# EQUIVALENT NORMS ON $L^P$ AND $E^P(T)$ SPACES

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## DECLARATION

I hereby declare that this dissertation is my own work and that it has not been previously submitted for degree purposes here or at any other University.

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# INTRODUCTION

In this paper we will be considering two measures on some measure space $(X, A)$. For general $p > o$ we can consider the two norms $||f||p,\mu$ and $||f||p,\nu$. We are interested in conditions on $\mu$ and $\nu$ that make $||f||p,\mu \simeq ||f||p,\nu$ for all $f$ in some class of functions, i.e. when there exist positive constants $k_1$ and $k_2$ such that

$$k_1||f||p,\nu \leq ||f||p,\mu \leq k_2\ ||f||p,\nu \qquad (1)$$

In chapter I we will give necessary and sufficient condition for (1) to hold for all measurable functions on an arbitrary measure space $(X, A)$. The techniques used in this will be standard techniques in measure theory and integration theory.

In Chapter II we will restrict ourselves to the real line with the Borel sigma field. The class of functions we are interested in is $E^p(T)$, entire functions of exponential type $T$ whose restrictions to $R$ are in $L^p(R,dx)$. We will give conditions on $\mu$ and $\nu$ that make (1) hold for all $f\varepsilon E^p(T)$.

The present work is largely an extension of LIN's work in [2]. He analyzed the $p = 2$ case in $N$ dimensions. We will consider arbitrary $p$ $(o < p < \infty)$ in the one dimensional case. Although we have not done so here, there are $N$ dimensional versions of all of our results.

When $p \neq 2$ we no longer have a Hilbert space and when $o < p < 1$, we are not even in a Banach space. The techniques used go back to Plancherel and Polya in [4]. Methods from functional analysis, complex analysis and real analysis will be used.

These types of problems are of interest in functional analysis and in prediction theory. In the former one is interested in classify spaces of functions and seeing when they have equivalent properties. In the latter there are spectral representation theorems that allow one to phrase prediction problems in some function space (see [3] for one example). The techniques and questions of each field are used to answer and motivate seemingly different problems in the other field.

# CHAPTER I

## EQUIVALENT NORMS ON $L^p$ SPACES

In this chapter, we shall answer the question "When is $||f||_{p,\mu} \simeq ||f||_{p,\nu}$ ? for all measurable functions f". $\mu$ and $\nu$ will be two measures on the same measure space $(X, A)$. Before answering the above question, we give the basic definitions.

**DEFINITION 1.1** Let $o < p < \infty$. The set $L^p(\mu)$ is the set of complex valued measurable functions $f$ mapping $X \to C$ such that $\int_X |f|^p d\mu < \infty$. Similary $L^p(\nu)$ is those functions for which $\int_X |f|^p d\nu < \infty$. The $L^p(\mu)$ norm of $f$ is $||f||_{p,\mu} = (\int_X |f|^p d\mu)^{1/p}$, $f \in L^p(\mu)$. A similar definition gives the $L^p(\nu)$ norm $||f||_{p,\nu}$.

$L^p(\mu) \subset L^p(\nu)$ means that every $f \in L^p(\mu)$ also belongs to $L^p(\nu)$. Thus every $f \in L^p(\mu)$ is also in $L^p(\nu)$, i.e. it has finite $L^p(\nu)$ norm. $L^p(\mu) = L^p(\nu)$ iff $L^p(\mu) \subset L^p(\nu)$ and $L^p(\mu) \supset L^p(\nu)$.

Note To save writing, we will sometimes write $||f||_{p,\mu} = +\infty$ to mean $f \notin L^p(\mu)$.

When $p = +\infty$, the definition 1.1 will change slightly.

**DEFINITION 1.1*** $L^\infty(\mu)$ is thespace of essentially bounded functions f. The $L^\infty(\mu)$ norm of $f$ is

$$||f||_{\infty,\mu} = \operatorname{ess}\sup_{x\in X} |f(x)| = \inf\{c \mid \mu\{|f|>c\} = o\}$$

Suppose we restrict two measures in such a way that $\mu(A) \leq b\,\nu(A)$ for all $A \in \mathcal{A}$ where $b > o$. The spaces $L^p(\nu)$ and $L^p(\mu)$, $||f||_{p,\mu}$ and $||f||_{p,\nu}$ and $\frac{d\mu}{d\nu}$ are related in the following way.

<u>THEOREM 1.2</u> The following are equivalent

(a) $\mu(A) \leq b\nu(A)$ for all $A \in \mathcal{A}$.

(b) For any $o < p < \infty$, $||f||_{p,\mu} \leq b^{1/p}\,||f||_{p,\nu}$ and thus $L^p(\nu) \subset L^p(\mu)$

(c) $\frac{d\mu}{d\nu} \leq b$ a.e $[\nu]$

<u>Proof</u> We will show that (a) => (b) => (a) => (c) => (a)

(a) => (b).

Let $\mu(A) \leq b\,\nu(A)$ for all $A \in \mathcal{A}$ and fix $p \in (o,\infty)$.

<u>Step 1</u> Let $f$ be a simple function in $L^p(\nu)$. Then $|f|^p$ is a simple function in $L^p(\nu)$. So $|f|^p = \sum_{j=1}^{n} a_j 1_{A_j}$ where $a_j \geq o$, $A_1, A_2, \ldots A_n$ are disjoint sets in $\mathcal{A}$ and $1_{A_j}$ is an indicator function. Then $\int_X |f|^p d\mu = \sum_{j=1}^{n} a_j \mu(A_j)$

By (a)

$$\sum_{j=1}^{n} a_j\, \mu(A_j) \leq b \sum_{j=1}^{n} a_j\, \nu(A_j)$$

Thus

$$\int_X |f|^p d\mu \leq b \int_X |f|^p d\nu$$

Taking $p^{th}$ roots shows

$$||f||_{p,\mu} \leq b^{1/p}\,||f||_{p,\nu}$$

Step 2. By step 1, if $s$ is a non negative simple function

$$\int_X s\,d\ddot{\mu} \le b \int_X s\,d\nu .$$

Then

$$\sup \{ \int_X s\,d\ddot{\mu} : o \le s \le |f|^p \} \le \sup \{ b \int_X s\,d\nu : o \le s \le |f|^p \}$$

$$= b \sup \{ \int_X s\,d\nu : o \le s \le |f|^p \}$$

So by definition of the integrals

$$\int_X |f|^p d\ddot{\mu} \le b \int_X |f|^p d\nu$$

Then $||f||_{p,\ddot{\mu}} \le b^{1/p} ||f||_{p,\nu}$.

By step (1) and (2) combined show that (a) => (b)

Step 3 We show that (b) => (a)

Fix $o < p < \infty$ and suppose $||f||_{p,\ddot{\mu}} \le b^{1/p} ||f||_{p,\nu}$. Then we show that $\ddot{\mu}(A) \le b\nu(A)$ for all $A \in \mathcal{A}$. Let $f = 1_A$, $A \in \mathcal{A}$. Then

$$\ddot{\mu}(A) = \int_X 1_A d\ddot{\mu} = ||f||^p_{p,\ddot{\mu}} \le b\, ||f||^p_{p,\nu} = b \int_X 1_A d\nu = b\nu(A)$$

Hence (b) => (a)

Step 4 We show (a) => (c)

Suppose $\ddot{\mu}(A) \le b\nu(A)$ for all $A \in \mathcal{A}$. Then $\nu(A) = o \Rightarrow \ddot{\mu}(A) = o$

Thus $\ddot{\mu} << \nu \Rightarrow \frac{d\ddot{\mu}}{d\nu}$ exists and $\ddot{\mu}(A) = \int_A \frac{d\ddot{\mu}}{d\nu} d\nu$.

Using (a)

$$\int_A \frac{d\ddot{\mu}}{d\nu} d\nu = \ddot{\mu}(A) \le b\, \nu(A) = \int_A b\,d\nu \quad \text{for all } A \in \mathcal{A}.$$

Since this is true for every $A \in \mathcal{A}$, we must have

$\frac{d\mu}{d\nu} \leq b$ a.e. $[\nu]$, which is (c)

Step 5 To show (c) => (a)

Suppose (c) $\frac{d\mu}{d\nu} \leq b$ a.e. $[\nu]$

Integrate with respect to any set $A \in \mathcal{A}$

$$\mu(A) = \int_A \frac{d\mu}{d\nu} d\nu \leq \int_A b d\nu = b \int_A d\nu = b\ \nu(A).$$

This completes the proof.

We now have the following necessary and sufficient conditions on two measures $\mu$ and $\nu$ so that $||f||_{p,\mu} \simeq ||f||_{p,\nu}$.

THEOREM 1.3 Let $a, b$ be positive real numbers. The following are equivalent

(a) $a\ \nu(A) \leq \mu(A) \leq b\ \nu(A)$ for all $A \in \mathcal{A}$.

(b) For any $0 < p < \infty$, $a^{1/p} ||f||_{p,\nu} \leq ||f||_{p,\mu} \leq b^{1/p} ||f||_{p,\nu}$

and thus $L^p(\mu) = L^p(\nu)$

(c) $a \leq \frac{d\mu(x)}{d\nu(x)} \leq b$ a.e $[\nu]$ (or equivalently a.e $[\mu]$)

Proof Use theorem 1.2 twice.

The $L^\infty$ case is slightly different.

**THEOREM 1.2*** The following are equivalent

(a) $\mu << \nu$

(b) $||f||_{\infty,\mu} \leq ||f||_{\infty,\nu}$ , thus $L^{\infty}(\nu) \subset L^{\infty}(\mu)$

**Proof** To show that (a) => (b)

Let $c = ||f||_{\infty,\nu}$ => for all $\varepsilon > 0$

$\nu\{x : |f(x)| > c + \varepsilon\} = 0$

=> $\mu\{x : |f(x)| > c + \varepsilon\} = 0$ for all $\varepsilon > 0$ since $\mu << \nu$.

This mean that $||f||_{\infty,\mu} \leq c = ||f||_{\infty,\nu}$ . To see that

(b) => (a), we assume $||f||_{\infty,\nu} \leq ||f||_{p,\mu}$. We show that any set of $\nu$ measure $0$ has $\mu$ measure $0$. Let $A$ have positive $\mu$ measure, then $||1_A||_{\infty,\mu} = 1$. But then

$||1_A||_{\infty,\nu} \geq ||1_A||_{\infty,\mu} = 1$. Thus $\nu(A) > 0$.

Thus $\mu(A) > 0$ implies $\nu(A) > 0$ ; the contrapositive shows $\mu << \nu$.

The analog of Theorem 1.3 is the following

**THEOREM 1.3*** The following are equivalent

(a) $\nu << \mu$ and $\mu << \nu$

(b) $||f||_{\infty,\nu} = ||f||_{\infty,\mu}$ , thus $L^{\infty}(\mu) = L^{\infty}(\nu)$

**Proof** Use Theorem 1.2* twice.

**EXAMPLE 1.4** Consider $(R, \mathcal{B})$, the real line with Borel Sigma field.

Let $\mu(dx) = \mu(x)dx$, $\nu(dx) = \nu(x)dx$ where $\mu(x)$ and $\nu(x)$ are $\mathcal{B}$ - measurable non negative functions and $dx$ is Lebesgue measure, i.e.

$$\ddot{\mu}(A) = \int_A \ddot{\mu}(x)dx$$

$$\nu(A) = \int_A \nu(x)dx$$

for all $A \in \mathcal{B}$. Let $0 < a < b < \infty$. Then $a\,\nu(x) \le \mu(x) \le b\,\nu(x)$ a . e dx if and only if $a^{1/p}\,||f||_{p,\nu} \le ||f||_{p,\mu} \le b^{1/p}\,||f||_{p,\nu}$ for all measurable function $f$.

<u>Proof</u> ( => ) Suppose $a\,\nu(x) \le \ddot{\mu}(x) \le b\,\nu(x)$ a.e. dx

Then for any $A \in \mathcal{A}$

$$\int_A a\,\nu(x)dx \le \int_A \ddot{\mu}(x)dx \le \int_A b\,\nu(x)dx$$

=> $a\nu(A) \le \ddot{\mu}(A) \le b\nu(A)$ $A \in \mathcal{B}$. Theorem 1.3 gives the the result.

Conversely, if $a^{1/p}\,||f||_{p,\nu} \le ||f||_{p,\mu} \le b^{1/p}\,||f||_{p,\nu}$, then by theorem 1.3

$$a \le \frac{d\mu(x)}{d\nu} \le b \qquad \nu - \text{a.e. (and } \mu - \text{a.e)}$$

$$a \le \frac{\mu(x)}{\nu(x)} \le b \qquad dx\text{-a.e. } x \text{ where } \nu(x)>0$$

Hence $a\,\nu(x) \le \mu(x) \le b\,\nu(x)$ dx-a.e where $\nu(x)>0$. Since $\mu(x) = 0$ for dx-a.e x when $\nu(x) = 0$, this last inequality holds dx - a.e

<u>EXAMPLE 1.5</u> This example shows that even on $L^p(\mu) \cap L^p(\nu)$, the $\mu$-norms and $\nu$-norms are not comparable in general.

Let $L^p(\nu) = L^1(\nu)$ where $\nu(dx) = e^{-x^2}dx$,

$L^p(\mu) = L^1(dx)$ and

$$f_n = 1_{[-n,n]}$$

Then $||f_n||_{1,dx} = 2n$, so $f_n \in L^1(dx)$

Also $||f_n||_{1,\nu} = \int_{-n}^{n} e^{-x^2}dx \leq \int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$,

so $f_n \in L^1(\nu)$.

Therefore $f_n \in L^1(dx) \cap L^1(\nu)$.

But

$$\operatorname*{Limit}_{n\to\infty} \frac{||f_n||_{1,dx}}{||f_n||_{1,\nu}} = \operatorname*{Limit}_{n\to\infty} \frac{2n}{\sqrt{\pi}} = +\infty,$$

So there is no constant $b$ such that $||f||_{1,dx} \leq b||f||_{1,\nu}$.

<u>EXAMPLE 1.6</u> (a) Let $\nu$ be counting measure on $Z$. Then $a^{1/p}||f||_{p,\nu} \leq ||f||_{p,\mu} \leq b^{1/p}||f||_{p,\nu}$ for all measurable $f$ if and only if $\mu$ is also supported on $Z$ and $a \leq \mu(n) \leq b$.

(b) More generally, if $\nu$ is supported on $Z$, then $||f||_{p,\nu} \simeq ||f||_{p,\mu}$ for all measurable $f$ iff $\mu$ is supported on $Z$ and $\nu\{n\} \simeq \mu\{n\}$ for all $n$.

# CHAPTER II

## EQUIVALENT NORMS ON $E^p(T)$

In this chapter, we will restrict ourselves to measures on the real line and find conditions on $\mu$ and $\nu$ which guarantee $||f||_{p,\mu} \simeq ||f||_{p,\nu}$ for all $f$ in a special class of functions. The class we are going to work with is $E^p(T)$. We now give some basic definition and properties.

### SECTION 1

To reduce the number of symbols, we will use the following conventions. We will be working on the real line $R$, with Borel set $\mathcal{B}$ and Lebesgue measure will again be denoted by $dx$.

(a) If an integral is over the entire real line, then instead of writing $\int_R |f|^p d\mu$, we will leave out the $R$ and write $\int |f|^p d\mu$.

(b) Likewise if we are summing over all the elements in a doubly infinite series, then we will leave out the bounds, i.e. $\Sigma\, a_n$ will mean $\sum_{n=-\infty}^{\infty} a_n$

(c) Lebesgue measure will be used frequently and the $L^p$ norm with respect to Lebesgue measure will be denoted by $||f||_p$, as an abbreviation for $||f||_{p,dx}$.

(d) $I(h,x) = [x - \frac{h}{2}, x + \frac{h}{2})$ = interval centred at $x$ of length $h$. Note that $\{I(h,nh)\}_{n=-\infty}^{\infty}$ is a disjoint collection of intervals of length $h$ whose union is $R$.

(e) For a fixed function $f$ and fixed $h>o$, we will let

$$f_n = \inf_{t\varepsilon I(h,nh)} |f(t)|$$

$$F_n = \sup_{t\varepsilon I(h,nh)} |f(t)|.$$

We make the same definitions for $f'$ : $f_n' = \inf_{t\varepsilon I(h,nh)} |f'(t)|$

and $F_n' = \sup_{t\varepsilon I(h,nh)} |f'(t)|$.

From basic definitions, if $\mu$ is any measure and $I$ is any interval, then for any measureable function $f$,

$$\inf_{x\varepsilon I} |f(x)|^p \mu(I) \leq \int_I |f|^p d\mu \leq \sup_{x\varepsilon I} |f(x)|^p \mu(I).$$

In particular, summing over the collection $\{I(h,nh)\}_{n=-\infty}^{\infty}$

$$\Sigma f_n^p \mu(I(h,nh)) \leq \int |f|^p d\mu \leq \Sigma F_n^p \mu(I(h,nh)) \qquad (2.1)$$

DEFINITION 2.1 $E(T)$ will be the set of entire functions of exponential type $T$, that is all entire (analytic on the whole complex plane) functions $f(z)$ which satisfy

$|f(z)| \leq k\, e^{(T + \varepsilon)|z|}$ for some constant $k>o$ and all $\varepsilon>o$.

We will actually be concerned with the restriction of $f(z)$ to the real line, but the fact that all these functions are analytic on the complex plane will be used at certain points. For convenience we will talk about functions $f(x)$ defined on the real line being in $E(T)$, rather than use the difficult phrasing "$f(x)$ is the restriction of the analytic function $f(z)$ in $E(T)$". The set of all functions in $E(T)$ whose restriction to $R$ are $p^{th}$ powers ~~integrable with~~ respect to Lebesgue measure will be called $\underline{E^p(T)}$. If we think of these functions being defined just on $R$ in the above way, then $E^p(T)$ is just the intersection of $E(T)$ and $L^p(R,dx)$. By $||f||_{p,\mu}$ we mean $(\int|f(x)|^p d\mu(x))^{1/p}$, i.e. the integral is over just the real line $R$.

DEFINITION 2.2 A real valued function $g(z)$ defined on the complex plane is **subharmonic** if it satisfies the inequality

$g(z_o) \leq \frac{1}{2\pi}\int_0^{2\pi} g(z_o + r\,e^{i\theta})d\theta$ for some $r>o$. and all $z_o \in C$

Comment If $g$ is analytic function, then $|g|^p$ is subharmonic for $p>o$. See page 329 of [5]

LEMMA 2.3 If $f$ is entire, then for any $r>o, p>o$

$$|f(x+iy)|^p \leq \frac{1}{\pi r^2}\iint_{D(x+iy,r)} |f(u+iv)|^p\,dudv$$

where $D(x+iy,r)$ is the disk centred at $x+iy$ with radius $r$.

Proof Since $f$ is entire, $|f|^p$ is subharmonic everywhere, so for $s\leq r$

$$|f(x+iy)|^p \leq \frac{1}{2\pi}\int_0^{2\pi}|f(x+iy+se^{i\theta})|^p d\theta$$

If we integrate the above inequality with respect to $sds$ from $o$ to $r$,

$$\int_0^r s|f(x+iy)|^p ds \leq \frac{1}{2\pi}\int_0^r\left(s\int_0^{2\pi}|f(x+iy+se^{i\theta})|^p d\theta\right)ds$$

Thus

$$\frac{r^2}{2}|f(x+iy)|^p \leq \frac{1}{2\pi}\int_0^r\left(s\int_0^{2\pi}|f(x+iy+se^{i\theta})|^p d\theta\right)ds$$

Divide by $\frac{r^2}{2}$ and use Fubini's theorem.

$$|f(x + iy)|^p \leq \frac{1}{\pi r^2} \int_0^{2\pi}\int_0^r |f(x + iy + se^{i\theta})|^p s\,ds\,d\theta.$$

The integration above is over the disk $D$ centred at $x+iy$ of radius $r$, which by change of coordinates gives the result.

LEMMA 2.4 (Plancherel - Polya) If $f \in E^p(T)$ then

$$\int_{-\infty}^{\infty} |f(x + iy)|^p dx \leq e^{pT|y|} \int_{-\infty}^{\infty} |f(x)|^p dx .$$

Proof See the Theorem 1.6, Page 93 of [6].

LEMMA 2.5 (Plancherel - Polya) If $f \in E^p(T)$ then $f' \in E^p(T)$ and $||f'||_p \leq c_1(p)T||f||_p$, where

$$c_1(p) = \left(\frac{4(2p)^p(p + 2)\,(e^{p+1} - 1)}{\pi(p + 1)^{p+1}}\right)^{1/p} .$$

Proof See Problem 7, page 99 of [6]. Choosing $\delta = (p + 1)/pT$ in that problem will yield the desired result.

LEMMA 2.6 Let $x_1$, $x_2$ be points in some real interval $I$ of length $h > 0$ and let $f$ be a complex valued differentiable function on $I$.

(a) If $0 < p \leq 1$, then

$$|f(x_1)|^p - |f(x_2)|^p \leq h^p \sup_{t\in I} |f'(t)|^p$$

(b) If $1 < p < \infty$, then

$$|f(x_1)|^p - |f(x_2)|^p \leq ph \sup_{t\in I} |f(t)|^{p-1} \sup_{s\in I} |f'(s)|.$$

<u>Proof</u> (a) $(0 < p \leq 1)$ If $0 < b < a$, then

$$a^p = (b + (a - b))^p \leq b^p + (a - b)^p$$

$$\Rightarrow a^p - b^p \leq (a - b)^p$$

If $|f(x_1)|^p < |f(x_2)|^p$, then (a) is trivial.

Otherwise we use the preceeding inequality to see that

$$\begin{aligned} |f(x_1)|^p - |f(x_2)|^p &\leq \big|\, |f(x_1)| - |f(x_2)|\,\big|^p \\ &\leq |\, f(x_1) - f(x_2)|^p \\ &\leq \big|\, h \sup_{t\in I} |f'(t)|\,\big|^p \\ &= h^p \sup_{t\in I} |f'(t)|^p . \end{aligned}$$

(b) $(p > 1)$ If $|f(x_1)|^p < |f(x_2)|^p$ then (b) is trivial,

so assume $|f(x_1)|^p > |f(x_2)|^p$. Then

$$\begin{aligned} |f(x_1)|^p - |f(x_2)|^p &\leq h \sup_{t\in I} \left|\frac{d}{dt} |f|^p(t)\right| \\ &= h \sup_{t\in I}(p|f(t)|^{p-1}|f'(t)|) \\ &\leq h\,p \sup_{t\in I} |f(t)|^{p-1} \sup_{s\in I} |f'(s)| . \end{aligned}$$

## SECTION 2

This Section deals with measures that are related to Lebesgue measure in the following sense.

DEFINITION 2.7   Let $h>o$. Two measures $\mu$ and $\nu$ on $R$ are h-equivalent if $\mu(I(x,h)) \simeq \nu(I(x,h))$ for all $x\varepsilon R$, (i.e. there exist $a,b>o$ such that

$$a\ \nu(I(x,h)) \leq \mu(I(x,h)) \leq b\nu(I(x,h)).$$

Note $\mu$ and $\nu$ are equivalent in the sense of chapter I if and only if a similar inequality holds for all Borel sets ~~are h-equivalent for all $h>o$ with fixed constants $a$ and $b$~~.

The main result of this Section is theorem 2.12 which says that if $\mu$ is h-equivalent to Lebesgue measure, then $||f||_p \simeq ||f||_{p,\mu}$ for all $f \varepsilon E^p(T)$ when $p$, $h$ and $T$ satisfy a certain condition.

We will prove Theorem 2.12 through a series of smaller results which we now begin.

PROPOSITION 2.8   Let $z_n = x_n + iy_n$ be a sequence of complex numbers such that $|Z_n - Z_j| \geq h$ for all $n \neq j$ and $|\ \mathrm{Im}\ (Z_n)\ | \leq M$. Then $\Sigma\ |f(Z_n)|^p \leq \dfrac{8(e^{pT(M+\frac{h}{2})}-1)}{\Pi\ PT\ h^2} \int |f(x)|^p dx$ for all $f \varepsilon E^p(T)$.

PROOF:   Let $D_n$ be the disk of radius $\frac{h}{2}$ centred at $Z_n$. By Lemma 2.3 with radius $r = \frac{h}{2}$

$$|f(z_n)|^p \leq \frac{4}{\Pi h^2} \iint_{D_n} |f(u+iv)|^p du dv$$

Thus

$$\Sigma \, |f(z_n)|^p \leq \frac{4}{\Pi h^2} \, \Sigma \iint_{D_n} |f(u+iv)|^p \, du dv$$

since the $D_n$ are disjoint and $\cup D_n \subset \{|\mathrm{Im}(u+iv)| \leq M + \frac{h}{2}\}$

$$\Sigma \, |f(z_n)|^p \leq \frac{4}{\Pi h^2} \int_{-M-\frac{h}{2}}^{M+\frac{h}{2}} \int_{-\infty}^{\infty} |f(u+iv)|^p du dv$$

By Lemma 2.4

$$\leq \frac{4}{\Pi h^2} \int_{-M-\frac{h}{2}}^{M+\frac{h}{2}} \left( e^{pT|v|} \int_{-\infty}^{\infty} |f(u)|^p du \right) dv$$

$$= \frac{8}{\Pi h^2} \left( \int_0^{M+\frac{h}{2}} e^{pTv} dv \right) \int_{-\infty}^{\infty} |f(u)|^p du$$

$$= \frac{8(e^{pT(M+\frac{h}{2})} - 1)}{\Pi P T h^2} \int_{-\infty}^{\infty} |f(u)|^p du.$$

<u>COROLLARY 2.9</u> (a) If $X_n$ is a sequence of real numbers such that $|X_n - X_j| \geq h$ for $n \neq j$, then

$$\Sigma \, |f(x_n)|^p \leq c_2 \int_{-\infty}^{\infty} |f(x)|^p dx$$

for all $f \in \varepsilon^p(T)$, where $c_2 = c_2(p,h,T) = \dfrac{8(e^{\frac{pTh}{2}} - 1)}{\Pi p h^2 T}$

(b) For any real $x$,

$$|f(x)|^p \leq PT \, \|f\|_p^p \quad \text{for all } f \in E^p(T).$$

PROOF (a) Take $M = 0$ in Proposition 2.8

(b) The constant in (a) is minimized when $phT \simeq 3.188$.

Define $h = \frac{3.188}{pT}$ and then using (a)

$$|f(x)|^p \le \Sigma|f(x+nh)|^p \le \frac{8(e^{1.594} - 1)pT}{\Pi(3.118)^2} \int_{-\infty}^{\infty} |f|^p dx$$

$$\le pT \, ||f||_p^p .$$

COROLLARY 2.10 Let $h>o$, then for all $f \in E^p(T)$

$$\sum_{n=-\infty}^{\infty} \sup_{t \in I(h,nh)} |f(t)|^p \le 2\, c_2 \int |f(x)|^p dx.$$

PROOF Let $t_n$ be the point in $I(h,nh)$ where

$F_n = \sup_{t \in I(h,nh)} |f(t)|^p$ is achieved. If $I(h,nh)$ and

$I(h,jh)$ are not adjacent, then $|t_n - t_j| \ge h$, so if

$n$ and $j$ are both even (or both odd), then $|t_n - t_j| \ge h$.

Now write $Z = E \cup O$ where $E$ = even integers and $O$ = odd integers.

Thus

$$\Sigma F_n^p = \Sigma|f(t_n)|^p$$

$$= \sum_{n\varepsilon E} |f(t_n)|^p + \sum_{n \varepsilon O} |f(t_n)|^p$$

By Corollary 2.9 (a)

$$\le c_2 \int |f|^p dx + c_2 \int |f|^p dx$$

$$= 2c_2 \int |f|^p dx.$$

This completes the proof.

For our next result, we will require that $p$ and $hT$ have a certain relationship. For convenience in stating the result, we define

$$\alpha = \alpha(p,hT) = \begin{cases} \dfrac{16\, c_1^{p}(p)}{\pi p}(hT)^{p-1}\left(e^{\frac{phT}{2}} - 1\right) & 0<p\leq 1 \\[2ex] \dfrac{16\, c_1(p)}{\pi}\left(e^{\frac{phT}{2}} - 1\right) & 1<p<\infty \end{cases} \qquad (2.2)$$

Where $c_1$ is the constant in Lemma 2.5 Note that $\alpha > o$ and for fixed $p$, $\alpha(p,.)$ is an increasing function of $hT$. Since $\alpha(p,o) = o$, $\alpha(p,hT) < 1$ whenever $hT$ is less than some value.

<u>PROPOSITION 2.11</u> Let $p, h, T$ be such that $\alpha < 1$. Then

$$\int|f|^{p}dx \leq \frac{h}{1-\alpha} \sum \inf_{t\varepsilon I(h,nh)} |f(t)|^{p} \quad \text{for all} \quad f\varepsilon E^{p}(T).$$

<u>Proof</u> Using the symbols $f_n$ and $F_n$ defined in convention (e) of section (2.1)

$$\begin{aligned} \int|f|^{p}dx &= \sum \int_{I(h,nh)} |f|^{p}dx \\ &\leq h \sum F_n^{\,p} \\ &= h \sum [f_n^{\,p} + (F_n^{\,p} - f_n^{\,p})] \\ &= h \sum f_n^{\,p} + h \sum (F_n^{\,p} - f_n^{\,p}) \qquad (2.3) \end{aligned}$$

Case 1 : $o < p \leq 1$. By Lemma 2.6 (a)

$$\Sigma (F_n^{\ p} - f_n^{\ p}) \leq h^p \Sigma F_n'^{\ p}$$

By Lemma 2.5, $f' \varepsilon E^p(T)$ and corollary 2.10 shows

$$\leq h^p 2c_2 \int |f'|^p dx$$

By the other part of Lemma 2.5

$$\leq h^p 2c_2\ c_1^p\ T^p \int |f|^p dx$$

Multiply by $h$ and substitute in (2.3) yields

$$\int |f|^p dx \leq h\ \Sigma\ f_n^{\ p} + \alpha \int |f|^p dx.$$

subtracting the last term from both sides and dividing by $1 - \alpha$

gives the result.

Case 2 : $1 < p < \infty$. By Lemma 2.6 (b)

$$\Sigma (F_n^{\ p} - f_n^{\ p}) \leq p\ h\ \Sigma\ F_n^{\ p-1} F_n'$$

Let $q$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. The sequence $\{a_n\} = \{F_n^{\ p-1}\}$

is in $\ell^q$ because $\Sigma\ |a_n|^q = \Sigma\ |a_n|^{\frac{p}{p-1}} = \Sigma F_n^{\ p}$

and by Corollory 2.10 this is

$$\leq 2c_2 \int |f|^p dx < \infty.$$

Also by Lemma 2.5 and Corollary 2.10

$\{ b_n \} = \{ F_n' \} \ \varepsilon\ \ell^p$.

Using Holder's inequality on $\ell^q$ and $\ell^p$

$$ph\ \Sigma\ F_n^{\ p-1} F_n' \leq ph(\Sigma(F_n^{\ p-1})^q)^{1/q} (\Sigma(F_n')^p)^{1/p}$$

$$\leq ph(\Sigma F_n^{\ p})^{1/q} (\Sigma(F_n')^p)^{1/p}$$

By Corollary 2.10

$$\leq ph\,(2c_2 \int |f|^p dx)^{1/q} (2c_2 \int |f'|^p dx)^{1/p}$$

By Lemma 2.5

$$\leq 2phc_2 (\int |f|^p dx)^{1/q}\; c_1 T (\int |f|^p dx)^{1/p}$$

$$= 2phc_1c_2\; T \int |f|^p dx$$

multiplying by h and continuing from (2.3) gives

$$\int |f|^p dx \leq h\, \Sigma f_n^{\,p} + \alpha \int |f|^p dx$$

Hence $\int |f|^p dx \leq \frac{h}{1-\alpha}\, \Sigma\, f_n^{\,p}$, finishing the proof.

Corollary 2.10 and Proposition 2.11 are key ingredients in the following theorem.

<u>THEOREM 2.12</u> Let $\ddot{\mu}$ be h-equivalent to Lebesgue measure and let p, h and T be such that $\alpha<1$. Then $||f||p \simeq ||f||p,\ddot{\mu}$ for all $f \varepsilon E^p(T)$.

<u>Proof</u> : Since $\ddot{\mu}$ is h-equivalent to Lebesgue measure there exist positive a and b such that for all x ,

$$ah \leq \ddot{\mu}\,(I(h,x)) \leq bh \qquad (2.4)$$

By proposition 2.11 (Using convention (c) of section 2.1)

$$\int |f|^p dx \leq \frac{1}{1-\alpha}\; \Sigma\, f_n^{\,p} h \quad \text{where} \quad o<\alpha<1$$

By (2.4)

$$\leq \frac{1}{a(1-\alpha)} \Sigma f_n^{\,p} \ddot{\mu} (I(h,n\dot{h}))$$

By (2.1)

$$\leq \frac{1}{a(1-\alpha)} \int |f|^p d\ddot{\mu}$$

Taking $c_3 = (a(1-\alpha))^{1/p}$ we have

$$c_3 \, ||f||p \leq ||f||p,\ddot{\mu}.$$

To get the other inequality, we have by (2.1)

$$\int |f|^p d\ddot{\mu} \leq \Sigma F_n^{\,p} \mu(I(h,n\dot{h}))$$

By (2.4)

$$\leq hb \, \Sigma F_n^{\,p}$$

By Corollary 2.10

$$\leq 2hbc_2 \int |f|^p dx$$

Taking $c_4 = (2hbc_2)^{1/p}$ yields

$$||f||p,\ddot{\mu} \leq c_4 ||f||p \, .$$

<u>Note</u> Theorem 1.3 showed that when $\ddot{\mu}$ is equivalent to Lebesgue measure (i.e. take $d\nu = dx$),

$$a^{1/p} \, ||f||p \leq ||f||p,\mu \leq b^{1/p} \, ||f||p$$

for all $f \in L^p(R, dx)$. The proceeding theorem shows that if $\mu$ is h-equivalent to Lebesgue measure and $T$ is sufficiently small, then

$$a^{1/p}(1-\alpha)^{1/p} \, ||f||p \leq ||f||p,\mu \leq b^{1/p}(2hc_2)^{1/p} ||f||p.$$

for all $f \varepsilon E^p(T)$. The first theorem holds for a smaller class of measures, but for a larger class of functions than the second theorem.

When $p = 2$, the Paley-Wiener Theorem [6] shows that every $f \in E^2(\Pi)$ is the Fourier transform of some $\hat{\phi} \in L^2([-\pi,\pi],dx)$. Using Plancerel's theorem on $f = \hat{\phi}$ shows that

$$\Sigma\, |f(n)|^2 = \int |f|^2 dx.$$

The following application of theorem 2.12 gives a similar result when $p \neq 2$.

COROLLARY 2.13 Let $p$ and $T$ be such that $\alpha(p,T)<1$. Then for all $f \in E^p(T)$

$$(1 - \alpha(p,T)) \int |f|^p dx \leq \Sigma\, |f(n)|^p \leq 2c_2(p,1,T) \int |f|^p dx$$

Proof Let $\mu$ be counting measure on $Z$. Then $\mu$ is 1-equivalent to Lebesgue measure (with $a = b = 1$) and theorem 2.12 establishes the corollary.

We note that a sharper condition and a sharper conclusion are possible in Corollary 2.13 (see section 31 and 33 of [4]. ) **However,** some condition on the product hT not getting large (in our case $\alpha < 1$) is necessary. For any $p>0$, any $\varepsilon > 0$ take some $\phi \in E^p(\varepsilon)$. Define $f(z) = \phi(z) \sin 2\pi z \in E^p(2\pi+\varepsilon)$. Then $f(n) = 0$ for all $n$, the middle term in the corollary is zero whereas the outer terms are clearly positive.

## SECTION 3

We will now generalize Theorem 2.12 to measures that are comparable near infinity. We shall see that the tail behaviour itself is enough to give equivalent norms. The following definition will allow us to state our main theorem.

DEFINITION 2.14 Two measures $\mu$ and $\nu$ are tail h-equivalent If $\mu(I(h,x)) \simeq \nu(I(h,x))$ for all $|x|$ sufficiently large, i.e. there exists $k$, $a,b>o$ such that

$$a\,\nu(I(h,x)) \leq \mu(I(h,x)) \leq b\,\nu(I(h,x)) \quad \text{for all } |x|\geq k.$$

$\mu$ and $\nu$ are tail equivalent if they are h-tail equivalent for every $h>o$.

THEOREM 2.15 Let $\mu$ be a measure on $\mathbb{R}$ that is finite on bounded sets and tail h- equivalent to Lebesgue measure. If $p$ and $T$ are such that $\alpha(p,hT)<1$, then

$$||f||_p \simeq ||f||_{p,\mu} \quad \text{for all } f \in E^p(T).$$

Proof Let $T_o>T$ be such that $\alpha_o = \alpha(p,hT_o)<1$.
Let $N$ be large enough so that $|n| \geq N$ implies

$$ah \leq \mu(I(h,nh)) \leq bh \qquad (2.5)$$

Set $K = \{|x| \leq Nh - \frac{h}{2}\}$.

<u>Step 1</u> For $f \in E^p(T_o)$

$$\int|f|^p dx < \frac{1}{1-\alpha_o}\left(\int_K |f|^p dx + \frac{1}{a}\int|f|^p d\mu\right).$$

To see this use $\alpha_o<1$ and proposition 2.11

$$\int|f|^p dx \leq \frac{h}{1-\alpha_o}\Sigma\, f_n^{\,p}$$
$$= \frac{1}{1-\alpha_o}\left(\sum_{|n|<N} f_n^{\,p}h + \sum_{|n|\geq N} f_n^{\,p}h\right)$$

By (2.5)

$$\leq \frac{1}{1-\alpha_o}\left(\sum_{|n|<N} f_n^{\,p}h + \frac{1}{a}\sum_{n=-\infty}^{\infty} f_n^{\,p}\mu(I(h,nh))\right)$$

By the reasoning in (2.1)

$$\leq \frac{1}{1-\alpha_o}\left(\int_K |f|^p dx + \frac{1}{a}\int|f|^p d\mu\right).$$

<u>Step 2.</u> If there exists a $g \in E^p(T)$ such that $||g||p,\mu = 0$ and $||g||p > 0$, then there exists a sequence $g_n$ in $E^p(T_o)$ such that $||g_n||p,\mu = 0$, $||g_n||p = 1$ and $||g_n - g_k||p \geq 1$ whenever $n \neq k$.

To see this, Let $T_1 = T_o - T$. Let $e_n$ be any ~~linearly~~ independent sequence in $E^p(T_1)$ and define $h_n = e_n g$. The sequence $h_n$ is in $E^p(T_o)$: $g$ and $e_n$ are entire so $e_n g$ is entire, $g \in E(T)$ and $e_n \in E(T_1)$ implies $e_n g \in E(T_1+T) = E(T_o)$ and $g$ is bounded on $R$ (by corollary 2.9 (b)) so $e_n g \in L^p(R, dx)$. The condition that $||g||p,\mu = 0$ implies that $g = 0$ $\mu$-a.e. and hence $h_n = e_n g = 0$ $\mu$ - a.e.

Define

$$g_1 = \frac{h_1}{||h_1||p}$$

By definition $||g_1||p = 1$ and since $h_1 = o\ \mu - a.e$,

$g_1 = o\ \mu - a.e$ and thus $||g_1||p,\mu = 0$. For $n \geq 2$, inductively define

$$g_n = \frac{h_n - \hat{h}_n}{||h_n - \hat{h}_n||p}$$

where $\hat{h}_n$ is a best $||\cdot||p$ - approximation to $h_n$ in $M_n = \text{span}\{g_1, \ldots g_{n-1}\}$. (such finite dimensional best approximations exists even when $o<p<1$.) By definition $||g_n||p = 1$. Since $g_j = o\ \mu - a.e.$ and $\hat{h}_n \in M_n$, we have $\hat{h}_n = 0\ \mu - a.e.$ Combined with $h_n = o\ \mu - a.e.$ this shows $g_n = o\ \mu - a.e.$ Therefore $||g_n||p,\mu = 0$. Finally, for $k<n$ $||g_n - g_k||_p \geq 1$ since $g_n$ was choosen to be at least one unit from $M_n$ and $g_k \in M_n$ for $k<n$.

<u>Step 3</u> For fixed $\tau > o$, $p > o$ the unit sphere $S = \{f \in E^p(\tau) : ||f||p = 1\}$ is precompact in the topology of uniform convergence on compact subsets of $C$.

By Theorem 12, Chapter 5 of [1] $S$ is precompact (normal) in this topology iff $S$ is locally bounded. Let $E$ be a compact subset of $C$ and let $M = \sup\{|Im(z)| : z \in E\}$. Taking $h = 2$ in proposition 2.8 shows

$$|f(z)|^p \leq \frac{2(e^{p\tau(M+1)} - 1)}{\Pi p\tau} \int |f|^p dx.$$

Since $||f||p = 1$ for all $f \in S$

$$|f(z)| \leq \text{constant } (p, \tau, M)$$

uniformly for $z \in E$ and $f \in S$. Thus $S$ is locally bounded and hence normal.

Step 4 $||g||_{p,\mu} = 0$ for $g \in E^p(T)$ iff $g = 0$. Suppose the only if part is not true, then there exists a $g \in E^p(T)$ with $||g||_{p,\mu} = 0$ and $||g||_p > 0$. By step 2, there exists a sequence $g_n$ in $E^p(T_0)$ such that $||g_n||_p = 1$, $||g_n||_{p,\mu} = 0$ and $||g_n - g_k||_p \geq 1$ for $n \neq k$. By Step 3 there is a subsequence of $\{g_n\}$ that converges uniformly on compact subsets. By relabeling the subsequence, we can assume that $\{g_n\}$ itself converges in that topology. Using step 1 on $f = g_n - g_k$ shows.

$$\int |g_n - g_k|^p dx \leq \frac{1}{1-\alpha_0} \left(\int_K |g_n - g_k|^p dx + \frac{1}{a} \int |g_n - g_k|^p d\mu\right).$$

Since $g_n - g_k = 0$ $\mu$ - a.e.

$$\leq \frac{1}{1-\alpha_0} \int_K |g_n - g_k|^p dx.$$

Since $\{g_n\}$ converges uniformly on compact subsets, $|g_n - g_k|$ can be made arbitrarily small on compact $K$. This shows that the last integral and hence $\int |g_n - g_k|^p dx$ can be made arbitrary small for $n$ and $k$ sufficiently large. But this contradicts $||g_n - g_k||_p \geq 1$. Therefore $||g||_{p,\mu} = 0$ implies $g = 0$. The converse is clear.

Step 5 There exists $c_5 > 0$ such that $f \in E^p(T)$ implies $||f||_p \leq c_5 ||f||_{p,\mu}$. If not, there exists a sequence $g_n$ in $E^p(T)$ such that $||g_n||_p = 1$ and $||g_n||_{p,\mu} \to 0$. By Step 3 and relabeling if necessary, we can assume $g_n$ converges to some function $g \in E^p(T)$. By Fatou's Lemma

$$||g||^p_{p,\mu} \leq \operatorname*{Limit}_{n \to \infty} ||g_n||^p_{p,\mu} = 0.$$

By step 4 $g = 0$. Therefore $g_n \to 0$ uniformly on every compact set.

Apply step 1 to $g_n$ to show

$$||g_n||_p^p \leq \frac{1}{1-\alpha_o}\left(\int_K |g_n|^p dx + \frac{1}{a}\int |g_n|^p d\mu\right)$$

By what we showed above $g_n \to 0$ uniformly on $K$, so that first integral can be made arbitrarily small by taking $n$ large. Also $\{g_n\}$ was chosen so that $||g_n||_{p,\mu} \to 0$ as $n \to \infty$, so the second integral can be made arbitrarily small. But this contradicts the fact that left hand side is 1 by assumption. Hence some $c_5$ must exist.

Step 6 There exists $c_6 > o$ such that $f \in E^p(T)$ implies $||f||_{p,\mu} \leq c_6 ||f||_p$

We always have

$$\int |f|^p d\mu = \int_K |f|^p d\mu + \int_{K^c} |f|^p d\mu$$ where $K^c$ is the complement of $K$.

By the reasoning in (2.1)

$$\leq \sup_{x \in K} |f(x)|^p \mu(K) + \sum_{|n| \geq N} F_n^p \mu(I(h,nh))$$

Using Corollary 2.9 (b) on the first term and (2.5) on the second term

$$\leq p\, T\, \mu(K)\, ||f||_p^p + bh \sum_{|n| \geq N} F_n^{\,p}$$

$$\leq p\, T\, \mu(K)\, ||f||_p^p + bh \sum_{n=-\infty}^{\infty} F_n^{\,p}$$

By Corollary 2.10

$$\leq p\,T\,\ddot{\mu}(K)\ \|f\|_p^p + bh\,2c_2\,\|f\|_p^p$$

$$= (pT\,\ddot{\mu}(K) + 2bhc_2)\,\|f\|_p^p$$

This completes the proof.

Corollary 2.16 If $\mu$ is tail equivalent to Lebesgue measure and finite on bounded sets, then for any $T>0$ $\|f\|_{p,\mu} \approx \|f\|_p$ for all $f \in E^p(T)$.

Proof Fix $p$ and $T$. Choose $h$ small enough so that $\alpha(p,hT)<1$. Then $\mu$ is $h$-equivalent to Lebesgue measure and Theorem 2.15 applies.

An easy application of this corollary is when $\ddot{\mu}$ is absolutely continuous with respect to Lebesgue measure. We state the conditions in the following.

Corollary 2.17 Let $\ddot{\mu}$ be absolutely continuous with respect to Lebesgue measure, i.e. $d\ddot{\mu}(x) = \ddot{\mu}(x)dx$. Suppose there exist $a,b,K>o$ such that

(a) $a \leq \ddot{\mu}(x) \leq b$ for all $|x| \geq K$ and

(b) $\int_{|x|\leq K}\ddot{\mu}(x)dx<\infty$.

Then for any $T>o$, $\|f\|_{p,\ddot{\mu}} \approx \|f\|_p$ for all $f \in E^p(T)$.

We note that the conditons of corollary 2.16 are weaker than those in corollary 2.17 The Cantor measure on R (i.e. the Lebesgue - Stieltjes measure generated by the Cantor function) is interval equivalent to Lebesgue measure. We can modify it on a compact set and get a measure that is tail equivalent to Lebesgue measure and a result like corollary 2.17 will hold.

Finally, it is clear that two measures that are tail related to Lebesgue measure will give equivalent norms. We state this formally in the following.

<u>DEFINITION 2.18</u> Let $M_h$ = {measures $\mu$ on $(R,B)$: $\mu$ is h-equivalent to Lebesgue measure and $\mu$ is finite on bounded sets}.

$M$ = {measures $\mu$ on $(R,B)$: $\mu$ is tail equivalent to Lebesgue measure and $\mu$ is finite on bounded sets}.

<u>Corollary 2.19</u> (a) If $\mu$ and $\nu$ are in $M_h$ and $T$ is sufficiently small, then $||f||_{p,\mu} \simeq ||f||_{p,\nu}$ for all $f \in E^p(T)$.

(b) If $\mu$ and $\nu$ are in $M$, then for any $T>o$,

$$||f||_{p,\nu} \simeq ||f||_{p,\mu} \quad \text{for all} \quad f \in E^p(T).$$

<u>Proof:</u> (a) By theorem 2.15, $||f||_{p,\mu} \simeq ||f||_p$ and $||f||_{p,\nu} \simeq ||f||_p$, so $||f||_{p,\mu} \simeq ||f||_{p,\nu}$. Note that the equivalence constants depend on $\mu$ and $\nu$, they are not uniform for $\mu, \nu \in M_h$ .

(b) By corollary 2.16, $||f||_{p,\mu} \simeq ||f||_p \simeq ||f||_{p,\nu}$. Again the constants depend on $\mu$ and $\nu$.

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