

Some Aspects of Hemodynamics On
Blood Flow

By
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A dissertation submitted in partial fulfilment of
the requirements for the degree of Master of Science
of the University of Zambia.

THE UNIVERSITY OF ZAMBIA

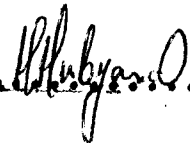
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DECLARATION:

I hereby declare that this dissertation is my own work and that it has not been previously submitted for degree purposes here or at any other University.

MASWABI M. MUBYANA

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PREFACE.

The material presented here make up a dissertation submitted in partial fulfilment of the requirements for the award of degree of Master of Science in Mathematics of the University of Zambia. It is a review of the available literature on some aspects of hemodynamics. Hemodynamics is a branch of Mechanics where concepts from fluid mechanics are used to study the dynamics of blood flow in the circulatory system.

The review will be on the literature that seeks to give a mathematical insight to the functional anatomy of the arterial and capillary beds of the circulatory system by making use of models which take into account the physical and rheological properties of blood as well as the properties of the blood vessels. The material is organised into five chapters. Chapter 1 is an introduction making a general view of the subject matter to be covered by subsequent chapters. Chapter 2 is on pulsatile blood flow. This is flow that takes account of the rhythmical pressure variations in the arterial bed. The reviewed materials on this chapter are Womersley's(1954) model of arterial blood flow and Kapur's (1983) Chapter entitled, 'Problems of Blood Flow in the Human System.' Chapter 3 is on Shukla, Parihar and Gupta's(1980) study on the effects of branching, variation of viscosity and stenosis (an abnormal growth on artery lining) on wall shear stress and blood flow resistance; assuming blood to be a Newtonian fluid. Chapter 4 is a study of blood flow in the arterioles using the non-Newtonian Casson and power law fluid models. Materials reviewed in this chapter are from Shukla, Parihar and Rao(1980). Chapter 5 is a review on Lighthill's

(1968) and Parihar's (1980) investigations on microcirculation. Microcirculation is a study of the motion of the red blood cells in narrow capillaries.

In each of these studies, simplifying assumptions are stated and suitable mathematical models made to investigate and give a mathematical insight to the blood flow problem at hand.

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CHAPTER I

GENERAL INTRODUCTION.

1.1 INTRODUCTION:

In recent years several workers have studied arterial blood flow using various mathematical models under varying simplifying assumptions. Early before this, blood circulatory phenomena had been known for centuries but the complexities of the blood circulatory system appear to have defied deeper analytical treatment until fairly recently, during the second half of the 19th century, when the idea of modelling arterial blood flow to propagation of disturbances along an elastic tube was established. This early work centred around determining a relationship between pulse wave velocity and the elastic properties of the arterial vessels. By the early fifties J.R. Womersley had developed his mathematical theory of arterial blood flow in mammals. This stimulated a rapid growth in the study of blood flow. The period that immediately followed saw a rapidly emerging trend that replaced the descriptive approach that was then prevalent in the biological sciences by analytical methods that involved more use of mathematical ideas. A decade later in April, 1963 the First International Symposium on Pulsatile Blood Flow was held in Philadelphia, U.S.A.

Numerous other workers have ever since been contributing to the subject. A number of investigators, Attinger (1964); Fry and Greenfield, Jr. (1964); Burton (1968); Copley (1968); Whitmore (1968); Huchaba and Hahn (1968); Young (1968); Lighthill (1968, 1972, 1974); Lih (1969, 1972); Lee and Fung

(1970); Bergel (1972); Fung (1969); Middleman (1972); Rosen (1972); Lightfoot (1974); McDonald (1979); Kapur (1983); Shukla and Parihar (1975); [REDACTED]; Rao (1976); Parihar (1976); Shukla, Parihar and Gupta (1980-a,b); Shukla, Parihar and Rao (1980); Parihar (1980-a,b) [REDACTED], have studied blood flow in the cardiovascular system under some simplifying assumptions by considering blood behaviour as either Newtonian or non-Newtonian.

1.2 THE CARDIOVASCULAR SYSTEM:

The cardiovascular system (blood circulatory system) is a blood distribution and collection system comprising of the heart, arteries, veins and tiny vessels known as the capillaries. All these together form an intricate and complex branched network of tubes with the heart as the pump providing the pressure forcing the blood to flow through the network. Initial flow of blood leaving the heart by the arteries is subdivided after about 20 to 30 separate branchings into hundreds of millions of small individual flows in the capillaries with diameters of less than 10 microns (1 micron = 10^{-4} cm.). The micron, sometimes known as the micrometer, will be denoted by μm . The capillaries eventually join up to form bigger vessels, the veins, which carry blood back to the heart.

The overall organisation of the cardiovascular system is diagrammatically illustrated below.

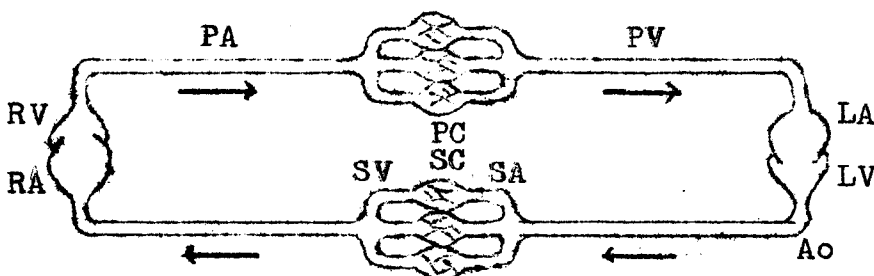


Figure 1.1: The Cardiovascular System, (Attlinger (1964), Pulsatile Blood Flow, pp.2)

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Key to figure: LA = left atrium; LV = left ventricle;
Ao = aorta; SA = systemic arteries; SC = systemic capillaries; SV = systemic veins; RA = right atrium; RV = right ventricle; PA = pulmonary artery; PC = pulmonary capillaries; PV = pulmonary veins.

The heart is divided into four chambers. The upper two chambers are called the right and the left atriums. The lower two chambers are called the right and the left ventricles. There is no direct flow communication between the right and the left side chambers of the heart. The two chambers are however connected through two parallel circulatory sub-systems. These are the systemic, feeding blood to the body trunk, the head and the limbs and the pulmonary, feeding the lungs.

Oxygenated blood from the lungs leaves the left ventricle by the pumping action of the heart and through the aorta to the arteries, arterioles and finally into the capillaries. Blood is in this way distributed to all parts of the body. As it passes through the capillaries it exchanges oxygen for carbon dioxide, food stuffs and water with the body cells. After this exchange the blood leaves the capillaries with a reduced oxygen saturation and a high carbon dioxide content. It is then collected in the venules (tiny veins), the venules collect into the veins and then it flows back to the right atrium from which it flows into the right ventricle. The right ventricle pumps the blood through the pulmonary arteries to the pulmonary capillaries (lungs) where it is re-oxygenated through an exchange of carbon dioxide with oxygen from the atmosphere, a process known as respiration. The re-oxygenated blood is then drained through the pulmonary venous system into the left atrium from which it passes on to the left ventricle. This completes the blood flow cycle through the cardiovascular

	Number	Dia- meter mm	Cross- Section cm ²	Length cm.	Volume ml.	Blood velocity cm/sec.	Reynolds number
Small arte- ries and arterioles	-----	-----	-----	-----	18	-----	-----
Pulmonary capillaries	6×10^2	0.008	300	0.05	16	0.14	0.006
Pulmonary veins	-----	-----	-----	-----	} 52	-----	-----
Large pulmo- nary veins	4	-----	-----	-----		-----	-----

Key: * indicates that the mean of the major and minor semiaxes of the elliptic cross-section are given.

As can be seen in the table above there is a substantial decrease in blood flow velocity from the aorta where it is 50 cm/sec. to the capillary bed where it is only 0.07 cm/sec. Several factors are responsible for this reduction in velocity but perhaps prominent amongst these could be the effect due to vessel bifurcations. To see how vessel bifurcations affect flow velocity we consider a special case of symmetrical bifurcation with half of the main vessel's flow going into each of the two branches. Letting β be the ratio of the combined cross-sectional area of the branches to the cross-sectional area of the main vessel, then velocity in the branches is $1/\beta$ of the velocity in the main vessel. The ratio β has been experimentally found to be less than one only in the aorta bifurcations, Caro, Schroter and Fitz-Gerald (1971), but consistently greater than one for all other bifurcations in the cardiovascular system. Referring to the table again it shows a big increase in the total vessel cross-sectional area

between the ventricular outflow tracts and the capillary bed, reflecting a corresponding decrease in velocity. Given the large number of bifurcations in the cardiovascular system one can now see how this affects blood flow velocity and in conclusion add that nature provided the numerous bifurcations to facilitate not only blood distribution but also to slow down flow so that physiologically vital gaseous and metabolic exchanges occurring in the capillary bed should be completed.

Another important observation from the table is the tremendous reduction in the Reynolds number from 2,500 in the aorta to only 0.003 in the capillaries. The Reynolds number of a fluid is a ratio of the inertial forces acting on a fluid under motion to the viscous resistance offered by the fluid. In tiny vessels, inertial forces are much smaller than viscous forces and this explains why the Reynolds number is so small in the capillaries. Flow in capillaries is solely determined by the balance between viscous forces and pressure gradients as inertial effects are totally negligible. In the larger vessels inertial forces are more predominant than viscous resistance.

1.4 BLOOD COMPOSITION:

Blood is a suspension containing about 50% by volume of small deformable bodies which are mainly the red blood cells. The volume percentage of red cells present in the blood is called the 'haematocrit'. For a normal healthy individual the haematocrit is about 45%. Red blood cells are disc shaped bodies of diameter about 8 μm , peripheral thickness about 2.5 μm and a dimpled centre of thickness about 1.0 μm . There are two other types of cells found in

whole blood and these are the white blood cells and very tiny other particles known as platelets. The white blood cells are about 0.17% by volume concentration hence their insignificant role in blood flow problems. The white blood cells are however much bigger in size than the red blood cells and have no fixed shape as they can flexibly assume any shape though they usually become spherical when freely moving in the blood stream. Platelets are very small in size compared to red blood cells and form roughly about 5% by volume of whole blood, Middleman (1972). The three types of cells are suspended in a transparent liquid, called the blood plasma.

Blood cells under flow tend to shun the walls of the conducting vessel and concentrate around the central region, Lih (1969). The viscosity (see next section) of blood depends on cell concentration being higher when the haematocrit or cell concentration is high. In small vessels of diameter a few times that of the cell, the cells may aggregate and form chains called rouleaux. These are like coin stacks along the vessel surrounded by a cell free plasma layer, Skalak and Branemark (1969). This cell free layer has significant effects on resistance to flow of blood as will be seen in Chapter 5 on Microcirculation.

1.5 RHEOLOGY OF BLOOD:

All fluids are resistive to deforming or shearing forces. This resistive power known as viscosity is an intrinsic property of any given fluid and varies from fluid to fluid. For many ordinary fluids viscosity is given as a ratio of stress to strain rate. However for many fluids the stress may not vary linearly with the strain rate.

Fluids are rheologically divided according to the manner

in which stress varies with their strain rate. The graph (Figure 1.2) shows stress-strain rate relationships for different kinds of fluids. Newtonian fluids are those fluids for which stress is linearly proportional to strain rate (Newtonian law of viscosity). The line marked (1) on the graph shows the stress-strain rate relationship for Newtonian fluids. The slope of the line which is constant gives the fluid viscosity. All other fluids exhibiting stress-strain rate relationships different from Newtonian are called non-Newtonian fluids.

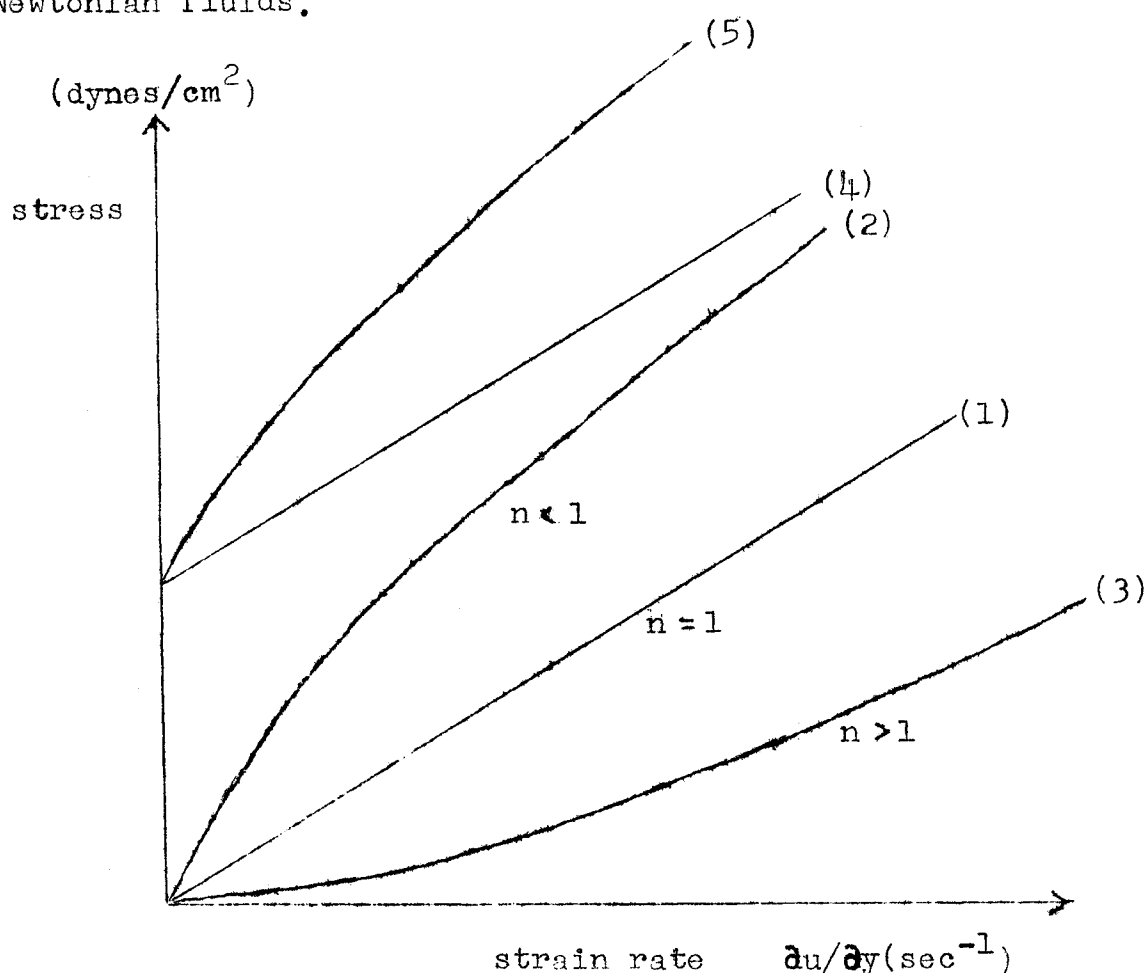


Figure 1.2: Stress-strain rate relationships for various fluid models.

Non-Newtonian fluids are of various types and the same graph (Figure 1.2) illustrates four different types. The lines

and curves on the graph are numbered (1) to (5). Considering one at a time the following observations are made, (see Kapur (1983), Knudsen and Katz (1958)):

Curve (2) shows that stress increases non-linearly with strain rate. The slope of the curve, which defines viscosity, decreases with increasing strain rate. The structure of the curve suggests that the stress is proportional to (strain rate)ⁿ where $n < 1$. Such fluids are called pseudoplastic. Curve (3) is for the case when $n > 1$ and here the slope of the curve (viscosity) increases with strain rate. The type of fluid depicted by curve (3) is called dilatant. When $n = 1$ we have the case for line (1) (Newtonian) thus (1) is a special case of (2) and (3). Cases (1), (2) and (3) all fall under the power law model for which the stress and strain rate relationship is given by the equation

$$\tau = - m \left| \frac{\partial u}{\partial y} \right|^{n-1} \left(\frac{\partial u}{\partial y} \right), \quad - (1.1)$$

where τ is the stress, $\partial u / \partial y$ is the strain rate, m is the consistency and n is the power law index (flow behaviour) which ranges between 0.1 and 2.0. When $n < 1$ in equation (1.1), we have the pseudoplastic case (curve (2)) and when $n > 1$, we have the dilatant case (curve (3)). If $n = 1$ then $m = \mu$ (viscosity) and equation (1.1) reduces to a Newtonian case (line (1)) for which the equation is

$$\tau = - \mu \frac{\partial u}{\partial y}. \quad - (1.2)$$

Line (4) indicates that the fluid can withstand a certain amount of stress, known as the yield stress, below which the fluid does not deform or strain. Beyond the yield stress the difference between stress and the yield stress

is directly proportional to strain rate and viscosity is constant. Such fluids are known as Bingham plastics and their rheological equation is given by

$$\tau \pm \tau_0 = \begin{cases} -\mu \frac{\partial u}{\partial y}, & \tau \geq \tau_0 \\ 0, & \tau \leq \tau_0, \end{cases} \quad - (1.3)$$

where τ_0 is the yield stress. When $\tau_0 = 0$ equation (1.3) reduces to (1.2).

Curve (5) has also a yield stress but unlike line (4) the relationship between stress and strain rate after the yield stress is non-linear. Fluids depicted by curve (5) fall under the Casson fluid model and their rheological equation is given by

$$\tau^{\frac{1}{2}} \pm \tau_0^{\frac{1}{2}} = \begin{cases} (-\mu \frac{\partial u}{\partial y})^{\frac{1}{2}}, & \tau \geq \tau_0 \\ 0, & \tau \leq \tau_0, \end{cases} \quad (1.4)$$

If $\tau_0 = 0$, equation (1.4) reduces to (1.2) which is the equation for the Newtonian case.

Since blood is a nonhomogeneous fluid with cell concentrations which may under certain conditions vary, it has no definitive rheological characteristics. Its rheology may under certain conditions be approximated to any of the four rheological models discussed above. Plasma may with little error be approximated to a Newtonian fluid for shear rates in the range 0.1 to 1,200 sec^{-1} , Copley and Stainsby (1960); Charm and Kurland (1962); Cokelet (1972); Merrill, Cokelet, Britten and Wells (1963). At high shear rates as that occurring in the arteries, blood may with little error be treated as a Newtonian fluid, Attinger (1964). At shear rates below 20 sec^{-1} several investigators have assumed blood to be a power law fluid, Charm and Kurland (1962); Hershey and

Cho (1966); Huchaba and Hahn (1968). However at shear rates above 20 sec^{-1} the power law model has been found to be not accurate enough, Charm and Kurland (1965). Blood has also been found to exhibit a small yield stress (about 0.005 dynes/cm^2), Cokelet (1972) and Kapur (1983). In this connection, it fits the Casson model over a wide range of shear rates (0.1 to $100,000 \text{ sec}^{-1}$), Casson (1959); Charm and Kurland (1962 and 1965); Merril, Cokelet, Britten and Wells (1963) and Cokelet (1972).

CHAPTER II
PULSATILE BLOOD FLOW.

2.1 INTRODUCTION:

A realistic and more accurate mathematical model of arterial blood flow should incorporate its pulsatile character. The pulsatile character of arterial blood flow is due to the regular beating of the heart, the result of its pumping action. Each time the heart beats a pressure wave (or pulse) is transmitted through the cardiovascular system. Pulsatile pressure variations in the arterial bed ranges from a maximum of 120 mmHg to a minimum of 80 mmHg, (Attinger (1964); pp.10). The transmission of the pressure pulse is a function of the physical properties of the blood and blood vessel walls. The pressure pulse is propagated at a speed which is ten to twenty times greater than blood flow velocity. When the heart beats flow in the aorta surges forward for about half of the cycle, transmitting a pressure and flow peak into the arterial bed, to be followed by a lower pressure and weaker flow during the remainder of the cycle. Small localized flow reversals do occur in the aorta and other proximal parts of the arterial bed during minimum pressure but valve action in the vessels prevents total flow from becoming negative. Both pressure and flow peaks become progressively weaker towards the peripheral due to valve action and the damping effect on the pulse by blood and the vessel wall. It is for this reason that pulsatile flow character of blood is of little or no importance in the capillary and venous flow.

This chapter, on pulsatile blood flow, is a survey on

two contributions to the subject by an author and two co-authors. The first part of the survey is on Womersley's model, which was a contribution by D.L. Fry and J.C. Greenfield Jr. to the First International Symposium on Pulsatile Blood Flow held in Philadelphia, U.S.A., (see Attinger (1964), Pulsatile Blood Flow, pp.85). The second part of the survey is on an article by J.N. Kapur (1983) of the Indian Institute of Technology in Kanpur, India.

2.2 SURVEY OF WOMERSLEY'S MODEL:

2.2.1 ASSUMPTIONS AND BASIC EQUATIONS:

Assumptions are important in reducing equations, describing a given physical system, to simpler form. However in applying assumptions to any prescribed system one has to be mindfull of the error introduced there by the assumptions. In constructing his model Womersley used the following assumptions:

- (1) Blood is an incompressible Newtonian fluid.
- (2) Flow is laminar, symmetrical and fully developed with no tangential velocities.
- (3) The blood vessel wall thickness is very small compared to its radius.
- (4) There is no slip velocity at the wall of the vessel.
- (5) Temperature is constant throughout the system under investigation.
- (6) The vessel is an elastic cylindrical tube which does not taper between branches.
- (7) Axial and radial displacements of the inner wall are small.
- (8) The physical properties of the vessel wall material are all linearly related and are the same in all

directions.

- (9) The vessel wall material is more elastic than it is viscous.
- (10) Radius of the vessel is small compared to the wavelength of the pulse wave.
- (11) Blood flow velocity is much smaller than pulse wave velocity.
- (12) There is no reflected wave.
- (13) Inertial forces are small compared to frictional or viscous forces.
- (14) The density of blood is nearly equal to that of the vessel wall material.

The two basic equations in problems of fluid flow are the momentum and the continuity equations. The momentum equation is an expression of Newton's second law of motion. It expresses the balance of forces at any point within the fluid and its boundaries. The continuity equation is an expression of the conservation of mass within the system. Under assumptions (1) and (2) the momentum and continuity equations are given by

$$\begin{aligned} \text{z-direction; } \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} = & - \frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{\mu}{\rho} \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial w}{\partial r} \right) \right. \\ & \left. + \frac{\partial^2 w}{\partial z^2} \right\}, \end{aligned} \quad (2.1)$$

$$\begin{aligned} \text{r-direction; } \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} = & - \frac{1}{\rho} \frac{\partial p}{\partial r} + \frac{\mu}{\rho} \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) \right. \\ & \left. - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2} \right\}, \end{aligned} \quad (2.2)$$

$$\frac{1}{r} \frac{\partial (ru)}{\partial r} + \frac{\partial w}{\partial z} = 0, \quad (2.3)$$

where r and z are the cylindrical coordinates, p is the blood pressure, μ is the blood viscosity, t is the time, w is the axial blood velocity component, u is the radial blood velocity component and ρ is the blood density. Some authors prefer using the symbol ν (kinematic viscosity) in place of μ/ρ but the latter shall here be maintained.

Using assumptions (3), (7) and (8) Womersley derived two equations describing the balance of forces at the boundaries of the flowing fluid. The equations are:

$$\rho_w H \frac{\partial^2 \xi}{\partial t^2} - p + \frac{E_c h}{1 - \sigma_c^2} \left(\frac{\sigma_c}{R} \frac{\partial \zeta}{\partial z} + \frac{\xi}{R^2} \right) = 0 \quad - (2.4)$$

and

$$\rho_w H \frac{\partial^2 \zeta}{\partial t^2} + K \zeta + \mu \left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} \right)_{r=R} - \frac{E_c h}{1 - \sigma_c^2} \left(\frac{\partial^2 \zeta}{\partial z^2} + \frac{\sigma_c}{R} \frac{\partial \xi}{\partial z} \right) = 0, \quad - (2.5)$$

where ξ and ζ are the radial and axial vessel wall displacements, respectively, h is the vessel wall thickness, ρ_w is the density of the wall substance, H is the weighted volume of the wall substance, K is the wall material spring constant, R is the radius of the vessel, σ_c is the complex Poisson's ratio and E_c is the complex elastic modulus of the vessel wall material. The real parts of σ_c and E_c are the usual Poisson's ratio and Young's modulus, respectively. The imaginary parts of σ_c and E_c represent the viscous constants of the wall material.

2.2.2 ANALYSIS:

By assumption (13), equation (2.1) and (2.2) are linearized and reduce to

$$\frac{\partial w}{\partial t} = - \frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{\mu}{\rho} \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial w}{\partial r} \right) + \frac{\partial^2 w}{\partial z^2} \right\}, \quad - (2.6)$$

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \frac{u}{\rho} \left\{ \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial u}{\partial r}) - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2} \right\}. \quad - (2.7)$$

Equations (2.3), (2.6) and (2.7) are nondimensionalised in the variable r by the substitution $y = r/R$, giving the result

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{u}{\rho} \left\{ \frac{1}{R^2 y} \frac{\partial}{\partial y} (y \frac{\partial w}{\partial y}) + \frac{\partial^2 w}{\partial z^2} \right\}, \quad - (2.8)$$

$$\frac{\partial u}{\partial t} = -\frac{1}{R\rho} \frac{\partial p}{\partial y} + \frac{u}{\rho} \left\{ \frac{1}{R^2 y} \frac{\partial}{\partial y} (y \frac{\partial u}{\partial y}) - \frac{u}{R^2 y^2} + \frac{\partial^2 u}{\partial z^2} \right\} \quad - (2.9)$$

$$\frac{1}{Ry} \frac{\partial}{\partial y} (yu) + \frac{\partial w}{\partial z} = 0. \quad - (2.10)$$

Womersley assumed both the pressure and flow velocities to be sinusoidal travelling wave functions having the same arguments. Under this assumption and using separation of variables, he obtained

$$w(z, y, t) = W(y) \exp\{in(t - z/c)\} \quad - (2.11)$$

$$u(z, y, t) = U(y) \exp\{in(t - z/c)\} \quad - (2.12)$$

$$p(z, t) = P \exp\{in(t - z/c)\}, \quad - (2.13)$$

where c is the complex wave velocity, n is the radian frequency (or wave number) and P is the sinusoidal pressure modulus. The terms $\partial^2 w / \partial z^2$ and $\partial^2 u / \partial z^2$ will be neglected in equations (2.8) and (2.9) because they contain $W(y)/c^2$ and $U(y)/c^2$ which by assumption (11) are too small. On neglecting the two terms equations (2.8) and (2.9) become

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{u}{\rho R^2 y} \frac{\partial}{\partial y} (y \frac{\partial w}{\partial y}), \quad - (2.14)$$

$$\frac{\partial u}{\partial t} = -\frac{1}{R\rho} \frac{\partial p}{\partial y} + \frac{u}{\rho R^2} \left\{ \frac{1}{y} \frac{\partial}{\partial y} (y \frac{\partial u}{\partial y}) - \frac{u}{y^2} \right\}. \quad - (2.15)$$

By substituting the expressions for w , u and p from equations (2.11), (2.12) and (2.13) into equations (2.10), (2.14) and (2.15), the result is

$$W''(y) + \frac{1}{y}W'(y) + (i^3 R^2 n_p / \mu)W(y) = - \frac{i R^2 n_p}{\mu c} , \quad - (2.16)$$

$$U''(y) + \frac{1}{y}U'(y) + (i^3 R^2 n_p / \mu - 1/y^2)U(y) = \frac{R}{\mu} \frac{\partial p}{\partial y} , \quad - (2.17);$$

and

$$U'(y) + \frac{1}{y}U(y) - \frac{i R n_p}{c}W(y) = 0 \quad - (2.18)$$

Note that the solutions of the homogeneous parts of equations (2.16) and (2.17) are Bessel functions of orders 0 and 1, respectively. The simultaneous solutions of equations (2.16), (2.17) and (2.18) are:

$$w(z,y,t) = \left[\frac{A J_0(\alpha i^{3/2} y)}{J_0(\alpha i^{3/2})} + \frac{P}{P_c} \right] \exp\{i n(t - z/c)\} , \quad - (2.19)$$

$$u(z,y,t) = \left[\frac{i n R \left\{ \frac{2 A J_1(\alpha i^{3/2} y)}{J_0(\alpha i^{3/2})} + \frac{y P}{P_c} \right\}}{2 c \alpha i^{3/2}} \right] \exp\{i n(t - z/c)\} , - (2.20)$$

$$\text{where } \alpha^2 = R^2 n_p / \mu , \quad - (2.21)$$

is the Bessel function parameter. The quantities $w(z,y,t)$ and $u(z,y,t)$ determined here are the complex blood flow velocities, the real parts of which give the instantaneous axial and radial blood flow velocities, respectively, at any time t and point (y,z) in the blood stream. However equations (2.19) and (2.20) do not completely determine w and u since the equations still contain the unknown constant A and pressure modulus P . Boundary conditions (2.4) and (2.5) are used to determine these unknown quantities.

At the boundaries, i.e. on the vessel wall, $y = 1$ and the velocities w and u become

$$w(z,1,t) = \left[A + \frac{P}{P_c} \right] \exp\{i n(t - z/c)\} , \quad - (2.22)$$

$$u(z, l, t) = \frac{inR}{2c} \left[\frac{2AJ_1(\alpha i^{3/2})}{\alpha i^{3/2} J_0(\alpha i^{3/2})} + \frac{P}{pc} \right] \exp\{in(t - z/c)\} . \quad (2.23)$$

Equations (2.22) and (2.23) give the axial and radial blood velocity at $y = 1$, i.e. on the vessel wall. Axial (ζ) and radial (ξ) wall displacements are also assumed sinusoidal travelling waves, given by

$$\zeta = B \exp\{in(t - z/c)\} \quad (2.24)$$

and

$$\xi = D \exp\{in(t - z/c)\} , \quad (2.25)$$

where B and D are arbitrary constants. Using equations (2.24) and (2.25) in equations (2.4) and (2.5), the result is

$$p_w Hn^2 D + P - \frac{E_c h}{1 - \sigma_c^2} \left(\frac{D}{R^2} - \frac{in \sigma_c B}{Rc} \right) = 0 , \quad (2.26)$$

$$p_w Hn^2 B - KB - \frac{\mu}{R} \left(\frac{n^2 R^2 P}{2c^3 p} - \frac{i^3 A \alpha^2 J_1(\alpha i^{3/2})}{\alpha i^{3/2} J_0(\alpha i^{3/2})} \right) - \frac{E_c h}{1 - \sigma_c^2} \left(\frac{n^2 B}{c^2} + \frac{in \sigma_c D}{Rc} \right) = 0 \quad (2.27)$$

At the vessel wall, we have the following:

$$w(z, l, t) \equiv \frac{\partial \zeta}{\partial t} = inB \exp\{in(t - z/c)\} , \quad (2.28)$$

$$u(z, l, t) \equiv \frac{\partial \xi}{\partial t} = inD \exp\{in(t - z/c)\} . \quad (2.29)$$

From equations (2.22), (2.23), (2.28) and (2.29), we have

$$inB = A + \frac{P}{pc} , \quad (2.30)$$

$$D = \frac{R}{2c} \left(\frac{2AJ_1(\alpha i^{3/2})}{\alpha i^{3/2} J_0(\alpha i^{3/2})} + \frac{P}{pc} \right) \quad (2.31)$$

Determination of the constants P (pressure modulus), c (complex wave velocity) and A (an integration constant)

will enable us to determine axial and radial velocities w and u entirely in terms of known quantities. Womersley first determined c as follows:

Equations (2.26), (2.27), (2.30) and (2.31) form up a system of four homogeneous linear equations in four unknowns A , B , D and P . From the theory on application of determinants to solve systems of linear equations, the coefficient determinant of the four equations should be equal to zero to satisfy the consistency requirement. The determinant contains c but not any of the four unknowns A , B , D and P . Solving this coefficient determinant for c , he obtained

$$\frac{1}{c} = \frac{1}{c_0} \left[\frac{E(1 - \sigma_c^2)X}{2E_0} \right]^{\frac{1}{2}} = \frac{1}{c_0} [X - iY], \quad - (2.32)$$

$$\text{where } c_0 = \left(\frac{hE}{2R_p} \right)^{\frac{1}{2}} \quad - (2.33 a)$$

$$(1 - \sigma_c^2)X = G_0 \pm \sqrt{G_0^2 - (1 - \sigma_c^2)^2 H'^2}, \quad - (2.33 b)$$

$$G_0 = \frac{5/4 - \sigma_c}{1 - F_{10}} + \frac{k'}{2} + \sigma_c - 1/4, \quad - (2.33 c)$$

$$H' = \frac{1 + 2k'}{1 - F_{10}} - 1, \quad - (2.33 d)$$

$$k' = \frac{H_p}{R_p} \left(1 - \frac{K}{p_w H_n^2} \right), \quad - (2.33 e)$$

$$F_{10} = \frac{2J_1(\alpha_1^{3/2})}{\alpha_1^{3/2} J_0(\alpha_1^{3/2})}, \quad - (2.33 f)$$

where E is Young's modulus and c_0 is the theoretical pulse wave velocity in a frictionless system.

Of the two velocity components w and u , only w , the axial velocity component, is significant. To determine w

we need an expression for A and this can be obtained using equations (2.27), (2.30) and (2.31), giving the result

$$A = \frac{P\eta}{\rho c}, \quad \text{-- (2.34)}$$

$$\text{where } \eta = \frac{2}{X(F_{10} - 2\delta_c)} - \frac{1 - 2\delta_c}{F_{10} - 2\delta_c}. \quad \text{-- (2.35)}$$

Substituting the expression for A from equation (2.34) into equation (2.19) and integrating the resultant expression between limits $y = 0$ and $y = 1$, gives

$$\bar{w} = \frac{P}{\rho c} (1 + \eta F_{10}) \exp\{in(t - z/c)\}, \quad \text{-- (2.36)}$$

where \bar{w} is the average axial blood velocity across the vessel at any time t and axial position z in the blood stream. The radial velocity u is also completely determined by substituting for A from equation (2.34) into (2.20), however as earlier indicated, u is small compared to w and is usually ignored in blood flow problems.

In arriving at equation (2.36), Womersley assumed no reflected pressure waves in the system, however Fry and Greenfield, Jr. (1964) argued that in a real situation both the forward and the reflected waves always exist. In the presence of a reflected wave and assuming a linear superimposition of the waves, the pressure p is given by

$$p = P_1 \exp\{in(t - z/c)\} + P_2 \exp\{in(t + z/c)\}, \quad \text{-- (2.37)}$$

where P_1 is the pressure modulus of the forward wave, P_2 is that of the reflected wave and $+z/c$ indicates that the reflected wave is travelling in the negative z -direction. On simplification and re-arrangement of terms, average axial velocity \bar{w} now becomes

$$\bar{w} = \frac{1}{in\rho} (1 + \eta F_{10}) \left\{ in/c \left[P_1 \exp\{in(t - z/c)\} - P_2 \exp\{in(t + z/c)\} \right] \right\} . \quad - (2.38)$$

The expression in the braces of equation (2.38) is $-\partial p/\partial z$ for p given by (2.37), which is the complex pressure gradient. At any given point $z = z_0$, the complex pressure gradient is given by

$$-\frac{\partial p}{\partial z} = A' \exp\{int\}, \quad - (2.39)$$

$$\text{where } A' = in/c \left[P_1 \exp\{-inz_0/c\} - P_2 \exp\{inz_0/c\} \right] . \quad - (2.40)$$

The quantity A' is the complex modulus of the complex pressure gradient at the given point $z = z_0$. Substituting the expression for $-\partial p/\partial z$ from equation (2.39) into (2.38), one obtains

$$\bar{w} = \frac{A'}{in\rho} (1 + \eta F_{10}) \exp\{int\} . \quad - (2.41)$$

By equation (2.40), Fry and Greenfield, Jr. (1964) established that given pressure gradient $-\partial p/\partial z$ at any point, one can determine the instantaneous average blood velocity in the presence of reflected waves at that point.

2.3 SURVEY OF KAPUR'S ARTICLE.

J.N. Kapur (1983) in an article entitled 'Problems of Blood Flow in the Human System' discussed pulsatile blood flow under assumptions already listed in Womersley's model. His article is in two sections. In the first section he assumed the arterial vessel wall to be rigid. Under this assumption he determined the axial velocity w as a function of y (nondimensionalized radial displacement) and time t . He also determined pulsatile volumetric flow rate Q . In the

second part of his article, he assumed the arterial vessel wall to be elastic. Under this assumption he analysed the harmonics separately and linearly combined the solutions for the different harmonics into series solutions for pressure and axial velocity.

2.3.1 RIGID VESSEL:

It is assumed that the vessel wall is rigid and that radial velocity is negligibly small. Under this condition equation (2.7) reduces to

$$\frac{\partial p}{\partial r} = 0 . \quad - (2.42)$$

Thus by equation (2.42) pressure p is a function of z only. The vessel is assumed to be a cylinder of uniform cross-section and flow is symmetrical. Under these assumptions, we have $\partial w / \partial z = 0$, implying w , the axial velocity, is independent of the axial displacement z . Kapur assumed both the pressure gradient and axial velocity to be sinusoidal waves with the same argument given by

$$\frac{\partial p}{\partial z} = - P \exp\{int\} , \quad - (2.43)$$

$$w(r,t) = W(r) \exp\{int\} , \quad - (2.44)$$

where n is the sinusoidal wave number and P is the pressure gradient modulus. From equations (2.6), (2.43) and (2.44), we have

$$W''(r) + 1/r W'(r) - (inp/\mu) W(r) = - P/\mu \quad - (2.45)$$

The homogeneous part of equation (2.45) is Bessel's equation of order zero. A general solution of (2.45) is

$$W(r) = AJ_0(i^{3/2}(np/\mu)^{1/2}r) + BY_0(i^{3/2}(np/\mu)^{1/2}r) - \frac{1P}{np} , \quad - (2.46)$$

where J_0 is the Bessel function of the first kind of order zero and Y_0 is the Bessel function of the second kind of order zero. For the velocity $W(r)$ to be finite at $r = 0$, B must be zero. Equation (2.46) becomes

$$W(r) = AJ_0(i^{3/2}(np/\mu)^{\frac{1}{2}}r) - \frac{iP}{np} . \quad - (2.47)$$

Using the no slip velocity on the vessel wall, assumption (4), i.e. $W = 0$ at $r = R$, Kapur obtained

$$A = \frac{iP}{npJ_0(i^{3/2}\alpha)} , \quad - (2.48)$$

where α is as given by equation (2.21). Nondimensionalizing the variable r by the substitution $y = r/R$, equation (2.47) becomes

$$W(y) = - \frac{iP}{np} \left(1 - \frac{J_0(i^{3/2}\alpha y)}{J_0(i^{3/2}\alpha)} \right) , \quad - (2.49)$$

Using equations (2.44), (2.47) and (2.21), we obtain

$$w(y,t) = - \frac{iPR^2}{n\alpha^2} \left(1 - \frac{J_0(i^{3/2}\alpha y)}{J_0(i^{3/2}\alpha)} \right) \exp\{int\} . \quad - (2.50)$$

Equation (2.50) determines the complex flow velocity the real part of which is the instantaneous axial flow velocity at any time t and radial position y . Flow is independent of axial displacement z .

Volumetric flow rate Q is given by

$$Q = 2\pi \int_0^R r w(r,t) dr = 2\pi R^2 \int_0^1 y w(y,t) dy \quad - (2.51)$$

which on substituting equation (2.50) for $w(y,t)$ and integrating, gives

$$Q = \frac{i\pi PR^4}{n\alpha^2} \left(1 - \frac{2J_1(i^{3/2}\alpha)}{i^{3/2}\alpha J_0(i^{3/2}\alpha)} \right) \exp\{int\} . \quad - (2.52)$$

Writing the Bessel functions in equations (2.52) in series form, we have

$$Q = \frac{\pi PR^4}{8\alpha^2} \left[\frac{1 - 1 - x^2/8 + x^4/384 - \dots}{1 - 1 - x^2/4 + x^4/64 - \dots} \right] \exp\{int\}, \quad - (2.53)$$

where $x = i^{3/2}\alpha$. The expression in the square brackets of equation (2.53) is denoted by $X(\alpha)$ and when this is expanded in a power series, the result is

$$X(\alpha) \approx \frac{1\alpha^2}{8} - \frac{7\alpha^4}{384} + o(\alpha^6). \quad - (2.54)$$

On substituting equation (2.54) into (2.53), the result is

$$Q = \left[\frac{PR^4}{8\mu} + o(\alpha^4) \right] \exp\{int\} \quad - (2.55)$$

As $n \rightarrow 0$, $\alpha \rightarrow 0$ and $Q \rightarrow Q_0$ where

$$Q_0 = \frac{\pi PR^4}{8\mu}, \quad - (2.56)$$

which is the volume flow rate under steady state and constant pressure, known as the Poiseuille equation. On the other hand, $X(\alpha)$ can be broken up into its real and imaginary parts and a graph of the real and imaginary parts of $X(\alpha)$ against α can be tabulated using tables of Bessel functions. Such a graph is given by Figure 2.1.

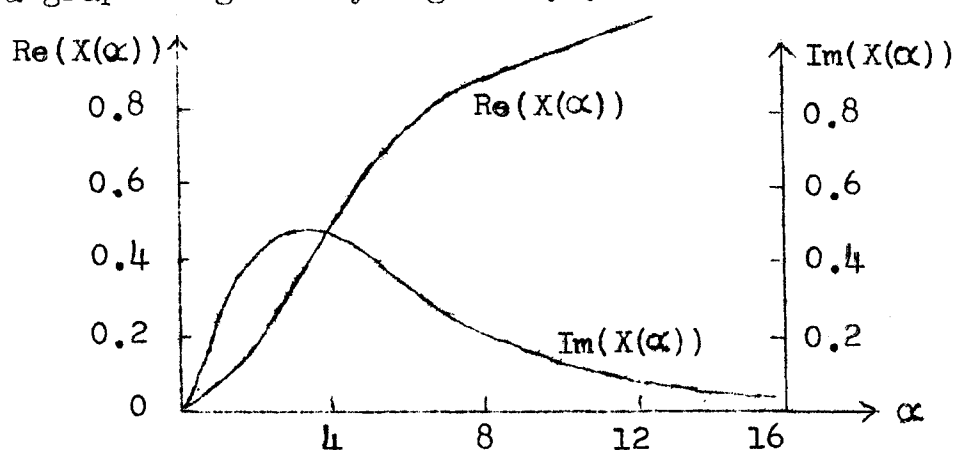


Figure 2.1: α against $Re(X(\alpha))$ and $Im(X(\alpha))$, (see 'Problems of Blood Flow in the Human System', Kapur (1983)).

Figure 2.1 shows that the difference in magnitude between the real and imaginary parts of $X(\alpha)$ increases with increasing α . Phase shift ϕ , between pulsatile flow rate Q and the pressure gradient is given by

$$\tan \phi = \frac{\operatorname{Re}(X(\alpha))}{\operatorname{Im}(X(\alpha))} \quad - (2.57)$$

For values of α beyond 4 on the graph, $\operatorname{Re}(X(\alpha))$ is increasing while $\operatorname{Im}(X(\alpha))$ is decreasing. By equation (2.57) this implies phase difference ϕ increases with α . Figure 2.2 is a graph of ϕ against α .

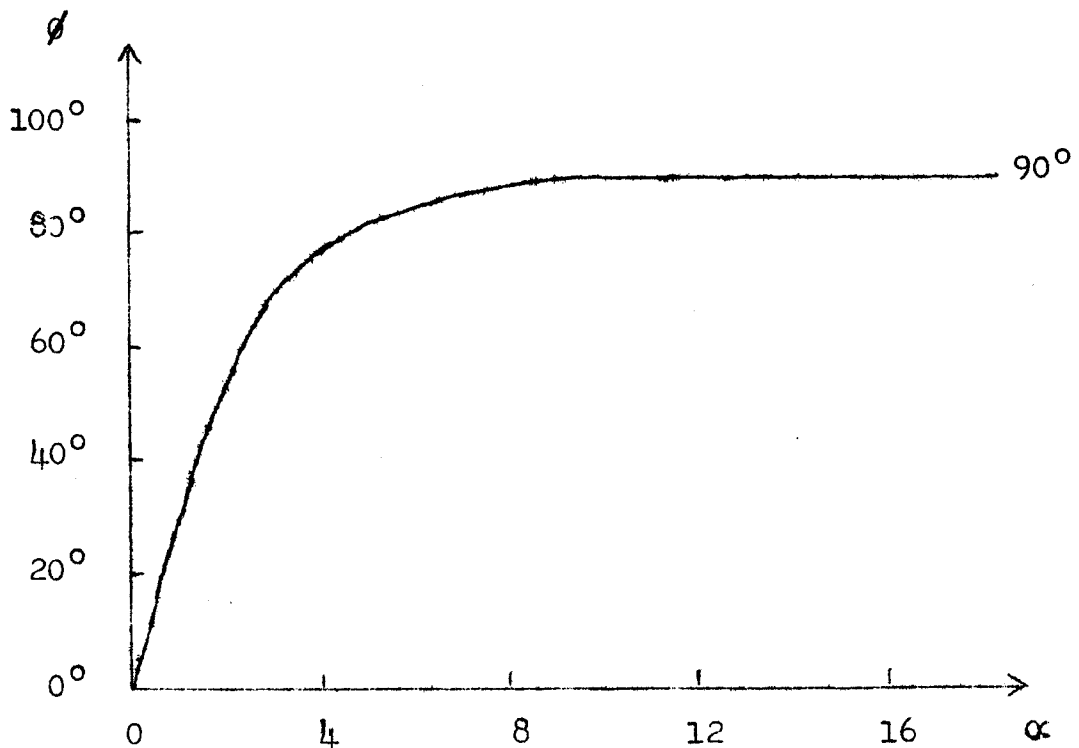


Figure 2.2: α against ϕ (Kapur (1983))

The graph shows that ϕ increases with α and levels off to 90° . Since by equation (2.21), α increases with vessel radius R , it is expected that phase difference ϕ should be higher in larger vessels.

Impedance Z is defined by

$$Z = - \frac{dp/dz}{Q} = P/Q \exp\{int\} . \quad - (2.58)$$

As $\alpha \rightarrow 0$, $n \rightarrow 0$ and $Z \rightarrow Z_0$, where

$$Z_0 = P/Q_0 . \quad - (2.59)$$

To see how volume flow rate Q varies with the pulsatile character of blood flow, consider a graph of $|Z|/Z_0$ against α , shown in Figure 2.3.

$$|Z|/Z_0 = Q_0/|Q| = \frac{8|X(\alpha)|}{\alpha^2} \approx \alpha^2/8 . \quad - (2.60)$$

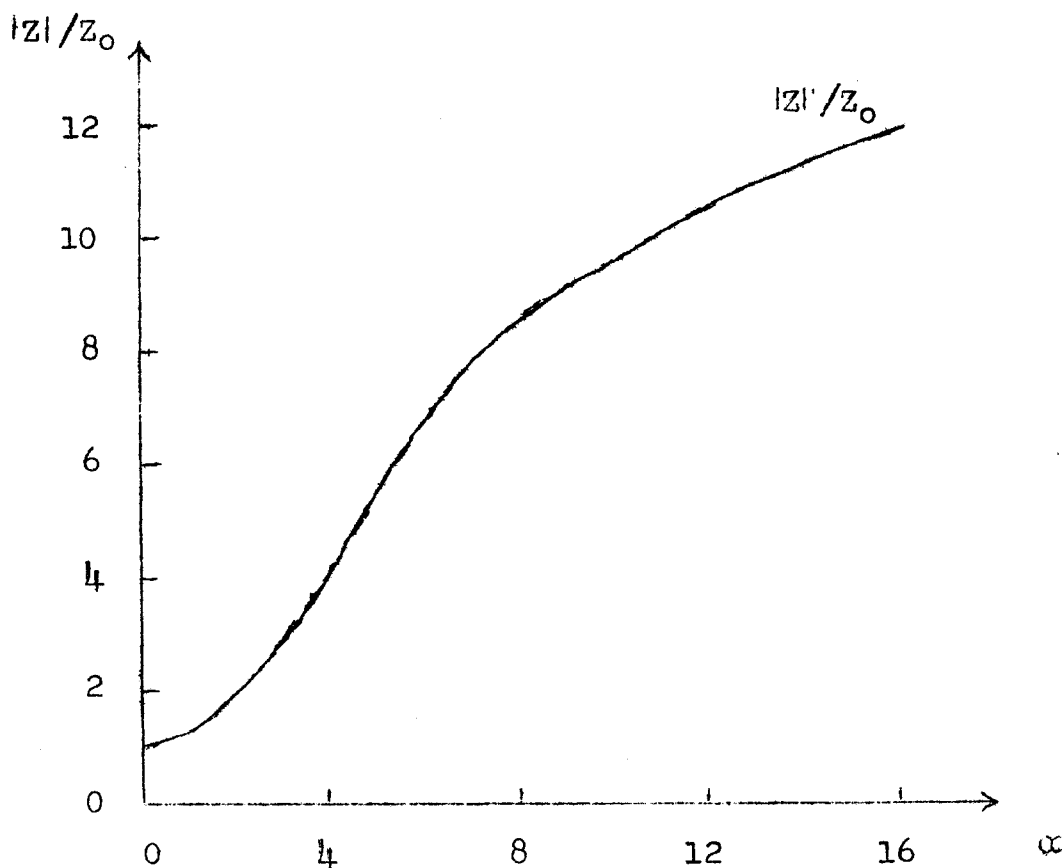


Figure 2.3: $|Z|/Z_0$ against α , (Kapur (1983)).

Figure 2.3 shows that $|Z|/Z_0$ increases with α which by equation (2.60) implies that $|Z|$ increases fast relative to Z_0 and $|Q|$ decreases fast relative to Q_0 , as α increases. Thus, Kapur was able to show that the pulsatile character of blood flow implies a reduction in volume flow rate compared

with steady state flow rate and equation (2.21) implies that this reduction is greater for large R , i.e. in the big vessels.

From equations (2.58) and (2.53), we have

$$Z = \left(-\frac{\mu}{\pi R^4} \right) \left(\frac{1}{X(\alpha)} \right) \alpha^2 . \quad - (2.61)$$

Dimensionless impedance is given by

$$\bar{Z} = \frac{1}{X(\alpha)} \alpha^2 = \{ 8 + o(\alpha^4) \} + i \{ 4/3 \alpha^2 + o(\alpha^4) \} . \quad - (2.62)$$

The approximation to $X(\alpha)$ given by equation (2.54) leading to the value given for \bar{Z} in equation (2.62) holds only for small values of α . For large values of α , Kapur used the asymptotic series of the Bessel functions and obtained

$$\bar{Z} = \{ \sqrt{2} \alpha + 3 + 2\sqrt{2}/\alpha + o(\alpha^{-2}) \} + i \alpha^2 \{ 1 - \sqrt{2}/\alpha + \frac{4\sqrt{2}}{\alpha^2} + o(\alpha^{-3}) \} . \quad - (2.63)$$

The real part of \bar{Z} in both equations (2.62) and (2.63) gives the resistance to flow contributed by viscous forces and the imaginary part is the resistance due to inertial forces.

Using equation (2.62) for small values of α , Kapur noted that for $\alpha^2 \leq 2/3$, viscous resistance is predominant being over 90% of total impedance effect. Using equation (2.63) for larger values of α , he noted that for $\alpha^2 \geq 162$, inertial resistance is predominant being over 90% of total impedance. For blood flow, $\mu/\rho = 0.04 \text{ cm}^2/\text{sec}$, $2\pi/n = 60/72$ and using equation (2.21) for $\alpha^2 = 2/3$ and 162 we have $2R \approx 1.3 \text{ mm}$ and 2 cm , respectively. Thus for small vessels of diameter less than 1.3 mm, i.e. arterioles down to the capillaries, inertial effects are small compared to viscous forces and energy dissipation mainly occurs there. For vessels of diameter greater than 2 cm inertial forces dominate

and much of viscous dissipation occurs in the large arteries.

2.3.3 ELASTIC VESSEL:

The vessel is here assumed to be elastic and Kapur used the relations between symmetric stress tensor and vessel wall displacements to derive equations governing fluid flow and wall movement at the fluid/wall interface. These equations are:

$$p \frac{\partial^2 \xi}{\partial t^2} = - \frac{\partial \Omega}{\partial r} + G \left\{ \frac{\partial^2 \xi}{\partial r^2} + \frac{1}{r} \frac{\partial \xi}{\partial r} - \frac{\xi}{r^2} + \frac{\partial^2 \xi}{\partial z^2} \right\}, \quad - (2.64)$$

$$p \frac{\partial^2 \zeta}{\partial t^2} = - \frac{\partial \Omega}{\partial z} + G \left\{ \frac{\partial^2 \zeta}{\partial r^2} + \frac{1}{r} \frac{\partial \zeta}{\partial r} + \frac{\partial^2 \zeta}{\partial z^2} \right\}, \quad - (2.65)$$

$$\frac{\partial \zeta}{\partial z} + \frac{\partial \xi}{\partial r} + \frac{\xi}{r} = 0, \quad - (2.66)$$

where ξ and ζ are as used previously, G is the shear modulus of the vessel wall material, Ω is the external force on the inner vessel wall. Due to continuity of motion at the fluid/wall interface, we have

$$u = \frac{\partial \xi}{\partial t}, \quad w = \frac{\partial \zeta}{\partial t} \quad \text{at } r = R_I, \quad - (2.67)$$

where R_I is the inner radius of the vessel. From continuity of shear stress and radial stress at the inner wall, we have

$$\left. \begin{aligned} \mu \left\{ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \right\} &= G \left\{ \frac{\partial \xi}{\partial z} + \frac{\partial \zeta}{\partial r} \right\} \\ \text{and} \\ -p + 2\mu \frac{\partial u}{\partial r} &= -\Omega + 2G \frac{\partial \xi}{\partial r} \end{aligned} \right\} \quad \text{at } r = R_I \quad - (2.68)$$

Assuming that the external wall is free of any external constraint, we have

$$G \left\{ \frac{\partial \xi}{\partial z} + \frac{\partial \zeta}{\partial r} \right\} = 0 \quad \text{and} \quad -\Omega + 2G \frac{\partial \xi}{\partial r} = 0 \quad \text{at } r = R_E \quad - (2.69)$$

where R_E is the external radius of the vessel.

To determine the k^{th} harmonic solutions of equations (2.3), (2.6) and (2.7), the method of separation of variables is applied as follows:

$$w_k(z, r, t) = W_k(r) \exp\{i(nkt - y_k z)\} , \quad - (2.70)$$

$$u_k(z, r, t) = U_k(r) \exp\{i(nkt - y_k z)\} , \quad - (2.71)$$

$$p_k(z, r, t) = P_k(r) \exp\{i(nkt - y_k z)\} , \quad - (2.72)$$

where y_k is the k^{th} harmonic propagation constant, w_k and u_k are the k^{th} harmonic axial and radial velocity components, respectively and p_k is the k^{th} harmonic pressure. Substituting expressions for w_k , u_k and p_k from equations (2.70), (2.71) and (2.72) into equations (2.3), (2.6) and (2.7), the result is

$$W_k''(r) + 1/r W_k'(r) - x_k^2 W_k(r) = -(iy_k/\mu) P_k(r) , \quad - (2.73)$$

$$U_k''(r) + 1/r U_k'(r) - (x_k^2 + 1/r^2) U_k(r) = 1/\mu P_k'(r) , \quad - (2.74)$$

and

$$U_k'(r) + 1/r U_k(r) - iy_k W_k(r) = 0 , \quad - (2.75)$$

where $x_k^2 = y_k^2 + ink\rho/\mu$. The solutions of the homogeneous parts of equations (2.73) and (2.74) are Bessel functions of orders 0 and 1, respectively. Let the tentative solutions be

$$W_k(r) = \alpha_1 J_0(iy_k r) + \alpha_2 J_0(ix_k r) \quad - (2.76)$$

and

$$U_k(r) = \beta_1 J_1(iy_k r) + \beta_2 J_1(ix_k r) . \quad - (2.77)$$

where α_1 , α_2 , β_1 and β_2 are arbitrary constants. The two given possible solutions satisfy

$$W_k''(r) + 1/r W_k'(r) - x_k^2 W_k(r) = -(i\alpha_1 nk\rho/\mu) J_0(iy_k r) \quad - (2.78)$$

and

$$U_k''(r) + 1/r U_k'(r) - (x_k^2 + 1/r^2)U_k(r) = -(i\beta_1 n k p / \mu) J_1(iy_k r). \quad - (2.79)$$

Solving equations (2.73) to (2.79) simultaneously, we have

$$W_k(r) = -i\{A_1 y_k J_0(iy_k r) + A_2 x_k J_0(ix_k r)\}, \quad - (2.80)$$

$$U_k(r) = -i\{A_1 y_k J_1(iy_k r) + A_2 x_k J_1(ix_k r)\} \quad - (2.81)$$

and

$$P_k(r) = -iA_1 n k p J_0(iy_k r). \quad - (2.82)$$

Kapur replaced the arbitrary constants α_1 and α_2 by $-iy_k A_1$ and $-iy_k A_2$, respectively. He then assumed the general solutions to be a linear combination of the harmonic solutions and obtained the results

$$w(z,r,t) = -\sum_{k=0}^{\infty} i\{A_1 y_k J_0(iy_k r) + A_2 x_k J_0(ix_k r)\} \exp\{i(nkt - y_k z)\}, \quad - (2.83)$$

$$u(z,r,t) = -\sum_{k=0}^{\infty} i\{A_1 y_k J_1(iy_k r) + A_2 x_k J_1(ix_k r)\} \exp\{i(nkt - y_k z)\} \quad - (2.84)$$

and

$$p(z,r,t) = -\sum_{k=0}^{\infty} iA_1 n k p J_0(iy_k r) \exp\{i(nkt - y_k z)\}. \quad - (2.85)$$

To determine the unknown constants A_1 and A_2 , consider equations (2.64) to (2.66) together with the boundary conditions (2.67) to (2.69). The structure of equations (2.64) to (2.66) suggests that solutions for $\partial \xi / \partial t$ and $\partial s / \partial t$ are similar to those for w and u . The solutions are of the following form:

$$\frac{\partial \xi}{\partial t} = \sum_{k=0}^{\infty} \{D_1 J_1(iy_k r) + E_1 J_1(ix_k r)\} \exp\{i(nkt - y_k z)\}, \quad - (2.86)$$

$$\frac{\partial \zeta}{\partial t} = \sum_{k=0}^{\infty} \left\{ D_2 J_0(i y_k r) + E_2 J_0(i x_k r) \right\} \exp\{i(nkt - y_k z)\} . \quad (2.87)$$

The six unknown constants A_1 , A_2 , D_1 , D_2 , E_1 and E_2 are to be determined using boundary conditions (2.67), (2.68) and (2.69). These boundary conditions generate six linear homogeneous equations in the six unknown constants, from which Cramer's rule can be applied to determine the constants.

It has already been noted that the pulsatile nature of arterial blood flow progressively becomes weaker away from the heart. It becomes negligibly small in the capillaries and veins. It may be desirable to determine the factor by which the pulse attenuates in travelling a given distance z in a given region of the arterial bed. To do this we need y_k , the complex propagation constant, which can be obtained from the consistency requirement of the coefficient matrix of the system of homogeneous equations in the unknown constants specified earlier. The resulting expression for y_k can be separated into real and imaginary parts. Suppose we have

$$y_k = B_k + i \delta_k \quad (2.88)$$

The axial variation of the pulse for the k^{th} harmonic is given by

$$\exp\{-i y_k z\} = \exp\{-i B_k z\} \exp\{\delta_k z\} . \quad (2.89)$$

In travelling a distance z the wave is damped by the factor $\exp\{\delta_k z\}$. In a wavelength λ_k , the damping factor is $\exp\{\delta_k \lambda_k\}$. The quantity B_k is called the wave number and is given by

$$B_k = 2\pi/\lambda_k . \quad (2.90)$$

The damping factor in a wavelength is then given by

$$\exp\{\delta_k \lambda_k\} = \exp\{2\pi \delta_k / B_k\} . \quad (2.91)$$

CHAPTER III.

EFFECTS OF STENOSIS ON ARTERIAL BLOOD FLOW.

3.1 INTRODUCTION:

Stenosis is an abnormal growth in the lumen of a blood vessel. The actual causes of stenosis are not well established, but its effects on blood flow in the cardiovascular system has been investigated by several workers, Texon (1957); May, DeWeese and Rob (1963); Fox and Hugo (1966); Rodbard (1966); Spain (1966); Fry (1968, 1972); Young (1968); Eklof and Schwartz (1971); Forrester and Young (1970); Lee and Fung (1970); Young and Tsai (1973); Lee (1974); Rodkiewicz (1974); Nerem (1974); Bergel, Nerem and Schwartz (1974); Morgan and Young (1974); Shukla and Parihar (1975); Parihar (1976); Richard, Young and Chalvin (1977); Shukla, Parihar and Gupta (1980-a,b); Shukla Parihar and Rao (1980).

This chapter is a review of the literature from Parihar (1976 and 1980) and from Shukla, Parihar and Gupta (1980-a) which is on effects of stenosis growth on arterial blood flow, considering blood as a Newtonian fluid. The effects of stenosis growth on resistance to flow and wall shear stress (ie. shearing stress at wall of vessel) were investigated by the named workers.

3.2 BASIC EQUATIONS:

Blood is assumed to be an incompressible, Newtonian fluid and that flow is steady, axially symmetrical, laminar and fully developed with no tangential velocities. Viscosity is assumed to vary only with radius. Under these conditions

and ignoring inertial forces, the Navier Stokes equations reduce to

$$\partial p / \partial r = 0, \quad - (3.1)$$

$$\partial p / \partial z = \frac{1}{r} \frac{\partial}{\partial r} \left(r \mu(r) \frac{\partial w}{\partial r} \right) \quad - (3.2)$$

and the continuity equation is

$$\frac{1}{r} \frac{\partial}{\partial r} (ru) + \frac{\partial w}{\partial z} = 0 \quad - (3.3)$$

where r and z are the cylindrical coordinates, w and u are the axial and radial velocity components, respectively, p is the blood pressure and $\mu(r)$ is the viscosity function. By equation (3.1) pressure p is a function of z only. Assuming no slip velocity at the wall of the vessel and also assuming Poiseuille flow profile the boundary conditions are

$$\left. \begin{array}{l} w = 0 \quad \text{at} \quad r = R(z) \\ \text{and} \\ \partial w / \partial r = 0 \quad \text{at} \quad r = 0 \end{array} \right\} \quad - (3.4)$$

where $R(z)$ is vessel radius to be specified later. Integrating equation (3.2) with respect to r and using boundary conditions (3.4), gives

$$w = \left(-\frac{1}{2} \frac{dp}{dz} \right) \int_0^{R(z)} \frac{r}{\mu(r)} dr. \quad - (3.5)$$

Flow flux Q is given by

$$Q = 2\pi \int_0^{R(z)} r w dr, \quad - (3.6)$$

which on integration by parts, gives

$$Q = \pi \int_0^{R(z)} r^2 \left(-\frac{\partial w}{\partial r} \right) dr \quad - (3.7)$$

and using equation (3.5) gives

$$Q = \left(-\frac{\pi}{2} \frac{dp}{dz} \right) \int_0^{R(z)} \frac{r^3}{\mu(r)} dr. \quad - (3.8)$$

By using the continuity equation (3.3), we show that Q is constant with respect to z . Solving for dp/dz , the pressure gradient, from equation (3.8), we obtain

$$\frac{dp}{dz} = -\frac{2Q}{\pi I(z)}, \quad - (3.9)$$

$$\text{where } I(z) = \int_0^{R(z)} \frac{r^3}{\mu(r)} dr. \quad - (3.10)$$

The boundary conditions for pressure at the end points of the vessel system are given as follows:

$$p = p_0 \text{ at } z = 0 \text{ and } p = p_L \text{ at } z = L, \quad - (3.11)$$

where L is the total length of the vessel system under investigation (see Figure 3.1). Integrating equation (3.9) using boundary conditions (3.11), gives

$$p_0 - p_L = \frac{2QF}{\pi}, \quad - (3.12)$$

$$\text{where } F = \int_0^L \frac{1}{I(z)} dz. \quad - (3.13)$$

Resistance to flow λ ; is given by, Burton (1968) and Young (1968)

$$\lambda = \frac{p_0 - p_L}{Q} \quad - (3.14)$$

Substituting for $p_0 - p_L$ from equation (3.12) into (3.14), the result is

$$\lambda = \frac{2F}{\pi}. \quad - (3.15)$$

It is clear from equation (3.14) that λ , the resistance to flow, is directly proportional to total pressure drop and inversely proportional to flow flux Q .

The shape and configuration of a stenosis growth is not well defined but it is here assumed to be axially symmetric as shown in Figure 3.1.

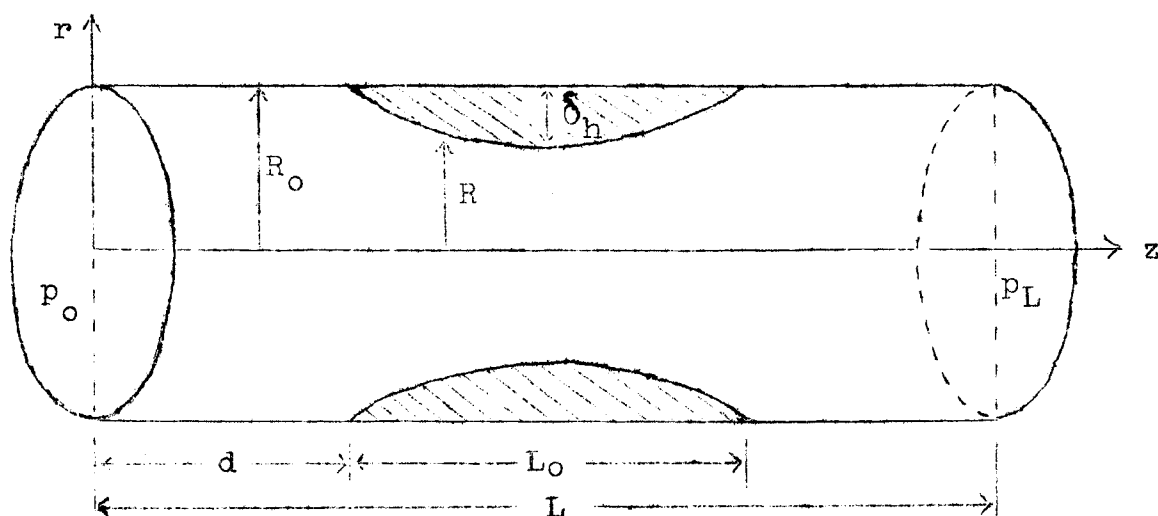


Figure 3.1: Geometry of Artery with Stenosis.

Variable radius of the vessel $R(z)$ is given by

$$\frac{R(z)}{R_0} = \begin{cases} 1 - \frac{\delta_h}{2R_0} \left\{ 1 + \cos \frac{2\pi(z-d-L_0)}{2L_0} \right\} & ; d \leq z \leq L_0 + d \\ 1 & ; \text{otherwise} \end{cases} \quad - (3.16)$$

where L_0 is the length of the stenosis, δ_h is the maximum height of the stenosis and R_0 is the constant radius of the vessel. Using equation (3.16) in equation (3.13), we have

$$F = \frac{L - L_0}{I_0} + \int_d^{d+L_0} \frac{dz}{I(z)}, \quad - (3.17)$$

$$\text{where } I_0 = \int_0^{R_0} \frac{r^3}{u(r)} dr. \quad - (3.18)$$

From equations (3.15) and (3.17), the resistance to flow λ is given by

$$\lambda = \frac{2}{\pi} \left\{ \frac{L - L_0}{I_0} + \left\{ \frac{dz}{dI(z)} \right\} \right\} . \quad - (3.19)$$

Resistance to flow λ was investigated under two different viscosity functions. Equation (3.19) was used to show the effects of these viscosity functions on λ .

Wall shear stress τ_R is given by

$$\tau_R = \left(-\mu(r) \frac{\partial w}{\partial r} \right)_{r=R(z)} . \quad - (3.20)$$

Using equations (3.5), (3.9) and (3.20), we get

$$\tau_R = \frac{RQ}{\pi I(z)} . \quad - (3.21)$$

Equation (3.21) was used to show the effects of the viscosity functions on wall shear stress.

3.3 LINEAR VARIATION OF VISCOSITY:

It is assumed that viscosity variation across the vessel is linear and is given by:

$$\mu(r) = \mu_0 \left(1 - \frac{\delta r}{R_0} \right) ; \quad \delta \ll 1 , \quad - (3.22)$$

where μ_0 is the viscosity of the fluid at $r = 0$ and δ is a constant parameter. Using $\mu(r)$ from equation (3.22) in the integrals $I(z)$ and I_0 from (3.10) and (3.18), respectively, we obtain

$$I(z) = \frac{R^4(z)}{4\mu_0} \left\{ 1 + \frac{4\delta R(z)}{5 R_0} \right\} \quad - (3.23)$$

and

$$I_0 = \frac{R_0^4}{4\mu_0} \left\{ 1 + \frac{4\delta}{5} \right\} \quad - (3.24)$$

3.3.1 EFFECT ON RESISTANCE:

Substituting the expressions for $I(z)$ and I_0 from equations (3.23) and (3.24) into (3.19), we have

$$\lambda = \frac{8\mu_0}{\pi R_0^4} \left\{ L - L_0 + \int_d^{d+L_0} \frac{dz}{(R/R_0)^4} - \frac{4\delta}{5} \left[L - L_0 + \int_d^{d+L_0} \frac{dz}{(R/R_0)^3} \right] \right\}$$

for $\delta \ll 1$. - (3.25)

Substituting for $R(z)/R_0$ from equation (3.16) into (3.25) and making suitable substitutions to facilitate integration, we have

$$\lambda = \frac{8\mu_0 L}{\pi R_0^4} \left\{ 1 - \frac{L_0}{L} + \frac{L_0}{2\pi L} \int_0^{2\pi} \frac{d\phi}{(a + b \cos \phi)^4} - \frac{4\delta}{5} \left[1 - \frac{L_0}{L} + \frac{L_0}{2\pi L} \int_0^{2\pi} \frac{d\phi}{(a + b \cos \phi)^3} \right] \right\} \quad - (3.26)$$

where $\phi = \pi - \frac{2\pi}{L_0}(z - d - \frac{L_0}{2})$, $a = 1 - \frac{\delta_h}{2R_0}$

and $b = \frac{\delta_h}{2R_0}$. - (3.27)

The two integrals in equation (3.26) can be evaluated by calculus of residues and the result is

$$\lambda = \frac{8\mu_0 L}{\pi R_0^4} \left\{ 1 - \frac{L_0}{L} + \frac{L_0}{L} \frac{a(2a^2 + 3b^2)}{2(a^2 - b^2)^{7/2}} - \frac{4\delta}{5} \left[1 - \frac{L_0}{L} + \frac{L_0}{L} \frac{(2a^2 + b^2)}{2(a^2 - b^2)^{5/2}} \right] \right\} \quad - (3.28)$$

Resistance to flow λ is nondimensionalised by dividing it by the quantity $8\mu_0 L / \pi R_0^4$. Nondimensionalised resistance $\bar{\lambda}$ is then given by

$$\bar{\lambda} = \frac{\lambda R_0^4}{8\mu_0 L} = \left\{ 1 - \frac{L_0}{L} + \frac{L_0}{L} \frac{a(2a^2 + 3b^2)}{2(a^2 - b^2)^{7/2}} - \frac{4\delta}{5} \left[1 - \frac{L_0}{L} + \frac{L_0(2a^2 + b^2)}{L 2(a^2 - b^2)^{5/2}} \right] \right\} . \quad (3.29)$$

When $\delta = 0$, the case without stenosis, equation (3.29) reduces to one obtained by Young (1968).

Equation (3.29) shows that for a given stenosis size, resistance to flow $\bar{\lambda}$ increases with decreasing δ . Since δ decreases with increasing viscosity at the vessel wall, it can thus be concluded that $\bar{\lambda}$ increases with increasing viscosity at the vessel wall. Graphs of the variation of resistance $\bar{\lambda}$ with δ_h/R_0 are given on Figure 3.2.

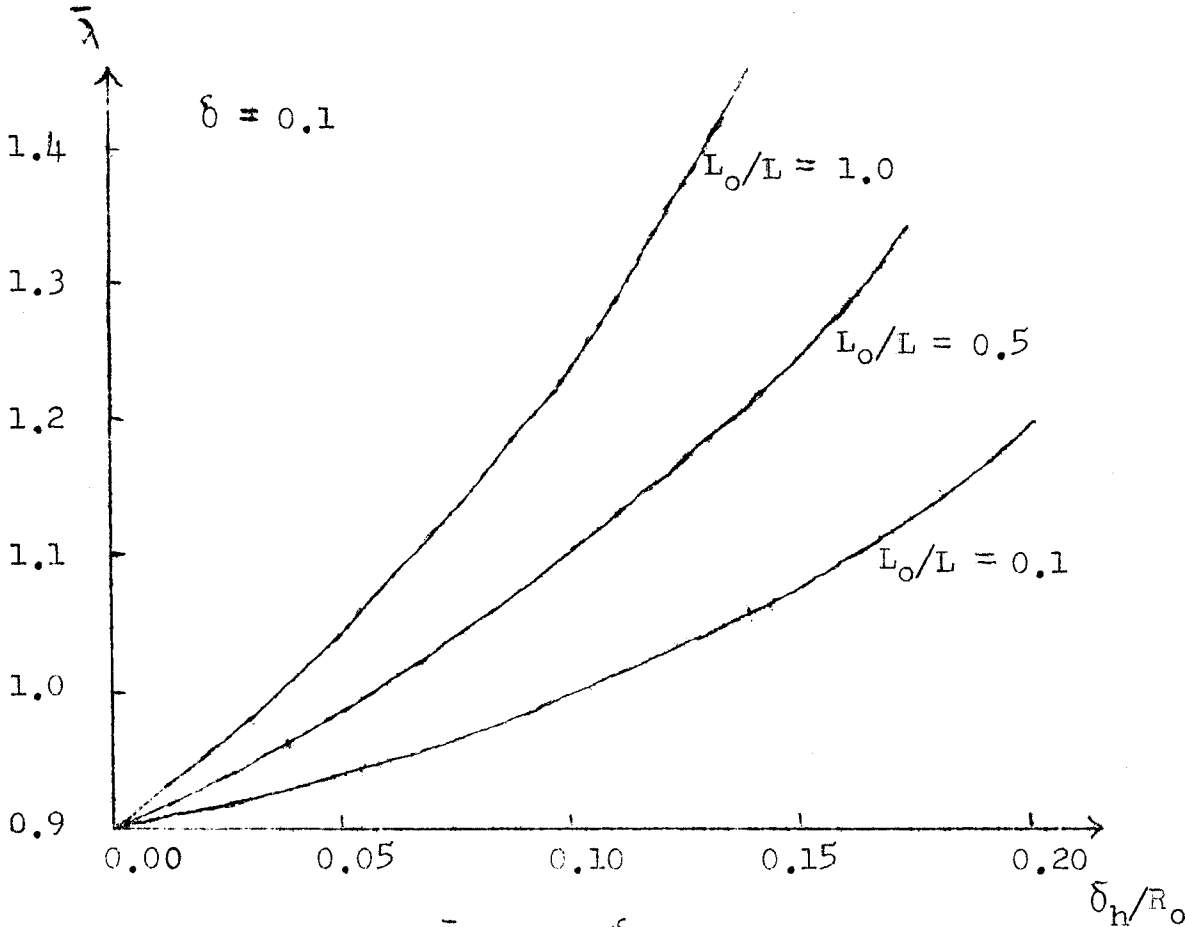


Figure 3.2: Variation of $\bar{\lambda}$ with δ_h/R_0 for different L_0/L .

From the graphs on Figure 3.2 it is noted that resistance to flow $\bar{\lambda}$ increases with increasing height and length of stenosis.

3.3.2 EFFECT ON WALL SHEAR:

At maximum height of stenosis (ie. at $z = d + L_0/2$) the wall shear stress τ_s is given by

$$\tau_s = \left(\frac{RQ}{\pi I(z)} \right)_{z = d + \frac{L_0}{2}} \quad - (3.30)$$

Substituting for $I(z)$ from equation (3.23) and using the expression for $R(z)/R_0$ from equation (3.16) in (3.30) gives

$$\tau_s = \frac{4\mu_0 Q}{\pi R_0^3} (1 - \frac{\delta_h}{R_0})^{-3} \left\{ 1 + \frac{4\delta}{5} (1 - \frac{\delta_h}{R_0}) \right\}^{-1} . \quad - (3.31)$$

Wall shear stress τ_s is nondimensionalized by dividing it by the term $4\mu_0 Q / \pi R_0^3$. The nondimensionalized wall shear stress $\bar{\tau}_s$ is given by

$$\bar{\tau}_s = \frac{s \pi R_0^3}{4\mu_0 Q} = (1 - \frac{\delta_h}{R_0})^{-3} \left\{ 1 + \frac{4\delta}{5} (1 - \frac{\delta_h}{R_0}) \right\}^{-1} \quad - (3.32)$$

For the case without stenosis $\delta = 0$ and the result reduces to that obtained by Young (1968).

Equation (3.32) shows that for a given stenosis size, wall shear stress $\bar{\tau}_s$ decreases with decrease in viscosity at the wall of the vessel and increases with δ_h/R_0 .

3.4 STEPWISE VARIATION OF VISCOSITY:

In small blood vessels, the tendency of the cells to concentrate at the centre of the vessel results in unequal distribution of viscosity across the vessel. The cell free layer near the wall contains only plasma which has lower viscosity and the central region with a higher cell concentration has a higher viscosity. In such small vessels a layered medium of different viscosities across the vessel can thus be justifiably assumed, Middleman (1972) and Lih

(1975).

In this section Parihar (1976); Shukla, Parihar and Gupta (1980) assumed that viscosity across the blood vessel varies as a step function as follows:

$$\begin{aligned} \mu &= \mu_1, \quad 0 \leq r \leq R_1(z) \\ \mu &= \mu_2, \quad R_1(z) \leq r \leq R_2(z) = R(z) \end{aligned} \quad - (3.33)$$

where μ_1 and μ_2 are the viscosities of the central and peripheral layers, respectively. It is assumed that $\mu_1 > \mu_2$. $R_1(z)$ is the shape of the central layer (see Figure 3.3) assumed to be given by

$$\frac{R_1(z)}{R_0} = \begin{cases} \alpha - \frac{\delta_1}{2R_0} \left\{ 1 + \cos \frac{2\pi(z-d-L_0)}{L_0} \right\} & ; d \leq z \leq d+L_0 \\ \alpha & ; \text{otherwise} \end{cases} \quad - (3.34)$$

where α is the ratio of the central core radius to the tube radius in the unobstructed region, δ_1 is the maximum bulging of the fluid layer interface at $z = d + L_0/2$ on the stenosis region.

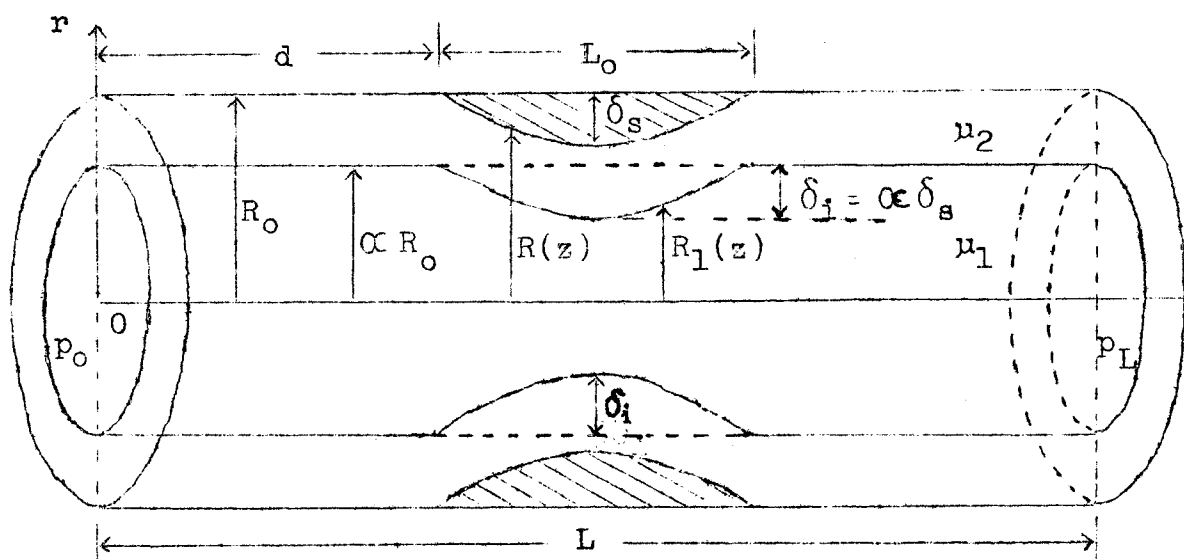


Figure 3.3: Geometry of arterial stenosis with peripheral layer.

A relation between δ_1 and δ_h is obtained as follows:

The two layers, central and peripheral, have different axial velocities denoted by w_c and w_p , respectively. By using equation (3.33) in (3.5) w_c and w_p are given by

$$w_c = \left(-\frac{1}{2} \frac{dp}{dz}\right) \left\{ \int_0^{R_1} \frac{r}{\mu_1} dr + \int_{R_1}^{R_2} \frac{r}{\mu_2} dr \right\}$$

$$= \left(-\frac{1}{4\mu_2} \frac{dp}{dz}\right) \left\{ R_2^2 - R_1^2 + \bar{\mu}_2(R_1^2 - r^2) \right\} \quad - (3.35)$$

$$w_p = \left(-\frac{1}{2} \frac{dp}{dz}\right) \int_0^{R_2} \frac{r}{\mu_2} dr = \left(-\frac{1}{4\mu_2} \frac{dp}{dz}\right) \left\{ R_2^2 - r^2 \right\}, \quad - (2.36)$$

where $\bar{\mu}_2 = \mu_2/\mu_1$. Flux for the central and peripheral layers, denoted by Q_c and Q_p , respectively, are given by

$$Q_c = 2\pi \int_0^{R_1} r w_c dr = \left(-\frac{\pi}{8\mu_2} \frac{dp}{dz}\right) 2R_1^2 \left\{ R^2 - (1 - \frac{\bar{\mu}_2}{2}) R_1^2 \right\}, \quad - (3.37)$$

$$Q_p = 2\pi \int_{R_1}^R r w_p dr = \left(-\frac{\pi}{8\mu_2} \frac{dp}{dz}\right) (R^2 - R_1^2)^2, \quad - (3.38)$$

The total flux Q is given by

$$Q = Q_c + Q_p = \left(-\frac{\pi}{8\mu_2} \frac{dp}{dz}\right) \left\{ R^4 - (1 - \bar{\mu}_2) R_1^4 \right\}. \quad - (3.39)$$

The expression for Q can as well be obtained by using equation (3.33) in equation (3.8). Further, by using the equation of continuity in separate layers, it can be shown that Q_c and Q_p are constants. Making dp/dz the subject of the formular in equations (3.37) to (3.39) and integrating the resultant equations along the vessel length using pressure boundary conditions (3.11), we have

$$Q_c = \frac{(p_o - p_L) \pi R_o^4}{4\mu_2 L} \cdot \frac{\alpha^2 \{1 - (1 - \bar{u}_2/2) \alpha^2\}}{1 - \frac{L_o}{L} + \alpha^2 \{1 - (1 - \bar{u}_2/2) \alpha^2\} G_1}, \quad - (3.40)$$

$$Q_p = \frac{(p_o - p_L) \pi R_c^4}{8\mu_2 L} \cdot \frac{(1 - \alpha^2)^2}{1 - \frac{L_o}{L} + (1 - \alpha^2)^2 G_2} \quad - (3.41)$$

and

$$Q = \frac{(p_o - p_L) \pi R_o^4}{8\mu_2 L} \cdot \frac{1 - (1 - \bar{u}_2) \alpha^4}{1 - \frac{L_o}{L} + \{1 - (1 - \bar{u}_2) \alpha^4\} G}, \quad - (3.42)$$

$$\text{where } G_1 = \frac{1}{L} \int_{d+L_o}^d \frac{dz}{(R_1/R_o)^2 \{ (R/R_o)^2 - (1 - \bar{u}_2/2) (R/R_o)^2 \}}, \quad - (3.43)$$

$$G_2 = \frac{1}{L} \int_{d+L_o}^d \frac{dz}{\{ (R/R_o)^2 - (R_1/R_o)^2 \}^2} \quad - (3.44)$$

and

$$G = \frac{1}{L} \int_{d+L_o}^d \frac{dz}{d(R/R_o)^4 - (1 - \bar{u}_2) (R_1/R_o)^4}. \quad - (3.45)$$

An expression for δ_i can be obtained by the condition

$Q = Q_c + Q_p$ using equations (3.37) to (3.39), giving

$$\frac{1 - (1 - \bar{u}_2) \alpha^4}{1 - \frac{L_o}{L} + \{1 - (1 - \bar{u}_2) \alpha^4\} G} = \frac{2\alpha^2 \{1 - (1 - \bar{u}_2/2) \alpha^2\}}{1 - \frac{L_o}{L} + \alpha^2 \{1 - (1 - \bar{u}_2/2) \alpha^2\} G_1} + \frac{(1 - \alpha^2)^2}{1 - \frac{L_o}{L} + (1 - \alpha^2)^2 G_2} \quad (3.46)$$

The relation $R_1(z) = \alpha R(z)$ satisfies equation (3.46) so that by equations (3.16) and (3.34), we have the relation

$$\delta_i = \alpha \delta_h. \quad - (3.47)$$

3.4.1 EFFECTS ON RESISTANCE TO FLOW:

Under the given definition of the viscosity function from equation (3.33), $I(z)$ and I_0 from equations (3.10) and (3.18) become

$$I(z) = \frac{1}{4\mu_2} \left\{ R_1^4(z) - (1 - \bar{\mu}_2) R_1^4(z) \right\} \quad - (3.48)$$

and

$$I_0 = \frac{1}{4\mu_2} \left\{ R_0^4 - (1 - \bar{\mu}_2) R_1^4(z) \right\} . \quad - (3.49)$$

Substituting expressions for $I(z)$ and I_0 from equations (3.48) and (3.49) into equation (3.19) and using the relation $R_1 = \alpha R$, the result is

$$\lambda = \frac{8\mu_1 \bar{\mu}_2 L}{\pi R_0^4 \{1 - (1 - \bar{\mu}_2) \alpha^4\}} \left\{ 1 - \frac{L_0}{L} + [1 - (1 - \bar{\mu}_2) \alpha^4] G \right\} , \quad - (3.50)$$

where G is now given in terms of α as

$$G = \frac{1}{L \{1 - (1 - \bar{\mu}_2) \alpha^4\}} \int_d^{d+L_0} \frac{dz}{(R/R_0)^4} . \quad - (3.51)$$

To nondimensionalise λ divide it by the quantity $8\mu_1 L / \pi R_0^4$. When this is done the result is

$$\bar{\lambda} = \frac{\bar{\mu}_2}{1 - (1 - \bar{\mu}_2) \alpha^4} \left\{ 1 - \frac{L_0}{L} + [1 - (1 - \bar{\mu}_2) \alpha^4] G \right\} . \quad - (3.52)$$

The integral in G of equation (3.52) has already been evaluated in equations (3.25) to (3.28). The final expression for $\bar{\lambda}$ is

$$\bar{\lambda} = \frac{\bar{\mu}_2}{1 - (1 - \bar{\mu}_2) \alpha^4} \left\{ 1 - \frac{L_0}{L} + \frac{L_0}{L} \frac{a(2a^2 + 3b^2)}{2(a^2 - b^2)^{7/2}} \right\} . \quad - (3.53)$$

When $\bar{\mu}_2 = 1$ equation (3.53) reduces to one obtained by Young (1968).

Graphs of $\bar{\lambda}$ against δ_h/R_0 for various values of L_0/L and \bar{u}_2 are given in Figure 3.4.

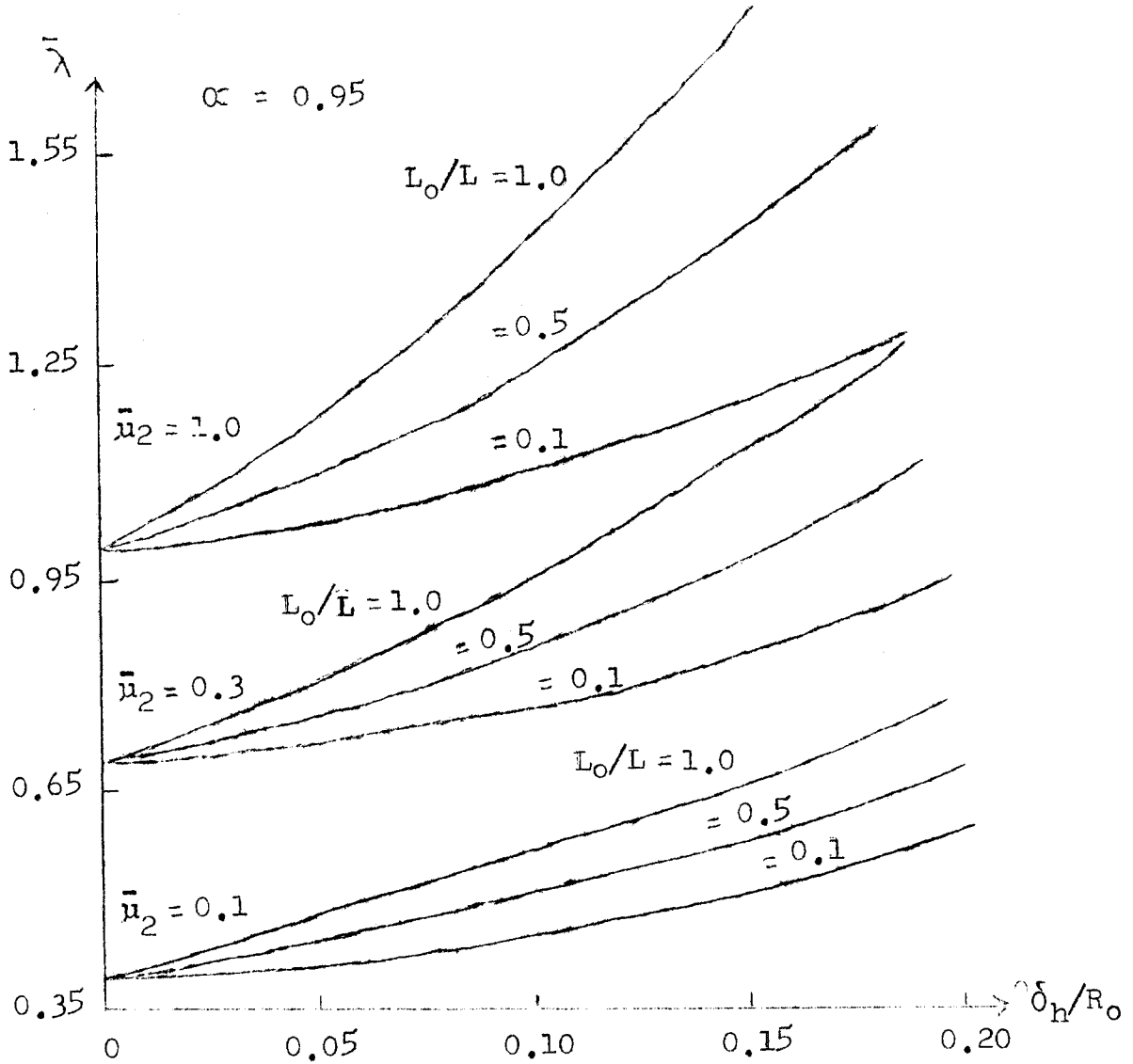


Figure 3.4: Variation of $\bar{\lambda}$ with δ_h/R_0 for different \bar{u}_2 and L_0/L .

The graph show that $\bar{\lambda}$ increases with δ_h/R_0 and L_0/L implying that resistance to flow increases with stenosis height and length. The same graph shows that $\bar{\lambda}$ decreases with \bar{u}_2 implying that resistance to flow decreases with viscosity of the peripheral layer.

3.4.2 EFFECT ON WALL SHEAR STRESS:

Substituting the expression for $I(z)$ from equation (3.48)

into equation (3.30), the result is

$$\tau_s = \left[\frac{4\mu_2 Q}{\pi R^3(z) \{1 - (1 - \bar{\mu}_2)(R_1/R)^4\}} \right]_{z=d+L_0/2} \quad - (3.54)$$

Using the relation $R_1 = \alpha R$ in equation (3.54), the shear stress τ_s becomes

$$\tau_s = \left[\frac{4\mu_2 Q}{\pi R_0^3 (R/R_0)^3 \{1 - (1 - \bar{\mu}_2)\alpha^4\}} \right]_{z=d+L_0/2} \quad - (3.55)$$

which on using equation (3.16), the final nondimensionalized expression for shear stress is given by

$$\bar{\tau}_s = \frac{\tau_s \pi R_0^3}{4\mu_1 Q} = \frac{\bar{\mu}_2}{(1 - \delta_h/R_0)^3 \{1 - (1 - \bar{\mu}_2)\alpha^4\}} \quad - (3.56)$$

When $\bar{\mu}_2 = 1$ equation (3.56) is the same as one obtained by Young (1968).

Graphs of $\bar{\tau}_s$ against δ_h/R_0 for various values of $\bar{\mu}_2$ are given by Figure 3.5.

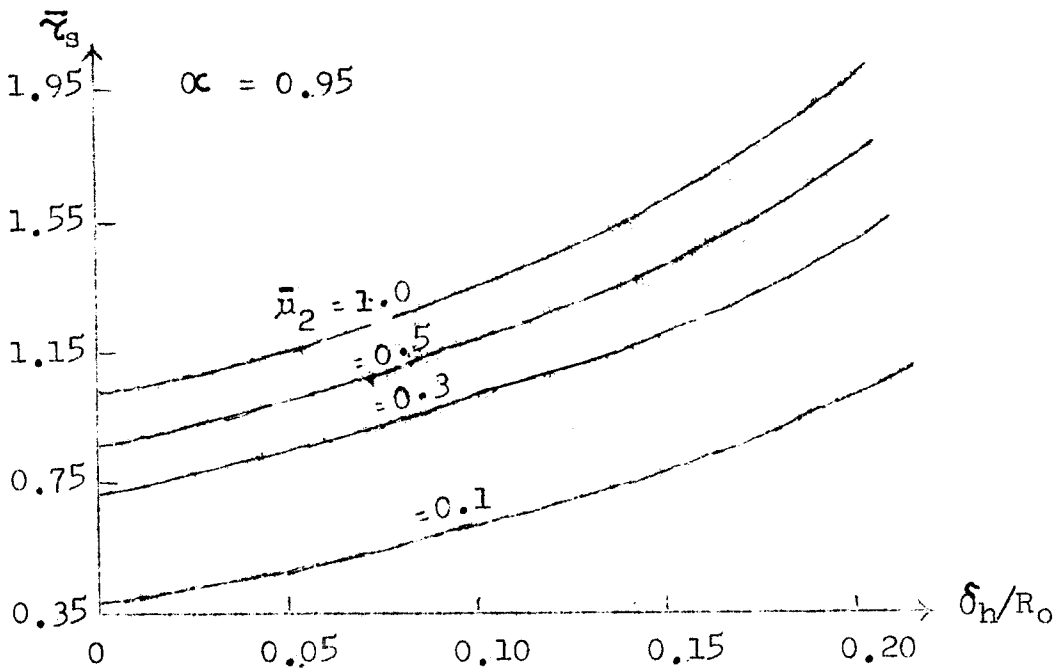


Figure 3.5: Variation of $\bar{\tau}_s$ with δ_h/R_0 for different $\bar{\mu}_2$.

The graph shows that $\bar{\tau}_s$ increases with δ_h/R_0 implying that wall shear stress increases with stenosis height and it also shows that $\bar{\tau}_s$ decreases with $\bar{\mu}_2$ implying that wall shear stress decreases with the peripheral layer viscosity.

3.5 BRANCHING EFFECTS:

Parihar (1976 and 1980) assumed the stenosis to be only in the main artery and not in the branches. Figure 3.6 is an illustration of the physical set up under investigation.

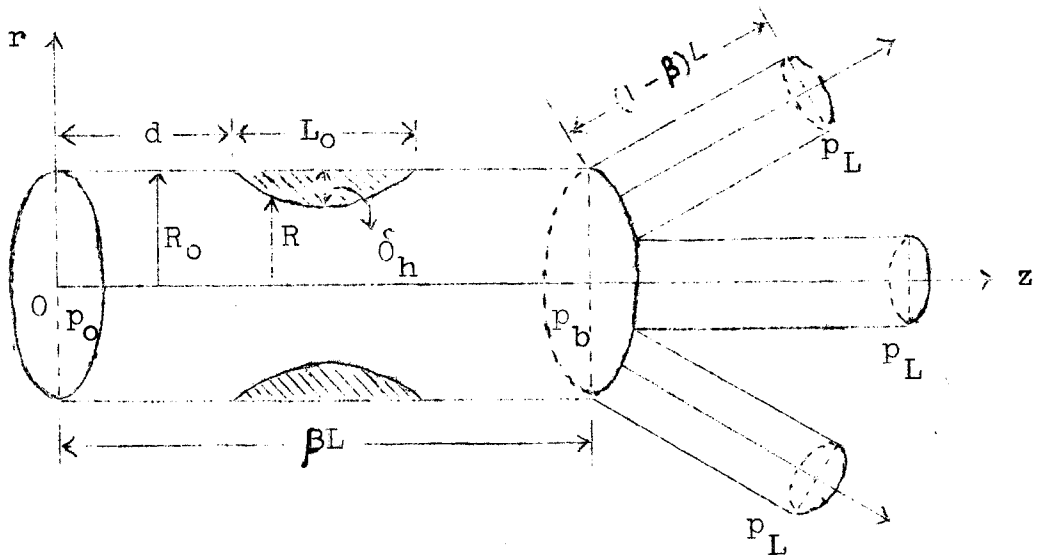


Figure 3.6: Geometry of a Branching System with stenosis, (see Parihar (1976), pp.32).

Pressure drop across the main vessel is obtained by using equation (3.9) and the result is

$$p_0 - p_b = \frac{2QF_1}{\pi R_0^4}, \quad - (3.57)$$

$$\text{where } F_1 = \int_0^{\beta L} \left(\frac{dz}{(1/R_0^4)} \right), \quad \beta \leq 1 \quad - (3.58)$$

and Q is the flux in the main artery and p_b is the pressure at $z = \beta L$. The pressure drop across the branchings is given by

$$p_b - p_L = \frac{2Q_j(1 - \beta)L}{\pi I_j}, \quad j = 1, 2, \dots, M, \quad - (3.59)$$

$$\text{where } I_j = \int_0^{R_j} \frac{r^3}{\mu(r)} dr \quad - (3.60)$$

and p_L is the outlet pressure of each branch, Q_j is flux of the j^{th} branch, R_j is the radius of the j^{th} branch and M is the total number of branches. By continuity of flux at the branching junction, we have

$$Q = \sum_{j=1}^M Q_j. \quad - (3.61)$$

Using equations (3.57), (3.59) and (3.61), an expression for the unknown pressure p_b is obtained as

$$p_b = \frac{p_o/F_1 + \frac{p_L R_o^{-4}}{(1-\beta)L} \sum_{j=1}^M (I_j)}{1/F_1 + \frac{R_o^{-4}}{(1-\beta)L} \sum_{j=1}^M (I_j)}. \quad - (3.62)$$

On substituting the expression for p_b from equation (3.62) into equation (3.57) and solving for the pressure drop $p_o - p_L$ across the system, we have

$$p_o - p_L = \frac{2QL}{\pi R_o^4} \left\{ \frac{F_1}{L} + \frac{(1 - \beta)R_o^4}{\sum_{j=1}^M (I_j)} \right\}. \quad - (3.63)$$

Resistance to flow is by equations (3.14) and (3.63) given by

$$\lambda = \frac{2L}{\pi R_o^4} \left\{ \frac{F_1}{L} + \frac{(1 - \beta)R_o^4}{\sum_{j=1}^M (I_j)} \right\} \quad - (3.64)$$

Using equations (3.58) and (3.16) in (3.64), we obtain

$$\lambda = \frac{2L}{\pi I_0} \left\{ \left(\beta - \frac{L_0}{L} \right) + \frac{1}{L} \int_0^{d+L_0} (I_0/I) dz + \frac{(1-\beta)I_0}{\sum_{j=1}^M (I_j)} \right\}. \quad - (3.65)$$

Equation (3.65) is applicable for any general viscosity function $\mu(r)$. The model considered linear variation of viscosity as given by equation (3.22). For the j^{th} branch, I_j is given by

$$I_j = \frac{R_j^4}{4\mu_0} \left(1 + \frac{4\delta}{5} \right). \quad - (3.66)$$

Using expressions for $I(z)$, I_0 and I_j from equations (3.23), (3.24) and (3.66) into equation (3.65), simplifying for $\delta \ll 1$, and nondimensionalizing the resultant expression for λ by dividing it by $\pi R_0^4 / 8\mu_0 L$, we have

$$\bar{\lambda} = \frac{\lambda \pi R_0^4}{8\mu_0 L} = \beta - \frac{L_0}{L} + \frac{L_0}{L} \frac{a(2a^2 + 3b^2)}{2(a^2 - b^2)^{7/2}} + \frac{M(1-\beta)}{(c^*)^2} - \frac{4\delta}{5} \left\{ \beta - \frac{L_0}{L} + \frac{L_0(2a^2 + b^2)}{2(a^2 - b^2)^{5/2}} + \frac{M(1-\beta)}{(c^*)^2} \right\}, \quad - (3.67)$$

where $c^* = MR_j^2/R_0^2$ is the ratio of total area of cross sections of the branches (taking all branches to be of equal radius) to the cross-sectional area of the main vessel. The average value of c^* is 1.28, McDonald (1979) and ranges over 0.75, 1.02, 1.29 in different parts of the cardiovascular system, Caro, Fitz-Gerald and Schroter (1971). When $\beta = 1$ equation (3.67) reduces to (3.29). Equation (3.67) shows that $\bar{\lambda}$, resistance to flow, decreases towards the wall of the vessel.

Graphs showing the variation of resistance to flow with δ_h/R_0 for different values of L_0/L and M are given by Figure 3.7.

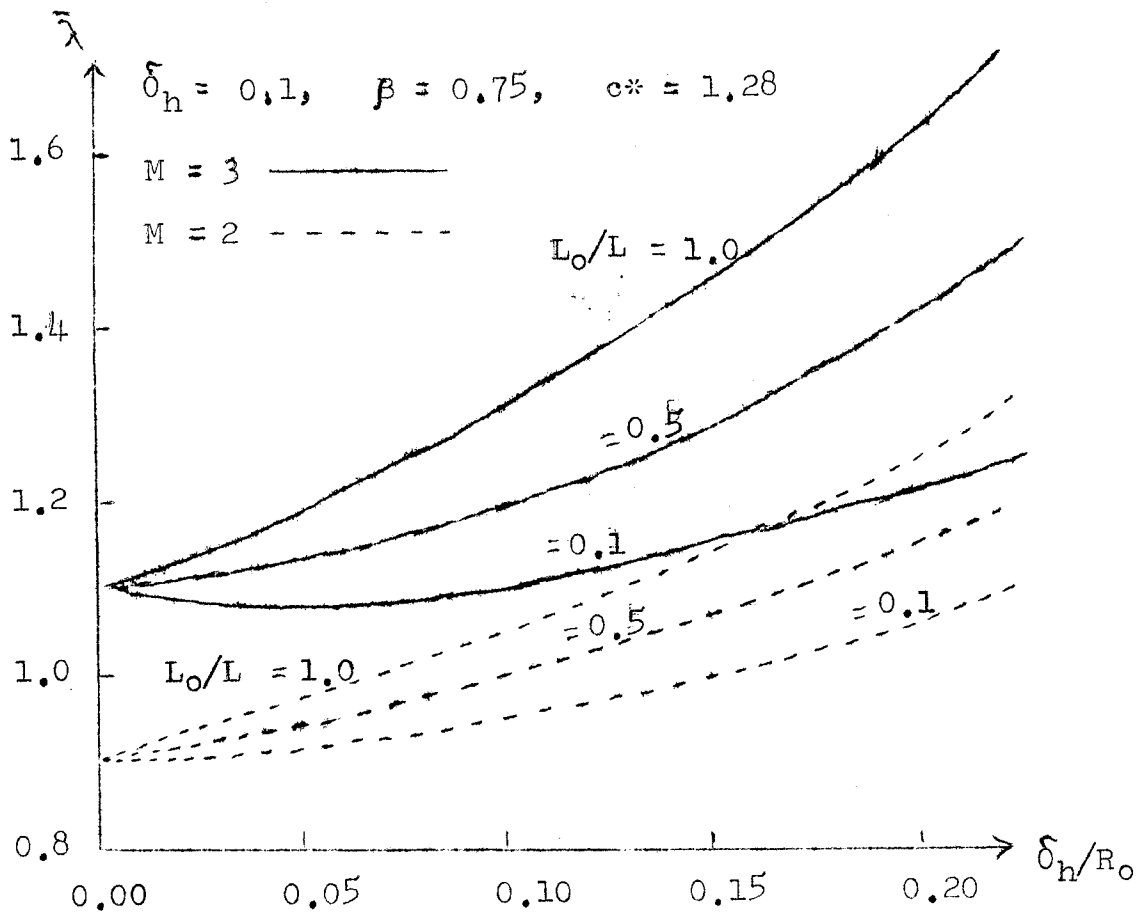


Figure 3.7: Variation of $\bar{\lambda}$ with δ_h/R_0 for different L_0/L and M .

Figure 3.7 shows that $\bar{\lambda}$, resistance to flow, increases with increasing δ_h/R_0 for fixed M . It also increases with L_0/L for a fixed M and with M for a fixed L_0/L . The general conclusion is that resistance to flow $\bar{\lambda}$ increases with the height δ_h/R_0 and length L_0/L of the stenosis and it also increases with the number of branches in the system.

3.6 CONCLUSION:

In this chapter Shukla, Parihar and Gupta investigated the effects of variation of viscosity of the blood across the artery with stenosis growth on its lining. The results have shown that resistance to flow and wall shear stress decrease as the viscosity of the blood decreases towards the wall of

the artery. The results further showed that the resistance to flow and wall shear stress increase as the size (ie. height and length) of **the** stenosis increases.

The effect of branching is to increase the resistance to flow in the system.

CHAPTER IV.

EFFECTS OF STENOSIS ON ARTERIAL BLOOD FLOW.

(NON-NEWTONIAN MODEL).

4.1 INTRODUCTION:

Chapter III is a review of literature on effects on blood flow by a stenosis growth on the lumen of an artery, considering blood as a Newtonian fluid. However, blood is a nonhomogeneous fluid and has thus no definitive rheological property. Hershey, Byrnes, Deddens and Rao (1964); Huchaba and Hahn (1968) have indicated that in tubes of diameter less than 0.2 mm, blood can be represented as a power law fluid. At low shear rates, Casson (1959); Reiner and Scott-Blair (1959); Charm and Kurland (1962 and 1965) have assumed blood to be a Casson fluid. Parihar (1976); Shukla, Parihar and Rao (1980) and Shukla, Gupta and Parihar (1980) investigated the effects on blood flow by a stenosis growth on an artery wall, considering blood under two models, the power law and Casson models. Under the two models they investigated effects of stenosis on resistance to flow and wall shear stress. This chapter is a review on the investigations by Parihar (1976); Shukla, Parihar and Rao (1980).

As in chapter III it is assumed that the stenosis is axially symmetric and its height and position depend on axial distance z . Refer to Figure 3.1, from the previous chapter, for an illustration of the physical set up. The radius of the artery over the stenosis region $R(z)$ is given by, Young (1968)

$$\frac{R(z)}{R_0} = \begin{cases} 1 - \frac{\delta_h}{2R_0} \left\{ 1 + \cos \frac{2\pi(z-d-L_0)}{2} \right\} ; & d \leq z \leq L_0 + d \\ 1 ; & \text{otherwise,} \end{cases} \quad - (4.1)$$

where L_0 is the length of stenosis and δ_h is the maximum height of the stenosis assumed to be much smaller in comparison to the radius of the artery, i.e. $\delta_h \ll R_0$.

4.2 POWER LAW MODEL.

4.2.1 EQUATIONS AND ANALYSIS:

Flow is assumed to be laminar, steady and axially symmetric. Neglecting inertial and entry region effects and also keeping in mind that $\delta_h \ll R_0$, the momentum equations for the power law model in the r and z directions are:

$$\frac{\partial p}{\partial r} = 0 \quad - (4.2)$$

and

$$\frac{dp}{dz} = \frac{1}{r} \frac{\partial(r\tau)}{\partial r}, \quad - (4.3)$$

where τ , the shear stress, is given by

$$\tau = -m \left| \frac{\partial w}{\partial r} \right|^{n-1} \left(\frac{\partial w}{\partial r} \right) \quad - (4.4)$$

in which m is the consistency and n is the flow behaviour index. For flow in a pipe, $\partial w / \partial r < 0$ and then equation (4.4) becomes

$$\tau = m \left(- \frac{\partial w}{\partial r} \right)^n. \quad - (4.5)$$

Assuming no slip velocity at the vessel walls, the boundary conditions are:

$$\frac{\partial w}{\partial r} = 0 \quad \text{at} \quad r = 0 \quad \text{and} \quad w = 0 \quad \text{at} \quad r = R(z) \quad - (4.6)$$

Equation (4.2) shows that under the given assumptions pressure p is a function of z only. Substituting the expression for τ from equation (4.5) into equation (4.3) and solving for w using boundary conditions (4.6), the result is

$$w = \left(-\frac{1}{2m} \frac{dp}{dz}\right)^{1/n} \cdot \frac{R^{1+1/n} - r^{1+1/n}}{1 + 1/n} . \quad - (4.7)$$

Volumetric flow rate Q , is given by

$$Q = 2\pi \int_0^{R(z)} r w dr , \quad - (4.8)$$

which on using equation (4.7), becomes

$$Q = \left(-\frac{1}{2m} \frac{dp}{dz}\right)^{1/n} \pi \frac{R^{3+1/n}}{3+1/n} . \quad - (4.9)$$

To show that Q is independent of z it suffices to show $dQ/dz = 0$ and can be established using the equation of continuity. From equation (4.9), we have

$$\frac{dp}{dz} = - 2m \left(\frac{3n+1}{n\pi} Q\right)^n \frac{1}{R^{3n+1}} . \quad - (4.10)$$

Integrating equation (4.10) and using the boundary conditions for pressure, given as

$$p = p_0 \text{ at } z = 0 \text{ and } p = p_L \text{ at } z = L , \quad - (4.11)$$

the result is

$$p_0 - p_L = \left(\frac{3n+1}{n\pi} Q\right)^n \frac{2mF_0}{R_0^{3n+1}} , \quad - (4.12)$$

$$\text{where } F_0 = \int_0^L \frac{dz}{(R/R_0)^{3n+1}} \quad - (4.13)$$

and R/R_0 is given by equation (4.1). Resistance to flow λ , is given by

$$\lambda = \frac{p_0 - p_L}{Q} \quad - (4.14)$$

which on using equation (4.12), gives

$$\lambda = \left(\frac{3n+1}{n\pi} Q \right)^n \frac{2mF_0}{QR_0^{3n+1}} \quad - (4.15)$$

In the case of a normal artery $\delta_h = 0$ and $R/R_0 = 1$. Then $F_0 = L$ and the resistance to flow for the normal artery, denoted by λ_N , is given by

$$\lambda_N = \left(\frac{3n+1}{n\pi} Q \right)^n \frac{2mL}{QR_0^{3n+1}} \quad - (4.16)$$

From equations (4.15) and (4.16) the ratio λ/λ_N of resistance to flow, is given by

$$\lambda/\lambda_N = F_0/L \quad - (4.17)$$

By using the expression for $R(z)/R_0$ from equation (4.1) in F_0 and using the resulting expression of F_0 in equation (4.17), the result is

$$\frac{\lambda}{\lambda_N} = 1 - \frac{L_0}{L} + \frac{L_0}{L} \left\{ \int_d^{d+L_0} \frac{dz}{\left[a - b \cos \frac{2\pi(z-d-L_0)}{L_0} \right]^{\frac{3n+1}{2}}} \right\} \quad - (4.18)$$

$$\text{where } a = 1 - \frac{\delta_h}{2R_0} \text{ and } b = \frac{\delta_h}{2R_0} \quad - (4.19)$$

To evaluate the integral in equation (4.19) the following substitution is made,

$$\phi = \pi - \frac{2\pi}{L_0}(z - d - \frac{L_0}{2}) \quad - (4.20)$$

Equation (4.18) becomes

$$\frac{\lambda}{\lambda_N} = 1 - \frac{L_0}{L} + \frac{L_0}{2\pi L} \int_0^{2\pi} \frac{d\phi}{(a + b \cos \phi)^{\frac{3n+1}{2}}} \quad - (4.21)$$

In the case when n is an integer the integral in equation (4.21) can be evaluated by the method of calculus of residues, giving the result

$$\frac{\lambda}{\lambda_N} = 1 - \frac{L_0}{L} + \frac{L_0}{(3n)! L} (b\sqrt{c^2-1})^{-3n-1} \sum_{s=0}^{3n} (-1)^s \frac{(3n)! (3n+s)! v^s}{(s!)^2 (3n-s)!}, \quad (4.22)$$

$$\text{where } c = a/b \text{ and } v = \frac{-c + (c^2 - 1)^{\frac{1}{2}}}{2(c^2 - 1)^{\frac{1}{2}}}. \quad (4.23)$$

When $n = 1$ in equation (4.22), the ratio λ/λ_N is the same as Young's (1968) result.

Figures 4.1 and 4.2 are graphs showing the variation of λ/λ_N with δ_h/R_0 . Figure 4.1 shows this variation for different values of n and Figure 4.2 shows it for different values of L_0/L .

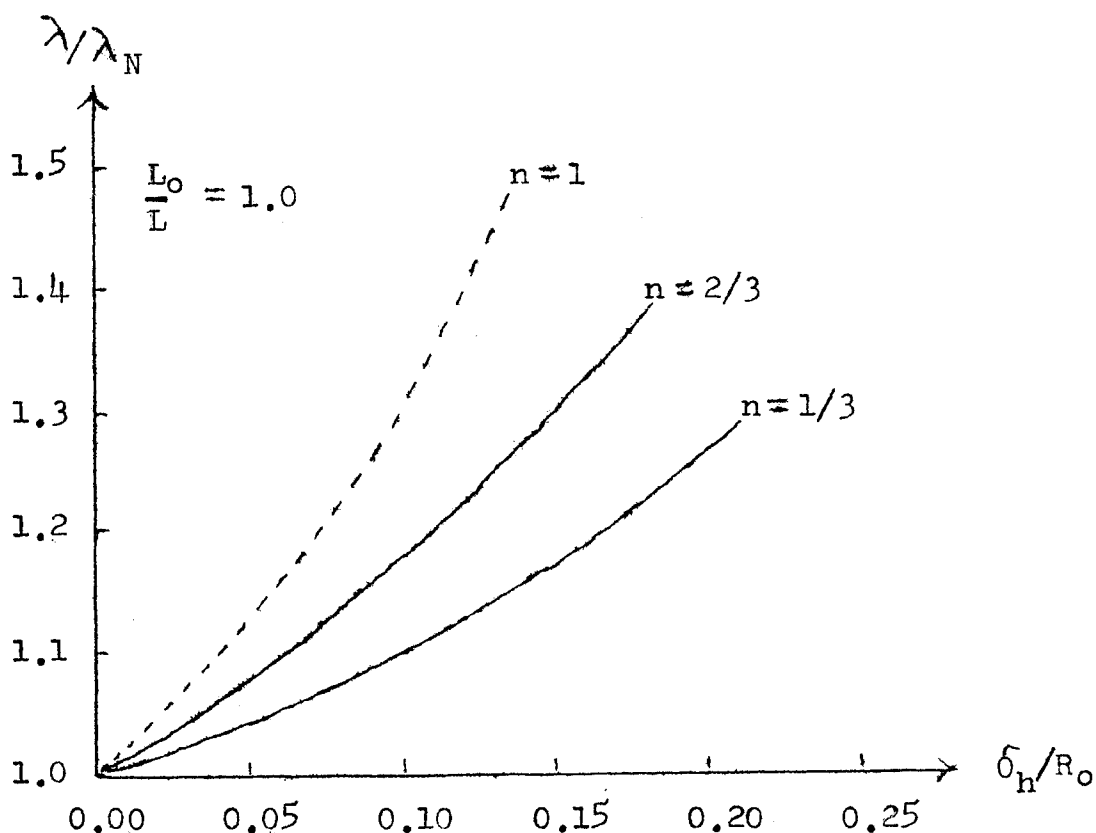


Figure 4.1: Variation of λ/λ_N with δ_h/R_0 for different n .

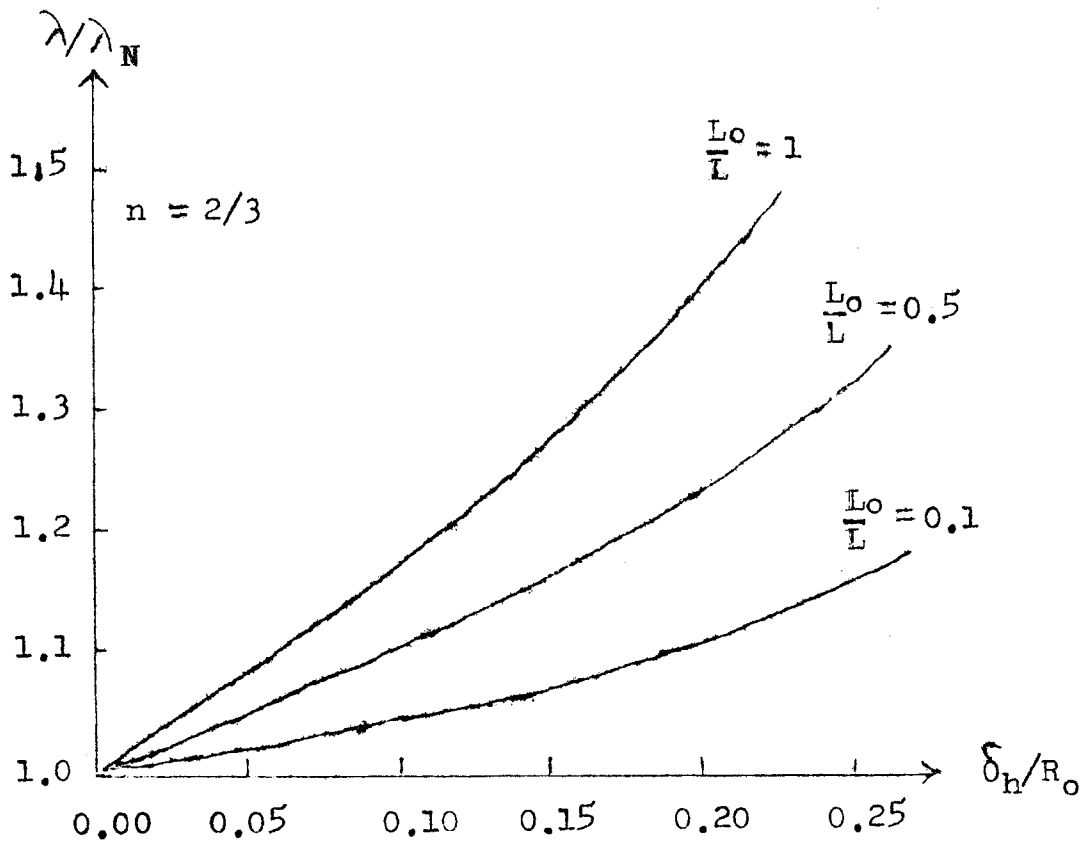


Figure 4.2: Variation of λ/λ_N with δ_h/R_0 for different L_0/L . The two graphs show that the ratio of resistances λ/λ_N increases with δ_h/R_0 . The variation of n on Figure 4.1 shows that λ/λ_N decreases with decreasing n . The variation of L_0/L on Figure 4.2 shows that the ratio λ/λ_N decreases with decreasing L_0/L . The general conclusion is that the resistance to flow increases with stenosis size (height and length) but the non-Newtonian character of blood tends to decrease this resistance.

The shearing stress, τ_s on the wall of an artery is given by

$$\tau_s = \left\{ m \left(- \frac{\partial w}{\partial r} \right)^n \right\}_{r=R} = - \frac{R}{2} \frac{dp}{dz} , \quad - (4.24)$$

which on using equation (4.10), becomes

$$\tau_s = m \left(\frac{3n+1}{n\pi} Q \right)^n \frac{1}{R^{3n}} . \quad - (4.25)$$

The ratio between the wall shears for the stenosis artery and normal artery τ_N , is given by

$$\tau_s / \tau_N = (R_0/R)^{3n} . \quad - (4.26)$$

Using the expression for R/R_0 from equation (4.1) in equation (4.26), we have

$$\tau_s / \tau_N = \left[1 - \frac{\delta_h}{2R_0} \left\{ 1 + \cos \frac{2\pi(z-d-L_0)}{L_0} \right\} \right]^{-3n} . \quad - (4.27)$$

At maximum stenosis height, $z = d + L_0/2$ and the ratio τ_s / τ_N , becomes

$$\tau_s / \tau_N = \left(1 - \frac{\delta_h}{R_0} \right)^{-3n} . \quad - (4.28)$$

When $n = 1$ (Newtonian) the result is the same as that obtained by Young (1968). The variation of the ratio τ_s / τ_N with δ_h / R_0 is plotted on Figure 4.3 for values of n decreasing from unity.

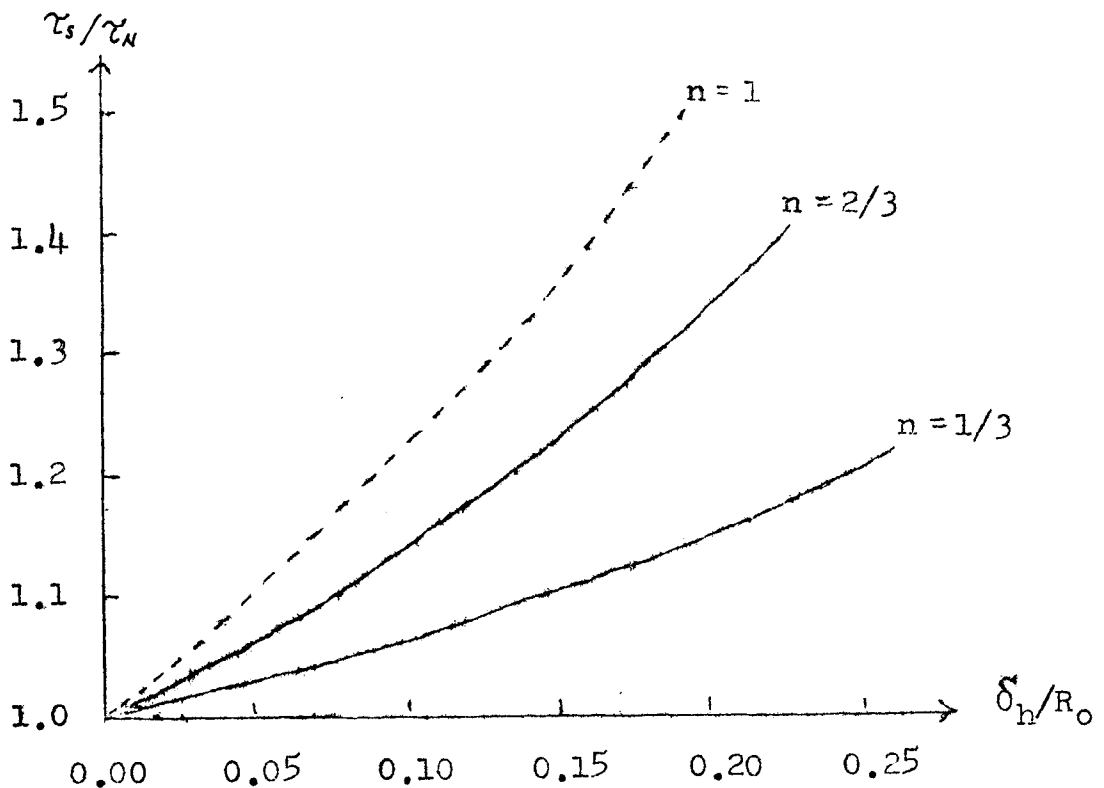


Figure 4.3: Variation of τ_s / τ_N with δ_h / R_0 for different n .

The graph shows that the ratio γ_s/γ_N increases from unity with δ_n/R_0 increasing from zero and that γ_s/γ_N decreases with n decreasing from unity. The general conclusion is that wall shear increases with stenosis size but the non-Newtonian character of blood has the effect of reducing wall shear,

4.2.2 EFFECTS OF BRANCHING:

It is assumed that stenosis only develops in the main artery and the branches are normal, Parihar (1976). For a diagrammatic illustration of the physical set up refer to Figure 3.6 from the previous chapter. Pressure drop across the main vessel is obtained by using equation (4.10) and the result is

$$p_0 - p_b = \frac{2mI(z)}{R_0^{3n+1}} \left(\frac{3n+1}{n\kappa} Q \right)^n, \quad - (4.29)$$

$$\text{where } I(z) = \int_0^{\beta L} \frac{dz}{(R/R_0)^{3n+1}}, \quad \beta \leq 1 \quad - (4.30)$$

and Q is the flux in the main artery and p_b is the pressure at $z = \beta L$. The pressure drop across the branchings is given by

$$p_b - p_L = \frac{2m(1-\beta)L}{R_j^{3n+1}} \left(\frac{3n+1}{n\kappa} Q_j \right)^n, \quad j = 1, 2, \dots, M \quad - (4.31)$$

where p_L is the outlet pressure of each branch, Q_j is the flux of the j^{th} branch, R_j is the constant radius of the j^{th} branch and M is the number of branches. By continuity of flux at the branching junction, we have

$$Q = \sum_{j=1}^M Q_j. \quad - (4.32)$$

On using equations (4.29), (4.31) and (4.32) an expression for the unknown pressure p_b is obtained as

$$p_b = \frac{\frac{p_o}{\beta} + \frac{p_L}{(1-\beta)} \left(\sum_{j=1}^M \alpha_j^{3+1/n} \right)^n}{\frac{1}{\beta} + \frac{1}{(1-\beta)} \left(\sum_{j=1}^M \alpha_j^{3+1/n} \right)^n}, \quad - (4.33)$$

where $\alpha_j = R_j/R_o$. On substituting the expression for p_b from equation (4.33) into equation (4.29) and solving for the pressure drop $p_o - p_L$ across the system, we have

$$p_o - p_L = \frac{2mL}{R_o^{3n+1}} \left(\frac{3n+1}{n\pi} Q \right)^n \left\{ \frac{I(z)}{L} + \frac{1-\beta}{\left(\sum_{j=1}^M \alpha_j^{3+1/n} \right)^n} \right\}, \quad - (4.34)$$

where $I(z)$ is still to be determined. It can be determined by using the expression for R/R_o from equation (4.1) in equation (4.30), Parihar (1976), the result is

$$\frac{I(z)}{L} = \beta - \frac{L_o}{L} + \frac{L_o}{L} \frac{1}{(3n)!} (b\sqrt{c^2-1})^{-3n-1} \sum_{s=0}^{3n} (-1)^s \frac{(3n)! (3n+s)!}{(s!)^2 (3n-s)!} \nu^s \quad - (4.35)$$

The resistance to flow across the system is thus given by

$$\lambda = \frac{2mL}{QR_o^{3n+1}} \left(\frac{3n+1}{n\pi} Q \right)^n \left\{ \frac{I}{L} + \frac{1-\beta}{\left(\sum_{j=1}^M \alpha_j^{3+1/n} \right)^n} \right\}. \quad - (4.36)$$

From equations (4.17) and (4.36) the ratio λ/λ_N is given by

$$\lambda/\lambda_N = \frac{I}{L} + \frac{1-\beta}{\left(\sum_{j=1}^M \alpha_j^{3+1/n} \right)^n}. \quad - (4.37)$$

When all the branches have the same radii ($\alpha_j = \alpha$), letting $M\alpha^2 = c^*$, McDonald (1979) and using the expression for $I(z)/L$ from equation (4.35), the ratio λ/λ_N becomes

$$\frac{\lambda}{\lambda_N} = \beta - \frac{L_o}{L} + \frac{L_o}{L} \frac{1}{(3n)!} (b\sqrt{c^2-1})^{-3n-1} \sum_{s=0}^{3n} (-1)^s \frac{(3n)! (3n+s)!}{(s!)^2 (3n-s)!} \nu^s + \frac{M^{(n+1)/2} (1-\beta)}{(c^*)^{(3n+1)/2}} \quad - (4.38)$$

When $\beta = 1$ equation (4.38) reduces to equation (4.22). The variation of λ/λ_N with δ_h/R_0 for different values of L_0/L is illustrated by Figure 4.4.

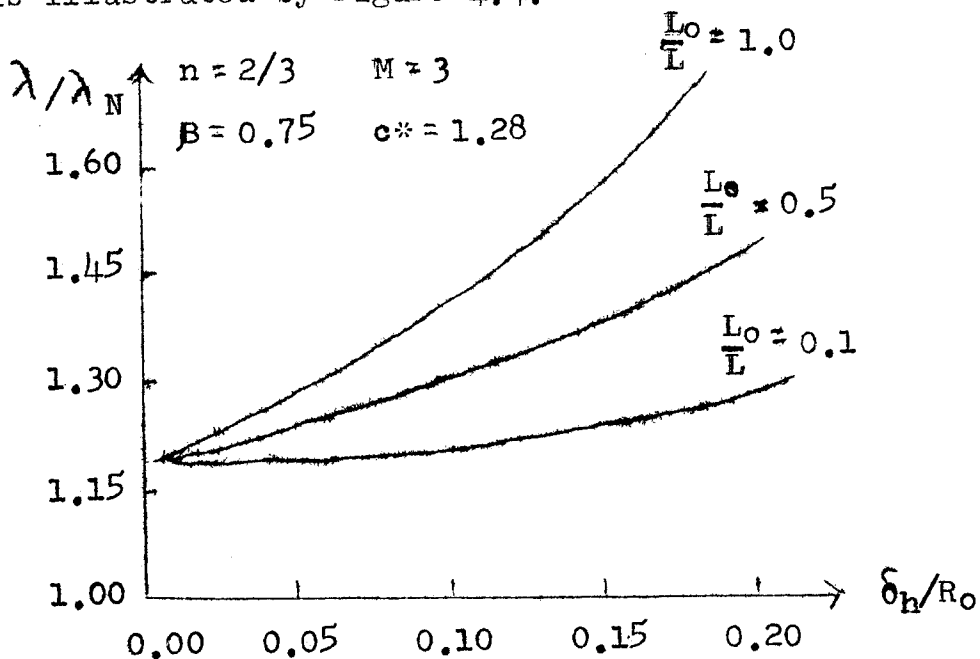


Figure 4.4: Variation of λ/λ_N with δ_h/R_0 for different L_0/L . The graph in Figure 4.4 shows that the ratio λ/λ_N increases with δ_h/R_0 and variations of L_0/L show that it also increases with L_0/L . The conclusion is that resistance to flow increases with stenosis height and length.

4.3 CASSON MODEL.

4.3.1 EQUATIONS AND ANALYSIS:

An illustration of the physical set up is shown in Figure 4.5.

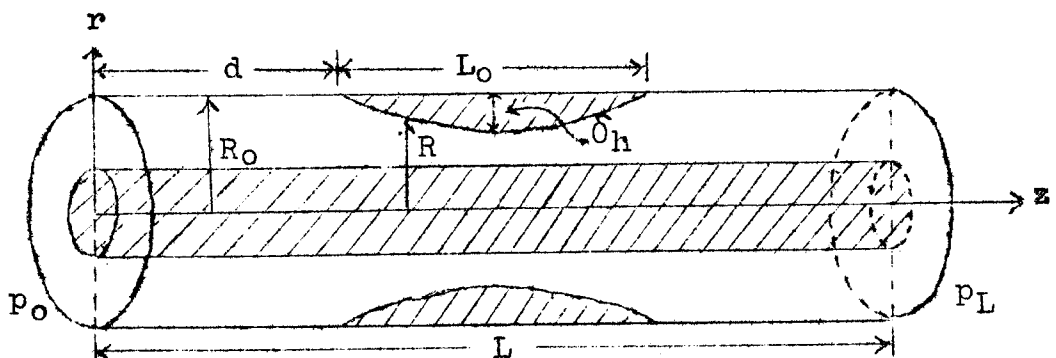


Figure 4.5: Geometry of tube with stenosis (Shukla, Parihar and Rao (1980))

Flow is assumed to be steady, laminar and axially symmetrical. Blood is assumed to be an incompressible Casson fluid. The stress-strain relation for the Casson fluid model is given by

$$\left. \begin{aligned} \left(-\frac{\partial w}{\partial r}\right)^{\frac{1}{2}} &= \frac{\tau^{\frac{1}{2}} - \tau_0^{\frac{1}{2}}}{\eta} ; \tau \geq \tau_0 \\ -\frac{\partial w}{\partial r} &= 0 ; \tau \leq \tau_0 \end{aligned} \right\} \quad - (4.39)$$

where τ_0 is the yield stress and η is the consistency of the fluid. The shear stress τ can be given in terms of pressure gradient as follows, Bird, Steward and Lightfoot (1960),

$$\tau = -\frac{r}{2} \frac{dp}{dz} . \quad - (4.40)$$

Denote the shear stress at the wall of the vessel by τ_w so that at the wall of the vessel equation (4.40) becomes

$$\tau_w = -\frac{R}{2} \frac{dp}{dz} . \quad - (4.41)$$

Eliminating dp/dz from equations (4.40) and (4.41), the result is

$$r = \frac{\tau}{\tau_w} R . \quad - (4.42)$$

Volumetric flow rate Q is given by

$$Q = 2\pi \int_0^{R(z)} r w dr \quad - (4.43)$$

which on integrating by parts and using boundary conditions $w = 0$ at $r = R(z)$, gives the result

$$Q = \pi \int_0^{R(z)} r^2 \left(-\frac{\partial w}{\partial r}\right) dr . \quad - (4.44)$$

Substituting the expression for $(-\partial w/\partial r)$ from equation (4.39)

into (4.44) and using equation (4.42) to express r in terms of γ , equation (4.44) becomes

$$Q = \frac{\pi R^3}{4\eta^2 \tau_w \tau_o} \int_{\tau_o}^{\tau_w} \gamma^2 (\gamma^{1/2} - \tau_o^{1/2})^2 d\gamma$$

$$= \frac{\pi R^3}{4\eta^2} \tau_w \left\{ 1 + \frac{4}{3} \left(\frac{\tau_o}{\tau_w} \right) - \frac{16}{7} \left(\frac{\tau_o}{\tau_w} \right)^{\frac{1}{2}} - \frac{1}{21} \left(\frac{\tau_o}{\tau_w} \right)^4 \right\} . \quad - (4.45)$$

For $\tau_o/\tau_w \ll 1$, the term $-\frac{1}{21} \left(\frac{\tau_o}{\tau_w} \right)^4$ in equation (4.45) can be neglected and the equation approximated to

$$Q \approx \frac{\pi R^3}{4\eta^2} \tau_w \left[1 - \frac{8}{7} \left(\frac{\tau_o}{\tau_w} \right)^{\frac{1}{2}} \right]^2 . \quad - (4.46)$$

Eliminating τ_w from equation (4.46) by means of equation (4.41), the result is

$$Q = \frac{\pi R^3}{4\eta^2} \left[\left(-\frac{R}{2} \frac{dp}{dz} \right)^{\frac{1}{2}} - \frac{8}{7} \tau_o^{\frac{1}{2}} \right]^2 . \quad - (4.47)$$

Solving for the pressure drop dp/dz from equation (4.47), the result is

$$\frac{dp}{dz} = -\frac{2}{R} \left[\frac{8}{7} \tau_o^{\frac{1}{2}} + 2\eta \left(\frac{Q}{R^3} \right)^{\frac{1}{2}} \right]^2 . \quad - (4.48)$$

Q is independent of z as can be shown by using the equation of continuity. Integrating equation (4.48) using pressure boundary conditions (4.11), the result is

$$p_o - p_L = \frac{128}{49R_o} \tau_o G_1 + \frac{8\eta^2 Q}{\pi R_o^4} G_2 + \frac{64\eta}{7R_o^{5/2}} \left(\frac{Q}{\pi} \right)^{\frac{1}{2}} G_3 , \quad - (4.49)$$

$$\text{where } G_1 = \int_0^L \frac{dz}{R/R_o} , \quad - (4.50 a)$$

$$G_2 = \int_0^L \frac{dz}{(R/R_0)^4} , \quad - (4.50 \text{ b})$$

$$G_3 = \int_0^L \frac{dz}{(R/R_0)^{5/2}} . \quad - (4.50 \text{ c})$$

Resistance to flow λ is given by

$$\lambda = \frac{p_0 - p_L}{Q} = fG_1 + gG_2 + hG_3 \quad - (4.51)$$

$$\text{where } f = \frac{128 \tau_0}{49QR_0} , \quad - (4.52 \text{ a})$$

$$g = \frac{8\eta^2}{\pi R_0^4} \quad - (4.52 \text{ b})$$

$$\text{and } h = \frac{64\eta \tau_0^{\frac{1}{2}}}{7(\pi R_0^5 Q)^{\frac{1}{2}}} . \quad - (4.52 \text{ c})$$

For the case without stenosis $\delta_h = 0$, $R = R_0$ and $G_1 = G_2 = G_3 = L$. Then resistance to flow for the case without stenosis, is given by

$$\lambda_N = (f + g + h)L . \quad - (4.53)$$

The ratio λ / λ_N is from equations (4.51) and (4.53) given by

$$\frac{\lambda}{\lambda_N} = \frac{fG_1 + gG_2 + hG_3}{(f + g + h)L} . \quad - (4.54)$$

Substituting the expression for R/R_0 from equation (4.1) into equations (4.50 a, b, c); the equations become

$$G_1 = L - L_0 + \int_d^{d+L_0} \frac{dz}{\left[a - b \cos \frac{2\pi(z-d-L_0)}{L_0} \right]} , \quad - (4.55 \text{ a})$$

$$G_2 = L - L_0 + \int_d^{d+L_0} \frac{dz}{\left[a - b \cos \frac{2\pi(z-d-L_0)}{L_0} \right]^4} , \quad - (4.55 \text{ b})$$

$$G_3 = L - L_0 + \int_d^{d+L_0} \frac{dz}{\left[a - b \cos \frac{2\pi(z-d-L_0)}{L_0} \right]^{5/2}} . \quad - (4.55 \text{ c})$$

Making the substitution given by equation (4.20) in the integrals of equations (4.55 a, b, c), the results are:

$$G_1 = L - L_0 + \frac{L_0}{2\pi_0} \int_0^{2\pi} \frac{d\phi}{a + b \cos \phi} , \quad - (4.56 \text{ a})$$

$$G_2 = L - L_0 + \frac{L_0}{2\pi_0} \int_0^{2\pi} \frac{d\phi}{(a + b \cos \phi)^4} , \quad - (4.56 \text{ b})$$

$$G_3 = L - L_0 + \frac{L_0}{2\pi_0} \int_0^{2\pi} \frac{d\phi}{(a + b \cos \phi)^{5/2}} . \quad - (4.56 \text{ c})$$

The integrals of equations (4.56 a, b) can now be evaluated by the method of calculus of residues, giving the result

$$G_1 = L - L_0 + \frac{L_0}{(a^2 - b^2)^{\frac{1}{2}}} , \quad - (4.57 \text{ a})$$

$$G_2 = L - L_0 + \frac{L_0 a(2a^2 + 3b^2)}{2(a^2 - b^2)^{7/2}} . \quad - (4.57 \text{ b})$$

To evaluate the integral in equation (4.56 c), make a binomial expansion of the integrand up to terms of $O(b/a)^4$. It is assumed that $b/a < 1$. Equation (4.56 c) becomes

$$G_3 = L - L_0 + \frac{L_0}{2\pi_0 a^{5/2}} \left\{ \int_0^{2\pi} \left(1 - \frac{5}{2}(b/a) \cos \phi + \frac{35}{8}(b/a)^2 \cos^2 \phi - \frac{315}{48}(b/a)^3 \cos^3 \phi + \frac{3465}{192}(b/a)^4 \cos^4 \phi + O(b/a)^5 \right) d\phi \right\}$$

$$= L - L_0 + \frac{L_0}{a^{5/2}} \left\{ 1 + \frac{35}{16} (b/a)^2 + \frac{3465}{1024} (b/a)^4 \right\} \quad - (4.57 \text{ c})$$

From equations (4.54), (4.57 a, b, c) the ratio λ/λ_N becomes

$$\frac{\lambda}{\lambda_N} = 1 - \frac{L_0}{L} + \frac{L_0}{L} \left\{ \frac{fG_1' + gG_2' + hG_3'}{f + g + h} \right\} \quad - (4.58)$$

$$\text{where } G_1' = \frac{1}{(a^2 - b^2)^{1/2}}, \quad - (4.59 \text{ a})$$

$$G_2' = \frac{a(2a^2 + 3b^2)}{2(a^2 - b^2)^{7/2}} \quad - (4.59 \text{ b})$$

$$\text{and } G_3' = \frac{1}{a^{5/2}} \left\{ 1 + \frac{35}{16} (b/a)^2 + \frac{3465}{1024} (b/a)^4 \right\}, \quad - (4.59 \text{ c})$$

when $\tau_0 = 0$ (Newtonian case), $f = h = 0$ and Parihar's result for the ratio λ/λ_N is the same as that obtained by Young (1968)

Figure 4.6 and 4.7 are graphs showing the variations of λ/λ_N with δ_h/R_0 . Figure 4.6 shows this variation for different values of τ_0 and Figure 4.7 shows the variation for different values of L_0/L .

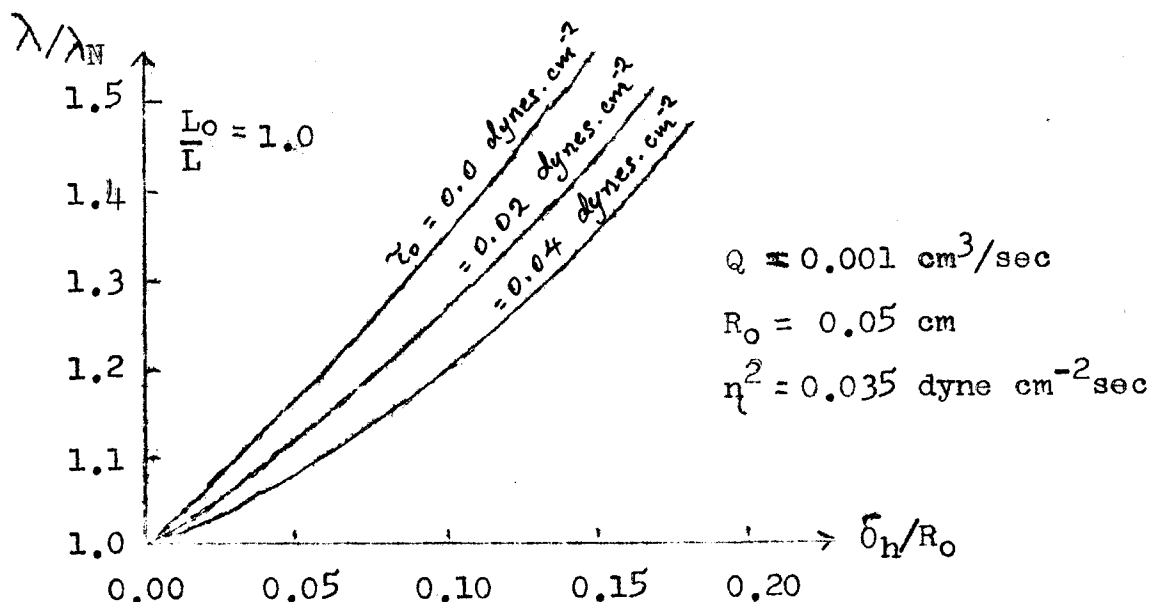


Figure 4.6: Variation of λ/λ_N with δ_h/R_0 for different τ_0 .

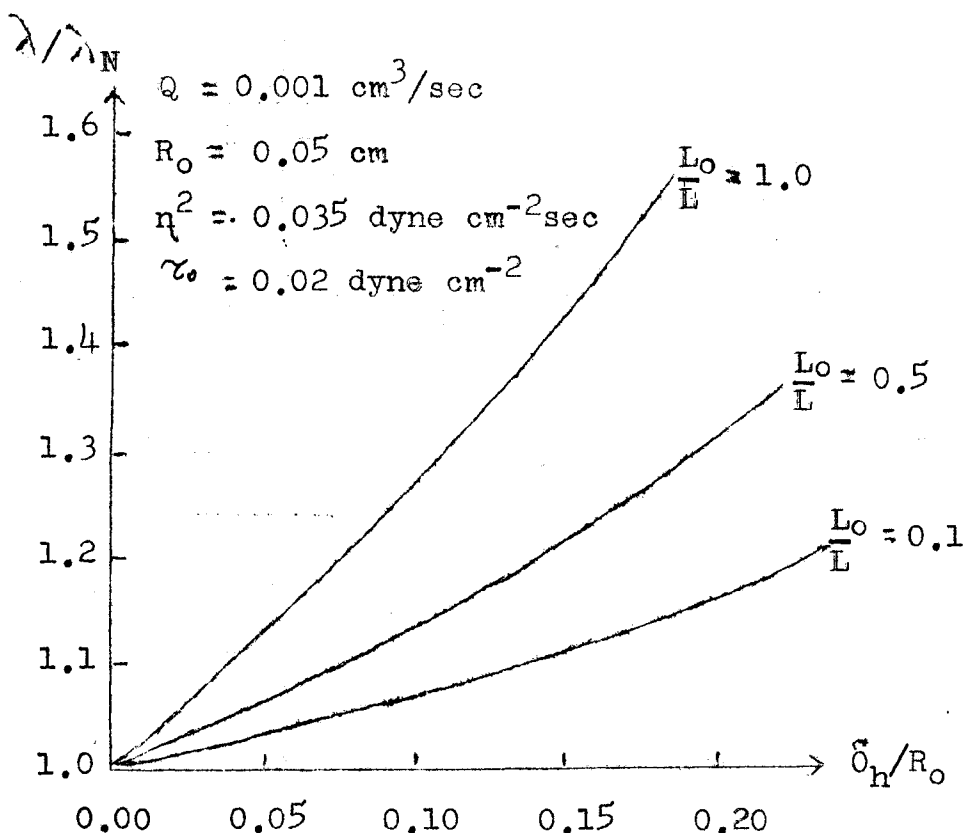


Figure 4.7: Variation of λ/λ_N with δ_h/R_0 for different L_0/L . The two graphs show that resistance to flow ratio λ/λ_N increases with δ_h/R_0 . The variations of τ_0 on Figure 4.6 shows that the ratio λ/λ_N decreases as τ_0 increases. Variations of L_0/L on Figure 4.7 shows that λ/λ_N increases with L_0/L . The general conclusion is that resistance to flow in the artery increases with increase in stenosis size, i.e. height and length and that the non-Newtonian character of blood (possession of yield stress) has the effect of reducing this resistance.

To determine the effects of stenosis on shearing stress he eliminated dp/dz from equations (4.41) and (4.48) and solved for τ_w , obtaining the result

$$\tau_w = \frac{QR_0}{2} \left[f + g(R_0/R)^3 + h(R_0/R)^{3/2} \right]. \quad (4.60)$$

In the case of no stenosis, $\delta_h = 0$ and $R = R_0$, the wall shear stress τ_N , is given by

$$\tau_N = \frac{QR_0}{2} [f + g + h] . \quad - (4.61)$$

The wall shear stress at maximum height of stenosis, (ie. at $z = d + L_0/2$), denoted by τ_s , is given by

$$\tau_s = \frac{QR_0}{2} \left\{ f + g \left(1 - \frac{\delta_h}{R_0}\right)^{-3} + h \left(1 - \frac{\delta_h}{R_0}\right)^{-3/2} \right\} . \quad - (4.62)$$

The ratio τ_s/τ_N can be determined by dividing equation (4.62) by equation (4.61) giving the result

$$\frac{\tau_s}{\tau_N} = \frac{1}{(f + g + h)} \left\{ f + g \left(1 - \frac{\delta_h}{R_0}\right)^{-3} + h \left(1 - \frac{\delta_h}{R_0}\right)^{-3/2} \right\} . \quad - (4.63)$$

Figure 4.8 is a graph showing the variation of τ_s/τ_N with δ_h/R_0 for different values of τ_0 .

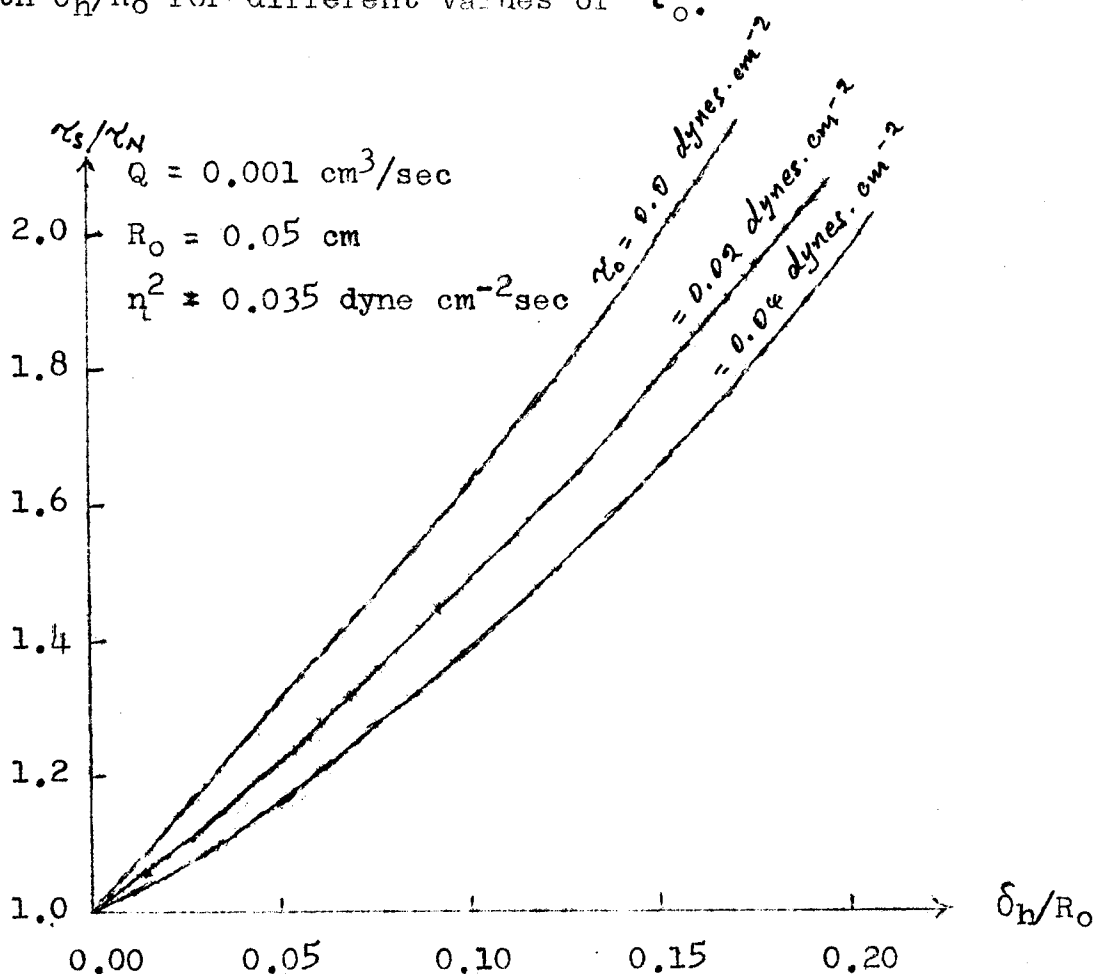


Figure 4.8: Variations of τ_s/τ_N with δ_h/R_0 for different τ_0 (Shukla, Parihar and Rao (1980)).

The graph shows that the shear stress ratio τ_s/τ_N increases with increase in δ_h/R_o . The graph shows that the ratio τ_s/τ_N decreases with increasing τ_o . It is thus concluded that shear stress increases with the height of stenosis.

4.3.2 EFFECTS OF BRANCHING:

As in section 4.2.2 it is assumed that only the main artery has stenosis growth and the branches have no stenosis. Figure 4.9 illustrates the physical set up.

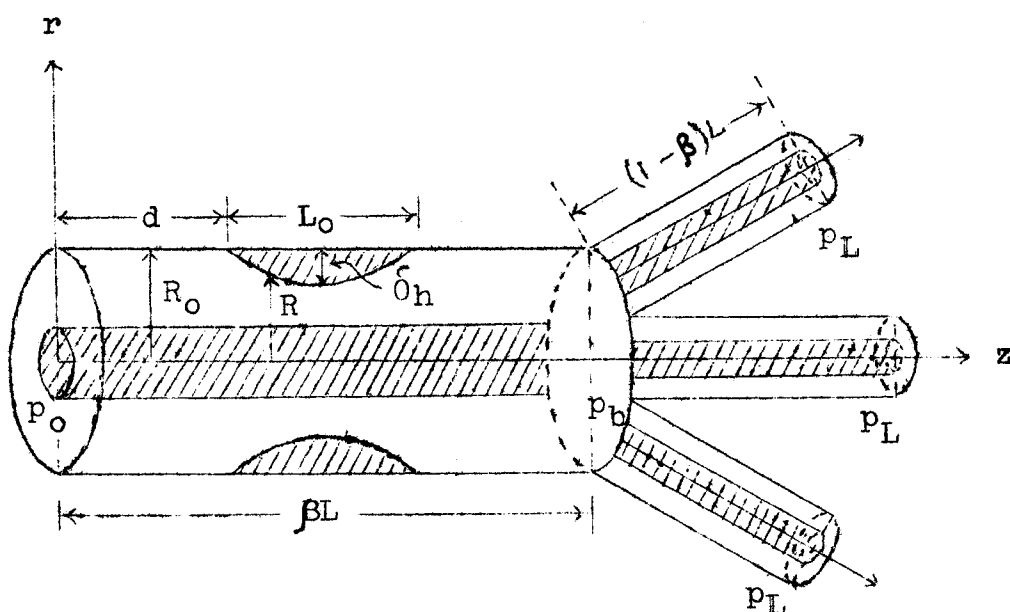


Figure 4.9: Geometry of a Branching System with stenosis.

As previously done Parihar determined the pressure drop across the main artery $p_o - p_b$ by using equation (4.48) and obtained

$$p_o - p_b = Q \{ f I_1 + g I_2 + h I_3 \} , \quad - (4.64)$$

$$\text{where } I_1 = \int_0^{\beta L} \frac{dz}{R/R_o} , \quad - (4.65 \text{ a})$$

$$I_2 = \int_0^{\beta L} \frac{dz}{(R/R_0)^4} , \quad - (4.65 \text{ b})$$

$$I_3 = \int_0^{\beta L} \frac{dz}{(R/R_0)^{5/2}} \quad - (4.65 \text{ c})$$

and Q is the flux in the main artery. On substituting the expressions for f , g and h from equations (4.52 a, b, c), using equation (4.1) in I_1 , I_2 and I_3 and integrating, the result is

$$\frac{p_o - p_b}{\beta L} = \frac{128 \tau_o a_1}{49 R_o} + \frac{8 \eta^2 Q b_1}{\pi R_o^4} + \frac{64 \eta c_1}{7 R_o^2} \left(\frac{\tau_o Q}{\pi R_o} \right)^{\frac{1}{2}} , \quad - (4.66)$$

$$\text{where } a_1 = \frac{I_1}{\beta L} = 1 - \frac{L_o}{\beta L} + \frac{L_o}{\beta L} \frac{1}{(a^2 - b^2)^{\frac{1}{2}}} , \quad - (4.67 \text{ a})$$

$$b_1 = \frac{I_2}{\beta L} = 1 - \frac{L_o}{\beta L} + \frac{L_o}{\beta L} \frac{a(2a^2 + 3b^2)}{2(a^2 - b^2)^{7/2}} \quad - (4.67 \text{ b})$$

$$\text{and } c_1 = \frac{I_3}{\beta L} = 1 - \frac{L_o}{\beta L} + \frac{L_o}{\beta L} \frac{1}{a^{5/2}} \left(1 + \frac{35}{16} (b/a)^2 + \frac{3465}{1024} (b/a)^4 \right) \quad - (4.67 \text{ c})$$

For the case without stenosis, $I_1 = I_2 = I_3 = \beta L$. Since the branches are assumed to be free of stenosis, the pressure drop across each branch having the same length $(1 - \beta)L$, is given by

$$p_b - p_L = (1 - \beta)L \left\{ \frac{128 \tau_o}{49 R_j} + \frac{8 \eta^2 Q_j}{\pi R_j^4} + \frac{64 \eta}{7 R_j^2} \left(\frac{\tau_o Q_j}{\pi R_j} \right)^{\frac{1}{2}} \right\} \quad - (4.68)$$

Solving equations (4.66) and (4.68) for Q_j and Q , the result is

$$Q = \frac{\pi R_o^4}{8 \eta^2 b_1} \left\{ \frac{128 \tau_o}{49 R_o} \left(\frac{2c_1^2 - a_1 b_1}{b_1} \right) + \frac{p_o - p_b}{\beta L} - \frac{16 c_1}{7} \left(\frac{2 \tau_o}{b_1 R_o} \right)^{\frac{1}{2}} \left[\frac{128 \tau_o}{49 R_o b_1} (2c_1^2 - a_1 b_1) + \frac{p_o - p_b}{\beta L} \right]^{\frac{1}{2}} \right\} , \quad - (4.69)$$

$$Q_j = \frac{\pi R_j^4}{8\eta^2} \left\{ \frac{128 \tau_0}{49R_j} + \frac{p_b - p_L}{(1 - \beta)L} - \frac{16}{7} \left[\frac{2\tau_0}{R_j} \frac{p_b - p_L}{(1 - \beta)L} \right]^{\frac{1}{2}} \right\} \quad (4.70)$$

Substituting expressions for Q and Q_j from equations (4.69) and (4.70) into the flux continuity equation (4.32), the result is

$$\begin{aligned} X = & \frac{H}{b_1 \beta \sum_{j=1}^M \alpha_j^4} + \frac{128 \tau_0 L}{49R_0} \cdot \frac{1}{\sum_{j=1}^M \alpha_j^4} \left\{ \frac{2c_1^2 - a_1 b_1}{b_1^2} - \sum_{j=1}^M \alpha_j^3 \right\} \\ & - \frac{16}{7} \left(\frac{2\tau_0 L}{R_0} \right)^{\frac{1}{2}} \cdot \frac{c_1}{b_1 \sum_{j=1}^M \alpha_j^4} \left[\frac{128 \tau_0 L}{49R_0} \cdot \frac{c_1^2 - a_1 b_1}{b_1^2} + \frac{H}{b_1 \beta} \right]^{\frac{1}{2}} \\ & + \frac{16}{7} \left(\frac{2\tau_0 L}{R_0} \right)^{\frac{1}{2}} \cdot \frac{X^{\frac{1}{2}}}{\sum_{j=1}^M \alpha_j^4} \cdot \sum_{j=1}^M \alpha_j^{7/2} \end{aligned} \quad (4.71)$$

$$\text{where } X = \frac{p_0 - p_L - H}{1 - \beta}, \quad (4.72)$$

$$\text{and } H = \beta L \left\{ \frac{128 \tau_0 a_1}{49R_0} + \frac{8\eta^2 b_1 Q}{\pi R_0^4} + \frac{64\eta}{7R_0^2} \left(\frac{\tau_0}{\pi R_0} \right)^{\frac{1}{2}} c_1 Q^{\frac{1}{2}} \right\}. \quad (4.73)$$

Equation (4.71) is a quadratic in X , from which the pressure drop $p_0 - p_L$ can be determined. If all the branches are of equal radius, pressure drop is given by

$$\begin{aligned} p_0 - p_L = & \frac{2(1 - \beta)L}{R_0} \left[\frac{8}{7} \left(\frac{\tau_0}{\alpha} \right)^{\frac{1}{2}} + \frac{2\eta}{\alpha^2} \left(\frac{Q}{M\pi R_0^3} \right)^{\frac{1}{2}} \right]^2 \\ & + \beta L Q (a_1 f + b_1 g + c_1 h). \end{aligned} \quad (4.74)$$

Resistance to flow is given by

$$\begin{aligned} \lambda = \frac{p_0 - p_L}{Q} = & \frac{2(1 - \beta)L}{R_0} \left[\frac{8}{7} \left(\frac{\tau_0}{Q\alpha} \right)^{\frac{1}{2}} + \frac{2\eta}{\alpha^2} \left(\frac{1}{M\pi R_0^3} \right)^{\frac{1}{2}} \right]^2 \\ & \beta L (a_1 f + b_1 g + c_1 h). \end{aligned} \quad (4.75)$$

The resistance to flow for a normal artery, denoted by λ_N , is given by

$$\lambda_N = L(f + g + h), \quad - (4.76)$$

since $\delta_h = 0$, $a_1 = b_1 = c_1 = 1$ and $\beta = 1$. Thus the ratio λ/λ_N , from equations (4.75) and (4.76) and using $MO^2 = c^*$, is given by

$$\frac{\lambda}{\lambda_N} = \frac{\frac{2(1-\beta)}{R_0} \left[\frac{8(M}{7c^*})^{\frac{1}{4}} \left(\frac{-Q}{\pi R_0^3}\right)^{\frac{1}{2}} + \frac{2\eta}{c^*} \left(\frac{M}{\pi R_0^3}\right)^{\frac{1}{2}} \right]^2 + \beta(a_1 f + b_1 g + c_1 h)}{(f + g + h)}, \quad - (4.77)$$

which reduces to equation (4.58) when $\beta = 1$ (no branching case).

The variation of resistance to flow ratio λ/λ_N with δ_h/R_0 for different values of L_0/L is plotted in Figure 4.10.

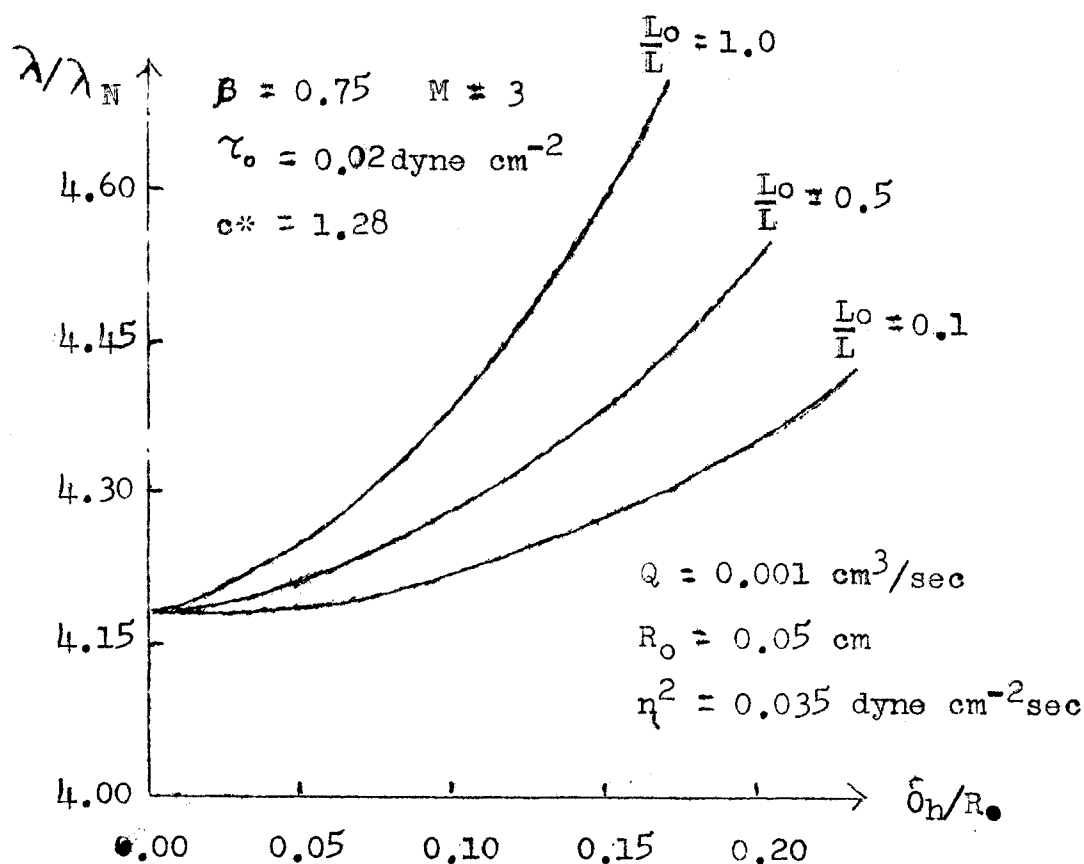


Figure 4.10: Variation of λ/λ_N with δ_h/R_0 for different L_0/L .

The graph shows that the ratio λ/λ_N increases with stenosis height δ_h/R_0 and with stenosis length L_0/L . It is therefore concluded that resistance to flow increases with stenosis size, i.e. height and length.

4.4 CONCLUSION:

In this chapter, the effects on blood flow by a stenosis growth in the lumen of an artery were investigated by considering blood as a non-Newtonian power law and Casson model. The results have shown that both the resistance to flow and wall shear stress increase with increase in the size of the stenosis in each of the two non-Newtonian models. In both models the non-Newtonian behaviour of blood tended to reduce resistance to flow and wall shear stress. Since reduction in these factors is desirable for normal functioning of the system it thus appears that the non-Newtonian behaviour of blood is helpful in the functioning of diseased arterial vessels.

Equations (4.38) and (4.77) show that in both models the effects of branching is to increase the resistance to flow in the system.

CHAPTER V.
MICROCIRCULATION.

5.1 INTRODUCTION:

Microcirculation is the term applied to blood flow in small vessels mainly the capillaries with diameters ranging from 50 μm down to 5 μm , ($1 \mu\text{m} = 10^{-6} \text{ cm}$). This is the region where, as earlier indicated, most of the pressure drop occurs in the cardiovascular system, Attinger (1964). It has been observed that blood flow through very narrow vessels, (capillaries) involves the passage of individual red blood cells in single file along the vessel, each separated from the one in front by plasma. It is expected that under these conditions both the cell and the vessel may undergo some elastic deformations to allow cell motion through the vessel.

This chapter reviews some materials on problems of blood flow in the capillaries. Many investigators have in past years studied microcirculation flow assuming either Newtonian or non-Newtonian behaviour for plasma, Green (1944); Haberman and Syre (1958); Reiner and Scott-Blair (1959); Prothero and Burton (1961); Wang and Skalak (1967); Whitmore (1968); Lighthill (1968, 1972, 1974); Fitzgerald (1969 - a,b); Skalak and Branemark (1969); Lew and Fung (1969, 1970); Fung (1969) and Parihar (1980) [REDACTED]. In most of these studies, the workers assumed that blood cells are of simple analytic shapes such as spheres, spheroids, ellipsoids and discs. Among some of the well known investigations are those by Wang and Skalak (1967) who analysed flow in a cylindrical tube containing a line of spherical bodies. Furthermore Skalak and some other co-workers demonstrated importance of the gap

between the particles and the tube wall on the drag on particles. The effect of the neighbouring particles was shown to be important only when the distance between the particles is sufficiently small. This chapter covers some of the material submitted to the Journal of Fluid Mechanics by Lighthill (1968) and also covers contributions to the same subject by Parihar (1980).

5.2 LIGHTHILL'S PAPER.

5.2.1 BASIC EQUATIONS:

Lighthill (1968) investigated the behaviour of deformable tightly fitting pellets being forced by a pressure difference to move slowly along a distensible tube filled with viscous Newtonian fluid. Using the assumptions of hydrodynamic lubrication theory, he analysed flow behaviour of the thin film of fluid between the pellet and the tube and finally discussed applications of his results to flow in the capillaries.

The pellet and the tube are assumed axis symmetrical with a reasonable approximation to a good fit between the two. It is only parts of the pellet surface at a distance from the axis nearly equal to the tube radius which influences the lubrication problem of the pellet. Assuming simple elastic properties for the pellet and tube, Lighthill approximated these parts by the relation

$$r_p = r_0 - \frac{1}{2}Kx^2 - \beta(p - p_0) \quad - (5.1)$$

where p is the local pressure, p_0 is the external pressure, r_p is the distance from the axis of the tube to the pellet surface, x is distance measured axially downstream from the point where the distance from the axis to the pellet surface

is a maximum denoted by r_0 , K is the curvature of the pellet's meridian section at the point where $r_p = r_0$ and finally β is the radial compliance of the pellet. The inner radius of the tube r_t is given by

$$r_t = r_0 + \alpha(p - p_0) \quad - (5.2)$$

where α is the radial compliance of the tube. The difference between r_p and r_t is the film thickness h and this can be determined using equations (5.1) and (5.2) giving the result

$$h = (\alpha + \beta)(p - p_0) + \frac{1}{2}Kx^2. \quad - (5.3)$$

The term $(\alpha + \beta)(p - p_0)$ in equation (5.3) is the clearance of the pellet in the tube which may be negative or positive at a local pressure p depending on the sign of $(p - p_0)$. Pressure distributions with $(p - p_0) > 0$ correspond to pellets whose maximum diameter is less than the internal diameter of the tube, but distributions with $(p - p_0) < 0$ correspond to tubes whose diameter is in places less than the pellet's maximum diameter.

The pellet is assumed to move at constant velocity U along the tube. If the frame of reference is taken to move together with the pellet, the tube wall is then seen to move at velocity $-U$ in the x -direction relative to the pellet. Denoting the distance across the film by y , with $y = 0$ on the pellet surface, the momentum equation without inertial terms takes the form

$$0 = -\frac{dp}{dx} - \frac{\partial \tau}{\partial y}, \quad - (5.4)$$

where τ is the shear stress given by

$$\gamma = -\mu \frac{\partial p}{\partial y} , \quad - (5.5)$$

μ is the fluid viscosity and u is the velocity component along the x -direction. From equations (5.4) and (5.5), the momentum equation may be written in the form

$$\mu \frac{\partial^2 u}{\partial y^2} = \frac{dp}{dx} . \quad - (5.6)$$

The boundary conditions are:

$$u = 0 \text{ at } y = 0 \text{ and } u = -U \text{ at } y = h . \quad - (5.7)$$

Equation (5.6) implies that u is a quadratic in y . We can thus write

$$u = A(x)y^2 + B(x)y + C(x) , \quad - (5.8)$$

where $A(x)$, $B(x)$ and $C(x)$ are suitable functions of x . Using boundary conditions (5.7) and substituting the second derivative of u with respect to y from equation (5.8) into (5.6), $A(x)$, $B(x)$ and $C(x)$ are determined giving the expression for u as

$$u = \left(\frac{1}{2\mu} \frac{dp}{dx} \right) y^2 - \left(\frac{U}{h} + \frac{h}{2\mu} \frac{dp}{dx} \right) y . \quad - (5.9)$$

The continuity equation may be given in the form

$$\int_0^h u dy = -Q \quad - (5.10)$$

where $2\pi r_0 Q$ is the rate of leakback of fluid past the pellet. It is shown that Q is independent of x as follows:

The usual form of the continuity equation is given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 , \quad - (5.11)$$

where v is the velocity component in the y direction. The

boundary conditions for v are

$$v = 0 \text{ at } y = 0 \text{ and } v = -U \frac{\partial h}{\partial x} \text{ at } y = h \quad - (5.12)$$

Integrating equation (5.11) with respect to y between limits $y = 0$ and $y = h$ and using boundary conditions (5.12), the result is

$$U \frac{\partial h}{\partial x} = \int_0^h \frac{\partial u}{\partial x} dy \quad - (5.13)$$

Now

$$\begin{aligned} \frac{d}{dx} \int_0^h u dy &= \int_0^h \frac{\partial u}{\partial x} dy + (u)_{y=h} \frac{\partial h}{\partial x} \\ &= \int_0^h \frac{\partial u}{\partial x} dy - U \frac{\partial h}{\partial x} \quad - (5.14) \end{aligned}$$

By equation (5.13) the R.H.S. of equation (5.14) is zero, i.e.

$$\frac{d}{dx} \int_0^h u dy = 0 \quad - (5.15)$$

Finally, equations (5.10) and (5.15) establishes the result that

$$\frac{dQ}{dx} = 0 \quad - (5.16)$$

Thus Q is independent of x .

Substituting equation (5.9) into (5.10) and integrating as indicated, the result is

$$-Q = \left(\frac{1}{2\mu} \frac{dp}{dx} \right) \frac{h^3}{3} - \left\{ \frac{U}{h} + h \left(\frac{1}{2\mu} \frac{dp}{dx} \right) \right\} \frac{h^2}{2} \quad - (5.17)$$

Eliminating $\left(\frac{1}{2\mu} \frac{dp}{dx} \right)$ between equations (5.9) and (5.17), the

result is

$$u = U \left\{ 2(y/h) - 3(y/h)^2 \right\} - \frac{6Q}{h} \left\{ (y/h) - (y/h)^2 \right\} . \quad - (5.18)$$

Substituting the expression for u from equation (5.18) into (5.6), the result is

$$\frac{dp}{dx} = - \frac{6uU}{h^2} + \frac{12uQ}{h^3} . \quad - (5.19)$$

The pellet is in equilibrium since it is moving at constant velocity U . To determine the unknown quantity Q use the equilibrium requirement demanding that the skin frictional or shear stress force F_s balance the axial force F_a . The skin frictional force is given by

$$F_s = 2\pi r_o \int_{-\infty}^{\infty} (\tau)_{y=0} dx , \quad - (5.20)$$

where $(\tau)_{y=0}$, the shear stress on the pellet's surface, is by equations (5.5) and (5.18) given by

$$(\tau)_{y=0} = -\mu \left(\frac{du}{dy} \right)_{y=0} = \frac{2uU}{h} + \frac{6uQ}{h^2} . \quad - (5.21)$$

From equations (5.20) and (5.21), the frictional force F_s is given by

$$F_s = 2\pi r_o \int_{-\infty}^{\infty} \left(-\frac{2uU}{h} + \frac{6uQ}{h^2} \right) dx . \quad - (5.22)$$

The axial force F_a is given by

$$F_a = \pi r_o^2 \int_{-\infty}^{\infty} \frac{dp}{dx} dx = \pi r_o^2 \int_{-\infty}^{\infty} \left(-\frac{6uU}{h^2} + \frac{12uQ}{h^3} \right) dx . \quad - (5.23)$$

Equilibrium requires that $F_a = F_s$, which from equations (5.22)

and (5.23) the condition becomes

$$\pi r_o^2 \int_{-\infty}^{\infty} \left(-\frac{6\mu U}{h^2} + \frac{12\mu Q}{h^3} \right) dx = 2\pi r_o \int_{-\infty}^{\infty} \left(-\frac{2\mu U}{h} + \frac{6\mu Q}{h^2} \right) dx . \quad (5.24)$$

The limits $x \rightarrow \infty$ and $x \rightarrow -\infty$ represent values of x far from the point of maximum pellet diameter. The range of values of h that satisfy equation (5.19) were found to be severely restricted by the equilibrium requirement. Pressure gradient dp/dx has an opposite sign to the shear stress $(\tau)_{y=0}$, since the two forces act in opposite directions. If $dp/dx < 0$ then $(\tau)_{y=0} > 0$. Applying these inequalities on the integrand of equation (5.24), the result is

$$2Q/U < h < 3Q/U \quad (5.25)$$

The integral on the L.H.S. of equation (5.24) is simply the difference $p(-\infty) - p(\infty)$ between the upstream and downstream pressures. The equation can thus be written as

$$\pi r_o^2 p(-\infty) - p(\infty) = 2\pi r_o \int_{-\infty}^{\infty} \left(-\frac{2\mu U}{h} + \frac{6\mu Q}{h^2} \right) dx . \quad (5.26)$$

5.2.2 REDUCTION TO NONDIMENSIONAL FORM:

The quantity $2Q/U$ is used as the typical measure of film thickness h and is here used in nondimensionalizing the variables h , p and x as follows:

$$H = \frac{h}{2Q/U} , \quad P = \frac{(\alpha + \beta)(p - p_o)}{2Q/U} , \quad X = \left(\frac{K}{2Q/U} \right)^{\frac{1}{2}} x \quad (5.27)$$

Equations (5.3) and (5.19) transform to

$$\frac{dP}{dX} = L(H^{-3} - H^{-2}) \quad \text{and} \quad H = P + \frac{1}{2}X^2 \quad (5.28)$$

$$\text{where } L = \frac{6\mu U(\alpha + \beta)}{(2Q/U)^{5/2} K^{\frac{1}{2}}} . \quad - (5.29)$$

We introduce a nondimensional quantity C determined by

$$C = \frac{2Q}{U r_0} , \quad - (5.30)$$

which is a ratio of standard film thickness $2Q/U$ to the maximum pellet radius r_0 . The differential equation to be solved for H and P is only accurate for $0 < C < 1$. In terms of C and nondimensionalizing, equation (5.26) becomes

$$P(-\infty) - P(\infty) = LC \int_{-\infty}^{\infty} (H^{-2} - \frac{2}{3} H^{-1}) dx \quad - (5.31)$$

Determination of the values of H and P at any given X requires solution of equation (5.28). The solution will be in terms of L and specification of U and Q completely determines this solution. If the downstream pressure $P(\infty)$ is specified then $P(-\infty) - P(\infty)$, the pressure difference motivating the motion of the pellet, can be determined. When this is used in equation (5.31) it fixes C . After this, Lighthill analyses the solution in the limiting cases when $L \rightarrow 0$ and $L \rightarrow \infty$ by means of perturbation theory and numerical analysis, to demonstrate how restrictions on the bounds of h satisfy the requirement on C . This model may throw some light on blood flow in narrow capillaries with red blood cells being squeezed through them in single file, lubricated by plasma. In applying results of his model to blood flow in the capillaries, taking the pellets as the red blood cells, he used results obtained by Rand and Burton (1964) on the estimation of β , the compliance of the red cell, which was estimated to be about 6 $\mu\text{m}/\text{mb}$. The value for α , the compliance of the

capillary vessel, was found to be much smaller compared to β so that its contribution in $\alpha + \beta$ can be ignored. For details of the analysis refer to Lighthill's article in Journal of Fluid Mechanics (1968), volume 34, part 1, pages 124 - 143.

5.3 PARIHAR'S (1980) PAPER.

5.3.1 BASIC EQUATIONS:

Whereas Lighthill's (1968) analysis assumed blood plasma to be a Newtonian fluid however, several other experimenters suggested it to be non-Newtonian, Whitmore (1968). Madow and Bloch (1956); Reiner and Scott-Blair (1959); Copley and Stainsby (1960); Charm and Kurland (1962); Bugliarello, Kepur and Hsiao (1965); Copley (1968); Scott-Blair (1969); Scott-Blair and Spanner (1974) and Parihar (1980) investigate plasma flow in the capillaries taking blood plasma as a power law model. Copley (1968); Whitmore (1968) and Scott-Blair and Spanner (1974) estimated that the parameter 'n' of the power law model for blood plasma lies between 0.95 and 0.995 and its viscosity was approximated to lie between 0.011 and 0.016 poise under normal conditions. What follows is a review on Parihar's Technical Report No. 5/1980 on microcirculation.

Assumptions made in this analysis are:

- (1) The suspended cell has its flat sides perpendicular to the vessel axis and that the axis of the cell coincides with that of the vessel.
- (2) Flow is laminar.
- (3) The Reynolds number is small enough so that the inertial terms can be neglected in the equations of motion.
- (4) Plasma is a non-Newtonian power law fluid.

$$\frac{\partial u}{\partial y} = 0 \quad \text{at} \quad y = \xi h \quad - (5.34 \text{ c})$$

$$u = -U \quad \text{at} \quad y = h, \quad - (5.34 \text{ d})$$

where u_m is the maximum velocity at $y = \xi h$. Boundary conditions (5.34 b,c) represent the continuity of velocity and shear stress at $y = \xi h$. In region I (see Figure 5.1), $0 \leq y \leq \xi h$ and $\partial u / \partial y > 0$. Equation (5.32) becomes

$$\tau = -m \left(\frac{\partial u}{\partial y} \right)^n. \quad - (5.35)$$

Using (5.33) and (5.35), we obtain

$$\frac{\partial}{\partial y} \left\{ -m \left(\frac{\partial u}{\partial y} \right)^n \right\} = -\frac{dp}{dx}. \quad - (5.36)$$

Integrating (5.36) with respect to y and using boundary condition (5.34 c), the result is

$$\frac{\partial u}{\partial y} = \left[\left(-\frac{1}{m} \frac{dp}{dx} \right) (\xi h - y) \right]^{1/n}. \quad - (5.37)$$

Again integrating (5.37) and using the boundary condition (5.34 a), the result is

$$u = \left(-\frac{1}{m} \frac{dp}{dx} \right)^{1/n} \left\{ \frac{(\xi h)^{1+1/n} - (\xi h - y)^{1+1/n}}{1 + 1/n} \right\}; \quad 0 \leq y \leq \xi h. \quad - (5.38)$$

In region II, $\xi h \leq y \leq h$ and $\partial u / \partial y < 0$. Equation (5.32) becomes

$$\tau = m \left(-\frac{\partial u}{\partial y} \right)^n. \quad - (5.39)$$

From equations (5.33) and (5.39), we have

$$\frac{\partial}{\partial y} \left\{ m \left(-\frac{\partial u}{\partial y} \right)^n \right\} = -\frac{dp}{dx} \quad - (5.40)$$

Similarly, on integrating (5.40) and using the boundary condition (5.34 c), we have the result

$$-\frac{\partial u}{\partial y} = \left[\left(-\frac{1}{m} \frac{dp}{dx} \right) (y - \varepsilon h) \right]^{1/n}, \quad - (5.41)$$

which on integrating with boundary condition (5.34 d), gives the result for velocity profile in region II as

$$u = \left(-\frac{1}{m} \frac{dp}{dx} \right)^{1/n} \left\{ \frac{(h - \varepsilon h)^{1+1/n} - (y - \varepsilon h)^{1+1/n}}{1 + 1/n} \right\} - U; \quad - (5.42)$$

$$\varepsilon h \leq y \leq h$$

To determine the unknown constant ε we use the maximum velocity u_m at $y = \varepsilon h$ which is common for regions I and II. Thus at $y = \varepsilon h$ the expressions for velocity profiles in the two regions given by equations (5.38) and (5.42) are equal and this equality yields an implicit expression for ε , given by

$$-U = \left(-\frac{1}{m} \frac{dp}{dx} \right)^{1/n} \frac{h^{1+1/n}}{1 + 1/n} \left\{ \varepsilon^{1+1/n} - (1 - \varepsilon)^{1+1/n} \right\} \quad - (5.43)$$

The physical set up shows that the point $y = \varepsilon h$ is closer to $y = h$. It is thus deduced that $\frac{1}{2} < \varepsilon < 1$ and that $(1 - \varepsilon) \ll 1$. Expanding the terms containing ε in equation (5.43) and retaining only the first powers of $(1 - \varepsilon)$, the result is

$$-U = \left(-\frac{1}{m} \frac{dp}{dx} \right)^{1/n} \frac{h^{1+1/n}}{1 + 1/n} \left\{ 1 - \left(\frac{1}{n} + 1 \right) (1 - \varepsilon) \right\} \quad - (5.44)$$

Equation (5.44) determines ε explicitly.

The volumetric flow flux per unit length Q is defined by

$$-Q = \int_0^h u dy = \int_0^{\varepsilon h} u dy + \int_{\varepsilon h}^h u dy \quad . \quad (5.45)$$

By using equations (5.38) and (5.42) for u in (5.45), the flow flux is given by

$$-Q = \left(-\frac{1}{m} \frac{dp}{dx}\right)^{1/n} \cdot \frac{h^{1+1/n}}{2+1/n} \left\{ \varepsilon^{2+1/n} + (1-\varepsilon)^{2+1/n} \right\} - Uh(1-\varepsilon) \quad , \quad (5.46)$$

Similarly expanding terms containing ε in equation (5.46) and retaining only first powers of $(1-\varepsilon)$, the result is

$$-Q = \left(-\frac{1}{m} \frac{dp}{dx}\right)^{1/n} \cdot \frac{h^{2+1/n}}{2+1/n} \left\{ 1 - \left(\frac{1}{n} + 2\right)(1-\varepsilon) \right\} - Uh(1-\varepsilon) \quad (5.47)$$

Eliminating ε between equations (5.44) and (5.47) gives the final expression for flow flux as

$$Q = \frac{U}{n+1} - \left(-\frac{1}{m} \frac{dp}{dx}\right)^{1/n} \frac{h^{2+1/n}}{(1+1/n)^2(2+1/n)} \quad , \quad (5.48)$$

which on making dp/dx , the pressure gradient, the subject of the formula, the equation becomes

$$\frac{dp}{dx} = -m \left[\frac{(n+1)(2n+1)}{n^3} \right]^n \left[\frac{Uh - (n+1)Q}{h^{2+1/n}} \right]^n \quad . \quad (5.49)$$

For the case when $m = \mu$ (viscosity) and $n = 1$, equation (5.49) reduces to the one obtained by Lighthill (1968).

The human red blood cell in the unstressed state is a biconcave disc of diameter about 8 μm , but under stress is rather easily deformed, Prothero and Burton (1961); Rand and Burton (1964). Many investigators have assumed the blood cell to be of a simple analytic shape. In this study the red blood cell is assumed to be either disc like or parabolic in shape and flow flux Q under each of the two assumptions is

calculated.

5.3.2 FLOW FLUX Q (DISC-LIKE CELL):

The shape of the cell is assumed to be disc-like and hence the film thickness $h = h_0$ is constant. Figure 5.2 gives an illustration of the physical set up.

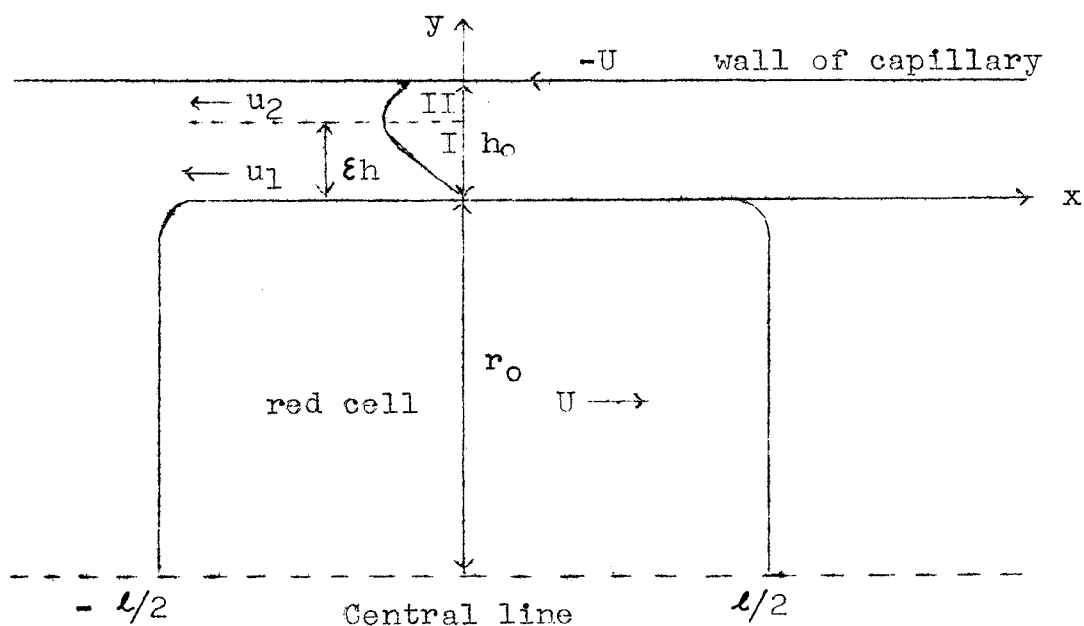


Figure 5.2: The red cell assuming disk-like shape.

The axial force F_a (due to pressure gradient dp/dx) acting on the red blood cell is given by

$$F_a = \pi r_o^2 \left(\frac{dp}{dx} \right)_{-l/2}^{l/2} \quad . \quad (5.50)$$

On using equation (5.49) to eliminate dp/dx from (5.50) and integrating, the result is

$$F_a = \pi r_{oml}^2 \left[\frac{(n+1)(2n+1)}{n^3} \right]^n \left[\frac{U_{h_0} - (n+1)Q}{2 + 1/n} \right]^n \quad (5.51)$$

The skin frictional (or shear stress) force on the surface of the cell is defined by

$$F_S = 2\pi r_0 \int_{-l/2}^{l/2} (\tau)_{y=0} dx \quad - (5.52)$$

where $(\tau)_{y=0}$ is the shear stress at $y = 0$. Using equation (5.35) and (5.37) at $y = 0$ gives

$$(\tau)_{y=0} = \xi h_0 \frac{dp}{dx} . \quad - (5.53)$$

Eliminating dp/dx from equations (5.53) and (5.49), the result is

$$(\tau)_{y=0} = - \xi h_0 m \left[\frac{(n+1)(2n+1)}{n^3} \right]^n \left[\frac{U h_0 - (n+1)Q}{h^2 + 1/n} \right]^n . \quad - (5.54)$$

Substituting equation (5.54) into (5.52) and then integrating, the expression for F_s , the frictional force, becomes

$$F_s = \frac{2\pi r_0 m l}{(n+1)h_0^{2n}} \left[\frac{(n+1)(2n+1)}{n^3} \right]^n \left[\frac{n^3 U h_0}{2n+1} - U h_0 + (n+1)Q \right] \cdot \left[U h_0 - (n+1)Q \right]^{n-1} \quad - (5.55)$$

As observed in Lighthill's analysis, we must have $F_a = F_s$ and this results in an explicit expression for Q given by

$$Q = \frac{U h_0}{n+1} \left[1 - \frac{2n^3}{(2n+1)(n+1) \frac{r_0}{h_0} + 2(2n+1)} \right] \quad - (5.56)$$

Equation (5.56) shows that Q increases as the velocity of the red cell U . It can also be shown by calculation that Q increases with h_0 and increases with n decreasing from unity.

5.3.2 FLOW FLUX Q (PARABOLIC CELL):

The undistorted meridian section of the cell is assumed to be parabolic. The blood vessel walls and the cell are both assumed to be rigid. Figure 5.1 gives an illustration of the physical set up. The lubricating film thickness h is given by

$$h = h_0 + \frac{1}{2} K x^2 \quad - (5.57)$$

where h_0 is the film thickness which is constant at $x = 0$. Equation (5.57) can be written as

$$h = h_0(1 + kx^2), \quad - (5.58)$$

where $k = K/2h_0$. Eliminating h from equations (5.49) and (5.58), the result is

$$\frac{dp}{dx} = -m \left[\frac{(n+1)(2n+1)}{n^3 h_0^2 + 1/n} \right]^n \left[U_{h_0} - (n+1)Q \right]^n \left[1 + Xkx^2 \right], \quad \text{for } k \ll 1, \quad - (5.59)$$

$$\text{where } X = \frac{nU_{h_0}}{U_{h_0} - (n+1)Q} - \frac{2n+1}{n}. \quad - (5.60)$$

Equation (5.59) is substituted into (5.50) to determine F_a and the result is

$$F_a = \frac{\pi r_{om}^2}{h_0^{2n+1}} \left[\frac{(n+1)(2n+1)}{n^3} \right]^n \left[U_{h_0} - (n+1)Q \right]^n \left[1 + \frac{Xk1^2}{12} \right]. \quad (5.61)$$

By means of equations (5.52), (5.53) and (5.59), the shear stress F_s is given by

$$F_s = \frac{2\pi r_{om}}{(n+1)h_0^{2n}} \left[\frac{(n+1)(2n+1)}{n^3} \right]^n \left[U_{h_0} - (n+1)Q \right]^{n-1} \cdot \left[\frac{n^3 U_{h_0}}{(2n+1)} - U_{h_0} + (n+1)Q + \frac{Yk1^2}{12} \right] \quad - (5.62)$$

$$\text{where } Y = \left[\frac{(n-1)U_{h_0}}{U_{h_0} - (n+1)Q} - 2n \right] \left[\frac{n^3 U_{h_0}}{2n+1} - \{U_{h_0} - (n+1)Q\} \right] + U_{h_0} \left[\frac{n^3}{2n+1} - 1 \right] \quad - (5.63)$$

Equilibrium condition requires that $F_a = F_s$ from which flow flux Q can be calculated using equations (5.61) and (5.62) as previously done.

This section has investigated the motion of a red blood cell in a narrow capillary by considering blood plasma as a

non-Newtonian power law fluid. The disc and parabolic approximations to the shape of the red cell were considered. Under each approximation the flow flux Q was determined. The results have shown that Q increases with the red cell velocity U and it also increases with h , the gap between the red cell and the wall of the capillary. Flow flux Q also increases as n , the flow behaviour index, decreases from unity showing that the non-Newtonian behaviour of blood has the effect of increasing Q . An obvious application of the expression we have obtained for Q is on the effects of sudden changes of atmospheric pressure and altitude on the human blood circulation system.

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