify that $f_A \in C(\mathscr{C}_0(X))$. Now, by Propositions 3.2.10 and 3.2.12, $d(x, A) - d(x, B) \leq d_H(A, B)$ and $d(x, B) - d(x, A) \leq d_H(A, B)$. Hence, $\mid d(x, A) - d(x, B) \mid \leq d_H(A, B)$ implying $f_A \in C(\mathscr{C}_o(X))$ since $d_H(A, B)$ is a metric independent of x.

Next, we prove that $\phi : A \longrightarrow (d(\cdot, A), X)$ is an isometry, that is $D(\phi(A), \phi(B)) = d_H(A, B)$. We have two cases to consider.

Case 1. If $d_H(A, B) = \infty$, then $A \neq B$, and so by the definition of D,

$$D((d(x, A), X), (d(x, B), X)) = \infty = d_H(A, B).$$

Case 2. On the other hand, if $d_H(A, B) < \infty$, then $A = B$ and so

$$D((d(x, A), X), (d(x, B), X)) < \infty.$$

Now, taking supremum over $x \in X$ on both sides of $\mid d(x, A) - d(x, B) \mid \leq d_H(A, B)$, the left side can not exceed $d_H(A, B)$, that is,

$$\sup_{x \in X} \mid d(x, A) - d(x, B) \mid \leq d_H(A, B).$$

Further, selecting $x \in A$ or $x \in B$, we get that $\sup\limits_{x \in A \cup B} \mid d(x, A) - d(x, B) \mid \geq d_H(A, B)$.

As a result, $(\mathscr{C}_0(X), d_H)$ can be isometrically embedded into (Δ_X, D). □

3.3. Bornology of metrically bounded subsets

In this section, we aim to present Beer's [7] characterizations of bornologies on a metrizable space denoted by $\mathscr{B}_d(X)$, for some compatible extended metric d of which are extensions of Hu's [**?**] construction and work by Garrido and Meroño's [15].

We first consider some general concepts of a bornology that will be useful throughout this section, and the third section of our third chapter.

Definition 3.3.1. [26] Let X be a Hausdorff space. If $\mathscr{A} \subseteq \mathscr{P}(X)$, we define;

(a) $\downarrow(\mathscr{A})$ by $\downarrow(\mathscr{A}) = \{B : B \subseteq A$, for some $A \in \mathscr{A}\}$ and,

(b) $\sum(\mathscr{A}) = \left\{ \bigcup_{i=1}^{n} A_i : i \leq n \in \mathbb{N}, A_i \in \mathscr{A} \right\}$.

Definition 3.3.2. [16] Let X be a nonempty set without any assumed structure. A family $\mathscr{B}$ of subsets of X is called a bornology on X provided the following conditions are satisfied:

(i) $\mathscr{B}$ forms a cover of X, i.e. $X = \bigcup_{B \in \mathscr{B}} B$;

(ii) $\mathscr{B}$ is hereditary under inclusion; whenever $B \in \mathscr{B}$ and A is a subset of X contained in B, then $A \in \mathscr{B}$;

(iii) $\mathscr{B}$ is stable under finite union; if $B_1, B_2, ..., B_n \in \mathscr{B}$ then $\bigcup_{i=1}^{n} B_i \in \mathscr{B}$.

A pair $(X, \mathscr{B})$ consisting of a set X and a bornology $\mathscr{B}$ is called a bornological space or universe, and the elements of $\mathscr{B}$ are called the bounded subsets of X. If (X, d) is an extended metric space, we denote its bornology of metrically bounded subsets by $\mathscr{B}_d(X)$ and call this the metric bornology determined by d.

The condition of $\mathscr{B}$ being stable under set inclusions implies that it is a hereditary family and thus must contain the empty set. Note that for any space X, the smallest bornology on X is the family of its finite subsets and the largest is its power set.

Definition 3.3.3. ([16]) Let X be a nonempty set without any assumed structure. A basis for a bornology $\mathscr{B}$ on X is a subfamily $\mathscr{B}_0 \subseteq \mathscr{B}$, such that every element of $\mathscr{B}$ is contained in some element of $\mathscr{B}_0$, that is, for each $A \in \mathscr{B}$ there exists a $B \in \mathscr{B}_0$ such that $A \subseteq B$.

Evidently, $\mathscr{B}_0$ is cofinal in $\mathscr{B}$. By being cofinal, it means $\mathscr{B}_0$ satisfies the property that: for a non-empty set X with a binary relation $\leq$, its subset $A \subseteq X$ is said to be cofinal if and only if for every $x \in X$, there exists some $a \in A$ such that $x \leq a$. $\mathscr{B}_0$ is called a closed (open) basis of $\mathscr{B}$, if each set $A \in \mathscr{B}_0$ is closed (open), [18].

Lemma 3.3.4. *([18, Theorem 4.2 and Theorem 4.3]) Let X be a topological space.*

(i) A family $\mathscr{B}_0 = \{A\}$ of subsets of X is a basis for some bornology in X, if and only if $\mathscr{B}_0$ covers X and the union of any two elements of $\mathscr{B}_0$ is contained in a member of $\mathscr{B}_0$.

(ii) If the bornology $\mathscr{B}$ in X is generated by $\mathscr{B}_0$, then the finite unions of elements in $\mathscr{B}_0$ form a basis of $\mathscr{B}$.

Proposition 3.3.5. *[7, 26] Let X be a topological space.*

(i) If $\mathscr{A}$ is a nonempty family of subsets of X, then the following properties are satisfied:

(a) $\mathscr{A} \subseteq \downarrow(\mathscr{A})$.

(b) $\mathscr{A} \subseteq \sum(\mathscr{A})$.

(c) $\downarrow(\sum(\mathscr{A})) = \sum(\downarrow(\mathscr{A}))$.

(ii) If $\mathscr{A} \subseteq \mathscr{P}(X)$ is a cover of X, $\sum(\downarrow(\mathscr{A})) = \downarrow(\sum(\mathscr{A}))$ is the smallest bornology on X containing $\mathscr{A}$.

Proof. (i) (a) For sufficiency, let A and B be arbitrary subsets of X. Then, for each $B \in \mathscr{P}(X)$ we can find a set $A \in \mathscr{A}$ such that $B \subseteq A$ by the definition of $\downarrow(\mathscr{A})$. Since B generated from $\mathscr{A}$ lies in $\downarrow(\mathscr{A})$ and A was arbitrary, we infer that, A belongs to $\downarrow(\mathscr{A})$ as such, $\mathscr{A}$ is cofinal. Hence, $\mathscr{A} \subseteq \downarrow(\mathscr{A})$.

(b) Set in (a) $B = \cup_{i=1}^{n} A_n$, $A_i \in \mathscr{A}$, $i = 1, 2, ..., n$, $n \in \mathbb{N}$. Now for each $i \leq n$, (a) guarantees that there exists $A \in \mathscr{A}$ such that $A_i \subseteq A$. Since each A_i is arbitrary and A can be one of the A_i, $\bigcup_{i=1}^{n} A_i \subseteq A$ and so, $\mathscr{A} \subseteq \sum(\mathscr{A})$.

(c) We have from $\downarrow(\mathscr{A}) = \{B : B \subseteq A$, for some $A \in \mathscr{A}\}$ and
$\sum(\mathscr{A}) = \left\{\bigcup_{i=1}^{n} A_i : n \in \mathbb{N}, i \leq n, A_i \in \mathscr{A}\right\}$ that

$$\begin{aligned}\textstyle\sum(\downarrow(\mathscr{A})) &= \{\cup_{i=1}^{n} A_i : A_i \in \downarrow(\mathscr{A})\} \\ &= \{\cup_{i=1}^{n} A_i : A_i \in \{B : B \subseteq A, \text{ for some } A \in \mathscr{A}\}\} \\ &= \{\cup_{i=1}^{n} A_i : A_i \subseteq C_i, \text{ for some } C_i \in \mathscr{A}\}\end{aligned}$$

$$
\begin{aligned}
&= \{\cup_{i=1}^n A_i : \cup_{i=1}^n A_i \subseteq \cup_{i=1}^n C_i, \text{ for some } C_i \in \mathscr{A}\} \\
&\subseteq \{B : B \subseteq A, \text{ for some } C_i \subseteq A = \cup_{i=1}^n C_i \in \textstyle\sum(\mathscr{A})\} \\
&= \downarrow (\textstyle\sum(\mathscr{A})
\end{aligned}
$$

Conversely,

$$
\begin{aligned}
\downarrow (\textstyle\sum(\mathscr{A})) &= \{B : B \subseteq A, \text{ for some } A \in \textstyle\sum(\mathscr{A})\} \\
&= \{B : B \subseteq A, \text{ for some } A \in \{\cup_{i=1}^n C_i : C_i \in \mathscr{A}\}\} \\
&= \{B : B \subseteq A, \text{ for some } A = \cup_{i=1}^n C_i \in \textstyle\sum(\mathscr{A})\} \\
&\subseteq \{\cup_{i=1}^n A_i : \cup_{i=1}^n A_i \subseteq \cup_{i=1}^n C_i, \text{ for some } C_i \in \mathscr{A}\} \\
&= \{\cup_{i=1}^n A_i : A_i \subseteq \cup_{1=1}^n C_i, \text{ for some } C_i \in \mathscr{A}\} \\
&= \{\cup_{i=1}^n A_i : A_i \in \{\cup_{1=1}^n C_i : C_i \subseteq E, \text{ for some } E \in \mathscr{A}\}\} \\
&= \{\cup_{i=1}^n A_i : A_i \in \{C_i : C_i \subseteq E, \text{ for some } E \in \mathscr{A}\}\} \\
&= \{\cup_{i=1}^n A_i : A_i \in \downarrow \mathscr{A}\} \\
&= \textstyle\sum (\downarrow (\mathscr{A})).
\end{aligned}
$$

(ii) For necessity, suppose the family $\mathscr{A}$ is a cover of X. We first need to show that $\sum(\downarrow (\mathscr{A}))$ satisfies bornological properties. Since $\mathscr{A}$ is a subset of both $\downarrow (\mathscr{A})$ and $\sum(\mathscr{A})$, $\mathscr{A}$ must be a subset of $\sum(\downarrow (\mathscr{A}))$ and so,

$$
X = \bigcup_{B \in \mathscr{A}} B \subseteq \bigcup_{B \subseteq C \in \sum(\downarrow(\mathscr{A})} C
$$

which implies that $\sum(\downarrow (\mathscr{A})$ covers X through $\mathscr{A}$.

Next, from $\sum(\downarrow (\mathscr{A})) = \{B : B \subseteq A, for\ \ some\ \ A \in \sum(\mathscr{A})\}$ it is clear that $\sum(\downarrow (\mathscr{A}))$ is hereditary under set inclusion. We also notice from $\sum (\downarrow (\mathscr{A})) = \{\cup_{i=1}^n A_i : A_i \in \downarrow \mathscr{A}, n \in \mathbb{N}\}$ that, $\sum (\downarrow (\mathscr{A}))$ is closed under finite union.

Finally, suppose $\mathscr{B}$ is another bornology containing the space $\mathscr{A}$. Then, $\downarrow (\mathscr{A}) \subseteq \mathscr{B}$ and so, $\sum (\downarrow (\mathscr{A})) \subseteq \mathscr{B}$ by (c) and the definition of a bornology. Hence, $\sum (\downarrow (\mathscr{A}))$ is the smallest bornology.

□

Remark 3.3.6. From Proposition 3.3.5, it's clear that if $\mathscr{A}$ is a cover of X then there is a smallest bornology that contains $\mathscr{A}$, namely the family of nonempty subsets of finite unions of members of the cover. Constructively, to begin with $\mathscr{A} \cup \mathscr{F}(X)$, form all finite unions of sets in this family, and then take all subsets. We call this the bornology generated by $\mathscr{A}$. From a cover of X directed by inclusion, we can obtain the bornology it generates simply by taking subsets. Such a cover is called a base for the bornology it generates.

We now consider the concepts of a bornology of metrically bounded sets. We start by stating [18, Hu's Theorem] that will be useful throughout this section, and the third section of chapter 3.

Definition 3.3.7. (cf. Definition 3.4 of Hu [18]) Let $(X, \mathscr{T})$ be a topological space. Then, a boundedness or bornology $\mathscr{B}$ in X will be called $\mathscr{T}$-proper if the universe $((X, \mathscr{T}), \mathscr{B})$ is proper, that is, for every $A \in \mathscr{B}$, there exist a $B \in \mathscr{B}$ such that $\text{cl}A \subseteq \text{int}B$.

Definition 3.3.8. (cf. Definition 10.1 of Hu, [18]) Let $((X, \mathscr{T}), \mathscr{B})$ be a bornological universe. We say that, $((X, \mathscr{T}), \mathscr{B})$ is metrizable if there exists a metric d on X such that $\mathscr{T} = \mathscr{T}_{(d)}$ and $\mathscr{B} = \{A \subseteq X : diam_d(A) < +\infty\}$. Moreover, $\mathscr{B}$ is the collection of all d-bounded sets.

Lemma 3.3.9. *([18, [Hu's Theorem]) Let X be a metrizable space. Then a bornology $\mathscr{B}$ on X can be realized as $\mathscr{B}_d(X)$ for some compatible metric d if and only if*

(i) $\forall B_1 \in \mathscr{B}$, $\exists B_2 \in \mathscr{B}$ with $cl(B_1) \subseteq int(B_2)$;

(ii) $\mathscr{B}$ has a countable base.

Conditions (i) indicates that the bornology has both a closed base and an open base.

Lemma 3.3.10. *[7] Let (X, d) be an extended metric space. Then*

(i) the family of all finite unions of open balls forms a base for $\mathscr{B}_d(X)$;

(ii) $\mathscr{B}_d(X)$ contains the bornology of relatively compact subsets of X;

(iii) whenever (x_n) is a Cauchy sequence in X, then $\{x_n : n \in \mathbb{N}\} \in \mathscr{B}_d(X)$.

Proof. (i) Let $\mathscr{B}_0^* = \left\{\bigcup_{i=1}^{n} B_d(x_i, \delta_i) : x_i \in X, \delta_i > 0, 1 \leq i \leq n \in \mathbb{N}\right\}$, a family of finite unions of open balls. Next, since each $B_d(x_i, \delta_i)$ is open and d bounded, so is $\bigcup_{i=1}^{n} B_d(x_i, \delta_i)$ as it is a union of open and d bounded sets. By the definition of a bornology, $\mathscr{B}_0^*$ generates $\mathscr{B}_d(X)$. Since the union of any two d-balls of $\mathscr{B}_0^*$ is contained in a set of $\mathscr{B}_0^*$, then the union of any finite number of d-balls of $\mathscr{B}_0^*$ is contained in a set of $\mathscr{B}_0^*$ by Lemma 3.3.4. Thus, every bounded d-ball of $\mathscr{B}_d(X)$ is a subset of some set in $\mathscr{B}_0^*$ and hence, we infer that $\mathscr{B}_0^*$ is a basis of $\mathscr{B}_d(X)$.

(ii) Let $\mathscr{B}_*$ be a bornology of relatively compact subsets of (X,d) and $C \in \mathscr{B}_*$. We aim to show that $\mathscr{B}_* \subseteq \mathscr{B}_d(X)$. For $\delta > 0$, it follows from the relatively compactness of C that there exists a finite set $K \subseteq C$ such that $C \subseteq \bigcup_{c\in K} B_d(c, \frac{\delta}{2}) \subseteq \bigcup_{c\in K} cl(B_d(c, \frac{\delta}{2}))$ and $cl(B_d(c, \frac{\delta}{2}))$ is compact. Then

$$[C]_d^{\frac{\delta}{2}} \subseteq \bigcup_{c\in K} \left[B_d\left(c, \frac{\delta}{2}\right)\right]^{\frac{\delta}{2}} \subseteq \bigcup_{c\in X} B_d(c,\delta).$$

Now, $\left\{\bigcup\limits_{c\in K} [B_d(c, \frac{\delta}{2})]^{\frac{\delta}{2}}\right\}$ forms a basis of $\mathscr{B}_*$ and $\{\bigcup\limits_{c\in X} B_d(c,\delta)\}$ forms a basis of $\mathscr{B}_d(X)$, we infer that $\mathscr{B}_* \subseteq \mathscr{B}_d(X)$.

(iii) Let (x_n) be a Cauchy-sequence in X. Then by definition, given an $\epsilon > 0$, there exists an $n_\epsilon \in \mathbb{N}$ such that for all $m, n \geq n_\epsilon$, $d(x_n, x_m) < \epsilon$. Now let us pick any $\epsilon > 0$; say, $\epsilon = 1$. Then by assumption, there exists some $n_1 \in \mathbb{N}$ such that for all $n, m \geq n_1$ we have that $d(x_n, x_m) < 1$. Now let $k \in \mathbb{N}$ and $k \geq n_1$. Since $d(x_k, x_m) < 1$, there are infinitely many terms of this sequence inside the ball of radius 1 centered at x_k; thus, these terms are bounded. In particular, we have $d(x_n, x_{n_1}) < 1$, where $x_n \in B(x_{n_1}, 1)$ for all $n \geq n_1$. Thus, $\{x_n : n \geq n_1 \in \mathbb{N}\}$ is bounded. Also $\{x_n : n < n_1 \in \mathbb{N}\}$ is bounded since its finite. Hence we conclude that, the entire range $\{x_n : n \in \mathbb{N}\}$ is a bounded subset of X and so lies in $\mathscr{B}_d(X)$.

□

Proposition 3.3.11. *([7]) Let (X,d) be an extended metric space and let E be a dense subset of X. Denoting the restriction of d to $E \times E$ by d_E, suppose $\mathscr{B}_0$ is a basis for $\mathscr{B}_{d_E}$. Then $\{cl(B) : B \in \mathscr{B}_0\}$ is a base for $\mathscr{B}_d(X)$.*

Proof. The proof is as a consequence of [5, Theorem 2.3.], [6, Lemma 3.3] and [15, Lemma 4.1]). Let $C \in \mathscr{B}_d(X)$. Since E is dense in X, we choose for each $n \in \mathbb{N}$ a finite subset E_n of E in $\mathscr{B}_{d_E}(E)$ such that $C \subseteq E_n^{\frac{1}{n}}$, with the enlargement being taken in X. This means that reciprocally, $E_n \subseteq C^{\frac{1}{n}}$. Let $T_n = E_n \cap C^{\frac{1}{n}}$ so that $T_n \in \mathscr{B}_{d_E}(E)$ and $d_H(C, T_n) \leq \frac{1}{n}$. Now for each $n \in \mathbb{N}$ set $B_n = \bigcup\limits_{j=1}^{n} T_j \in \mathscr{B}_{d_E}(E)$. We need to prove that $C \cup \bigcup\limits_{n=1}^{\infty} T_n$ is bounded in X. Let $\delta > 0$ be arbitrary; we claim that whenever $\frac{1}{n} < \frac{\delta}{2}$ then

$$C \cup \bigcup_{j=1}^{\infty} T_j \subseteq B_n^{\delta}.$$

First $d_H(C, T_n) \leq \frac{1}{n} < \frac{\delta}{2}$ gives

$$C \subseteq T_n^{\frac{\delta}{2}} \subseteq B_n^{\frac{\delta}{2}}.$$

If $j \leq n$ then $T_j \subseteq B_n \subseteq B_n^{\delta}$. On the other hand, if $j > n$ then $d_H(C, T_j) < \frac{\delta}{2}$ and so,

$$d_H(T_n, T_j) \leq d_H(T_n, C) + d_H(C, T_j) < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Thus in this case $T_j \subseteq T_n^{\delta} \subseteq B_n^{\delta}$. Since $\delta > 0$ was arbitrary we infer that $C \cup \left(\bigcup_{n=1}^{\infty} T_n\right)$ is d-bounded, and so its subset $\bigcup_{n=1}^{\infty} T_n$ is d-bounded. Our construction gives $B \subseteq C \subseteq cl_X\left(\bigcup_{n=1}^{\infty} T_n\right)$ and so, $B \subseteq cl_E\left(\bigcup_{n=1}^{\infty} T_n\right)$. Let $\mathscr{B}_0 = \{B_n\}_{n\in\mathbb{N}}$, then it is clear that $cl_X B_n \in \mathscr{B}_d(X)$, for every $B_n \in \mathscr{B}_{d_E}(E)$. □

The following characterization gives properties of a generating cover for a metric bornology.

Theorem 3.3.12. [7] Let X be a metrizable space and let $\mathscr{B}$ be a bornology on X. Then $\mathscr{B} = \mathscr{B}_d(X)$ for some compatible extended metric d if and only if there exist $\mathscr{A} \subset \mathscr{B}$ such that $\downarrow(\Sigma(\mathscr{A})) = \mathscr{B}$ and a partition $\{\mathscr{A}_i : i \in I\}$ of $\mathscr{A}$ with the following properties:

(i) each $\mathscr{A}_i$ contains a nonempty subset of X;

(ii) $\forall i \in I, \forall A_1 \in \mathscr{A}_i, \exists A_2 \in \mathscr{A}_i$ with $\text{cl}(A_1) \subseteq int(A_2)$;

(iii) whenever $A_i \in \mathscr{A}_i$ and $A_j \in \mathscr{A}_j$ for $i \neq j$, then $A_i \cap A_j = \emptyset$;

(iv) each $\mathscr{A}_i$ has a countable subfamily which is cofinal in $\mathscr{A}_i$ with respect to inclusion.

Proof. $\Longrightarrow$ Suppose X is metrizable. For necessity, let d be a compatible extended metric on X with metric components $\{X_i : i \in I\}$. For each $i \in I$, let $\mathscr{A}_i = \{B_d(x, \alpha) : x \in X_i, \alpha > 0\}$ and let $\mathscr{A}$ be the collection of open balls in X. Since a subset of X is d-bounded if and only if it is contained in a finite union of open balls and $\mathscr{A}$ is a cover of X, we have

$$\mathscr{B}_d(X) = \mathscr{B}_d(\cup_{i\in I} X_i) = \downarrow\left(\sum(\cup_{i\in I}\mathscr{A}_i)\right) = \downarrow\left(\sum(\mathscr{A})\right).$$

(i) This condition clearly holds, because from $\mathscr{A}_i$'s construction, we can always find a neighborhood A_i of $x \in X_i$ such that $A_i \subseteq B_d(x_i, \delta_i)$ where we can take $\alpha = \max\{\delta_i - d : i \in I\}$. That is equivalently to say $\mathscr{A}_i$ covers an A_i.

(ii) Since each $\mathscr{A}_i$ contains a nonempty bounded subset of X, we have for every arbitrary $A \in \mathscr{A}_i$ that there exists an open ball $B_i(y)$ such that $A \subseteq B_i(y)$ so, $\overline{A} \subset \overline{B_i(y)} \subset B_{i+1}(y)$, taking $cl(A_1) = \overline{A}$ and $int(A_2) = B_{i+1}(y)$.

(iii) Let $A_1 \in \mathscr{A}_i$ and $A_2 \in \mathscr{A}_j$ $\forall i \neq j, i, j \in \{1,2\}$. Since $\{\mathscr{A}_i : i \in I\}$ partitions $\mathscr{A}$, we have that, $A_1 \cap A_2 = \emptyset$. Generalizing this to all but finitely many arbitrary $\mathscr{A}_i$'s we have for the indices $i \neq j, i, j \in I$ that $A_i \cap A_j = \emptyset$.

(iv) Finally, choose a point $y \in X_i$, $i \in I$. For some positive integer n the collection $\mathscr{A}_{i_n} = \{B_d(y,n) : n \in \mathbb{N}\} = \{B_n(y) = \{x \in X_i : d(y,x) < n\} : i \in I, 1 \leq i \leq n, n \in \mathbb{N}\}$ of open balls forms a subfamily for $\mathscr{A}_i$ which is countable. Since from our construction, $B_1(y) \subseteq B_2(y) \subseteq ... \subseteq B_n(y)$, then $\sum(\{B_n(y) = \{x \in X_i : d(y,x) < n\} : i \in I, 1 \leq i \leq n, n \in \mathbb{N}\})$ contains a countably cofinal subfamily for $\mathscr{A}_i$.

$\Longleftarrow$ Conversely, suppose the partition $\{\mathscr{A}_i : i \in I\}$ of $\mathscr{A}$ has the asserted properties. For each $i \in I$, let $X_i = \cup\mathscr{A}_i$. By property (i), each X_i is nonempty, and by (ii), each X_i is open since each $\mathscr{A}_i$ has an open basis. Since $X_i = \cup\mathscr{A}_i$, (iii) guarantees that $\{X_i : i \in I\}$ is a pairwise disjoint family and thus partitions X. Since $\{X_i : i \in I\}$ are metrc components and (ii) implies there is both a closed base and an open base, we infer that each X_i is clopen. For each $i \in I$, put $\mathscr{B}_i :=\downarrow (\Sigma(\mathscr{A}_i))$ where $\downarrow (\Sigma(\mathscr{A}_i)) =\downarrow \{\cup_{i=1}^n A_i : n \in \mathbb{N}, A_i \in \cup_{i\in I}\mathscr{A}_i = \mathscr{A}\} = \{A_i : A_i \subset \cup_{i=1}^n A_i \subseteq B_i \in \mathscr{A}_i$, for some $B_i \in \mathscr{A}_i \subseteq \mathscr{A}\}$. Since $\mathscr{A}_i$ is a cover of X_i, $\mathscr{B}_i$ is a bornology on X_i. Since the closure of a finite union is the union of the closures, property (ii) holds with $\mathscr{B}_i$ replacing $\mathscr{A}_i$. Property (iii) holds as well if $\mathscr{B}_i$ replaces $\mathscr{A}_i$ and $\mathscr{B}_j$ replaces $\mathscr{A}_j$. Finally, if $\mathscr{A}_i^*$ is countable and cofinal in $\mathscr{A}_i$,then $\Sigma(A_i^*)$ is countable and cofinal in $\mathscr{B}_i$ and so, (iv) holds with $\mathscr{B}_i$ replacing $\mathscr{A}_i$. Thus by Lemma 3.3.9, for each positive $i \in I$, there is a metric d_i on X_i such that $\mathscr{B}_i = \mathscr{B}_{d_i}(X_i)$.

As $\downarrow (\Sigma(\mathscr{A}))$ is assumed to be a cover of X, we conclude that $\mathscr{A}$ is a cover of X and so $\{X_i : i \in I\}$ covered by the partition $\{\mathscr{A}_i : i \in I\}$ is a cover of X. Let us define $d : X \times X \to [0,\infty]$ by $d(x,w) = d_i(x,w)$ if $\exists i$ with $\{x,w\} \subseteq X_i$, and $d(x,w) = \infty$ otherwise. Then,

$$\mathscr{B}_d(X) = \sum(\cup_{i\in I}\mathscr{B}_i) =\downarrow \left(\sum(\cup_{i\in I}\mathscr{A}_i)\right) =\downarrow \left(\sum(\mathscr{A})\right) = \mathscr{B}.$$

□

The second result is based on this strong defining property of metrically bounded subsets.

Definition 3.3.13. [7] Let (X, d) be an extended metric space and $A \subseteq X$ be d-bounded. Then, A is said to be a member of $\mathscr{B}_d(X)$ if its intersection with each metric component lies in the relative metric bornology and all but finitely many of these intersections are empty.

Theorem 3.3.14. [7] Let X be a metrizable space and let $\mathscr{B}$ be a bornology on X. Then $\mathscr{B} = \mathscr{B}_d(X)$ for some compatible extended metric d if and only if there exists a collection $\{\mathscr{B}_i : i \in I\}$ of families of subsets of X with the following properties:

(i) $\forall i \in I$, $\mathscr{B}_i$ has at least two subsets of X as elements including $\emptyset$;

(ii) $\forall i \in I$, $\forall B_1 \in \mathscr{B}_i$, $\exists B_2 \in \mathscr{B}_i$ with $\mathrm{cl}(B_1) \subseteq \mathrm{int}(B_2)$;

(iii) whenever $B_i \in \mathscr{B}_i$ and $B_j \in \mathscr{B}_j$ for $i \neq j$, then $B_i \cap B_j = \emptyset$;

(iv) each $\mathscr{B}_i$ has a countable subfamily which is cofinal in $\mathscr{B}_i$ with respect to inclusion;

(v) $\forall A \in \mathscr{P}(X)$, $A \in \mathscr{B}$ if and only if $\forall i \in I$, $A \cap (\cup \mathscr{B}_i) \in \mathscr{B}_i$ and for all but finitely many i ,$A \cap (\cup \mathscr{B}_i) = \emptyset$.

Proof. $\Longrightarrow$ For necessity let d be a compatible extended metric on X such that $\mathscr{B} = \mathscr{B}_d(X)$. Let $\{X_i : i \in I\}$ be the metric components of X and let d_i be the restriction of d on $X_i \times X_i$. Set $\mathscr{B}_i := \mathscr{B}_{d_i}(X_i)$, then

(i) From the definition of a metric component of X, each X_i contains some element of X implying that each is nonempty and $\mathscr{B}_{d_i}(X_i)$ contains minimally the finite subsets of X_i.

(ii) Fix $z_i \in X_i$. Since each $\mathscr{B}_i$ contains at least two subsets of X as elements, then for every arbitrary bounded subset B of X_i in $\mathscr{B}_i$ there exists an open ball $B_i(z_i)$ such that $B \subseteq B_i(z_i)$ so, $\overline{B} \subset \overline{B_i(z_i)} \subset B_{i+1}(z_i)$. Hence, $clB_1 = \overline{B} \subset B_{i+1}(z_i) = intB_2$.

(iii) Suppose $B_1 \in \mathscr{B}_i$ and $B_2 \in \mathscr{B}_J$, $\forall i \neq j, i, j \in \{1, 2\}$. Since d_i the restriction of d assumes values of infinite, let $x_1, x_2 \in X_i$ be such that $x_1 \neq x_2$. Then $q(x_1, x_2) > 0$ in X_i and so for an $\alpha > 0$, the α-open balls $B_d(x_1, \alpha)$ and $B_d(x_2, \alpha)$ are disjoint. Indeed, suppose there exists a $z \in X_i$ such that $z \in B_d(x_1, \alpha)$ and $z \in B_d(x_2, \alpha)$, then $d(x_1, z) < \alpha$ and $d(z, x_2) < \alpha$. Put $\alpha = \frac{q(x_1,x_2)}{2}$, we have that $d(x_1, z) + d(z, x_2) < 2\alpha = d(x_1, x_2)$. This is a contradiction of the definition of a metric, so there can be no such a point z. Since B_1 and B_2 are bounded, taking $\bigcup B_d(x_1, \delta) \supseteq B_1$ and $\bigcup B_d(x_2, \alpha) \supseteq B_2$, we infer that B_1 and B_2 are disjoint. Generalizing this to many arbitrary $\mathscr{B}_i$'s for all indices $i \neq j, i, j \in I$ we have, $B_i \cap B_j = \emptyset$.

(iv) Choose a point $z_i \in \{X_i : i \in I\}$. For each positive integer n the collection of open balls $\{B(z_i, n) = \{x \in X_i : d(z_i, x) < n\} : i \in I, n \in \mathbb{N}\}$ is a countable sequence of bounded sets which in turn must form a countable subfamily for $\mathscr{B}_i$. Thus, $\sum(\{B(z_i, n) = \{x \in X_i : d(z_i, x) < n\} : i \in I, n \in \mathbb{N}\})$ is also countably bounded and hence must contain a countable cofinal subfamily for $\mathscr{B}_i$.

(v) Finally, let $A \in \mathscr{P}(X)$ be such that $A \in \mathscr{B}$. As $\{\mathscr{B}_i : i \in I\}$ partitions $\mathscr{B}$, $A \in \mathscr{B}_d(X)$ must be contained in some $\mathscr{B}_i$. Now, since the $\mathscr{B}_i$s are bornologies on metric components X_i which are clopen subsets of X, $A \cap \mathscr{B}_i$ is just an intersection of A with the X_i and so, by the definition of a bornology, $A \cap (\cup \mathscr{B}_i) = \cup(A \cap \mathscr{B}_i) \in \mathscr{B}_i$. Since A is arbitrary in $\mathscr{P}(X)$ and d assumes the infinite value, we have by applying (iii) and (i) for all but finitely many indices i that $A \cap \mathscr{B}_i = \emptyset$ and so, $A \cap (\cup \mathscr{B}_i) = \cup_i (A \cap \mathscr{B}_i) = \emptyset$, (One could also use Proposition 3.2, Theorem 4.1 of Beer [4]).

$\Longleftarrow$ Conversely, suppose $\{\mathscr{B}_i : i \in I\}$ satisfies the properties above. For each $i \in I$, put $X_i := \cup \mathscr{B}_i$; by condition (i) each X_i is a nonempty subset of X. By conditions (ii) and (iii), $\{X_i : i \in I\}$ partitions X into clopen subsets. Next, we need to show that each family $\mathscr{B}_i$ is a bornology on X_i.

Fix $i \in I$ and set $\mathscr{U}_i := \{B \cap X_i : B \in \mathscr{B}\}$. Since each $X_i = \cup \mathscr{B}_i$, we have by (v) that $\mathscr{U}_i$ is contained in $\mathscr{B}_i$. Thus, $\mathscr{U}_i$ is clearly a basis for a bornology on X_i since for each arbitrary $B \in \mathscr{B}$, there is an arbitrary $B \cap X_i \in \mathscr{U}_i$ with $B \subseteq B \cap X_i \in \mathscr{U}_i$. So, $\mathscr{U}_i$ is however a bornology on X_i. Conversely, we need to show that $\mathscr{B}_i \subseteq \mathscr{U}_i$. Let $B_i \in \mathscr{B}_i$ be arbitrary, then its enough to show that $B_i \in \mathscr{U}_i$ which suffices to show that $B_i \in \mathscr{B}$. The structure $X_i = \cup \mathscr{B}_i$ implies $B_i \cap X_i \in \mathscr{B}_i$. Taking $j \neq i$, condition (iii) gives $B_i \cap X_j = \emptyset$. And by condition (i), we have

$$B_i \cap X_j = \emptyset \in \mathscr{B}_j.$$

Now, in view of condition (v), $B_i \in \mathscr{B}$ so that $B_i = B_i \cap X_i \in \mathscr{U}_i$. Hence, the family $\mathscr{B}_i$ is established as a bornology.

Further, applying Lemma 3.3.9, conditions (ii) and (iv) implies that for each $i \in I$, there exists a metric d_i on X_i such that $\mathscr{B}_i = \mathscr{B}_{d_i}(X_i)$ that is to say $\mathscr{B}_i$ is a metric bornology on X_i induced by some metric d_i. With the extended metric d on X defined by $d(x_1, x_2) = d_i(x_1, x_2)$ if $\exists i$ with $\{x_1, x_2\} \subseteq X_i$ and $d(x_1, x_2) = \infty$ otherwise, condition (iv) guarantees that $\mathscr{B} = \mathscr{B}_d(X)$.

□

The concept of uniform metrization of bornological universes was studied by Garrido and Meroño in [15]. In their results they used the concepts of a characteristic function as it plays a crucial role in the study of bornologies. The use of a characteristic function in order to obtain metrization for bornological universes was first introduced by Hu in [18], as such, the results of Garrido and Meroño [15] was an extension of Hu's. The next result gives concepts of uniform metrization of a bornology on sets equipped with an extended metric, which is a natural extension of results by Garrido and Meroño [15].

Definition 3.3.15. [18, Definition 13.1] Let $(X, \mathscr{B})$ be a bornological universe. A characteristic function of $(X, \mathscr{B})$ is a real-valued non negative continuous function $\chi : X \longrightarrow [0, \infty)$ such that a subset $E \subset X$ is bounded if $\chi(E)$ is bounded in $[0, \infty)$, that is,

$$\mathscr{B} = \{E \subset X : \sup\{\chi(x) : x \in E\} < +\infty\}.$$

Lemma 3.3.16. *[15] Let (X, d) be a metric space and $\mathscr{B}$ a bornology on it such that $X \notin \mathscr{B}$. Then, $(X, \mathscr{B})$ admits a uniformly continuous characteristic function if and only if $\mathscr{B}$ has a countable base $\{B_n : n \in \mathbb{N}\}$, such that for some $\delta > 0$,*

$$B_n^\delta \subseteq B_{n+1}, \forall n \in \mathbb{N}.$$

Proof. The proof comes from [15]. First, suppose that $\mathscr{B}$ has a countable base satisfying the above hypotheses. For $n \in \mathbb{N}$ define the function $\phi_n : X \longrightarrow [0, 1]$ by $\phi_n(x) = \min\{1, \frac{1}{\delta} d(x, B_n)\}$. Then $\phi_n(B_n) = 0$, $\phi_n(X - B_{n+1}) = 1$, $0 \leq \phi_n \leq 1$ and ϕ_n is uniformly continuous. Now if we put $B_0 = \emptyset$ and $\phi_0(x) = 1$, for all $x \in X$, we can consider a function $\chi : X \longrightarrow [0, \infty)$, defined by

$$\chi(x) = \sum_{i=2}^{n-1} \phi_i(x) = n - 2 + \phi_{n-1}(x)$$

when $x \in B_n - B_{n-1}, n \in \mathbb{N}$. In this way, χ is a uniformly continuous characteristic function of $(X, \mathscr{B})$. Indeed, $\forall \epsilon > 0$, $\exists \delta' > 0$ such that $\delta' < \delta$ and $\delta' < \epsilon\delta$, and $\forall x, y \in X$ with $d(x, y) < \delta'$, there exists $n \in \mathbb{N}$ such that both x, y are in $B_n - B_{n-1}$, or $x \in B_n - B_{n-1}$ and $y \in B_{n+1} - B_n$. In any case, we have

$$\begin{aligned}| \chi(x) - \chi(y) |=| \phi_{n-1}(x) - \phi_{n-1}(y) + \phi_n(x) - \phi_n(y) | &< \frac{1}{\delta}d(x,y) + \frac{1}{\delta}d(x,y) \\ &= \frac{2}{\delta}d(x,y).\end{aligned}$$

Now, let $E \subset \mathscr{B}$, there exists $n \in \mathbb{N}$ such that $E \subset B_n$. So $\chi(E) = \sum_{i=1}^{n-1} \phi_i(E) \subseteq [0, n-1]$. On the other hand, if $\chi(E)$ is bounded then $E \subset B_n$ for some $n \in \mathbb{N}$, because otherwise for every $n \in \mathbb{N}$ there will be $x \in E$ such that $\chi(x) \geq n$. Thus, χ is a characteristic function for $(X, \mathscr{B})$.

Conversely, suppose that $(X, \mathscr{B})$ admits a uniformly continuous characteristic function χ. Then from the uniform continuity of χ, there exists some $\delta > 0$ such that $d(x,y) < \delta$ implies $| \chi(x) - \chi(y) | \leq 1$. Now if we take, $B_n = \chi^{-1}([0,n])$, $n \in \mathbb{N}$, then it is easy to see that the family $\{B_n\}_n$ is a countable base for $\mathscr{B}$, satisfying the required property for this δ. □

Theorem 3.3.17. [7] Let (X,d) be an extended metric space. The following conditions are equivalent:

(i) the set of metric components induced by d is countable;

(ii) there exists a compatible metric d' such that $\mathscr{B}_{d'}(X) = \mathscr{B}_d(X)$.

Proof. The proof of our characterization is a consequence of [15, Theorem 2.4].

(i) ⇒ (ii) Let $\{mc_d(x_i) : i \in I\}$ be the set of distinct metric components induced by d where I is countable. For each positive integer n, the collection $\{B_n(x_i) = \{B_d(x_i, n) : x_i \in mc_d(x_i), i \in I\} : n \in \mathbb{N}\}$ is a countable sequence of bounded sets and so is $\{\sum(\{B_d(x_i, n) : i \in I, n \in \mathbb{N}\})\}$ since it is a finite union of countable sequences and thus forms a countable basis for $\mathscr{B}_d(X)$. Now, let B denote an arbitrary subset of X and choose an integer n satisfying $n \geq d(B, x_i)$, then we have $B \subset B_d(x_i, n)$ implying $B \subset \sum(\{B_d(x_i, n) : i \in I, n \in \mathbb{N}\})$. For this $B \in \mathscr{B}_d(X)$, there exists $B_n(x_i)$ such that $B \subset B_n(x_i)$ so, $\overline{B} \subset \overline{\mathscr{B}_n(x_i)} \subseteq \mathscr{B}_{n+1}(x_i)$. Hence the closure of each bounded set is contained in the interior of another, and this is true for each finite union of open balls.

Now, if each $mc_d(x_i) \in \mathscr{B}_d(X)$, the proof is of course trivial because $\mathscr{B} = \mathscr{P}(mc_d(x_i) \equiv \mathscr{P}(X)$ and we have $d' = \min\{1, d\}$. So now, suppose $mc_d(x) \notin \mathscr{B}_d(X)$. Since X is metrizable, there exists a bounded metric $d^* : X \times X \longrightarrow \mathbb{R}$ which is topologically equivalent to d defined by $d^* = \min\{1, d\}$. Applying Lemma 3.3.16 we define a new metric $d' : X \times X \longrightarrow \mathbb{R}$, by

$$d'(x,y) = d^*(x,y) \vee \mid \chi(x) - \chi(y) \mid$$

where χ is a characteristic function associated to the base of $\mathscr{B}_d(X)$. To show that d' and d are equivalent it is enough to prove that d' and d^* are equivalent. On one hand, $d^* \leq d'$, so if $d'(x,y) < \epsilon$ then $d^*(x,y) < \epsilon$ and then $\mathscr{T}_d$ is coarser than $\mathscr{T}_{d^*}$. On the other hand, using the uniform continuity of χ, that is, using the property of Lemma 3.3.16 that $d(x,y) < \delta$ imply $\mid \chi(x) - \chi(y) \mid < \frac{2}{\delta} \cdot d(x,y)$, we have that

$$d'(x,y) = d(x,y) \vee \mid \chi(x) - \chi(y) \mid < \max\left\{d(x,y), \frac{2}{\delta}d(x,y)\right\} = \max\left\{1, \frac{2}{\delta}\right\} \cdot d(x,y)$$

whenever $d(x,y) < \min\{1, \delta\}$, and we have that the equivalence of topologies follows and also the equivalence of the metrics.

Finally, if $B \in \mathscr{B}$ then it is clearly bounded for the quasi-metric d^* and $\chi(B)$ is bounded in $\mathbb{R}$ by Lemma 3.3.16, and so $B \in \mathscr{B}_{d'}(X)$ since d^* and d' are uniformly equivalent. Conversely, if $B \in \mathscr{B}_{d'}(X)$ then there exists a $M \in \mathbb{R}$ such that $\forall x, y \in B$, $d'(x,y) < M$. Since d^* is a bounded metric, $\chi(B)$ is bounded in $\mathbb{R}$ so then by Lemma 3.3.16, $B \in \mathscr{B}$. Consequently $\mathscr{B}_d(X) = \mathscr{B}_{d'}(X)$ of which is an extension of Lemma 3.3.9.

(ii) $\Rightarrow$ (i) Let $\{mc_d(x_i) : i \in I\}$ denote the distinct metric components. If $\mathscr{B}_d(X) = \mathscr{B}_d(X)$, fix a certain $x \in \{mc_d(x_i) : i \in I\}$. By Lemma 3.3.9, we have for some $n \in \mathbb{N}$ that the collection

$$\{B_d(x_i, n) = \{x_i \in mc_d(x_i) : d(x, x_i) < n\}, i \in I, n \in \mathbb{N}\}$$

forms some countable base for $B_d(X)$. Hence, $\Sigma(\{B_d(x_i, n) : i \in I, n \in \mathbb{N}\})$ must contain a countable cofinal family within $\mathscr{B}_d(X)$. This implies that I is countable as $\{\{x_i\} : i \in I\}$ is a family of d-bounded sets. Therefore, $\{mc_d(x_i) : i \in I\}$ is countable. □

We now investigate concepts of a bornology generated by compact subsets of X called compact bornology.

Definition 3.3.18. Let $(X, \mathscr{T})$ be a topological space and $B \subseteq X$. Then, B belongs to a compact bornology if it is contained in some compact subset of X, or equivalently, when $cl_X B$ is compact. A compact bornology is metrizable when there exists a metric d on X, in such a way that (X, d) satisfies the Heine-Borel property. We say that the metric space

(X, d) satisfies the Heine-Borel property when every closed and bounded subset of (X, d) is compact (See, [15], [18]).

Since locally compact Hausdorff spaces are completely regular [14, p. 137-139], [41, p. 136], a second countable locally compact Hausdorff space is immediately metrizable by the Urysohn metrization theorem [41, p. 166]. The problem of metrizability of the compact bornology was studied by Vaughan in [38]. In the proof of our next result, we will make use of Vaughan's result [38].

Lemma 3.3.19. *(Vaughan's result [38]) A Hausdorff space X admits a compatible metric such that closed and bounded sets are compact if and only if X is locally compact and second countable (or equivalently, separable).*

For simplicity purposes, we will call an extended metric space whose closed and bounded sets are compact boundedly compact.

Theorem 3.3.20. [7] Let X be a Hausdorff space that is a free union of metric components $\{X_i : i \in I\}$. The following conditions are equivalent:

(i) each X_i in its relative topology is second countable and locally compact;

(ii) there exists a boundedly compact compatible extended metric d with metric components $\{X_i : i \in I\}$.

Proof.

(i) $\Rightarrow$ (ii) Suppose (i) holds. As each X_i is Hausdorff, let d_i for each $i \in I$ be a metric compatible with the relative topology for X_i and $\{B_{d_i}(x_i, n) : i \in I, 1 \leq i \leq n, n \in \mathbb{N}\}$ be its countable base. By locally compactness of X_i, the open d_i-ball is a bounded compact neighborhood for x_i and so, each closed d_i-ball is compact by Lemma 3.3.19. Since X is a free union of the X_i's, define the extended metric d induced by $\{d_i : i \in I\}$ on X by $d(x_1, x_2) = d_i(x_1, x_2)$ if $\exists i$ with $\{x_1, x_2\} \subseteq X_i$ and $d(x_1, x_2) = \infty$ otherwise. Then, $\{B_d(x_i, n) : i \in I, n \in \mathbb{N}\}$ forms a compact countable base for X and hence each closed and d-bounded subset must lie in a finite union of compact closed balls and is thus compact.

(ii) $\Rightarrow$ (i) Suppose (ii) holds; since each X_i is closed, the restriction of d to $X_i \times X_i$ is boundedly compact as well. Again by Lemma 3.3.19, each X_i in the relative topology is both second countable and locally compact. $\square$

Given an extended metric space (X, d), the topology on X is the free union topology determined by its metric components $\{mc_d(x_i) : i \in I\}$. Analogously, we could have defined a subset A of X to be bounded if its intersection with each metric component is metrically bounded. This is a weaker requirement than A belonging to $\mathscr{B}_d(X)$. We next investigate concepts of a bornology generated by weakly bounded subset of X.

Definition 3.3.21. Let (X, d) be an extended metric space and $\mathscr{B}$ be its bornology. Then, we will say that $A \subseteq X$ is d-weakly bounded if for each $i \in I$, we have $A \cap mc_d(x_i) \in \mathscr{B}_d(X)$ whenever $x_i \in X$. We denote the bornology of d-weakly bounded sets by $\mathscr{B}_d^w(X)$.

As just mentioned, $\mathscr{B}_d^w(X) \supseteq \mathscr{B}_d(X)$, with equality if X has at most finitely many metric components (cf,. [5], [16]). Clearly, a subset of a particular metric component is bounded if it is weakly bounded. The proof of Theorem 3.3.14 with very slight modifications yields the following characterization of bornologies of weakly bounded sets.

Theorem 3.3.22. [7] Let X be a metrizable space and let $\mathscr{B}$ be a bornology on X. Then $\mathscr{B} = \mathscr{B}_d^w(X)$ for some compatible extended metric d if and only if there exists a collection $\{\mathscr{B}_i : i \in I\}$ of families of subsets of X with the following properties:

(i) $\forall i \in I$, B_i has at least two subsets of X as elements including $\emptyset$;

(ii) $\forall i \in I$, $\forall B_1 \in \mathscr{B}_i$, $\exists B_2 \in \mathscr{B}_i$ with $\text{cl}(B_1) \subseteq \text{int}(B_2)$;

(iii) whenever $B_i \in \mathscr{B}_i$ and $B_j \in \mathscr{B}_j$ for $i \neq j$, then $B_i \cap B_j = \emptyset$;

(iv) each $\mathscr{B}_i$ has a countable subfamily which is cofinal in $\mathscr{B}_i$ with respect to inclusion;

(v) $\forall A \in \mathscr{P}(X)$, $A \in \mathscr{B}$ if and only if $\forall i \in I$, $A \cap (\cup \mathscr{B}_i) \in \mathscr{B}_i$.

Proof. $\Longrightarrow$ The metrizability of X implies that $\mathscr{B} = \mathscr{B}_d(X)$ for some compatible extended metric d on X. Now for this d, suppose $\mathscr{B} = \mathscr{B}_d^w(X)$. Let $\{X_i : i \in I\}$ be the metric components of X and let d_i be the trace of d on $X_i \times X_i$. Set $\mathscr{B}_i = \mathscr{B}_{d_i}^w(X_i)$. Clearly condition (i), (ii), (iii) and (iv) holds in a similar manner as those of Theorem 3.3.14. Now, for condition (v), let $A \in \mathscr{P}(X)$ be an element of $\mathscr{B}$, then A is a weakly bounded subset of X. Since $\{\mathscr{B}_i : i \in I\}$ induces $\mathscr{B}$, A must be contained in some $\mathscr{B}_i$. As A is weakly bounded, $A \cap \mathscr{B}_i \in \mathscr{B}_d(X_i)$ and the definition of a bornology implies $A \cap (\cup \mathscr{B}_i) = \cup(A \cap \mathscr{B}_i) \subset \mathscr{B}_i$. Since A is arbitrary in $\mathscr{P}(X)$ and d assumes values of infinite, we have by applying (iii) and (i) for all but finitely many indices i that, $A \cap (\cup \mathscr{B}_i) = \cup_i (A \cap \mathscr{B}_i) = \emptyset$ and so (v) holds.

$\Longleftarrow$ Conversely, suppose $\{\mathscr{B}_i : i \in I\}$ satisfies the properties above. For each $i \in I$, put $X_i := \cup \mathscr{B}_i$ which by condition (i) is a nonempty subset of X as it contains an element of x.

By conditions (ii) and (iii), $\{X_i : i \in I\}$ partitions X into clopen subsets. We next aim to show that each family $\mathscr{B}_i$ is a weak bornology on X_i.

Fix $i \in I$ and set $\mathscr{U}_i := \{B \cap X_i : B \in \mathscr{B}\}$. Since $X_i = \cup \mathscr{B}_i$ and the arbitrary set B is bounded, we have $B \cap X_i \in \mathscr{B}_d(X)$ so that B is weakly bounded. Evidently $\mathscr{U}_i$ is a base bornology of weakly bounded sets on X_i since for each arbitrary weakly bounded set $B \in \mathscr{B}$, there is an arbitrary $B \cap X_i \in \mathscr{U}_i$ with $B \subseteq B \cap X_i \in \mathscr{U}_i$. By (v), $\mathscr{U}_i$ is contained in $\mathscr{B}_{d_i}(X) = \mathscr{B}_i$. So, $\mathscr{U}_i$ is basically a weak bornology on X_i. Conversely, suppose $\mathscr{U}$ is a weak bornology on X_i, then we need to show that $\mathscr{B}_i \subseteq \mathscr{U}_i$. Let $B_i \in \mathscr{B}_i$ be an arbitrary weakly bounded set, then its enough if we show that $B_i \in \mathscr{U}_i$ by first showing that $B_i \in \mathscr{B}$. Now, as $X_i = \cup \mathscr{B}_i$ it is clear that $B_i \cap X_i \in \mathscr{B}_{d_i}(X_i) = \mathscr{B}_i$. Taking $i \neq j$, condition (iii) gives $B_i \cap X_j = \emptyset$. And by condition (i), we have

$$B_i \cap X_j = \emptyset \in \mathscr{B}_j.$$

Now, in view of condition (v), $B_i \in \mathscr{B}$ so that $B_i = B_i \cap X_i \in \mathscr{U}_i$, where the B_i can be empty. Hence, the family $\mathscr{B}_i$ is established as a weak bornology.

Further by conditions (ii) and (iv), invoking Hu's Theorem, for each $i \in I$, there exists a metric d_i on X_i such that $\mathscr{B}_i = \mathscr{B}^w_{d_i}(X_i)$ that is, $\mathscr{B}_i$ is a weak metric bornology on X_i induced by some metric d_i. Finally, define the extended metric d induced by d_i on weakly bounded subsets in X by $d(x_1, x_2) = d_i(x_1, x_2)$ if $\exists i$ with $\{x_1, x_2\} \subseteq X_i$ and $d(x_1, x_2) = \infty$ otherwise. Then, condition (v) guarantees that $\mathscr{B} = \mathscr{B}^w_d(X)$. □

Theorem 3.3.23. [7] Let (X, d) be an extended metric space. The following conditions are equivalent:

(i) all but finitely many of the metric components for d are d-bounded that is, they belong in $\mathscr{B}_d(X)$;

(ii) there exists a compatible metric d' such that $\mathscr{B}_{d'}(X) = \mathscr{B}^w_d(X)$.

Proof. Denote the metric components determined by d by $\{mc_d(x_i) : i \in I\}$, so we will apply facts of Theorem 3.3.17 throughout the proof.

(i) ⇒ (ii) Suppose (i) holds, let $I_1 \subseteq I$ be those indices for which $mc_d(x_i) \in \mathscr{B}_d(X)$. To show (ii) holds, we need only to produce a countable base for $\mathscr{B}^w_d(x)$. For each $n \in \mathbb{N}$, define $\mathscr{A}$ by

$$\mathscr{A} = \left\{ \cup_{i \in I_1} mc_d(x_i) \bigcup \cup_{i \in I - I_1} \mathscr{B}_d(x_i, n) : n \in \mathbb{N} \right\}.$$

Since each $mc_d(x_i)$ for $i \in I_1$ is d-bounded, $\cup_{i \in I_1} mc_d(x_i)$ is also d-bounded and intersects no $\cup_{i \in I - I_1} \mathscr{B}_d(x_i, n)$ for all but finitely many indices i. But $(\cup_{i \in I_1} mc_d(x_i)) \bigcap (\cup_{i \in I - I_1} \mathscr{B}_d(x_i, n)) \in \mathscr{B}_d(X) \equiv \mathscr{B}_{d'}(X)$, which implies $\cup_{i \in I_1} mc_d(x_i)$ is weakly bounded and so is $mc_d(x_i)$. Hence, the collection $\mathscr{A}$ forms a countable base for $\mathscr{B}_d^w(X)$. Further, $\{mc_d(x_i) : i \in I_1\} \subseteq \{B_d(x_i, n) : i \in I_1, n \in \mathbb{N}\}$ and so,

$$\begin{aligned}
\overline{\{mc_d(x_i) : i \in I_1\}} &\subseteq \{mc_d(x_i) : i \in I - I_1\} \\
&\subseteq \{B_d(x_i, n) : i \in I - I_1, n \in \mathbb{N}\} \\
&\subset \overline{\{B_d(x_i, n) : i \in I_1, n \in \mathbb{N}\}} \\
&\subset \overline{\{B_d(x_i, n) : i \in I - I_1, n \in \mathbb{N}\}} \\
&\equiv int\{B_d(x_i, n) : i \in I - I_1, n \in \mathbb{N}\}.
\end{aligned}$$

Evidently the closure of each weakly bounded subset is in the interior of another. Hence, by Lemma 3.3.9, there exists a compatible d' such that $\mathscr{B}_{d'}(X) = \mathscr{B}_d^w(X)$.

(ii) $\Rightarrow$ (i) Suppose a compatible metric d' exists with $\mathscr{B}_{d'}(X) = \mathscr{B}_d^w(X)$, yet for some countably infinite set of indices $\{i_j : j \in \mathbb{N}\}$, $mc_d(x_{i_j})$ is not d-bounded. Fix $p \in X$. For each $n \in \mathbb{N}$, put $B_n = \{B_{d'}(p, n) : n \in \mathbb{N}\}$ and let $\delta \in (0, 1)$ where we take $\delta < \frac{1}{2}$. Since $[B_n]_{d'}^{\frac{1}{2}} \subseteq B_{n+1}$ and $[B_n]_{d'}^{\frac{1}{2}} \cap B_{d'}(p, n) \in \mathscr{B}_{d'}(X)$, B_n must be cofinal in $\mathscr{B}_d^w(X)$.

Now for each $j \in \mathbb{N}$, $\exists A_j \in \mathscr{B}_d(X)$ such that $A_j \subseteq mc_d(x_{i_j})$ and that $A_j \nsubseteq B_{d'}(p, j)$. Then $\cup_{j=1}^{\infty} A_j \cup \{\{x_i\} : i \in I\}$ is weakly d-bounded but is not contained in any d'-ball with center p, which is a contradiction since $\mathscr{B}_d(X) = \mathscr{B}_{d'}(X)$. Hence, $mc_d(x_{i_j})$ must be d-bounded for some countably infinite set of indices $\{i_j : j \in \mathbb{N}\}$ and so, $\{mc_d(x_i) : I \in I\}$ is d bounded. □

Theorem 3.3.24. [7] Let X be a Hausdorff space that is a free union of $\{X_i : i \in I\}$. The following conditions are equivalent:

(i) I is finite and X is locally compact and second countable;

(ii) there exists a compatible extended metric d with metric components $\{X_i : i \in I\}$ such that each closed and weakly bounded subset is compact.

Proof.

(i) $\Rightarrow$ (ii) If I is finite, then for any extended metric d with metric components $\{X_i : i \in I\}$, we have $\mathscr{B}_d^w(X) = \mathscr{B}_d(X)$. We now apply the facts of Theorem 3.3.20: let d_i be the metric compatible with the relative weak topology for X_i and $\{B_{d_i}(x_i, n) : i \in I, n \in \mathbb{N}\}$ be its countable base from weakly d_i-bounded sets. As each open subspace of a locally compact and second countable space retains these properties, each open and weakly bounded d_i-ball is a compact neighborhood for x_i and so, by Lemma 3.3.19 each closed and weakly bounded d_i-ball is compact. Since X_i induces X, define a restriction d of $\{d_i : i \in I\}$ on $X_I \times X_i$ by $d(x_1, x_2) = d_i(x_1, x_2)$ if $\exists i$ with $\{x_1, x_2\} \subseteq X_i$ and $q(x_1, x_2) = \infty$ otherwise. Then, the collection $\{B_d(x_i, n) : i \in I, n \in \mathbb{N}\}$ of weakly open d-balls forms a compact countable base for X and hence each closed and d-weakly bounded subset must lie in a finite union of weak compact closed balls and is compact.

(ii) $\Rightarrow$ (i) Assume (ii) holds. Clearly, I must be finite, all else choosing $x_i \in X_i$, the discrete set $\{\{x_i\} : i \in I\}$ would be closed and weakly d-bounded but not compact. Again, $\mathscr{B}_d^w(X) = \mathscr{B}_d(X)$. Since a free union of finitely many second countable spaces is second countable and each X_i is closed, the restriction of d to $X_i \times X_i$ is weakly boundedly compact as well. And so by Lemma 3.3.19, each X_i in the relative weak topology is both second countable and locally compact. □

CHAPTER 4 : THE STRUCTURE OF EXTENDED REAL-VALUED QUASI-METRIC SPACES

Various characterizations on metric spaces, normed vector spaces, quasi-metric spaces, topological spaces and extended metric spaces have been done [3,6,7], [13], [15], [16], [?],[19,20], [21], [26], [29], [31], [36], [37], [39] as such, in this chapter we aim to establish some of these characterizations to sets equipped with an extended real-valued T_0-quasi-metric with an emphasis on large structure.

4.1. Definitions and basic properties

Definition 4.1.1. Let X be a set without an assumed structure and $q : X \times X \longrightarrow [0, \infty]$ be a function mapping into the set $[0, \infty]$ of the nonnegative reals. Then q is called an extended real-valued quasi-pseudometric on X if

(q_1) $q(x, x) = 0$, whenever $x \in X$,

(q_2) $q(x, z) \leq q(x, y) + q(y, z)$, whenever $x, y, z \in X$,

Moreover if

(q_3) $q(x, y) = 0 = q(y, x) \Rightarrow x = y$ for each $x, y \in X$ then q is said to be an extended T_0 quasi metric. The latter condition is referred to as the T_0 separation axiom.

A nonempty set X endowed with an extended real-valued quasi-pseudometric q is called an extended real-valued quasi-pseudometric space denoted by (X, q). Moreover, if q is an extended real-valued T_0-quasi-metric on X, the pair (X, q) is called an extended real-valued T_0-quasi-metric space.

Remark 4.1.2. Any extended real-valued T_0-quasi-metric q on X generates its conjugate $q^{-1} : X \times X \longrightarrow [0, \infty]$ defined by $q^{-1}(x, y) = q(y, x)$, $\forall x, y \in X$. If $q = q^{-1}$, then q is called an extended real-valued metric in the sense of Beer [7]. Further, whenever q is an extended real-valued T_0-quasi-metric, it generates an associated extended real-valued metric q^s on $X \times X$ defined by $q^s(x, y) = \max\{q(x, y), q^{-1}(x, y)\}$.

Note that, all results valid for general T_0-quasi-metric spaces also apply to extended real-valued T_0-quasi-metric spaces as each of the properties in both cases are similar.

Below are some examples of extended real-valued T_0-quasi metrics.

Example 4.1.3. Let $X = \mathbb{R}$. For each $x, y \in X$, define $q : X \times X \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ by

$$q(x,y) = \begin{cases} 0 & if \;\; x \geq y \\ \\ \infty & if \;\; x < y \end{cases}$$

Then, q as defined is an extended T_0 quasi-metric.

Proof. It is clear that $q : X \times X \longrightarrow [0, +\infty]$, $q(x,x) = 0$, $q(x,y) = 0 = q(y,x)$ if and only if $x = y$.

For all $x, y, z \in X$, we have two cases:

Case 1. $x \geq y$. We have $q(x,y) = 0$.

If $z < y$, then $q(x,z) = 0$ and $q(z,y) = \infty$.

If $y \leq z \leq x$, then $q(x,z) = 0$ and $q(z,y) = 0$.

If $x \leq z$, then $q(x,z) = \infty$ and $q(z,y) = 0$.

So we have $q(x,y) \leq q(x,z) + q(z,y)$.

Case 2. $x < y$. We have $q(x,y) = \infty$.

If $z < x$, then $q(x,z) = 0$ and $q(z,y) = \infty$.

If $x \leq z < y$, then $q(x,z) = \infty$ and $q(z,y) = \infty$.

If $y \leq z$, then $q(x,z) = \infty$ and $q(z,y) = 0$.

So we have $q(x,y) \leq q(x,z) + q(z,y)$.

and since for example, $q(2,0) = 0 \neq \infty = q(0,2)$, q is an extended T_0-quasi-metric but not an extended metric. □

Example 4.1.4. (Sorgenfrey line) Let $X = \mathbb{R}$. Define a T_0-quasi-metric $q : X \times X \longrightarrow [0, \infty]$ for each $x, y \in X$ by

$$q(x,y) = \begin{cases} x - y & if \;\; y \leq x \\ \\ \infty & otherwise \end{cases}$$

Then q as defined is an extended T_0-quasi-metric.

Proof. It is clear that $q : X \times X \longrightarrow [0, +\infty]$, $q(x, x) = 0$ and $q(x, y) = 0 = q(y, x)$ if and only if $x = y$.

For all $x, y, z \in X$, we consider the following two cases.

Case 1. $x \geq y$. We have $q(x, y) = x - y$.

If $z < y$, then $q(x, z) = x - z$ and $q(z, y) = \infty$.

If $y \leq z \leq x$, then $q(x, z) = x - z$ and $q(z, y) = z - y$.

If $x \leq z$, then $q(x, z) = \infty$ and $q(z, y) = z - y$.

So we have $q(x, y) \leq q(x, z) + q(z, y)$.

Case 2. $x < y$. We have $q(x, y) = \infty$.

If $z < x$, then $q(x, z) = x - y$ and $q(z, y) = \infty$.

If $x \leq z < y$, then $q(x, z) = \infty$ and $q(z, y) = \infty$.

If $y \leq z$, then $q(x, z) = \infty$ and $q(z, y) = z - y$.

So we have $q(x, y) \leq q(x, z) + q(z, y)$.

By the above, q is an extended quasi-metric on X. But, $q(0, 2) = \infty \neq 2 = q(2, 0)$, q is not an extended metric on $\mathbb{R}$.

A basis of open q-neighborhoods of a point $y \in \mathbb{R}$ is formed by the family $\{[y, y+\delta), 0 < \delta < 1\}$. The family of intervals $\{(y - \delta, y], 0 < \delta < 1\}$, forms a basis of open q^{-1}-neighborhoods of x. Obviously, $q^s(x, y) =\mid x - y \mid$ for $x = y$, and $q^s(x, y) = \infty$ for $x \neq y$ so that $\mathscr{T}_{q^s}$ is the metric topology of $\mathbb{R}$. However, for any $x \in \mathbb{R}$ and $\delta \geq 1$, $B_q(x, \delta) = \mathbb{R} \cup \{\infty\}$ [25]. □

Example 4.1.5. (Compare [22,36]). Let $x, y \in X = \mathbb{R}$. Then $q : X \times X \longrightarrow [0, \infty]$ defined by

$$q(x, y) = \begin{cases} \max\{x - y, 0\} & if \quad y \leq x \\ \\ \infty \quad otherwise \end{cases}$$

is an extended T_0-quasi metric on $\mathbb{R}$ and its conjugate is $q^{-1}(x, y) = \max\{y - x, 0\}$ if $x \leq y$, and $q^{-1}(x, y) = \infty$ otherwise. A basis for open q-neighborhoods of a point $y \in \mathbb{R}$ is formed by the family $\{[y, y + \delta), 0 < \delta < \infty\}$. The family of intervals $(y - \delta, y]$, $0 < \delta < \infty$, forms a basis of open q^{-1} -neighborhoods of y. Obviously, $q^s(x, y) =\mid x - y \mid$ for $x, y \in \mathbb{R}$. so that $\mathscr{T}_{q^s}$ is the usual Euclidean topology of $\mathbb{R}$.

Example 4.1.6. (Compare [21, Example 5.4]) Let $X = \mathbb{R}$ and q be a compatible quasi-metric defined on X. For each $x, y \in X$, let d be the metric for $\mathbb{R}$ defined by $d(x, y) = \min\{1, \mid y - x \mid\}$ and put $q : X \times X \longrightarrow [0, \infty]$ by $q(x, y) = d(x, y)$ if $x \geq y$ and $q(x, y) = \infty$ otherwise. Then, q is an extended quasi-metric.

The topology of the space (X, q) has for any $x \in X$, the local base for $\mathscr{T}_q$-neighborhoods of a point x formed by the family $\{[x, x + \delta) : 0 < \delta < \infty\}$ and the family of intervals $\{(x - \delta, x] : 0 < \delta < \infty\}$ forms a local basis of $\mathscr{T}_{q^{-1}}$ -neighborhoods.

In the sequel, we shall call an extended real-valued T_0-quasi-metric q on a quasi-metrizable space X simply an extended quasi-metric, and a set equipped with such a quasi-metric (X, q) an extended quasi-metric space. We next consider characterizations in relation to quasi components following ([8, 11, 13, 31]).

Definition 4.1.7. (Compare Definition 3.1.2) Let (X, q) be an extended quasi-metric space. Then for each $x, y \in X$, we define a natural relation $\mathrm{R_q}$ (or $\sim$) on X by x $\mathrm{R_q}$ y (or $x \sim y$) provided $q(x, y) < \infty$ and $q(y, x) = q^{-1}(x, y) < \infty$.

Lemma 4.1.8. *(Compare Lemma 3.1.4) If (X, q) is an extended quasi-metric space, then the natural relation $\mathrm{R_q}$ is an equivalence relation on X.*

Proof. Let (X, q) be an extended quasi-metric space on which is a defined relation $\mathrm{R_q}$. Then, for each $x \in X$, $q(x, x) = 0 < \infty$ and $q^{-1}(x, x) = 0 < \infty$ so, x $\mathrm{R_q}$ x and x $\mathrm{R_{q^{-1}}}$ x respectively implying reflexivity holds. Next, for each $x, y \in X$ let x $\mathrm{R_q}$ y, then by the definition, we have that $q(x, y) < \infty$ and $q^{-1}(x, y) < \infty$ which also implies $q(y, x) < \infty$ and $q^{-1}(y, x) < \infty$ giving $y\mathcal{R}_q x$. Thus, x $\mathrm{R_q}$ y if and only if y $\mathrm{R_q}$ x. Finally, for every $x, y, z \in X$ let's assume x $\mathrm{R_q}$ y and y $\mathrm{R_q}$ z holds. Then we have $q(x, y) < \infty$, $q(y, x) < \infty$, $q(y, z) < \infty$ and $q(z, y) < \infty$. We aim to show that x $\mathrm{R_q}$ z. By the triangle inequality, we have $q(x, z) \leq q(x, y) + q(y, z) < \infty + \infty = \infty$ and $q(z, x) \leq q(z, y) + q(y, x) < \infty + \infty = \infty$ so that $q(x, z) < \infty$ and $q(z, x) < \infty$. Hence, x $\mathrm{R_q}$ z, which implies transitivity. Therefore, $\mathrm{R_q}$ is an equivalence relation. □

Remark 4.1.9. Note that if $q = q^{-1}$ then the relation $\mathrm{R_q}$ is exactly the equivalence relation in the sense of Beer [7].

Definition 4.1.10. (Compare Definition 3.1.5) Let $((X, \mathscr{T}_1, \mathscr{T}_2), q)$ be an extended quasi-metric space.

(i) We say that a subset A of X is clopen if A is both open and closed in X. More formally, a subset A of a bitopoplogical space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is called an (i, j)-clopen set if $A \in \mathscr{T}_i \cap co\mathscr{T}_j$, or $A \in co\mathscr{T}_i \cap \mathscr{T}_j$, $i, j \in \{1, 2\}, i \neq j$ where $co\mathscr{T}_j$ means $\mathscr{T}_j$-closed and vice versa.

(ii) $(X, \mathscr{T}_1, \mathscr{T}_2)$ is said to be pairwise connected if X could not be represented as the union of two nonempty disjoint sets A and B such that A is $\mathscr{T}_1$-open ($A \in \mathscr{T}_1 \setminus \{\emptyset\}$) and B is $\mathscr{T}_2$-open ($B \in \mathscr{T}_2 \setminus \{\emptyset\}$) with $(A \cap cl_{\mathscr{T}_1} B) \cup (cl_{\mathscr{T}_2} A \cap B) = \emptyset$, [31].

(iii) The quasi-component Q_x of $x \in X$ is the intersection of all the clopen subsets of X containing x .i.e, $Q_x = \bigcap\{U_{(x)} : x \in U_{(x)} \subset (i, j) - clop(X)\}$. We will denote the quasi-component of $x \in X$ endowed by q by $qmc_q(x)$.

Remark 4.1.11. As $\mathrm{R_q}$ is a natural equivalence relation on the extended quasi-metric space (X, q), it provides a partition of X into equivalence classes. These equivalence classes are clopen subsets of X which are called quasi-components of X [31]. The components of X are pairwise closed and disjoint. Every component is contained in a quasi-component and every quasi-component is a union of components, as a result quasi-components fill out or partitions X [31]. Note that if a quasi-component happens to be pairwise connected, it is already a pairwise connected component of X thus, the quasi-components are maximal pairwise connected subsets (Berarducci A. et al.[8], De-Groot[11]).

Further, in a quasi-metrizable space X, any quasi-component Q_x satisfies the equation $Q_x = cl_{\mathscr{T}_q} Q_x \cap cl_{\mathscr{T}_{q^{-1}}} Q_x$ in the sense of Pervin [28] and, Reilly and Young [31]. So, it's easy to see that $qmc_{q^s}(x) \subseteq qmc_q(x)$ and $qmc_{q^s}(x) \subseteq qmc_{q^{-1}}(x)$ whenever $x \in X$ where $qmc_{q^s}(x)$ is a metric component in the sense of Beer [7].

If we equip each quasi-component with the relative asymmetric topologies, the asymmetric topologies of X then are the free union of the relative asymmetric topologies and this leads us to;

Proposition 4.1.12. *(Compare Proposition 3.1.7) Let $(X, \mathscr{T}_1, \mathscr{T}_2)$ be a (Pairwise Hausdorff)-bitopological space. If X can be partitioned into nonempty clopen sets $\{X_i : i \in I\}$ and each X_i is quasi-metrizable, then choosing a compatible quasi-metric q_i for X_i, the function q on $X \times X$ defined by*

$$q(x, w) = \begin{cases} q_i(x, w) & if \;\; \exists i \;\; with \;\; \{x, w\} \subseteq X_i \\ \\ \infty \;\; otherwise \end{cases}$$

is an extended quasi-metric compatible with the asymmetric topology of X and has $\{X_i : i \in I\}$ as quasi-components.

Proof. Let $x, y, z \in X$. If $\{x, y\} \subseteq X_i$, it is clear that $q : X \times X \longrightarrow [0, +\infty]$, $q(x, y) \geq 0$ and $q(x, y) = q_i(x, y) = 0 = q_i(y, x) = q(y, x)$ if and only if $x = y$ since q_i is a T_0-quasi metric. Also $q(x, y) = \infty$ for $\{x\} \subseteq X_i$, $\{y\} \subseteq X_j$ $\forall i \neq j$.

Next, for all $x, y, z \in X$, if $\{x\} \subseteq X_i$, $\{y\} \subseteq X_j$ for all $i \neq j$, then either $\{z\} \subseteq X_j$ and $\{x\} \subseteq X_i$ or $\{z\} \subseteq X_i$ and $\{y\} \subseteq X_j$. Now,

- if $\{x\} \subseteq X_i$ and $\{z\} \subseteq X_j$, $i \neq j$, then $q(x, z) = \infty$ and $q(z, y) = q_i(x, y)$. And so, $\infty = q(x, y) + q(y, z) \geq q(x, z)$.
- if $\{z\} \subseteq X_i$ and $\{y\} \subseteq X_j$, $i \neq j$, then $q(x, z) = q_i(x, y)$ and $q(z, y) = \infty$, which clearly implies that $\infty = q(x, y) + q(y, z) \geq q(x, z)$. Hence, q as defined is an extended quasi-metric.

Finally, since q_i is a restriction of q on $X_i \times X_i$ and the family of nonempty clopen sets $\{X_i : i \in I\}$ partitions X, we infer that q has $\{X_i : i \in I\}$ as quasi-metric components □

From the concepts above, we have the following characterization that uses Lemma 2.1.25 (i).

Proposition 4.1.13. *(Compare Proposition 3.1.9) A quasi-metrizable space X is pairwise-connected if and only if each extended quasi-metric on X is a quasi-metric.*

Proof. On the contrary, suppose X is not pairwise connected, that is X admits a partition into two nonempty clopen sets X_1 and X_2. Then the quasi-metric q defined by $q(x_1, x_2) = q_i(x_1, x_2)$ if $\exists i \in \{1, 2\}$ with $\{x_1, x_2\} \subseteq X_i$ and $q(x_1, x_2) = \infty$ otherwise, produces an extended quasi metric on X that assumes values of infinity since $X_1 \cap X_2 = \emptyset$, and if $x_1 \in X_1$ and $x_2 \in X_2$, then we have $q(x_1, x_2) = \infty$. Now, let $x \in X$ be arbitrary and $qmc_q(x)$ be the quasi-metric component of x with respect to q. We claim that $\{qmc_q(x)\}$ and $\{X \setminus qmc_q(x)\}$ are separated. Now, $qmc_q(x)$ is $\mathscr{T}_q$-closed implies $\overline{qmc_q(x)} = cl_{\mathscr{T}_q}(qmc_q(x)) = qmc_q(x)$, and $qmc_q(x)$ is $\mathscr{T}_q$-open implies $X \setminus qmc_q(x)$ is $\mathscr{T}_{q^{-1}}$-closed so that $\overline{(X \setminus qmc_q(x))} = cl_{\mathscr{T}_{q^{-1}}}(X \setminus qmc_q(x)) = X \setminus qmc_q(x)$ as well (See [13, Proposition 2.1, Theorem 3.1]). Then

$$\overline{(qmc_q(x))} \cap (X \setminus qmc_q(x)) = qmc_q(x) \cap (X \setminus qmc_q(x)) = qmc_q(x) \cap \overline{(X \setminus qmc_q(x))} = \emptyset.$$

Since $X = (qmc_q(x)) \cup (X \setminus qmc_q(x))$, we see that $\{qmc_q(x), X \setminus qmc_q(x)\}$ partitions X into two nonempty clopen sets, that is, X has been written as a union of two nonempty separated sets, and so X is not pairwise connected. □

4.2. A universal space for extended quasi-metric spaces

In this section, our goal is to produce a universal space for the family of extended quasi-metric spaces based on X using paired partial functions, of which is a generalization of results by Beer [7] and also an extension of the universal quasi-metric space constructed by Stojmirović in [36].

We first construct an extended quasi metric and look at its properties of bicompleteness and universality.

Definition 4.2.1. (Compare Definition 3.2.2)

Let (X, q) be an extended quasi-metric (Hausdorff) space, $A \subseteq X$ and let $\mathcal{FP}(X, q)$ be a space of paired partial functions $f = (f_1, f_2)$ where $f_i : X \longrightarrow [0, \infty)$, $(i = 1, 2)$. In the sequel, $C(X, q)$ a space of continuous real valued functions on (X, q) is defined by

$$C(X, q) = \{f = (f_1, f_2) \in \mathcal{FP}(X, q) : f_1 \text{ and } f_2 \text{ are continuous } \}.$$

Furthermore, $C^b(X, q)$ a space of continuous real-valued functions which are bounded on (X, q) is defined by

$$C^b(X, q) = \{f \in C(X, q) : f_1 \text{ and } f_2 \text{ are bounded with respect to } q^s \}$$

Then, whenever $f, g \in C(X, q)$ with $f : X \longrightarrow \mathbb{R}$, we can define a supremum quasi-metric on X by

$$Q_q(f, g) = \sup_{x \in X}(f_1(x) \dot{-} g_1(x)) \vee \sup_{x \in X}(g_1(x) \dot{-} f_2(x))$$

Obviously, $(C^b(X, q), Q_q)$ is a quasi-metric space, where Q_q is just $Q_q = Q_q|_{C^b(X,q)}$.

Remark 4.2.2. We observe that

$$Q_q^s(f, g) = \sup_{x \in A} \mid f_1(x) - g_1(x) \mid \vee \sup_{x \in A} \mid g_2(x) - f_2(x) \mid,$$

which is a metric.

Definition 4.2.3. (Compare Definition 3.2.3 and [22, Definition 7]) Let X be a quasi-metrizable space and let $\Delta_{(X,q)} = \{(f, A) : f \in C(A, q) \text{ and } A \in \mathscr{C}_0(X, q^s)\}$. Then, for any compatible quasi-metric q on X, we define an extended quasi-metric Q on $\Delta_{(X,q)}$ by

$$Q((f,A),(g,B)) = \begin{cases} \infty \quad if \quad A \neq B \\ \\ \sup_{x\in A}(f_1(x)\dot{-}g_1(x)) \vee \sup_{x\in A}(g_1(x)\dot{-}f_2(x)) \quad if \quad A = B \end{cases}$$

Remark 4.2.4. We observe that

$$Q^s((f,A),(g,B)) = \begin{cases} \infty \quad if \quad A \neq B \\ \\ \sup_{x\in A} \mid f_1(x) - g_1(x) \mid \vee \sup_{x\in A} \mid g_2(x) - f_2(x) \mid \quad if \quad A = B \end{cases}$$

is an extended metric whenever $f, g \in \mathcal{FP}(X,d)$. The extended metric space $(\Delta(X,q), Q^s)$ is the universal space in the sense of Beer [7]. Furthermore, $(\Delta(X,q), Q^s)$ is a complete extended metric space in the sense of [7, Proposition 3.1].

Proposition 4.2.5. *(Compare Proposition 3.2.5 and [34, Theorem 2.1]) Let X be a pairwise Hausdorff space. If we do a corestriction of f_i $(i = 1,2)$ to $[0,\infty)$, the extended quasi-metric space $(\Delta_{(X,q)}, Q)$ is bicomplete.*

Proof. The proof is a consequence of [22, Remark 3]. Let $f_n = ((f_n)_1, (f_n)_2)_{n\in I}$ be a family of points in $\mathcal{FP}(X)$ such that (f_n, A_n) is a Cauchy sequence in the space $(\Delta_{(X,q)}, Q)$. Let us assume without loss of generality that the partial functions $(f_n)_i$,$(i = 1,2)$ all have a common domain A. Then, for each $t \in A$, $\forall\delta > 0$, $\exists N \in \mathbb{N}$ such that $\forall j,n > N$

$$Q((f_n,A_n),(f_j),A_j)) = \sup_{t\in A}((f_n)_1(t)\dot{-}(f_j)_1(t)) \vee \sup_{t\in A}((f_j)_2(t)\dot{-}(f_n)_2(t)) < \delta,$$

where $A \in \mathscr{C}_0(X)$ $(A_n = A_j = A)$, $f_n, f_j \in C(A,q)$. Now, for any fixed $t = t_o \in A$ I have

$$\sup_{t_o\in A}((f_n)_1(t_o)\dot{-}(f_j)_1(t_o)) \leq \delta, \quad \sup_{t_o\in A}((f_j)_2(t_o)\dot{-}(f_n)_2(t_o)) \leq \delta \quad \forall j,n > N,$$

and so,

$$\sup_{t_o\in A}((f_n)_1(t_o) - (f_j)_1(t_o)) \leq \delta, \quad \sup_{t_o\in A}((f_j)_2(t_o) - (f_n)_2(t_o)) \leq \delta \quad \forall j,n > N.$$

This shows that, $(f_n, A_n)_{n\in\mathbb{N}}$ is a quasi-Cauchy sequence of real numbers and so, converges point-wise to some real number say, (f, A).

I now show that, this point-wise convergence is actually quasi-uniform in $t \in A \in \mathscr{C}_0(X, q)$, that is, $\forall \delta > 0$, $\exists N_\delta > N \in \mathbb{N}$ such that $\forall n > N_\delta$

$$\max\left\{\sup_{t\in A}((f_n)_1(t)\dot{-}f_1(t)), \sup_{t\in A}(f_2(t)\dot{-}(f_n)_2(t))\right\} < \delta.$$

Given $\delta > 0$ choose N such that $\forall n, j > N$, $Q^s((f_n, A_n), (f_j, A_j)) < \frac{\delta}{2}$. Then, by the triangle inequality and $\forall n > N_\delta$, we have

$$\begin{aligned}\sup_{t\in A}((f_n)_1(t)\dot{-}f_1(t)) &\leq \sup_{t\in A} \mid (f_n)_1(t) - f_1(t) \mid \\ &\leq \sup_{t\in A} \mid (f_n)_1(t) - (f_j)_1(t) \mid + \sup_{t\in A} \mid (f_j)_1(t) - f_1(t) \mid\end{aligned}$$

and

$$\begin{aligned}\sup_{t\in A}((f_2(t)\dot{-}(f_n)_2(t)) &\leq \sup_{t\in A} \mid f_2(t) - (f_n)_2(t) \mid \\ &\leq \sup_{t\in A} \mid (f_2(t) - (f_j)_2(t) \mid + \sup_{t\in A} \mid (f_j)_2(t) - (f_n)_2(t) \mid .\end{aligned}$$

Similarly ,

$$\begin{aligned}\sup_{t\in A}(f_1(t)\dot{-}(f_n)_1(t)) &\leq \sup_{t\in A} \mid f_1(t) - (f_n)_1(t) \mid \\ &\leq \sup_{t\in A} \mid (f_1(t) - (f_j)_1(t) \mid + \sup_{t\in A} \mid (f_j)_1(t) - (f_n)_1(t) \mid\end{aligned}$$

and

$$\begin{aligned}\sup_{t\in A}((f_n)_2(t)\dot{-}(f_2(t)) &\leq \sup_{t\in A} \mid (f_n)_2(t) - (f_2(t) \mid \\ &\leq \sup_{t\in A} \mid (f_n)_2(t) - (f_j)_2(t) \mid + \sup_{t\in A} \mid (f_j)_2(t) - f_2(t) \mid .\end{aligned}$$

Hence, $Q((f_n, A_n), (f, A)) \leq Q^s((f_n, A_n), (f_j, A_j)) + Q^s((f_j, A_j), (f, A))$ and $Q((f, A), (f_n, A_n)) \leq Q^s((f_n, A_n), (f_j, A_j)) + Q^s((f_j, A_j), (f, A))$. By choosing j sufficiently large (j may depend on t), each term on the right-hand side of the equations can be made less than $\frac{\delta}{2}$ and so, $Q((f_n, A_n), (f, A)) < \delta$ and $Q((f, A), (f_n, A_n)) < \delta$.

Hence, the sequence is both left and right Q-uniformly Cauchy on A and thus is quasi-uniformly convergent to f. Since the f_ns are quasi-continuous on A and the convergence is quasi-uniform, it follows that, $f \in C(A, q)$. Clearly, $\lim_{n \longrightarrow \infty} Q((f_n, A_n), (f, A)) = 0$ and $\lim_{n \longrightarrow \infty} Q((f, A), (f_n, A_n)) = 0$. Thus, $(\Delta_{(X,q)}, Q)$ is a bicomplete extended quasi-metric space. $\square$

Next, we look at concepts involving universality of an extended quasi-metric space.

Proposition 4.2.6. *Let X be a quasi-metrizable space, $A \in \mathscr{P}_0(X)$ and $f = (f_1, f_2) \in \mathcal{FP}(X, q)$. For each compatible T_0 quasi-metric q and some fixed $a \in X$, set $(f_a)_1(x) = q(x, a)$ and $(f_a)_2(x) = q(a, x)$ whenever $x \in X$. Let the set $\mathbb{R}$ be equipped by the quasi-metric $\mu(x, y) = x \dot{-} y$ whenever $x, y \in \mathbb{R}$. Then, the mappings $(f_a)_1 : (X, q) \longrightarrow (\mathbb{R}, \mu)$ and $(f_a)_2 : (X, q^{-1}) \longrightarrow (\mathbb{R}, \mu)$ are continuous.*

Proof. For each $a \in A$, define respectively $(f_a)_1 : (X, q) \longrightarrow (\Delta_{(X,q)}, Q)$ and $(f_a)_2 : (X, q^{-1}) \longrightarrow (\Delta_{(X,q)}, Q)$ by $(h_a)_1 = q(x, a) - q(y, a)$ and $(h_a)_2 = q(y, a) - q(x, y)$ whenever $x, y \in X$. Then, by the triangle inequality, we have $q(x, a) \leq q(x, y) + q(y, a)$ and $q(a, x) \leq q(a, y) + q(y, x)$, so that

$$q(x, a) - q(y, a) \leq q(x, y) \text{ and } q(a, x) - q(a, y) \leq q(y, x) = q^{-1}(x, y).$$

Since, $q(x, y)$ and $q^{-1}(y, x)$ are real numbers independent of $a \in X$, $f_a \in C(X, q)$. $\square$

Lemma 4.2.7. *Let X be a quasi-metrizable space. Then, for a compatible quasi-metric q and any $a, b \in X$, set $f_a(x) = (q(a, x), q(x, a))$ and $f_b(x) = (q(b, x), q(x, b))$ whenever $x \in X$. Then the map $\eta_X : (X, q) \longrightarrow (C^b(X, q), Q_q)$ defined by $\eta_X(x) = (f_a, f_b)$ yields an isometric embedding.*

Proof. For any $a, b \in X$ let η_X be the map $a \longmapsto \eta_X(a) = f_a$ defined by $(\eta_X)_1(a) = q(a, x) - q(b, x)$ and $(\eta_X)_2(a) = q(x, b) - q(x, a)$ whenever $x \in X$. By the triangle inequality, we have $q(a, x) \leq q(a, b) + q(b, x)$ and $q(x, b) \leq q(x, a) + q(a, b)$, so that $q(a, x) - q(b, x) \leq q(a, b)$ and $q(x, b) - q(x, a) \leq q(a, b)$. Hence,

$$\eta_X(a) = ((\eta_X)_1(a), (\eta_X)_2(a)) \leq q(a, b)$$

and so, $\eta_X(a)$ is continuous and bounded with respect to $q(a, b)$.

Next, we aim to show that η_X is an isometry, that is, showing that $Q_q(f_a, f_b) = q(a,b)$. Indeed, for any $a, b \in X$, it follows by setting $x = b$ that $\sup_{x\in X}(q(a,x) - q(b,x)) = q(a,b)$. Similarly, $\sup_{x\in X}(q(x,b) - q(x,a)) = q(a,b)$ by setting $x = a$. Hence, η_X is an isometric map. □

Theorem 4.2.8. (Compare Theorem 3.2.9) Let (X, q) be an extended quasi-metric space. For any $a \in X$, set $\eta_a(x) = (q(x,a), q(a,x))$ whenever $x \in X$. For any $a, b \in X$ we have that $q(a,b) = Q((\eta_a|_{qmc_q(a)}, qmc_q(a)), (\eta_b|_{qmc_q(b)}, qmc_q(b)))$. Therefore, the map $\theta : (X, q) \longrightarrow (\Delta_{(X,q)}, Q)$ defined by

$$\theta(x) = (\eta_x|_{qmc_q(x)}, qmc_q(x))$$

whenever $x \in X$ is an isometry. Moreover, θ is injective.

Proof. Let $x, y \in X$ such that $x \neq y$. Let $\theta : (X, q) \longrightarrow (\Delta_{(X,q)}, Q)$ be the map $x \longmapsto \theta(x) = \eta_x = ((\eta_x)_1, (\eta_x)_2)$ where $(\eta_x)_1 : (X, q) \longrightarrow (\Delta_{(X,q)}, Q)$ and $(\eta_x)_2 : (X, q^{-1}) \longrightarrow (\Delta_{(X,q)}, Q)$ and are defined by $(\eta_x)_1(a) = q(x,a) - q(y,a)$ and $(\eta_x)_2(a) = q(a,y) - q(a,x)$ respectively whenever $x \in X$. Clearly, $\theta \in C^b(X, q^s)$ by Proposition 4.2.6 and Lemma 4.2.7.

Next, we prove that θ is an isometry, by showing that $Q(\theta(x), \theta(y)) = q(x,y)$ which suffices to show that $\sup\limits_{a\in qmc_q(x)} (q(x,a) - q(y,a)) = q(x,y)$ and $\sup\limits_{a\in qmc_q(x)} (q(a,y) - q(a,x)) = q(x,y)$, $\forall x, y \in X$. We have four cases to consider

Case 1. If $q(x,y) = \infty$ and $q(y,x) = \infty$. We have $qmc_q(x) \neq qmc_q(y)$. Then, the definition of Q gives

$$Q(\theta(x), \theta(y)) = Q((\eta_x|_{qmc_q(x)}, qmc_q(x)), (\eta_y|_{qmc_q(y)}, qmc_q(y))) = \infty = q(x,y).$$

Case 2. If $q(x,y) = \infty$ and $q(y,x) < \infty$. Then $qmc_q(x) \neq qmc_q(y)$ which implies by the definition of Q that

$$Q((\eta_x|_{qmc_q(x)}, qmc_q(x)), (\eta_y|_{qmc_q(y)}, qmc_q(y))) = \infty = q(x,y).$$

Case 3. If $q(x,y) < \infty$ and $q(y,x) = \infty$. We have that $qmc_q(x) \neq qmc_q(y)$. So

$$Q((\eta_x|_{qmc_q(x)}, qmc_q(x)), (\eta_y|_{qmc_q(y)}, qmc_q(y))) = \infty = q(y,x).$$

Case 4. If $q(x,y) < \infty$ and $q(y,x) < \infty$. Then $qmc_q(x) = qmc_q(y)$ and so,

$$Q((\eta_x|_{qmc_q(x)}, qmc_q(x)), (\eta_y|_{qmc_q(y)}, qmc_q(y))) < \infty.$$

Since $q(x,a) - q(y,a) \leq q(x,y)$ and $q(a,y) - q(a,x) \leq q(y,x)$ for each a in the common quasi-metric component $qmc_q(x)$, supremum taken over a can not make the left hand sides exceed $q(x,y)$ or $q(y,x)$, that is

$$\sup_{a \in qmc_q(x)} (q(x,a) - q(y,a)) \leq q(x,y) \text{ and } \sup_{a \in qmc_q(x)} (q(a,y) - q(a,x)) \leq q(x,y).$$

On the other hand, letting $a = x$ and/or $a = y$, we have

$$\sup_{a \in qmc_q(x)} (q(x,a) - q(y,a)) \geq q(x,y) \text{ and } \sup_{a \in qmc_q(x)} (q(a,y) - q(a,x)) \geq q(y,x).$$

Thus, $Q(\theta(x), \theta(y)) = q(x,y)$, implying that θ as defined is an isometry.

Finally, if for any $x, y \in X$, $\theta(x) = \theta(y)$, then $Q(\theta(x), \theta(y)) = 0 = Q(\theta(y), \theta(x))$ which implies $0 = q(x,x) = q(x,y)$ and $0 = q(y,y) = q(y,x)$ so that $q(x,y) = 0 = q(y,x)$ since θ is an isometry. Consequently $x = y$ since (X,q) is a T_0-quasi-metric space, (see [25, Lemma 4], [32, Theorem 8]). We have shown that θ is injective. □

In the following result, we shall mainly consider extended T_0-quasi-metric spaces (X,q). We show next that a generalization of the kind of pair functions used in Theorem 4.2.8 from singletons to nonempty subsets of X leads to the extended Hausdorff quasi-metric. Indeed this fact motivated the definition of the distance Q on $\Delta_{(X,q)}$ with respect to $\mathcal{FP}(X,q)$.

Proposition 4.2.9. *(Compare Proposition 3.2.10) Let (X,q) be a quasi-metric space and $f_1, f_2 \in \mathcal{FP}(X,q)$. Let the set $\mathbb{R}$ be equipped with the quasi-metric $\mu(a,b) = a \dot{-} b$ whenever $a, b \in \mathbb{R}$. For each $A \in \mathscr{P}_0(X)$, set $f_A(x) = (q(A,x), q(x,A))$ whenever $x \in X$. Then the maps $(f_A)_1 : (X, q^{-1}) \longrightarrow (\mathbb{R}, \mu)$ and $(f_A)_2 : (X,q) \longrightarrow (\mathbb{R}, \mu)$ are nonexpansive.*

Proof. The proof comes from [22]. For each $x, y \in X$ and $A \in \mathscr{P}_0(X)$, define $(f_A)_1$ and $(f_A)_2$ by $(f_A)_1(x) = q(A,x) - q(A,y)$ and $(f_A)_2(x) = q(x,A) - q(y,A)$. Then by the triangle inequality, we have $q(A,x) \leq q(A,y) + q(y,x)$ and $q(x,A) \leq q(x,y) + q(y,A)$ so that $q(A,x) - q(A,y) \leq q(y,x) = q^{-1}(x,y)$ and $q(x,A) - q(y,A) \leq q(x,y)$. Hence, f_A is nonexpasive. □

The next result can be found in [22], and we include a simple proof which motivates our definition of Q.

Proposition 4.2.10. *(Compare Proposition and 3.2.12, [22, Remark 4]) Let (X, q) be an extended T_0-quasi-metric space and let $f, g \in \mathcal{FP}(X, q)$. For $A, B \in \mathscr{P}_0(X)$, set $f_A(x) = ((f_A)_1(x), (f_A)_2(x)) = (q(A, x), q(x, A))$ whenever $x \in X$. Then,*

$$q_H(A, B) = Q_q(f_A, f_B)$$

where $q_H(A, B) = \sup_{b \in B} q(A, b) \vee \sup_{a \in A} q(a, B)$ is an extended Hausdorff quasi-metric on $\mathscr{P}_0(X)$.

Proof. Fix $x \in X$. It suffices to prove that for any given $A, B \in \mathscr{P}_0(X)$, $\sup_{b \in B} q(A, b) = \sup_{x \in X}(q(A, x) - q(B, x))$ and $\sup_{a \in A} q(a, B) = \sup_{x \in X}(q(x, B) - q(x, A))$. By the triangle inequality, we have $q(A, x) \leq q(A, b) + q(b, x)$. Since $b \in B$ is arbitrary,

$$q(A, x) \leq q(A, b) + \inf_{b \in B} q(b, x) \leq \sup_{b \in B} q(A, b) + \inf_{b \in B} q(b, x) \leq q_H^-(A, B) + q(B, x)$$

so that, $q(A, x) - q(B, x) \leq q_H^-(A, B)$. Similarly, $q(x, B) - q(x, A) \leq q_H^+(A, B)$. Now, supremum taken over $x \in X$, the left hand sides can not exceed $q_H^-(A, B)$ nor $q_H^+(A, B)$. So,

$$\sup_{x \in X}(q(A, x) - q(B, x)) \leq q_H^-(A, B) \quad \textit{and} \quad \sup_{x \in X}(q(x, B) - q(x, A)) \leq q_H^+(A, B).$$

Conversely, for any $x \in X$ and arbitrary $b \in B$

$$q_H^-(A, B) = \sup_{b \in B} q(A, b) = \sup_{b \in B}(q(A, b) - q(B, b)) \leq \sup_{x \in X}(q(A, x) - q(B, x)).$$

Similarly for an arbitrary $a \in A$,

$$q_H^+(A, B) = \sup_{a \in A} q(a, B) = \sup_{a \in A}(q(a, B) - q(a, A)) \leq \sup_{x \in X}(q(x, B) - q(x, A)).$$

Consequently,

$$q_H^-(A, B) = \sup_{x \in X}(q(A, x) - q(B, x)) \text{ and } q_H^+(A, B) = \sup_{x \in X}(q(x, B) - q(x, A)).$$

□

There exist alternative definitions of a Hausdorff quasi-metric. For instance, Vitolo [39] defines an (extended) Hausdorff quasi-metric over the collection of all nonempty closed subsets of a metric space say (Y,d) by $e_d(A,B) = \sup_{a\in A} d(a,B)$, that is, in our notation, his quasi-metric corresponds to q_H^+. His main goal was to show that extended quasi-metric spaces with respect to closed sets are somewhat universal quasi-metric spaces: every quasi-metric space (X,q) is (isomorphic to) a subspace of $\mathscr{C}_0(Y,d)$, for suitable set Y and metric d. In our next results we extend Vitolo's ([39]) characterization to structure of sets equipped with an extended quasi-metric with an emphasis on large structure.

Proposition 4.2.11. *(Compare Proposition 3.2.14) Let X be a quasi-metrizable space and $f,g \in \mathcal{FP}(X,q)$. For each compatible quasi-metric q on X and $A,B \in \mathscr{C}_0(X,q)$, set $f_A(x) = ((f_A)_1(x),(f_A)_2(x)) = (q(A,x),q(x,A))$ whenever $x \in X$. Then, $(\mathscr{C}_0(X,q),q_H)$ can be isometrically embedded in $(\Delta_{(X,q)},Q)$.*

***Proof*.** Let q be a compatible T_0-quasi metric and $A,B \in \mathscr{C}_0(X,q)$. By Proposition 4.2.10, $q_H(A,B) = \sup_{x\in X}(q(A,x)-q(B,x)) \vee \sup_{x\in X}(q(x,B)-q(x,A))$. For each $A \in \mathscr{C}_0(X,q)$, let $\phi : (\mathscr{C}_0(X,q),q_H) \longrightarrow (\Delta_{(X,q)},Q)$ be the map $A \longmapsto \phi_X(A) = f_A = ((q(A,\cdot),X),(q(\cdot,A),X))$ defined by $(\phi_1)_{(\mathscr{C}_0(X,q))}(A) = q(A,x)-q(B,x)$ and $(\phi_2)_{(\mathscr{C}_0(X,q))}(A) = q(x,B)-q(x,A)$ whenever $x \in X$. Clearly, $\phi_X \in C(\mathscr{C}_0(X,q),q_H)$ by Proposition 4.2.9 and 4.2.10.

We aim to prove that ϕ is an isometry, that is $Q(\phi(A),\phi(B)) = q_H(A,B)$ which surfaces to show that $\sup_{x\in X}(q(A,x)-q(B,x)) \vee \sup_{x\in X}(q(x,B)-q(x,A)) = q_H^-(A,B) \vee q_H^+(A,B)$. Four cases arise:

Case 1. If $q_H^-(A,B) = \infty$ and $q_H^+(A,B) = \infty$, $A \neq B$ and by the definition of Q, we obtain

$$Q((q(A,x),A),(q(x,B),B)) = \infty = q_H^-(A,B).$$

Case 2. If $q_H^-(A,B) = \infty$ and $q_H^+(A,B) < \infty$ then $A \neq B$ and so we have

$$Q((q(A,x),A),(q(x,B),B)) = \infty = q_H^-(A,B).$$

Case 3. If $q_H^-(A,B) < \infty$ and $q_H^+(A,B) = \infty$. We have that $A \neq B$ so that

$$Q((q(A,x),A),(q(x,B),B)) = \infty = q_H^+(A,B).$$

Case 4. If $q_H^-(A,B) < \infty$ and $q_H^+(A,B) < \infty$, then $A = B$ and the definition of Q implies

$$
\begin{aligned}
Q(\phi(A), \phi(B)) &= Q((q(A,x), A), (q(B,x), B)) \\
&= \sup_{x \in B}(q(A,x) - q(B,x)) \vee \sup_{x \in A}(q(x,B) - q(x,A)) \\
&= \max\{q_H^-(A,B), q_H^+(A,B)\} \\
&= q_H(A,B) < \infty.
\end{aligned}
$$

Clearly, ϕ is an isometric function form $(\mathscr{C}_0(X,q), q_H)$ into $(\Delta_{(X,q)}, Q)$. □

4.3. On the bornology of quasi-metrically bounded subsets

This section has been inspired by the necessary and sufficient conditions for a bornology to be (uniformly) metrizable from [7],[15] and [18], and of a bornology to be (uniformly) quasi-metrizable in [29]. We aim to present two separate characterizations of bornologies on an extended quasi-metrizable space which are generalization of results constructed by Beer[7].

Definition 4.3.1. ([29, Definition 1.3]) Let $(X, \mathscr{T}_1, \mathscr{T}_2)$ be a bitopological space and $\mathscr{B}$ be its bornology. We say that $\mathscr{B}$ is $(\mathscr{T}_1, \mathscr{T}_2)$-proper if for each $A \in \mathscr{B}$, there exists $B \in \mathscr{B}$ such that $cl_{\mathscr{T}_2} A \subseteq int_{\mathscr{T}_1}(B)$.

Remark 4.3.2. Let us notice that if $\mathscr{T} = \mathscr{T}_1 = \mathscr{T}_2$ and the boundedness $\mathscr{B}$ is $(\mathscr{T}_1, \mathscr{T}_2)$-proper, we will say that $\mathscr{B}$ is $\mathscr{T}$-proper in the sense of [18, Definition 3.1 and Definition 3.4] of Hu.

Definition 4.3.3. ([29, Definition 1.4]) A bornological biuniverse is defined to be an ordered pair $((X, \mathscr{T}_1, \mathscr{T}_2), \mathscr{B})$ where $(X, \mathscr{T}_1, \mathscr{T}_2)$ is a bitopological space and $\mathscr{B}$ is a bornology of X.

Remark 4.3.4. If (X,q) is a quasi-metric space, we denote the bornology of quasi-bounded subsets of X by $\mathscr{B}_q(X)$ and call this the quasi-metric bornology determined by q. Note that any bornology on a quasi-metrizable space that agrees with $\mathscr{B}_q(X)$ for some admissible quasi-metric q compatible with the asymmetric topologies is also called a quasi-metric bornology.

Definition 4.3.5. ([29, Definition 1.7]) Let $((X, \mathscr{T}_1, \mathscr{T}_2), \mathscr{B})$ be a bornological biuniverse. We say that $((X, \mathscr{T}_1, \mathscr{T}_2), \mathscr{B})$ is quasi-metrizable if there exists a quasi-metric q on X such that $\mathscr{T}_1 = \mathscr{T}_{(q)}$, $\mathscr{T}_2 = \mathscr{T}_{(q^{-1})}$ and $\mathscr{B} = \mathscr{B}_q(X)$.

Lemma 4.3.6. *Let (X,q) be an extended quasi-metric space and $\mathscr{B}_q(X)$ be the quasi-metric bornology. Then, a basis generated by $\mathscr{T}_{q^s}$ open (closed) balls is a subset of the basis generated by $\mathscr{T}_q$ and also $\mathscr{T}_{q^{-1}}$ open (closed) balls respectively.*

Proof. For every $x \in X$, $B_{q^s}(x,\delta) \subset B_q(x,\delta)$ and $B_{q^s}(a,\delta) \subset B_{q^{-1}}(x,\delta)$ by Lemma 2.1.25. Now, since any subset A of X is q^s-bounded if and only if it is both q and q^{-1} bounded, the result then follows. □

Lemma 4.3.7. *(Compare Lemma 3.3.10). Let (X,q) be an extended quasi-metric space. Then*

(i) the family of all finite intersections of quasi-open balls forms a base for $\mathscr{B}_q(X)$;

(ii) $\mathscr{B}_q(X)$ contains the bornology of pairwise-relatively compact subsets of X;

(iii) whenever x_n is a quasi-Cauchy sequence in X, then $\{x_n : n \in \mathbb{N}\} \in \mathscr{B}_q(X)$.

Proof. (i) Let $[(B_q(x_i,\epsilon_i))_{i\in I};(B_{q^{-1}}(x_i,\delta_i))_{i\in I}]$ with $x_i \in X$, $\epsilon_i,\delta_i \in [0,\infty)$ be a family of double balls in $\mathscr{B}_q(X)$ such that it satisfies the binary intersection property. Since $B_q(x,\delta) \cap B_{q^{-1}}(x,\delta) = B_{q^s}(x,\delta)$ by Lemma 2.1.25, we set

$$\mathscr{B}_0 = \left\{ \bigcap_{i=1}^{n} B_{q^s}(x_i,\delta_i) : n \in \mathbb{N}, x_i \in X, \delta_i > 0, 1 \le i \le n \right\}.$$

Then, the definition of a q-bounded set, guarantees that each $B_{q^s}(x_i,\delta_i)$ $\forall i = 1,2,...,n$ is open and bounded, as each is a finite intersection of $(\mathscr{T}_q,\mathscr{T}_{q^{-1}})$-open balls. We aim to show that $\mathscr{B}_0$ generates $\mathscr{B}_q(X)$ by showing that the family $\{B_q(x,\epsilon) \cap B_{q^{-1}}(x,\delta) : x \in X, \epsilon,\delta > 0\}$ of $(\mathscr{T}_q,\mathscr{T}_{q^{-1}})$-open balls in X can serve as a base for $\mathscr{B}_q(X)$. Take any $x \in X$ and $\epsilon,\delta > 0$ such that $B_q(x,\epsilon) \cap B_{q^{-1}}(x,\delta) \neq \emptyset$. If for $z \in X$, $z \in B_q(x,\epsilon) \cap B_{q^{-1}}(x,\delta)$, set $\alpha = \min\{\epsilon - q(x,z), \delta - q(z,x)\}$, then

$$z \in B_{q^s}(z,\alpha) \subseteq B_q(x,\epsilon) \cap B_{q^{-1}}(x,\delta) \tag{4.1}$$

which is a formal consequence of the triangle inequality with respect to q^s. By Lemma 2.1.25 (ii), $B_{q^s}(z,\alpha) \subseteq B_q(x,\epsilon)$ and $B_{q^s}(z,\alpha) \subset B_{q^{-1}}(x,\delta)$. By generalizing (4.1) and applying the mixed binary intersection property, we have for $x_i \in X$, $\epsilon_i,\delta_i > 0$, $z \in \bigcap_{i=1}^{n} (B_q(x_i,\epsilon_i) \cap B_{q^{-1}}(x_i,\delta_i))$ and putting $\alpha = \min\{\epsilon_i - q(x_i,z), \delta_i - q(z,x_i) : n \in \mathbb{N}, \forall i \le n\}$ that

$$B_{q^s}(z,\alpha) \subseteq \bigcap_{i=1}^{n} (B_q(x_i,\epsilon_i) \cap B_{q^{-1}}(x_i,\delta_i)).$$

Since the intersection of these balls belongs to $\mathscr{B}_0$, $\mathscr{B}_0$ generates $\mathscr{B}_q(X)$ since every $B_{q^s}(z, \alpha) = B_q(z, \alpha) \cap B_{q^{-1}}(z, \alpha) \in \mathscr{B}_q(X)$ is contained in some arbitrary member of $\mathscr{B}_0$ (See also [16, Chapter 2], [27, Section 2]).

(ii) Let $\mathscr{B}^*$ be a bornology of pairwise-relatively compact subsets in (X, q). Let K be an arbitrary set of $\mathscr{B}^*$ and $\delta > 0$. Since K is pairwise relatively compact, there is a finite set $\{x_1, x_2, ..., x_n\}$ in K such that

$$K \subseteq \bigcup_{i=1}^{n} B_q(x_i, \frac{\delta}{2}) \subseteq \bigcup_{i=1}^{n} \left(cl_{\mathscr{T}_{q^{-1}}}(B_q(x_i, \frac{\delta}{2})) \right) \subseteq \bigcup_{i=1}^{n} B_q(x_i, \delta).$$

This is equivalent to saying that, "since K is pairwise compact, finitely many of these $\mathscr{T}_q$-open balls cover K and this cover has $\mathscr{T}_{q^{-1}}$ closure which is also pairwise relatively compact" ([1], [32]). Since $\{\bigcup_{i=1}^{n} \left(cl_{\mathscr{T}_{q^{-1}}}(B_q(x_i, \frac{\delta}{2})) \right) : x_i \in K\}$ and $\{\bigcup_{i=1}^{n} B_q(x_i, \delta) : x_i \in X\}$ forms a basis of $\mathscr{B}^*$ and $\mathscr{B}_q(X)$ respectively, we infer that

$$\mathscr{B}^* \subseteq \mathscr{B}_q(X).$$

(iii) Let (x_n) be a quasi Cauchy-sequence in X. Then, given any $\delta > 0$, there exists an $n_\delta \in \mathbb{N}$ such that $q^s(x_n, x_m) < \delta$ for all $n, m \geq n_\delta$ and so $q(x_m, x_n) < \delta$ and $q(x_n, x_m) < \delta$, $\forall n \geq m \geq n_\delta$, implying that (x_n) is both left and right K-Cauchy and hence, both left and right q-Cauchy by Definition 2.2.2. Let the arbitrary $\delta = 1$. Then by assumption, there exists some $n_1 \in \mathbb{N}$ such that for all $n \geq m \geq n_1$, $q(x_m, x_n) < 1$ and $q(x_n, x_m) < 1$. Now let $k \in \mathbb{N}$ and $k \geq n_1$. Observe that $q(x_m, x_k) < 1$ and $q(x_k, x_m) < 1$, so there are infinitely many terms of these sequences inside all quasi balls of radius 1 centered at x_k; thus, by Proposition 2.2.3, these terms are bounded. In particular, we have $q(x_{n_1}, x_n) < 1$ and $q(x_n, x_{n_1}) < 1$, whence

$$x_n \in B_q(x_{n_1}, 1) \cap B_{q^{-1}}(x_{n_1}, 1)$$

for all $n \geq n_1$. Thus, $\{x_n : n \geq n_1 \in \mathbb{N}\}$ is bounded with respect to q^s. Hence, conclude that the entire range $\{x_n : n \in \mathbb{N}\} \subseteq \mathscr{B}_q(X)$.

□

Proposition 4.3.8. *(Compare Proposition 3.3.11) Let (X, q) be an extended quasi-metric space, let B be a bounded subset of X, and let E be a dense subset of X. Denoting the restriction of q to $E \times E$ by q_E, suppose $\mathscr{B}_0$ is a base for $\mathscr{B}_{q_E}$. Then $\{cl_{\mathscr{T}_{q^{-1}}}(B) : B \in \mathscr{B}_0\}$ is a base for $\mathscr{B}_q(X)$.*

Proof. Let $C \in \mathscr{B}_q(X)$. Then, for every $n \in \mathbb{N}$, choose a finite subset F_n in X such that $C \subseteq F_n^{\frac{1}{2n}} = \bigcup_{j=1}^{n} B_q(y, \frac{1}{2n})$ and $B_{q^{-1}}(y, \frac{1}{2n}) \cap C \neq \emptyset$, $\forall y \in F_n$.

Since E is dense in X, we choose for each point $y \in F_n$ some point $x \in E$, with $q(y, x) < \frac{1}{2n}$. In that way, we get a finite set B_n in E, such that

$$F_n \subseteq B_n^{\frac{1}{2n}} \text{ and } B_n \subseteq F_n^{\frac{1}{2n}}$$

for every $n \in \mathbb{N}$. If we define $B = \bigcup_{j=1}^{n} B_n$, then it is not difficult to see that $C \subseteq cl_{\mathscr{T}_{q^{-1}}} B$. Indeed, let $y \in C$, $\epsilon > 0$ and take $n \in \mathbb{N}$ with $\frac{1}{n} < \epsilon$. Since $C \subseteq F_n^{\frac{1}{2n}} \subseteq B_n^{\frac{1}{n}}$, it follows that

$$q(y, B) \leq q(y, B_n) < \frac{1}{n} < \epsilon,$$

We nest show that $B \in \mathscr{B}_{d_E}(E)$. To see this, let $\epsilon > 0$ and choose $k \in \mathbb{N}$ with $\frac{1}{k} < \frac{\epsilon}{2}$. Since $B_n \subseteq F_n^{\frac{1}{2n}} \subseteq C^{\frac{1}{2n}}$, for every $n \in \mathbb{N}$, then if $n \geq k$, we have $B_n \subseteq C^{\frac{1}{2n}} \subseteq C^{\frac{1}{2k}} \subseteq (B_k)^{\frac{3}{2k}} \subseteq (B_k)^{\epsilon}$. Hence, $B \in \mathscr{B}_{d_E}(E)$ since $B_1 \cup ... \cup B_k$ is a finite set and,

$$B = \bigcup B_n \subseteq \left(\bigcup_{n=1}^{k} B_n \right)^{\epsilon}.$$

Our construction gives $B \subseteq C \subseteq cl_{\mathscr{T}_{q^{-1}}}(\bigcup B_n)$ and so, $B \subseteq cl_{\mathscr{T}_{q_E^{-1}}}(\bigcup B_n)$. Let $\mathscr{B}_0 = \{B_n\}_{n \in \mathbb{N}}$, then, it is clear that $cl_{\mathscr{T}_{q^{-1}}} B \in \mathscr{B}_q(X)$, for every $B \in \mathscr{B}_{q_E}(E)$. □

Our next characterization gives properties of a generating cover for a quasi-metric bornology.

Theorem 4.3.9. (Compare Theorem 3.3.12) Let X be a quasi-metrizable space and let $\mathscr{B}$ be a bornology on X. Then $\mathscr{B} = \mathscr{B}_q(X)$ for some compatible extended quasi-metric q if and only if there exist $\mathscr{A} \subseteq \mathscr{B}$ such that $\downarrow (\Sigma(\mathscr{A})) = \mathscr{B}$ and a partition $\{\mathscr{A}_i : i \in I\}$ of $\mathscr{A}$ with the following properties:

(i) each $\mathscr{A}_i$ contains a nonempty subset of X;

(ii) $\forall i \in I, \forall A_1 \in \mathscr{A}_i, \exists A_2 \in \mathscr{A}_i$ with $\mathrm{cl}_{\mathscr{T}_{q^{-1}}}(A_1) \subseteq int_{\mathscr{T}_q}(A_2)$;

(iii) whenever $A_i \in \mathscr{A}_i$ and $A_j \in \mathscr{A}_j$ for $i \neq j$, then $A_i \cap A_j = \emptyset$;

(iv) each $\mathscr{A}_i$ has a countable subfamily which is cofinal in $\mathscr{A}_i$ with respect to inclusion.

Proof. $\Longrightarrow$ For necessity, suppose q is a compatible extended quasi-metric on X with quasi-metric components $\{X_i : i \in I\}$ such that $\mathscr{B}_q(X) = \mathscr{B}$. For each $i \in I$, let $\mathscr{A}_i = \{B_{q^s}(x, \alpha) :$

$x \in X_i, \alpha > 0\}$ and let $\mathscr{A}$ be the collection of $\mathscr{T}_{q^s}$-open balls in X. Since a subset of X is bounded if and only if it is contained in an intersection of quasi open balls and $\mathscr{A}$ is a cover of X, we have

$$\mathscr{B}_q(X) = \mathscr{B}_q(\cup_{i\in I} X_i) =\downarrow \left(\sum(\cup_{i\in I}\mathscr{A}_i)\right) =\downarrow \left(\sum(\mathscr{A})\right).$$

With respect to our choice of $\mathscr{A}_i$, we have that

(i) Since each $\mathscr{A}_i$ covers at-least a X_i, there exists a $\mathscr{T}_{q^s}$-neighborhood A of $x \in X_i$ such that $A \subseteq B_{q^s}(x, \alpha)$. Hence, each $\mathscr{A}_i$ is nonempty.

(ii) Since for every $i \in I$, each $\mathscr{A}_i$ contains a nonempty subset of X, let $A_1 \subseteq X$ and $A_2 \subseteq X$ with $A_1, A_2 \in \mathscr{A}_i$ be such that, for some $\delta > 0$, the inclusion $[A_1]^\delta_q \subseteq A_2$ holds. Let $x \in cl_{\mathscr{T}_{q^{-1}}} A_1$, that is, x is an accumulation point of A_1. There exists $y \in A_1 \cap B_{q^{-1}}(x, \delta)$. Then $q(y, x) < \delta$, so $x \in [A_1]^\delta_q$. Therefore $cl_{\mathscr{T}_{q^{-1}}} A_1 \subseteq [A_1]^\delta_q$. Now, since $[A_1]^\delta_q \subseteq A_2$, we have $cl_{\mathscr{T}_{q^{-1}}} A_1 \subseteq [A_1]^\delta_q \subseteq int_{\mathscr{T}_q} A_2$. As a result, $cl_{\mathscr{T}_{q^{-1}}} A_1 \subseteq int_{\mathscr{T}_q} A_2$.

(iii) Let $A_1, A_2 \in \mathscr{A}_i \ \forall i \neq j, i, j \in \{1, 2\}$. Since $\{\mathscr{A}_i : i \in I\}$ partitions $\mathscr{A}$, we have from $\mathscr{A}_i = \{B_{q^s}(x, \alpha) : x \in X_i, \alpha > 0\}$ that, $A_1 \cap A_2 = \emptyset$. Generalizing this to all but finitely many arbitrary $\mathscr{A}_i$'s we have for the indices $i \neq j, i, j \in I$ that $A_i \cap A_j = \emptyset$.

(iv) Finally, choose a point $x_i \in X_i$, $\forall i \in I \subseteq \mathbb{N}$, that is we assume I to be countable. For some positive integer n the collection $\{B_{q^s}(x_i, n_i) : i \leq n \in \mathbb{N}\}$ forms a subfamily for $\mathscr{A}_i$ which must be countable and hence, $\sum(\{B_{q^s}(x_i, n_i) : i \leq n \in \mathbb{N}\})$ must contain the countably cofinal subfamily for $\mathscr{A}_i$.

$\Longleftarrow$ Conversely, suppose the partition $\{\mathscr{A}_i : i \in I\}$ of $\mathscr{A}$ has the asserted properties. For each $i \in I$, let $X_i = \cup\mathscr{A}_i = \bigcup\{B_{q^s}(x, \alpha) : x \in X_i, \alpha > 0\}$. By property (i), each X_i is nonempty, and by (ii), each X_i is open. As $X_i = \cup\mathscr{A}_i$, condition (iii) guarantees that $\{X_i : i \in I\}$ is a pairwise disjoint family. Thus, each X_i is in fact clopen since condition (ii) guarantees the presence of a closed and open basis. Now, for each $i \in I$, put $\mathscr{B}_i :=\downarrow (\Sigma(\mathscr{A}_i))$ where $\downarrow (\Sigma(\mathscr{A}_i)) =\downarrow \{\cup_{i=1}^n A_i : n \in \mathbb{N}, A_i \in \cup_{i\in I}\mathscr{A}_i = \mathscr{A}\} = \bigcup_{i=1}^{n} \{A_i : A_i \subseteq B_i, \exists B_i \in \mathscr{A}, 1 \leq i \leq n \in \mathbb{N}\}$. Then $\mathscr{B}_i$ is a bornology on X_i because $\mathscr{A}_i$ is a cover of X_i with respect to $\mathscr{T}_{q^s}$. Since the closure of a finite union is the union of the closures, property (ii) holds with $\mathscr{B}_i$ replacing $\mathscr{A}_i$. Property (iii) holds as well if $\mathscr{B}_i$ replaces $\mathscr{A}_i$ and $\mathscr{B}_j$ replaces $\mathscr{A}_j$. Finally, if $\mathscr{A}_i^*$ is countable and cofinal in $\mathscr{A}_i$, then $\Sigma(A_i^*)$ is countable and cofinal in $\downarrow (\Sigma(\mathscr{A}_i)) = \mathscr{B}_i$, that is, (iv) holds with $\mathscr{B}_i$ replacing $\mathscr{A}_i$. Thus generalizing Lemma 3.3.9, for each positive $i \in I$, there is a quasi-metric d_i on X_i such that $\mathscr{B}_i = \mathscr{B}_{q_i}(X_i)$.

As $\downarrow(\Sigma(\mathscr{A}))$ is assumed to be a cover of X, we infer that $\mathscr{A}$ is a cover of X and so $\{X_i : i \in I\} = \{\cup \mathscr{A}_i : i \in I\}$ is a cover of X. Let us define a restriction q of q_i on $X_i \times X_i$ for every $i \in I$ by $q(x, w) = q_i(x, w)$ if $\exists i$ with $\{x, w\} \subseteq X_i$ and $q(x, w) = \infty$ otherwise. By condition (iv), we have

$$\mathscr{B}_q(X) = \sum(\cup_{i \in I}\mathscr{B}_i) = \downarrow \left(\sum(\cup_{i \in I}\mathscr{A}_i)\right) = \downarrow \left(\sum(\mathscr{A})\right) = \mathscr{B}.$$

□

Our second result is based on the strong defining property of quasi-metrically bounded sets.

Definition 4.3.10. Let (X, q) be an extended quasi-metric space and $A \subseteq X$. Then, A is said to be a member of $\mathscr{B}_q(X)$ if its intersection with each quasi-metric component lies in the relative quasi-metric bornology and all but finitely many of these intersections are empty.

Theorem 4.3.11. (Compare Theorem 3.3.14) Let X be a quasi-metrizable space and let $\mathscr{B}$ be a bornology on X. Then $\mathscr{B} = \mathscr{B}_q(X)$ for some compatible extended quasi-metric q if and only if there exists a collection $\{\mathscr{B}_i : i \in I\}$ of families of subsets of X with the following properties:

(i) $\forall i \in I$, $\mathscr{B}_i$ has at least two subsets of X as elements including $\emptyset$.

(ii) $\forall i \in I$, $\forall B_1 \in \mathscr{B}_i$, $\exists B_2 \in \mathscr{B}_i$ with $cl_{\mathscr{T}_{q^{-1}}}(B_1) \subseteq int_{\mathscr{T}_q}(B_2)$;

(iii) whenever $B_i \in \mathscr{B}_i$ and $B_j \in \mathscr{B}_j$ for $i \neq j$, then $B_i \cap B_j = \emptyset$;

(iv) each $\mathscr{B}_i$ has a countable subfamily which is cofinal in $\mathscr{B}_i$ with respect to inclusion;

(v) $\forall A \in \mathscr{P}(X)$, $A \in \mathscr{B}$ if and only if $\forall i \in I$, $A \cap (\cup \mathscr{B}_i) \in \mathscr{B}_i$ and for all but finitely many i ,$A \cap (\cup \mathscr{B}_i) = \emptyset$.

Proof. $\Longrightarrow$ For necessity, let q be a compatible extended quasi-metric on X such that $\mathscr{B} = \mathscr{B}_q(X)$. Let $\{X_i : i \in I\}$ be the quasi-metric components of X and let q_i be the trace of q on $X_i \times X_i$. Set $\mathscr{B}_i := \mathscr{B}_{q_i}(X_i)$, then;

(i) Since each X_i as a quasi-metric component is nonempty and $\mathscr{B}_{q_i}(X_i)$ contains minimally the finite subsets of X_i which includes the empty set, property (i) clearly holds.

(ii) Fix $x_i \in X_i$, $\forall i \in I$. By property (i), let us assume without loss of generality that for every arbitrary $\mathscr{T}_q$ set B_1 in $\mathscr{B}_i$ there exists a $\mathscr{T}_q$ set $B_2 \in \mathscr{B}_i$ such that for some $\delta > 0$, the inclusion $[B_1]_q^\delta \subseteq B_2$ holds. Let $x \in cl_{\mathscr{T}_{q^{-1}}}(B_1)$, that is x is an accumulation point of B_1 with respect to $\mathscr{T}_{q^s}$. There exists $y \in B_1 \cap B_{q^{-1}}(x, \delta)$. Then $q(y, x) = q^{-1}(x, y) < \delta$, so that $x \in [B_1]_q^\delta$. Therefore $cl_{\mathscr{T}_{q^{-1}}} B_1 \subseteq [B_1]_q^\delta$. Of course, since $[B_1]_q^\delta \subseteq B_2$, we have $[B_1]_q^\delta \subseteq int_{\mathscr{T}_q} B_2$. As a consequence, $cl_{\mathscr{T}_{q^{-1}}} B_1 \subseteq int_{\mathscr{T}_q} B_2$ by applying [29, Theorem 4.7].

(iii) Suppose $B_1, B_2 \in \mathscr{B}_i$, $\forall i \neq j, i, j \in \{1, 2\}$. Let $x_1, x_2 \in X_i$ be such that $x_1 \neq x_2$. Then $q(x_1, x_2) > 0$ and $q(x_2, x_1) > 0$ in X_i. So, for a $\delta > 0$, the δ-open balls $B_{q^s}(x_1, \delta) = B_q(x_1, \delta) \cap B_{q^{-1}}(x_1, \delta)$ and $B_{q^s}(x_2, \delta) = B_q(x_2, \delta) \cap B_{q^{-1}}(x_2, \delta)$ are disjoint. Indeed, suppose there exists a $z \in X_i$ such that $z \in B_{q^s}(x_1, \delta)$ and $z \in B_{q^s}(x_2, \delta)$, then $q(x_1, z) < \delta$ and $q(z, x_2) < \delta$. Put $\delta = \min\{\frac{q(x_1,x_2)}{2}, \frac{q(x_2,x_1)}{2}\}$. Then, $q(x_1, z) + q(z, x_2) < 2\delta \leq q(x_1, x_2)$ and $q(x_2, z) + q(z, x_1) < 2\delta \leq q(x_2, x_1)$ which is a contradiction of a quasi-metric and so, there can be no such a point z. Hence, taking $B_1 \subseteq B_{q^s}(x_1, \delta)$ and $B_2 \subseteq B_{q^s}(x_2, \delta)$ implies $B_1 \cap B_2 = \emptyset$. Generalizing this to many $\mathscr{B}_i$'s for the indices $i \neq j, i, j \in I$ we have, $B_i \cap B_j = \emptyset$.

(iv) Choose a point $x_i \in X_i$, $\forall i \in I \subseteq \mathbb{N}$. For each positive integer n the collection of open balls $\{B_{q^s}(x_i, n) : i \leq n, n \in \mathbb{N}\}$ is a countable sequence of $\mathscr{T}_{q^s}$-bounded sets and forms a countable subfamily for $\mathscr{B}_i$. Thus, $\sum(\{B_{q^s}(x_i, n) : i \leq n, n \in \mathbb{N}\})$ is also countably bounded as it is a finite union of the intersection of $(\mathscr{T}_q, \mathscr{T}_{q^{-1}})$-bounded sets and hence must contain a $(\mathscr{T}_q, \mathscr{T}_{q^{-1}})$-countable cofinal subfamily for $\mathscr{B}_i$.

(v) Finally, suppose $A \in \mathscr{P}(X)$ is such that $A \in \mathscr{B}$. By our choice of $\{\mathscr{B}_i : i \in I\}$, A is contained in a $\mathscr{B}_i$. Since $\{\mathscr{B}_i : i \in I\}$ partitions $\mathscr{B}$ and each $\mathscr{B}_i$ is a bornology of each quasi-metric component X_i, $A \cap (\cup \mathscr{B}_i) = \cup(A \cap \mathscr{B}_i) \in \mathscr{B}_i$. Since A is arbitrary and q assumes the value infinite, we have by conditions (i) and (iii) for all but finitely many indices $i \in I$ that, $A \cap \mathscr{B}_i = \emptyset$ and so, $A \cap (\cup \mathscr{B}_i) = \cup_i (A \cap \mathscr{B}_i) = \emptyset$.

$\Longleftarrow$ Conversely, suppose $\{\mathscr{B}_i : i \in I\}$ satisfies the properties above. For each $i \in I$, put $X_i = \cup \mathscr{B}_i$ which by condition (i) is a nonempty subset of X. By condition (ii), each family $\mathscr{B}_i$ has a closed basis and an open basis, implying that $\{X_i : i \in I\}$ is a family of clopen subsets in X and by (iii), $\{X_i : i \in I\}$ partitions X. We next need to show that each family $\mathscr{B}_i$ is a bornology on X_i.

Fix $i \in I$ and set $\mathscr{E}_i = \{B \cap X_i : B \in \mathscr{B}\}$. Evidently $\mathscr{E}_i$ is a bornology on X_i which, by condition (v), is contained in $\mathscr{B}_i$. Indeed, for $B \in \mathscr{B}$, $B \cap X_i \subseteq B \cap (\cup \mathscr{B}_i) = \cap(B \cup \mathscr{B}_i) \in \mathscr{B}_i$. Hence, $\mathscr{E}_i \subseteq \mathscr{B}_i$. Conversely, we need to show that $\mathscr{B}_i \subseteq \mathscr{E}_i$. Let $B_i \in \mathscr{B}_i$ be arbitrary. Then, its enough to show that $B_i \in \mathscr{E}_i$ which suffices to show that $B_i \in \mathscr{B}$. Since $X_i = \cup \mathscr{B}_i$, we have $B_i \cap X_i \in \mathscr{B}_i$ by condition (v). Taking $j \neq i$, condition (iii) gives $B_i \cap X_j = \emptyset$ and so, by condition (i) we have

$$B_i \cap X_j = \emptyset \in \mathscr{B}_j.$$

condition (v) implies, $B_i \in \mathscr{B}$ so that $B_i = B_i \cap X_i \in \mathscr{E}_i$. Hence, the family $\mathscr{B}_i$ as established is a bornology.

Further, by conditions (ii) and (iv), generalizing Hus Theorem, we infer that for each $i \in I$, there exists a quasi-metric q_i on X_i such that $\mathscr{B}_i = \mathscr{B}_{q_i}(X_i)$ that is to say $\mathscr{B}_i$ is a quasi-metric bornology on X_i induced by some quasi-metric q_i. Defining a restriction q of q_i on $X_i \times X_i$ by $q(x,w) = q_i(x,w)$ if $\exists i$ with $\{x,w\} \subseteq X_i$ and $q(x,w) = \infty$ otherwise, condition (v) guarantees that

$$\mathscr{B} = \sum(\cup_{i\in I}\mathscr{B}_i) = \sum(\cup_{i\in I}B_{q_i}(X_i)) = \mathscr{B}_q(X).$$

□

Our next results is on uniformly quasi-metrization of a bornology on sets equipped with large structure, which is a generalization of uniform-metrization of a bornology in [7] and [15], and an extension of uniform quasi-metrization of a bornology in [29].

Definition 4.3.12. ([29, Definition 2.1]) Let $(X, \mathscr{T}_1, \mathscr{T}_2)$ be a bitopological space. Then a $(\mathscr{T}_1, \mathscr{T}_2)$-characteristic function for a bornology $\mathscr{B}$ in X, is a bicontinuous function $\chi : (X, \mathscr{T}_1, \mathscr{T}_2) \longrightarrow (\mathbb{R}_+, \mathscr{T}_l, \mathscr{T}_u)$ such that $\mathscr{B} = \{E \subseteq X : \sup\{\chi(x) : x \in E\} < +\infty\}$.

Lemma 4.3.13. *(Compare Lemma 3.3.16) Let (X, q) be an extended quasi-metric space and $\mathscr{B}$ be the bornology of X. Then, $\mathscr{B}$ admits a $(\mathscr{T}_1, \mathscr{T}_2)$-characteristic function if and only if $\mathscr{B}$ has a countable base $\{B_n : n \in N\}$ such that for some $\delta > 0$, $[B_n]^\delta \subseteq B_{n+1}$ for every $n \in \mathbb{N}$.*

Proof. The proof comes from [29]. Suppose that $\mathbb{B}$ has a countable base satisfying the above hypotheses. Let us assume that $\delta > 0$ and that $\{B_n : n \in I\}$ is a base for $\mathscr{B}$ such that $B_0 = \emptyset$, $B_1 \neq \emptyset$ and $[B_n]_q^\delta \subseteq B_{n+1}$ for each $n \in I$. For each $x \in X$ we defined the function $\phi_n : X \longrightarrow [0,1]$ by

$$\phi_n(x) = \begin{cases} 1 \quad if \quad n = 0 \\ \\ \min\left\{1, \frac{1}{\delta}q(B_n, x)\right\} \quad if \quad n \in I \setminus \{0\} \end{cases}$$

Clearly, $\phi_n(B_n) \subseteq \{0\}$, $\phi_n(X \setminus B_{n+1}) \subseteq \{1\}$, $0 \leq \phi_n \leq 1$ and ϕ_n is uniformly quasi-continuous. Now if we put $B_0 = \emptyset$ and $\phi_0(x) = 1$, for all $x \in X$, we can consider the function $\chi : X \longrightarrow [0, +\infty)$ defined by

$$\chi(x) = n - 2 + \phi_{n-1}(x)$$

when $x \in B_n \setminus B_{n-1}$ and for each $n \in I \setminus \{0\}$. To prove that χ is uniformly quasi-continuous, let us consider an arbitrary pair x, y of points in X such that $q(x,y) < \delta$. Let $n \in I$ be the unique natural number such that $x \in B_n \setminus B_{n-1}$. If $z \in X \setminus B_{n+1}$, then $q(x,z) \geq \delta$. This implies that $y \in B_{n+1} \setminus B_n$. Let $m \in I \setminus \{0\}$ be the unique natural number such that $y \in B_m \setminus B_{m-1}$. Then $m \leq n+1$. We have $\chi(y) - \chi(x) = m - n + \phi_{m-1}(y) - \phi_{n-1}(x)$. If $m = n+1$, then $\chi(y) - \chi(x) = 1 + \phi_n(y) - \phi_{n-1}(x) = \phi_n(y) - \phi_n(x) + \phi_{n-1}(y) - \phi_{n-1}(x) \leq \frac{2}{\delta} q(x,y)$ If $m = n$, then $\chi(y) - \chi(x) = \phi_{n-1}(y) - \phi_{n-1}(x) \leq \frac{1}{\delta} q(x,y)$. Suppose that $m < n$. Then $m - n + 1 \leq 0$, $x \notin B_m$, $y \in B_{n-1}$ and $\chi(y) - \chi(x) = m - n + 1 + \phi_{m-1}(y) - \phi_{m-1}(x) + \phi_{n-1}(y) - \phi_{n-1}(x) \leq \frac{2}{\delta} q(x,y)$ In consequence, χ is uniformly quasi-continuous. Therefore, $\chi : (X, \mathscr{T}_q, \mathscr{T}_{q^{-1}}) \longrightarrow (\mathbb{R}, \mathscr{T}_l, \mathscr{T}_u)$ is bicontinuous.

Finally, if $A \in X$ is such that $A \in \mathscr{B}$, then there exists an $n \in \mathbb{N}$ such that $A \subset B_n$. So $\chi(A) \subseteq [0, n-1]$. Conversely if $\chi(A)$ is bounded then for some $n \in \mathbb{N}$, $A \subseteq B_n$ because otherwise for every $n \in \mathbb{N}$ there exists an $x \in A$ such that $\chi(x) \geq n$. Hence, the map χ is a $(\mathscr{T}_q, \mathscr{T}_{q^{-1}})$-characteristic function for $\mathscr{B}$.

Conversely, suppose that $\mathscr{B}$ admits a $(\mathscr{T}_1, \mathscr{T}_2)$-uniformly continuous characteristic function χ. Then from the uniform continuity of χ, there exists some $\delta > 0$ such that $q(x,y) < \delta$ implies $\max\{\chi(y) - \chi(x), 0\} \leq 1$. Now if we take, $B_n = \chi^{-1}([0,n])$, $n \in \mathbb{N}$, then it is easy to see that the family $\{B_n\}_{n \in \mathbb{N}}$ is a countable base for $\mathscr{B}$, satisfying the required property for this δ. □

Theorem 4.3.14. (Compare Theorem 3.3.17) Let (X, q) be an extended quasi-metric space. The following conditions are equivalent:

(i) the set of quasi-metric components induced by q is countable;

(ii) there exists a compatible quasi-metric q' such that $\mathscr{B}_{q'}(X) = \mathscr{B}_q(X)$.

Proof. The construction of our proof will be a consequence of [29, Theorem 6.5].

(i) ⇒ (ii) Let $\{qmc_q(x_i) : i \in I\}$ be the set of distinct quasi-metric components induced by q where $I \subseteq \mathbb{N}$. For each positive integer n, the collection $B_n = \{B_{q^s}(x_i, n) : i \in I, n \in \mathbb{N}\}$ is a countable sequence and $\sum(\{B_{q^s}(x_i, n) : i \in I, n \in \mathbb{N}\})$ is a finite union of countable sequences and hence, forms a countable basis for $\mathscr{B}_q(X)$. Let B denote an arbitrary subset of X and choose a positive integer n satisfying $n \geq q(B, x_i)$ and $n \geq q(x_i, B)$, $\forall x_i \in qmc_q(x_i)$. Then there exists B_n such that $B \subseteq B_n$, and so $B \subset \sum(\{B_{q^s}(x_i, n) : i \in I, n \in \mathbb{N}\})$. Hence, $cl_{\mathscr{T}_{q^{-1}}} B \subseteq cl_{\mathscr{T}_{q^{-1}}} B_n \subseteq B_{n+1}$, which implies that, the closure of each bounded set is contained in the interior of another, as this is true for each finite union of open balls.

Next, the case $\{qmc_q(x_i) : i \in I\} \in \mathscr{B}_q(X)$ is of course trivial because $\mathscr{B}_q(X) = \mathscr{P}(qmc_q(x_i)) = \mathscr{P}(X)$ and we have $q' = \min\{1, q\}$. Suppose $\{qmc_q(x_i) : i \in I\} \notin \mathscr{B}_q(X)$ and take the bounded quasi-metric $q^* = \min\{1, q\}$ which is uniformly equivalent to q. By applying Lemma 4.3.13, we define a new quasi-metric $q' : X \times X \longrightarrow \mathbb{R}$, by

$$q'(x, y) = q^*(x, y) \vee \frac{\delta}{2}[\chi(y) \dot{-} \chi(x)].$$

To show that q' and q are uniformly equivalent it is enough to prove that q' and q^* are uniformly equivalent.Indeed, since $q^* \leq q'$, we have for $\epsilon = 1$ there exists $\delta > 0$ such that $q'(a, b) < 1$ whenever $q^*(a, b) < \delta$ and so, $\mathscr{T}_{(q^*)^s}$ is coarser than $\mathscr{T}_{(q')^s}$. Conversely, using the property in the proof of Lemma 4.3.13 that $q(x, y) < \delta$ imply $\chi(y) \dot{-} \chi(x) < \frac{2}{\delta} q(x, y)$, we have

$$q'(a, b) = q(a, b) \vee (\chi(y) \dot{-} \chi(x)) \leq \max\left\{1, \frac{2}{\delta}\right\} \cdot q(x, y)$$

whenever $q^*(x, y) < \min\{1, \delta\}$, and we have that the equivalence of asymmetric topologies follows and also the uniform equivalence of the quasi-metrics.

Finally, we want to show that $B \in \mathscr{B}$ if and only if $B \in \mathscr{B}_{d'}(X)$. If $B \in \mathscr{B}$ then it is clearly bounded for the quasi-metric q^* and $\chi(B)$ is quasi-bounded in $\mathbb{R}$ by Lemma 4.3.13, so obviously $B \in \mathscr{B}_{q'}(X)$ as q^* and q' are uniformly equivalent. Conversely, if $B \in \mathscr{B}_{q'}(X)$ then there exists a $k \in \mathbb{R}$ such that $\forall x, y \in B$, $(q')^s(x, y) < k$. As q^* is a bounded quasi-metric then $\chi(B)$ is bounded in $\mathbb{R}$ and so, $B \in \mathscr{B}$. Consequently, $\mathscr{B}_q(X) = \mathscr{B}_{q'}(X)$, of which is a generalization of Lemma 3.3.9.

(ii) $\Rightarrow$ (i) Suppose that $\mathscr{B}_q(X) = \mathscr{B}_{q'}(X)$, for some quasi-metric q' uniformly equivalent to q. Denote the distinct quasi-metric components by $\{qmc_q(x_i) : i \in I\}$. Then, for a fixed $x_i \in \{qmc_q(x_i) : i \in I\}$ and for $n \in \mathbb{N}$, the family

$$B_n = \{B_{(q')^s}(x_i, n+1) : i \in I, n \in \mathbb{N}\} = \{x \in qmc_q(x_i) : (q')^s(x, x_i) < n+1, i \in I, n \in \mathbb{N}\}$$

is a countable base for $\mathscr{B}$. On the other hand, since q and q' are uniformly equivalence, for $\epsilon = \frac{1}{2}$ we can choose $\delta \in (0, +\infty)$ such that $q'(x,y) < \frac{1}{2}$ whenever $q(x,y) < \delta$. So, $[B_n]_q^{\delta} \subseteq [B_n]_{q'}^{\frac{1}{2}} \subseteq B_{n+1}$ for each $n \in \mathbb{N}$. Thus, $\{B_{q^s}(x_i, n) : i \in I, n \in \mathbb{N}\}$ must be a countable base for $B_q(X)$ and hence $\sum(\{B_{q^s}(x_i, n) : i \in I, n \in \mathbb{N}\})$ must contain a countable cofinal family within $\mathscr{B}_q(X)$. This implies that I is countable as $\{\{x_i\} : i \in I\}$ is a family of quasi-bounded sets. Therefore, $\{qmc_q(x_i) : i \in I\}$ is countable. □

We next consider bornologies generated by piarwise compact subsets of X with an emphasis of large structures. Since pairwise locally compact Hausdorff spaces are completely pairwise regular [31, p. 407-411], [21, p. 71–89], a second countable pairwise locally compact Hausdorff space is immediately quasi-metrizable a generalization of the Urysohn metrization theorem in [31, p 407-411], [21, p 71–89]. In the proof of our next result, we will generalize Lemma 3.3.19 a Vaughan [38] result: a pairwise Hausdorff space X admits a compatible quasi-metric such that closed and bounded sets are pairwise compact if and only if X is pairwise locally compact and second countable. We will call an extended quasi-metric space whose closed and bounded sets are pairwise-compact boundedly pairwise compact.

Theorem 4.3.15. (Compare Theorem 3.3.20) Let X be a pairwise-Hausdorff space that is a free union of quasi-metric components $\{X_i : i \in I\}$. The following conditions are equivalent:

(i) each X_i in its relative asymmetric-topology is second countable and pairwise locally compact;

(ii) there exists a boundedly pairwise compact compatible extended quasi-metric q with quasi-metric components $\{X_i : i \in I\}$.

Proof.

(i) ⇒ (ii) Suppose (i) holds. Let q_i be a quasi-metric compatible with the relative asymmetric topologies for each pairwise Hausdorff space X_i and $\{B_{q_i{}^s}(x, n) : i \in I, n \in \mathbb{N}\}$ be its countable base $\forall x \in X_i$. Then, by the pairwise locally compactness of each X_i, each open ball $B_{q_i}(x, n)$ is boundedly pairwise locally compact and so is $cl_{\mathscr{T}_{q_i^{-1}}}(B_{q_i}(x, n))$ by applying Lemma 2.2.11 and generalization of Lemma 3.3.19. Since X is a free union of the quasi-components $\{X_i : i \in I\}$, lets define an extended quasi-metric q induced by $\{q_i : i \in I\}$ on X by $q(x, w) = q_i(x, w)$ if $\exists i$ with $\{x, w\} \subseteq X_i$ and $q(x, w) = \infty$ otherwise. Then, $\{B_{q^s}(x, n) : i \in I, n \in \mathbb{N}\}$ forms a pairwise compact countable basis for X and hence each closed and quasi-bounded subset must lie in a finite union of pairwise-compact closed balls and is thus pairwise-compact (See [1, 9]).

(ii) ⇒ (i) Suppose (ii) holds; lets define q on X by $q(x,w) = q_i(x,w)$ if $\exists i$ with $\{x,w\} \subseteq X_i$ and $q(x,w) = \infty$ otherwise. Since q is boundedly pairwise compact and each quasi-metric component X_i is closed, q_i the restriction of q to $X_i \times X_i$ is boundedly pairwise-compact and so does all the q_i quasi-balls from points in $\{X_i : i \in I\}$. By Lemma 2.2.12, we infer that the q_i-balls form a countable basis for $\{X_i : i \in I\}$. Now, generalizing Lemma 3.3.19, each X_i in its relative asymmetric topology is second countable and pairwise locally compact (see and compare [29, Theorem 8.5]). □

Just as in metric spaces, given an extended quasi-metric space (X,q), the asymmetric topologies on X are the free union asymmetric topologies determined by its quasi components $\{qmc_q(x_i) : i \in I\}$. Thus, we can define a subset A of X to be bounded if its intersection with each quasi component is quasi-metrically bounded. As this is a weaker requirement than A belonging to $\mathscr{B}_q(X)$, we will say that A is weakly bounded if for each $i \in I$, we have $A \cap qmc_q(x_i) \in \mathscr{B}_q(X)$. We denote the bornology of quasi-weakly bounded sets by $\mathscr{B}_q^w(X)$. Note that, $\mathscr{B}_q^w(X) \supseteq \mathscr{B}_q(X)$, with equality if X has at most finitely many quasi components. Clearly, a subset of a particular quasi component is bounded if is weakly bounded ([27]).

Below are some results on bornologies of weakly bounded sets in a quasi-metric space.

Theorem 4.3.16. (Compare Theorem 3.3.22) Let X be a quasi-metrizable space and let $\mathscr{B}$ be a bornology on X. Then $\mathscr{B} = \mathscr{B}_q^w(X)$ for some compatible extended quasi-metric q if and only if there exists a collection $\{\mathscr{B}_i : i \in I\}$ of families of subsets of X with the following properties:

(i) $\forall i \in I$, $\mathscr{B}_i$ has at least two subsets of X as elements including $\emptyset$;

(ii) $\forall i \in I$, $\forall B_1 \in \mathscr{B}_i$, $\exists B_2 \in \mathscr{B}_i$ with $cl_{\mathscr{T}_{q^{-1}}}(B_1) \subseteq int_{\mathscr{T}_q}(B_2)$;

(iii) whenever $B_i \in \mathscr{B}_i$ and $B_j \in \mathscr{B}_j$ for $i \neq j$, then $B_i \cap B_j = \emptyset$;

(iv) each $\mathscr{B}_i$ has a countable subfamily which is cofinal in $\mathscr{B}_i$ with respect to inclusion;

(v) $\forall A \in \mathscr{P}(X)$, $A \in \mathscr{B}$ if and only if $\forall i \in I$, $A \cap (\cup \mathscr{B}_i) \in \mathscr{B}_i$.

Proof. $\implies$ The quasi-metrizability of X implies that $\mathscr{B} = \mathscr{B}_q(X)$ for some compatible extended quasi-metric q on X.Now, for this q, suppose $\mathscr{B} = \mathscr{B}_q^w(X)$. By applying the facts of Theorem 4.3.11, denote the family of quasi-metric components of X by $\{qmc_q(x_i) : i \in I\}$ and for each $i \in I$, set $\mathscr{B}_i = \{B_{q^s}(x,\delta) : x \in qmc_q(x_i), \delta > 0\}$. Let q_i be the trace or restriction of q on $X_i \times X_i$ and put $\mathscr{B}_i = \mathscr{B}_{q_i}^w(X_i)$. Then, conditions (1), (ii), (iii) and (iv) clearly hold by applying Theorem 4.3.11. For condition (v), let $A \in \mathscr{P}(X)$ be such that $A \in \mathscr{B}$. Then,

A is a weakly bounded subset of X, that is, $A \cap qmc_q(x_i) \in \mathscr{B}_q(X)$ and so, since $\{B_i : i \in I\}$ is a family of subsets of X, A must be contained in some $\mathscr{B}_i$. As A is weakly bounded, $A \cap \mathscr{B}_i \in \mathscr{B}_{q_i}(X_i)$ and the definition of a bornology implies $A \cap (\cup \mathscr{B}_i) = \cup (A \cap \mathscr{B}_i) \subset \mathscr{B}_i$. Since A is arbitrary in $\mathscr{P}(X)$ and q assumes values of infinite, we have by applying (i) and (iii) for all but finitely many indices i that, $A \cap (\cup \mathscr{B}_i) = \cup_i (A \cap \mathscr{B}_i) = \emptyset$ and so (v) holds.

$\Longleftarrow$ Conversely, suppose $\{\mathscr{B}_i : i \in I\}$ satisfies the properties above. For each $i \in I$, put $qmc_q(x_i) := \cup \mathscr{B}_i$ which by condition (i) is a nonempty subset of X as it contains an element of x. By conditions (ii) and (iii), $\{qmc_q(x_i) : i \in I\}$ partitions X into clopen subsets. We next show that each family $\mathscr{B}_i$ is a weak bornology on $qmc_q(x_i)$.

Fix $i \in I$ and set $\mathscr{E}_i := \{B \cap qmc_q(x_i) : B \in \mathscr{B}\}$. Since $qmc_q(x_i) = \cup \mathscr{B}_i$ and the arbitrary set B is quasi-bounded, we have $B \cap qmc_q(x_i) \in \mathscr{B}_q(X)$ and so, B is weakly bounded. Clearly, $\mathscr{E}_i$ is a bornology of weakly bounded sets on $qmc_q(x_i)$. Indeed, if $B \in \mathscr{B}$ is any arbitrary weakly quasi-bounded set, there is an arbitrary $B \cap qmc_q(x_i) \in \mathscr{E}_i$ with $B \subseteq B \cap qmc_q(x_i) \in \mathscr{E}_i$. By (v), $\mathscr{E}_i$ is contained in $\mathscr{B}_i$. So, $\mathscr{E}_i$ is basically a weak bornology on $qmc_q(x_i)$. Conversely, suppose $\mathscr{E}$ is a weak bornology on $qmc_q(x_i)$, then we need to show that $\mathscr{B}_i \subseteq \mathscr{E}_i$. Let $B_i \in \mathscr{B}_i$ be an arbitrary weakly bounded set. It is enough if we show that $B_i \in \mathscr{E}_i$ by first showing that $B_i \in \mathscr{B}$. Since $qmc_q(x_i) = \cup \mathscr{B}_i$ it's clear that $B_i \cap qmc_q(x_i) \in \mathscr{B}_i$. Taking $i \neq j$, condition (iii) gives $B_i \cap qmc_q(x_j) = \emptyset$. And by condition (i), we have

$$B_i \cap qmc_q(x_j) = \emptyset \in \mathscr{B}_j.$$

Now, in view of condition (v), $B_i \in \mathscr{B}$ and so, $B_i = B_i \cap qmc_q(x_i) \in \mathscr{E}_i$, with the possibility that one of the B_i can be empty. Hence, the family $\mathscr{B}_i$ is established as a weak bornology.

Further by conditions (ii) and (iv), generalizing Lemma 3.3.9, there exists a quasi-metric q_i on $qmc_q(x_i)$ for each $i \in I$ such that $\mathscr{B}_i = \mathscr{B}^w_{q_i}(X_i)$ that is, $\mathscr{B}_i$ is a weak quasi-metric bornology on $qmc_q(x_i)$ induced by some quasi-metric q_i. Finally, define the extended quasi-metric q induced by q_i on weakly bounded subsets in X by $q(x, w) = q_i(x, w)$ if $\exists i$ with $\{x, w\} \subseteq qmc_q(x_i)$ and $q(x, w) = \infty$ otherwise. Then, condition (v) guarantees that

$$\mathscr{B} = \mathscr{B}^w_q(X).$$

□

Theorem 4.3.17. (Compare Theorem 3.3.23) Let (X, q) be an extended quasi-metric space. The following conditions are equivalent:

(i) all but finitely many of the quasi-metric components for q are quasi-bounded that is, they belong to $\mathscr{B}_q(X)$;

(ii) there exists a compatible quasi-metric q' such that $\mathscr{B}_{q'}(X) = \mathscr{B}^w_q(X)$.

Proof. Denote the quasi-metric components for q by $\{qmc_q(x_i) : i \in I\}$, so we will apply facts of Theorem 4.3.14 throughout the proof.

(i) $\Rightarrow$ (ii) . Suppose (i) holds, let $I_1 \subseteq I$ be those indices for which $qmc_q(x_i) \in \mathscr{B}_q(X)$. To show (ii) holds, we only need to produce a countable base for $\mathscr{B}^w_q(x)$. For each $n \in \mathbb{N}$, define $\mathscr{A}$ by

$$\mathscr{A} = \left\{ \cup_{i \in I_1} qmc_q(x_i) \bigcup \cup_{i \in I - I_1} B_{q^s}(x_i, n) : n \in \mathbb{N} \right\}.$$

Since each $qmc_q(x_i)$ for $i \in I_1$ is quasi-bounded, $\cup_{i \in I_1} qmc_q(x_i)$ is also quasi-bounded and intersects no $\bigcup\limits_{i \in I - I_1} B_{q^s}(x_i, n)$ for all but finitely many indices i.

But $(\cup_{i \in I_1} qmc_q(x_i)) \bigcap (\cup_{i \in I - I_1} B_{q^s}(x_i, n)) \in \mathscr{B}_q(X) \equiv \mathscr{B}_{q'}(X)$. Hence, $\cup_{i \in I_1} qmc_q(x_i)$ is weakly quasi-bounded and so is $qmc_q(x_i)$. Thus, the collection $\mathscr{A}$ forms a countable base for $\mathscr{B}^w_q(X)$. Further, $\{qmc_q(x_i) : i \in I_1\} \subseteq \{B_{q^s}(x_i, n) : i \in I_1, n \in \mathbb{N}\}$ and so,

$$\begin{aligned} cl_{\mathscr{T}_{q^{-1}}} \{qmc_q(x_i) : i \in I_1\} &\subset cl_{\mathscr{T}_{q^{-1}}} \{B_q(x_i, n) : i \in I_1, n \in \mathbb{N}\} \\ &\subseteq cl_{\mathscr{T}_q} \{qmc_q(x_i) : i \in I - I_1\} \\ &\subset int_{\mathscr{T}_q} \{B_q(x_i, n) : i \in I - I_1, n \in \mathbb{N}\}. \end{aligned}$$

Evidently the closure of each weakly bounded subset is in the interior of another. Hence, generalizing Hu's Theorem, there exists a compatible q' such that $\mathscr{B}_{q'}(X) = \mathscr{B}^w_q(X)$.

(ii) $\Rightarrow$ (1) Suppose a compatible quasi-metric q' exists with $\mathscr{B}_{q'}(X) = \mathscr{B}^w_q(X)$, yet for some countably infinite set of indices $\{i_j : j \in \mathbb{N}\}$, $qmc_q(x_{i_j})$ is not quasi-bounded (or simultaneously q-bounded and q^{-1}-unbounded and vice versa). Fix $p \in X$. For each $n \in \mathbb{N}$, put $B_n = \{B_{(q^s)'}(p, n) : n \in \mathbb{N}\}$ and let $\delta > 0$. Since for a $\delta < \frac{1}{2}$, $[A_n]^{\frac{1}{2}}_{(q^s)'} \subseteq B_{n+1}$ and $[B_n]^{\frac{1}{2}}_{(q^s)'} \cap B_{(q^s)'}(p, n) \in \mathscr{B}_{q'}(X)$, B_n is a collection of quasi-balls formed by bounded subsets of X and so, B_n is cofinal in $\mathscr{B}_q^w(X)$, (see [27,29]).

Now for each $j \in \mathbb{N}$, $\exists A_j \in \mathscr{B}_q(X)$ such that $A_j \subseteq qmc_q(x_{i_j})$ and such that $A_j \nsubseteq B_{(q^s)'}(p,j)$. Then $\cup_{j=1}^{\infty} A_j \cup \{\{x_i\} : i \in I\}$ is weakly quasi-bounded but is contained in no any $(q^s)'$-ball with center p, which is a contradiction since $\mathscr{B}_q(X) = \mathscr{B}_{q'}(X)$. Hence, $qmc_q(x_{i_j})$ must be quasi-bounded for some countably infinite set of indices $\{i_j : j \in \mathbb{N}\}$ and so, $\{qmc_q(x_i) : i \in I\}$ is quasi-bounded, (See [1,9,21,27,32]). □

Theorem 4.3.18. (Compare Theorem 3.3.24) Let X be an extended pairwise-Hausdorff space that is a free union of $\{X_i : i \in I\}$. The following conditions are equivalent:

(i) I is finite and X is pairwise locally compact and second countable;

(ii) there exists a compatible extended quasi-metric q with quasi-metric components $\{X_i : i \in I\}$ such that each closed and weakly quasi-bounded subset is pairwise compact.

Proof.

(i) ⇒ (ii) . If I is finite, then for any extended quasi-metric q with quasi-metric components $\{X_i : i \in I\}$, we have $\mathscr{B}_q^w(X) = \mathscr{B}_q(X)$. Now by Theorem 4.3.15, let q_i be the quasi-metric compatible with the relative weak bitopology for X_i and $\{B_{q_i^s}(x_i, n) : i \in I, n \in \mathbb{N}\}$ be its countable base from weakly quasi-bounded sets, [27]. Since each open subspace of a pairwise locally compact and second countable space retains these properties, each open and weakly quasi-bounded $B_{(q_i)^s}(x_i, n)$-ball is a pairwise-compact neighborhood for x_i and so, by generalizing Lemma 3.3.19 each closed and weakly quasi-bounded ball is pairwise-compact. Since X_i induce X, define the extended quasi-metric q induced by $\{q_i : i \in I\}$ by $q(x,w) = q_i(x,w)$ if $\exists i$ $\{x,w\} \subseteq X_i$ and $q(x,w) = \infty$ otherwise. Then, the collection $\{B_{q^s}(x_i, n) : i \in I, n \in \mathbb{N}\}$ of weakly open quasi-balls forms a pairwise compact countable base for X and hence each closed and weakly quasi-bounded subset must lie in a finite union of weak pairwise-compact closed balls and is thus pairwise-compact (compare [1,21,27,32])).

(ii) ⇒ (i) Assume (ii) holds. Clearly, I must be finite for the closed and weakly quasi-bounded subset in $\{X_i : i \in I\}$ to be quasi-compact, all else choosing $x_i \in X_i$, the discrete set $\{\{x_i\} : i \in I\}$ would be closed and weakly quasi-bounded but not pairwise-compact. Again, $\mathscr{B}_q^w(X) = \mathscr{B}_q(X)$. Since a free union of finitely many second countable spaces is second countable and each X_i is closed, the restriction of q to $X_i \times X_i$ is weakly quasi-boundedly compact as well. And so by generalizing Lemma 3.3.19, each X_i in the relative weak bitopology is both second countable and pairwise locally compact. □

CHAPTER 5 : CONCLUSION

In this MSc dissertation, we have successfully generalized the results of Beer [7] on the structure of extended real-valued metric spaces to extended real-valued T_0-quasi-metric spaces with an emphasis on large structure. In this last part of our work, we present a summary of our investigation.

In the first part of our investigations, we have generalized Beer's extension of the Urysohn universal metric space to sets equipped with an extended real-valued T_0 quasi metric with emphasis on large structure.

In the second part of our work, we first provided some properties of a bornological biuniverse. Then we used the conditions of Beer's metrization Theorem of a bornology on sets equipped with an extended real-valued metric to develop a quasi-metrization theorem of a bornology on sets equipped with an extended real-valued T_0-quasi-metric with an emphasis on large structure. After that, we generalized the uniform metrization theorem of a bornology to unform T_0-quasi-metrization theorem on sets assuming infinity. Thereafter, we generalised the result of a bornology generated by compact sets in an extended metric space to pairwise compact sets in an extended T_0-quasi-metric space.

Our conclusion leads us to list some open problems that we hope to study in future work.

Problem 1. *Is it possible to generalize the classical results of Garrido and Meroño [15] to asymmetric normed spaces?*

Problem 2. *Is it possible to extend the classical results of Piękosz and Wajch [29] to asymmetric normed spaces?*

Problem 3. *Under what assumptions can we generalize the results of Beer [7] to sets equipped with an extended quasi-pseudo-metric with an emphasis on large structure?*

Problem 4. *Under what conditions can the results of Beer [7] be generalized to sets equipped with an extended asymmetric norm with an emphasis on large structure?*

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