ON PROJECTIVE CHARACTERS OF ROTATION SUBGROUPS OF WELY GROUPS OF TYPES D_6 and D_7

by

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I declare that this dissertation represents my own work and that it has not previously been submitted for a degree at this or another University.

I dedicate this Dissertation to Mama Sungata, Seddy, Brother B. Muchima, my Wife Edith and my son Chiyeji.

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ABSTRACT

Let Φ be a root system in a ℓ-dimensional real Euclidean space V with Weyl group $W(\Phi)$, and let $W^+(\Phi)$ denote its rotation subgroup. In [17], the projective representations of the rotation subgroup $W^+(\Phi)$ have been determined from those of $W(\Phi)$ for each root system Φ. This is done by constructing non-trivial central extensions of $W^+(\Phi)$ via the double coverings of the rotation groups $SO(\ell)$. This adaptation gives a unified way of obtaining the basic projective representations of $W^+(\Phi)$ from those of $W(\Phi)$, determined in [9]. In particular, formulae giving irreducible characters of these representations are explicitly determined in each case.

Our object here is to apply the fore-going results to Rotation subgroups of Weyl groups of types D_6 and D_7, that is, those groups which have Schur multiplier $(Z_2)^2 \circledcirc Z_3$. In particular, we give the α-regular classes for the factor set α considered in [16], as well as obtain basic projective characters for these groups.

The following is a brief description of the individual chapters of this dissertation. In chapter 1, we give basic ideas of factor sets and projective representations of finite groups and some of the properties of these representations. In chapter 2, we present the concept of Schur multipliers and give the relationship between central extensions and projective representations of finite groups. Projective characters of finite groups and some of their properties, are given in chapter 3. Chapter 4 is mainly concerned with Weyl groups and their Rotation subgroups, and Schur

multipliers of these subgroups. The work in these chapters is applied in chapter 5, to obtain the basic projective characters of the Rotation subgroups of the Weyl groups of types D_6 and D_7. The results are summarized in Tables II and III.

CHAPTER ONE

PROJECTIVE REPRESENTATIONS OF FINITE GROUPS

Let G be a finite group, K an algebraically closed field of characteristic zero, and let K^* denote the multiplicative group of K.

1.1. Factor sets

1.1.1. Definition

A mapping $\alpha: G \times G \to K^*$ is called a factor set of G if, for all $x, y, z \in G$,

$$\alpha(x,y)\,\alpha(xy,z) = \alpha(x,yz)\,\alpha(y,z)$$

and

$$\alpha(x,e) = 1 = \alpha(e,x)$$

where e is the identify element of G.

1.1.2 Definition

Two factor sets α, β of G are equivalent if there exists a mapping $\mu: G \to K^*$ such that for all $x, y \in G$,

$$\alpha(x,y) = \mu(x)\,\mu(y)\,\mu(xy)^{-1}\beta(x,y)$$

and

$$\mu(e) = 1.$$

1.1.3. Definition

A factor set of G is said to be normalised if for all $x \in G$,

$$\alpha(x,x^{-1}) = 1.$$

1.1.4. Lemma.

If α is a normalised factor set, then for all $x, y \in G$,

$$\alpha^{-1}(x,y) = \alpha(y^{-1},x^{-1}).$$

Proof:

By definitions 1.1.1 and 1.1.3,

$$\alpha(x,y)\,\alpha(xy,y^{-1}) = \alpha(x,yy^{-1})\,\alpha(y,y^{-1})$$

$$= \alpha(x,e)\,\alpha(y,y^{-1}))$$

$$= 1 \cdot 1 = 1$$

and

$$\alpha(xy,y^{-1})\,\alpha(x,x^{-1}) = \alpha(xy,\ y^{-1}x^{-1})\,\alpha(y^{-1},x^{-1}).$$

Therefore, $\alpha(xy,y^{-1}) = \alpha(y^{-1},x^{-1})$ and hence

$\alpha(x,y)\,\alpha(y^{-1},x^{-1}) = 1$, so that $\alpha^{-1}(x,y) = \alpha(y^{-1},x^{-1})$.

1.1.5. Definition

Let α be any factor set of G. An element $g \in G$ is said to be <u>α-regular</u> if

$\alpha(x,g) = \alpha(g,x)$, for all $x \in C_G(g)$, the centralizer of g in G.

1.1.6. Definition

A conjugacy class in G is said to be an <u>α-regular class</u> if all its elements are α-regular.

1.1.7. Lemma.

If g is an α-regular element of G, then so is every element in the conjugacy class containing g.

Proof.

Let g be in the same conjugacy class as h. Then $x^{-1}gx = h$ for some $x \in G$.

If g is α-regular in G, then

$$\alpha(y,g) = \alpha(g,y), \text{ for all } y \in C_G(g).$$

Thus $\alpha(y, xhx^{-1}) = \alpha(xhx^{-1}, y)$, for all $y \in C_G(xhx^{-1})$, and hence xhx^{-1} is α-regular. Therefore h is α-regular in G.

1.1.8. Lemma

If g is an α-regular element of G, then it is also a β-regular element, for all factor sets β equivalent to α.

Proof.

Let g be α-regular in G. Then $\alpha(g,x) = \alpha(x,g)$ for all $x \in C_G(g)$.

If α is equivalent to β, then by 1.1.2,

$$\alpha(g,x) = \mu(g)\mu(x)\mu(gx)^{-1}\beta(g,x), \text{ for some map } \mu : G \to K^*.$$

Since g is α-regular, we have

$$\begin{aligned}\alpha(g,x) &= \mu(g)\mu(x)\mu(gx)^{-1}\beta(g,x) = \alpha(x,g)\\ &= \mu(x)\mu(g)\mu(xg)^{-1}\beta(x,g).\end{aligned}$$

Thus $\mu(g)\mu(x)\mu(gx)^{-1}\beta(g,x) = \mu(x)\mu(g)\mu(xg)^{-1}\beta(x,g)$ so that $\beta(g,x) = \beta(x,g)$ since $\mu(g), \mu(x)$ and $\mu(gx)$ are non-zero numbers in K^*. Therefore g is α-regular in G.

Let α be any factor set of G and define

$$f_\alpha(x,g) = \alpha(x,g)\alpha^{-1}(xgx^{-1}, x) \tag{1.1}$$

for all α-regular $g \in G$, all $x \in G$.

1.1.9. Definition

The factor set α is said to be simple if

(i) $f_\alpha(x,g) = 1$, for all α-regular $g \in G$, all $x \in G$.

(ii) $\alpha(x,x^{-1}) = 1$, for all $x \in G$.

We now state and prove some results which will be needed in proving that each factor set of a group G is equivalent to a simple factor set (see Theorem 1.1.13).

1.1.10 Lemma

Let α be any factor set of G and g any α-regular element. Let $x,y \in G$ be such that $xgx^{-1} = ygy^{-1}$. Then $f_\alpha(x,g) = f_\alpha(y,g)$.

Proof

Let $x,y,g \in G$ satisfy the above conditions. Then

$$f_\alpha(x,g)f_\alpha(y,g)^{-1} = \frac{\alpha(x,g)\alpha(ygy^{-1},y)}{\alpha(y,g)\alpha(xgx^{-1},x)}$$

$$= \frac{\alpha(x,g)\alpha(xg,x^{-1})\alpha(xgx^{-1},y)}{\alpha(x,x^{-1})\alpha(y,g)}$$

$$= \frac{\alpha(x,g)\alpha(x^{-1},y)\alpha(xg,x^{-1}y)}{\alpha(x^{-1},x)\alpha(y,g)}$$

$$= \frac{\alpha(x^{-1},y)\alpha(g,x^{-1}y)\alpha(x,gx^{-1}y)}{\alpha(x^{-1},x)\alpha(y,g)}.$$

Now $x^{-1}y \in C_G(g)$ and g is α-regular. Thus $\alpha(g,x^{-1}y) = \alpha(x^{-1}y,g)$ and $\alpha(x,gx^{-1}y) = \alpha(x,x^{-1}yg)$.

Hence

$$f_\alpha(x,g)\ f_\alpha(y,g)^{-1} = \frac{\alpha(x^{-1},y)\alpha(x^{-1}y,g)\alpha(x,x^{-1}yg)}{\alpha(x^{-1},x)\alpha(y,g)}$$

$$= \frac{\alpha(x^{-1},yg)\alpha(y,g)\alpha(x,x^{-1}yg)}{\alpha(x^{-1},x)\alpha(y,g)}$$

$$= \frac{\alpha(x^{-1},yg)\alpha(x,x^{-1}yg)}{\alpha(x^{-1},x)} = \frac{\alpha(x,x^{-1}}{\alpha(x^{-1},x)} = 1.$$

1.1.11. Lemma

Let α be any factor set of G, and g be α-regular. If $f_\alpha(x,g) = 1$ for all $x \in G$, then $f_\alpha(x,ygy^{-1}) = 1$ for all $x,y \in G$.

Proof.

Let $x,y,g \in G$ satisfy the above conditions of the lemma. Then

$$\alpha(x,ygy^{-1})\alpha(xygy^{-1},y) = \alpha(ygy^{-1},y)\alpha(x,yg)$$

and

$$\alpha(xygy^{-1}x^{-1},x)\alpha(xygy^{-1},y) = \alpha(x,y)\alpha(xygy^{-1}x,\ xy).$$

Thus
$$f_\alpha(x,ygy^{-1}) = \frac{\alpha(ygy^{-1},y)\alpha(x,yg)}{\alpha(x,y)\alpha(xygy^{-1}x^{-1},xy)}$$

$$= \frac{\alpha(ygy^{-1},y)\alpha(xy,g)}{\alpha(xygy^{-1}x^{-1},\ xy)\alpha(y,g)}$$

$$= f_\alpha(xy,g)f_\alpha(xy,g)^{-1}$$

$$= 1.$$

1.1.12. Lemma

Let α be any factor set of G, and let $f_\alpha(x,g) = 1$ for all $x \in G$, all α-regular $g \in G$. Then $\alpha(g,g^{-1}) = \alpha(xgx^{-1},\ xg^{-1}x^{-1})$ for all α-regular $g \in G$, all $x \in G$.

Proof.

For all $x,y \in G$ satisfying the conditions of the lemma, we have

$$\alpha(x\,gx^{-1},\ xg^{-1}x^{-1})\alpha(x,x) = \alpha(xg^{-1}x^{-1},x)\alpha(xgx^{-1},xg^{-1}).$$

Since g is α-regular, g^{-1} is also α-regular, for $x \in C_G(g^{-1})$ implies $g^{-1}x \in C_G(g)$ and

$$\alpha(x,g)\ \alpha(g^{-1},x)^{-1} = \alpha(g,g^{-1}x)\alpha(g^{-1}x,g)^{-1} = 1.$$

Thus $f_\alpha(x,g^{-1}) = 1$ and $f_\alpha(xg^{-1},g) = 1$ for all $x \in G$, and hence

$$\begin{aligned}\alpha(xgx^{-1},xg^{-1}x^{-1}) &= \alpha(xg^{-1},g)\alpha(x,g^{-1})\\ &= \alpha(g^{-1},g)\\ &= \alpha(g,g^{-1}).\end{aligned}$$

1.1.13. Theorem

Let α be any factor set of G. Then there exists a simple factor set γ of G equivalent to α.

Proof.

We define $\mu : G \to K^*$ as follows.

Let $\{g_1,\ldots,g_t\}$ be an α-regular class of G, and let

$$G = \bigcup_{i=1}^{t} x_i C_G(g_1) \quad \text{such that} \quad g_i = x_i g_1 x_i^{-1} \quad (i=1,\ldots,t).$$

We call g_1 the representative element of the α-regular class containing it. Define $\mu(g_i) = f_\alpha(x_i,g_1)$ for $i=1,\ldots,t$. By lemma 1.1.10, μ is well defined for any choice of

$\{x_1,\ldots,x_t\}$. Similarly, we define μ on the other α-regular classes. Further, set $\mu(x) = 1$ if x is not α-regular.

Define $\beta(x,y) = \mu(x)\mu(y)\mu(xy)^{-1}\alpha(x,y)$ for all $x,y \in G$. Then for all $z \in G$, we have $f_\beta(z,g_1) = 1$, and hence by lemma 1.1.11, $f_\beta(z,g) = 1$, for all β-regular $g \in G$, all $z \in G$.

Now define $\delta(z) = \beta(z,z^{-1})^{-\frac{1}{2}}$ for all $z \in G$; and set $\gamma(x,y) = \delta(x)\delta(y)\delta(xy)^{-1}\beta(x,y)$ for all $x,y \in G$. Then

$$\begin{aligned}\gamma(z,z^{-1}) &= \delta(z)\,\delta(z^{-1})\delta(e)^{-1}\beta(z,z^{-1})\\ &= \beta(z,z^{-1})\,(\beta(z,z^{-1})\beta(z,z^{-1}))^{-\frac{1}{2}}\\ &= 1, \text{ for all } z \in G,\end{aligned}$$

and if g is α-regular, $f_\gamma(z,g) = 1$ for all $z \in G$, by lemma 1.1.12. Thus, γ is a simple factor set of G equivalent to α.

1.2. Projective representations

In this section, we define a projective representation of a finite group G and consider certain properties of these representations.

Let V be a vector space over K, $GL(V)$ be a group of non-singular linear transformations on V and $GL(n,K)$, a group of non-singular $n \times n$ matrices over K.

1.2.1. Definition

A mapping $P:G \to GL(V)$ is called a <u>projective representation</u> of G with factor set α and representation space V over K if for all $x,y \in G$,

$$P(x)P(y) = \alpha(x,y)P(xy)$$

and

$$P(e) = I_V,$$

where I_V is the identity linear transformation. The properties of factor sets in Definition 1.1.1 are now clear from our definition of a projective representation of a group.

The following is an alternative definition to 1.2.1.

1.2.2. Definition.

A matrix projective representation P of G with factor set α and degree n is a mapping $P:G \to GL(n,K)$ such that

$$P(x)P(y) = \alpha(x,y)P(xy), \text{ for all } x,y \in G$$

and

$$P(e) = I_n,$$

where I_n is the nxn identity matrix.

If α is a trivial factor set of G, that is, if $\alpha(x,y) = 1$ for all $x,y \in G$, then the above representation associated with α is the linear (ordinary) representation.

1.2.3 Definition

Two projective representations P_1 and P_2 of G with factor sets α_1 and α_2 are said to be projectively equivalent if there exists a non-singular matrix T such that for all $x \in G$,

$$T^{-1} P_1(x)T = P_2(x).$$

If P_1 and P_2 are equivalent, we shall write $P_1 \approx P_2$.

1.2.4. Defintion

A projective representation P of G with representation space V and factor set α is said to be reducible if there exists a proper subspace U of V such that for $u \in U$,

$$P(x)u \in U.$$

Otherwise, P is irreducible.

Let α be a factor set of G and define $(KG)_\alpha = \{ \sum_{x \in G} \nu_x \gamma(x) \mid \nu_x \in K \}$ where $\{\gamma(x) \mid x \in G\}$ is a set of elements in 1-1 correspondence with the elements of G. Define addition and scalar multiplication on $(KG)_\alpha$ componentwise and multiplication of elements of $(KG)_\alpha$ by

$$\left(\sum_{x \in G} \nu_x \gamma(x) \right) \left(\sum_{y \in G} \eta_y \gamma(y) \right) = \sum_{x,y \in G} \nu_x \eta_y \alpha(x,y) \gamma(xy).$$

We prove the following simple result.

1.2.5. Lemma.

$(KG)_\alpha$ as defined above is an associative algebra over K, with identity $\gamma(e)$.

Proof

$(KG)_\alpha$ is an algebra (e.g. see[5]).

To show that $(KG)_\alpha$ is associative, let $\sum_{x\in G} \nu_x \gamma(x)$, $\sum_{y\in G} \eta_y \gamma(y)$ and $\sum_{z\in G} \xi_z \gamma(z)$ be any elements in $(KG)_\alpha$.

Then

$$\left(\sum_{x\in G} \nu_x \gamma(x)\right)\left[\left(\sum_{y\in G} \eta_y \gamma(y)\right)\left(\sum_{z\in G} \xi_z \gamma(z)\right)\right]$$

$$= \left(\sum_{x\in G} \nu_x \gamma(x)\right)\left[\sum_{y,z\in G} \eta_y \xi_z \alpha(y,z)\gamma(yz)\right]$$

$$= \sum_{x,y,z\in G} \nu_x \eta_y \xi_z \alpha(y,z)\alpha(x,yz)\gamma(xyz).$$

Also,

$$\left[\left(\sum_{x\in G} \nu_x \gamma(x)\right)\left(\sum_{y\in G} \eta_y \gamma(y)\right)\right]\left(\sum_{z\in G} \xi_z \gamma(z)\right)$$

$$= \left[\sum_{x,y\in G} \nu_x \eta_y \alpha(x,y)\gamma(xy)\right]\left(\sum_{z\in G} \xi_z \gamma(z)\right)$$

$$= \sum_{x,y,z\in G} \nu_x \eta_y \xi_z \alpha(x,y)\alpha(xy,z)\gamma(xyz).$$

Hence, $(KG)_\alpha$ is associative over K.

1.2.6. Definition

$(KG)_\alpha$ as defined above is called the <u>twisted group algebra</u> associated with the factor set α of G.

We now prove the following result.

1.2.7. Lemma

There is a 1-1 correspondence between projective representations of G with factor set α and the ordinary representations of $(KG)_\alpha$.

Furthermore, there exists a 1-1 correspondence between representations of $(KG)_\alpha$ and finite-dimensional left $(KG)_\alpha$-modules V.

Proof

Let α be a factor set of G, and T the linear representation of $(KG)_\alpha$. Define $P:G\to GL(V)$ by

$$P(x) = T(\gamma(x)) \text{ for all } x\in G.$$

Then
$$\begin{aligned} P(x)P(y) &= T(\gamma(x))T(\gamma(y)) \\ &= T(\gamma(x)\gamma(y)) \\ &= T(\alpha(x,y)\gamma(xy)) \\ &= \alpha(x,y)\,T(\gamma(xy)) \\ &= \alpha(x,y)P(xy), \end{aligned}$$

and $P(e) = T(\gamma(e)) = I_V$.

Thus, P is a projective representation of G with factor set α.

If now P is a projective representation of G with factor set α, let $T(\gamma(x)) = P(x)$, $x\in G$. Then T is a representation of $(KG)_\alpha$ as an algebra.

Now, if V is a finite-dimensional $(KG)_\alpha$-module, define $P(x)v = \gamma(x)v$, for all $x\in G$, $v\in V$.

Then $P(x)P(y)v = \alpha(x,y)\gamma(xy)v$

$$= \alpha(x,y)P(xy)v,$$

so that P is a projective representation of G with factor set α.

Following directly from (1.2.7), we remark that the problem of classifying projective representations of G with factor set α reduces to that of classifying all finite dimensional $(KG)_\alpha$ - modules.

1.2.8. Definition

$(KG)_\alpha$ - modules V_1 and V_2 are said to be isomorphic if there exists a vector space isomorphism

$S: V_1 \to V_2$ such that for all $x \in G$ and $v \in V_1$,

$$S(\gamma(x)v) = \gamma(x)S(v).$$

1.2.9. Definition

A $(KG)_\alpha$-module V is said to be completely reducible if for each subspace V_1 of V, there exists a subspace V_2 such that $V = V_1 \oplus V_2$. Otherwise, V is indecomposable.

Having defined indecomposable and completely reducible $(KG)_\alpha$-modules, we can now generalise Maschke's theorem to $(KG)_\alpha$-modules as follows.

1.2.10. Theorem

Every $(KG)_\alpha$-module V is completely reducible.

Proof.

Let V_1 be a non-trivial subspace of V. Then there exists some U such that $V=V_1\oplus U$. Therefore, there exists a homomorphism $\lambda\in Hom(V,V_1)$ such that if $v=v_1+u$, $v_i\in V$, $u\in U$, then $\lambda v=v_1$.

Let $P:V\to V_1$ be defined by

$$Pv = |G|^{-1} \sum_{x\in G} \gamma(x)\lambda\gamma(x^{-1})v.$$

Then $Pv = |G|^{-1}.|G|.\lambda v = \lambda v = v_1$. i.e. $P_v=V_1$.

Now, let $v_2 = \{v-Pv \mid v\in V\}$.

Then V_2 is a submodule of V and $V = V_1\oplus V_2$, so that V is completely reducible.

1.2.10. Corollary

If $P:G\to GL(n,k)$ is a projective representation of G with factor set α, there exists a matrix $T\in GL(n,K)$ such that

$$T^{-1}P(x)T = \begin{pmatrix} S_1(x) & & & \\ & S_2(x) & & 0 \\ & & \ddots & \\ 0 & & & S_r(x) \end{pmatrix},$$

where each $S_i (i=1,\dots,r)$ is an irreducible projective representation of G with factor set α. We write $P\simeq S_1\dotplus S_2\dotplus\dots\dotplus S_r$.

Let $M_{n_i}(K)$ $(i=1,\dots,s)$ be the full matrix algebra of $n_i\times n_i$ matrices over K. Then (see [5])

$$(KG)_\alpha = M_{n_1}(K)\oplus\dots\oplus M_{n_s}(K) \qquad (1.2)$$

Furthermore, there exists s non-isomorphic $(KG)_\alpha$ -modules $N_1,\ldots,N_s$, such that each $M_{n_i}(K)$ is isomorphic to a direct sum of n_i copies of N_i; where n_i is the dimension of N_i over K. That is

$$M_{n_i}(K) \cong N_1 \oplus \ldots \oplus N_i \ . \qquad (1.3)$$

That is the irreducible projective representation of G of degree n_i which is afforded by the $(KG)_\alpha$ - module N_i, appears n_i times as an irreducible component of the projective representation of G with factor set α, which is afforded by the $(KG)_\alpha$ -module $(KG)_\alpha$.

1.2.11 Lemma (see [5])

Let Z be the centre of $(KG)_\alpha$. Then

$Z = Z(M_{n_1}(K)) \oplus \ldots \oplus Z(M_{n_s}(K))$ and hence

$(Z:K) = S$.

Proof

It is apparent from above considerations that

$Z = Z(M_{n_1}(K)) \oplus \ldots \oplus Z(M_{n_s}(K))$.

Furthermore, the only matrices which will commute with all the matrices $M_{n_i}(K)$ are the scalar mutiples of I_{n_i}. Thus $(Z(M_{n_i}(K):K) = 1$, and hence $(Z:K) = S$.

1.2.12 Lemma

Let α be a simple factor set of G, and let $\{\mathcal{K}_1,\ldots\mathcal{K}_t\}$ be the set of α-regular classes of G. Define $C_i = \sum_{x \in \mathcal{K}_i} \gamma(x)$,

$i = 1,\ldots,t$.

Then $\{C_1,\ldots,C_t\}$ is a K-basis for Z.

Proof.

Let $g\in G$ and $x\in \mathcal{H}_j$. Then

$$\gamma(g^{-1})C_j\gamma(g)=\sum_{x\in \mathcal{H}_j}\gamma(g^{-1})\gamma(x)\gamma(g)$$

$$=\sum_{x\in \mathcal{H}_j}\alpha(g^{-1},x)\gamma(g^{-1}x)\gamma(g)$$

$$=\sum_{x\in \mathcal{H}_j}\alpha(g^{-1},x)\alpha(g^{-1}x,g)\gamma(g^{-1}xg)$$

$$=C_j \quad \text{since } \alpha(g^{-1},x)\alpha(g^{-1}x,g)\in K^*.$$

Furthermore, since each C_j is a sum of disjoint sets of group elements, $\{C_1,\ldots,C_t\}$ is linearly independent over K.

Now, if $y=\sum_{g\in G}\xi_g\gamma(g)\in Z$. Then we show that if $\xi_g\neq 0$, then g is α-regular in G. That is we show that we may sum over the C_i's. That is we show that if $h\in C_G(g)$, then $f_\alpha(h^{-1},g)=\alpha(h^{-1},g)\alpha(h^{-1}g,h)=1$.

Now since $y\in Z$,

$\gamma(h^{-1})y\gamma(h)=y$, for all $h\in G$, or

$$\gamma(h^{-1})y\gamma(h)=\gamma(h^{-1})\Big(\sum_{g\in G}\xi_g\gamma(g)\Big)\gamma(h)$$

$$=\xi_g\gamma(h^{-1})\gamma(g)\gamma(h)+\sum_{g'\neq g\in G}\xi_{g'}\gamma(h^{-1})\gamma(g')\gamma(h)$$

$$=\xi_g\alpha(h^{-1},g)\gamma(h^{-1}g)\gamma(h)+\sum_{g'\neq g\in G}\xi_{g'}\alpha(h^{-1},g')\gamma(h^{-1}g')\gamma(h)$$

$$=\xi_g\alpha(h^{-1},g)\alpha(h^{-1}g,h)\gamma(h^{-1}gh)$$

$$+\sum_{g'\neq g\in G}\xi_{g'}\alpha(h^{-1},g')\alpha(h^{-1}g',h)\gamma(h^{-1}g'h)$$

$$= \xi_g f_\alpha(h^{-1},g)\gamma(h^{-1}gh) + \sum_{g' \neq g \in G} \xi_{g'} f_\alpha(h^{-1},g')\gamma(h^{-1}g'h)$$

$$= \xi_g \gamma(g) + \sum_{g' \neq g} \xi_{g'} \gamma(g') \text{ since } y = \sum_{g \in G} \xi_g \gamma(g).$$

Now, since $\xi_g \neq 0$, we have $f_\alpha(h^{-1},g) = 1$ and thus g is α-regular.

Thus, if $y = \sum_{g \in G} \xi_g \gamma(g) \in Z$, then the summation is over α-regular elements of G.

Also, $\gamma(h^{-1})y\gamma(h) = y$ implies that

$\sum_{g \in G} \xi_{h^{-1}gh} f_\alpha(h^{-1},g)\gamma(h^{-1}gh) = \sum_{g \in G} \xi_g \gamma(g)$ and since $f_\alpha(h^{-1},g)=1$ for all α-regular $g \in G$, then $\xi_{h^{-1}gh} = \xi_g$.

In particular, $\xi_{h^{-1}g'h} = \xi_g$ whenever $g' \in G$ is conjugate to g. Thus g' is in the same α-regular class as g, and $y \in Z$ is a linear combination of $\{C_1, C_2, \ldots, C_t\}$. Hence $\{C_1, \ldots, C_t\}$ is a K-basis for Z.

Given a factor set α, we need to determine the number of non-isomorphic irreducible $(KG)_\alpha$-modules; hence, the number of inequivalent irreducible projective representations of G with factor set α. This is given by the following result.

1.2.13. Theorem

The number of non isomorphic irreducible $(KG)_\alpha$-modules equals the number of α-regular classes in G.

Proof.

By lemma 1.2.11, the dimension of $(KG)_\alpha$ is S and from lemma 1.2.12, the number of elements in the K-basis for $(KG)_\alpha$ is t; which equals the number of α-regular classes in G. Therefore s=t.

We now prove our last result in this section.

1.2.14. Lemma.

If α , β are equivalent factor sets of G, then the number of inequivalent irreducible projective representations of G with factor set α equals the number of inequivalent irreducible projective representations of G with factor set β .

Proof:

Let $\{P_1,\ldots P_t\}$ be a complete set of inequivalent irreducible projective representations of G with factor set α . Since β is equivalent to α , then by 1.1.2, $\alpha(x,y) = \mu(x)\mu(y)\mu(xy)^{-1}\beta(x,y)$, for all x,y G.

For each $i\varepsilon\{1,\ldots,t\}$ and all $x\varepsilon G$, define $P_i'(x) = \mu(x)\ P_i(x)$. Then

$$\begin{aligned} P_i'(x)P_i'(y) &= \mu(x)\mu(y)\ P_i(x)\ P_i(y) \\ &= \mu(x)\mu(y)\alpha(x,y)P_i(xy) \\ &= \mu(x)\mu(y)\alpha(x,y)\mu(xy)^{-1}P_i'(xy) \\ &= \beta(x,y)P_i'(xy), \end{aligned}$$

and P_i' is a projective representation of G with factor set β.

Clearly, $\{P_1', \ldots, P_t'\}$ is a set of inequivalent irreducible representations of G, for if P_i' was equivalent to P_j' for some $i,j \in \{1,\ldots,t\}$, then for all $x \in G$, we would have

$$Q^{-1} P_i'(x)\, Q = P_j'(x),$$

where Q is some non-singular matrix, so that P_i and P_j are equivalent; which is a contradiction.

CHAPTER TWO

SCHUR MULTIPLIERS AND CENTRAL EXTENSIONS

In this chapter, we are concerned with central extensions of finite groups. In particular, we consider the relationship between central extensions and projective representations. For a more complete treatment of the subject, we refer the reader to Suzuki [15, p. 245-268].

We begin with the concept of Schur Multipliers.

2.1. Schur Multipliers

Let α and α' be two factor sets of a group G, and define a function $\alpha\alpha':G\times G\to K^*$ by

$$\alpha\alpha'(x,y) = \alpha(x,y)\,\alpha'(x,y) \tag{2.1}$$

for all $x,y\in G$. Then for all $x,y,z\in G$,

$$\begin{aligned}\alpha\alpha'(x,y)\alpha\alpha'(xy,z) &= \alpha(x,y)\alpha'(x,y)\alpha(xy,z)\,\alpha'(xy,z)\\ &= \alpha(x,y)\alpha(xy,z)\,\alpha'(x,y)\,\alpha'(xy,z)\\ &= \alpha(x,yz)\,\alpha(y,z)\,\alpha'(x,yz)\,\alpha'(y,z)\\ &= \alpha(x,yz)\alpha'(x,yz)\alpha(y,z)\alpha'(y,z)\\ &= \alpha\alpha'(x,yz)\,\alpha\alpha'(y,z)\end{aligned}$$

so that $\alpha\alpha'$ is also a factor set of G.

For a factor set α of G, define its inverse α^{-1} for all $x,y\in G$, by

$$\alpha^{-1}(x,y) = (\alpha(x,y))^{-1}, \tag{2.2}$$

so that α^{-1} is also a factor set of G.

Then the set of all factor sets of a group G forms a group under the law of composition given in (2.1).

2.1.1. Definition

The group of factor sets of G defined above is called the group of 2-cocycles, denoted $Z^2(G,K^*)$.

Let $\delta : G \to K^*$ be a function on G and define a function $\mu_\delta : G \times G \to K^*$ by

$$\mu_\delta(x,y) = \delta(x)\delta(y)\delta(xy)^{-1} \tag{2.3}$$

where $\mu : G \to K^*$ is arbitrary.

Since $\mu_\delta(x,y)\mu_\delta(xy,z) = \delta(x)\delta(y)\delta(z)\delta(xyz)^{-1}$ and

$\mu_\delta(x,yz)\mu_\delta(y,z) = \delta(x)\delta(y)\delta(z)\delta(xyz)^{-1}$ implies

that $\mu_\delta(x,y)\mu_\delta(xy,z) = \mu_\delta(x,yz)\mu_\delta(y,z)$, it follows that μ_δ is a factor set of G.

We note that μ is a homomorphism from a group of K^*-valued functions on G to $Z^2(G,K^*)$. We now consider the subgroup $\mathrm{im}\,\mu \subseteq Z^2(G,K^*)$. Denote the image $\mathrm{im}\,\mu$ of μ by $B^2(G,K^*)$.

2.1.2. Definition

$B^2(G,K^*)$ is called a group of 2-Coboundaries. We now denote the factor group $Z^2(G,K^*)/B^2(G,K^*)$ by $H^2(G,K^*)$. Then $H^2(G,K^*)$ is the so called the second cohomology group of K (see, e.g. [15, p. 201]). The group $H^2(G,K^*)$ is usually known as the Schur Multiplier of G.

Let α and β be equivalent factor sets of G in the sense of definition 1.1.2. Then in view of the above discussion, α and β are congruent modulo $B^2(G,K^*)$. It is easily seen that equivalence of factor sets is an equivalence relation. Thus if $[\alpha]$ denotes the equivalence class containing α, then this class also contains every factor set of G which lies in $\alpha B^2(G,K^*)$. The set of all such classes $[\alpha]$ can thus be identified with $H^2(G,K^*)$.

For any two such classes $[\alpha]$, $[\gamma]$ in $H^2(G,K^*)$, define $[\alpha][\gamma] = [\alpha\gamma]$.

Also, let $[1]$ denote the class containing the trivial factor set; and define $[\alpha]^{-1} = [\alpha^{-1}]$. With this multiplication, it is an easy matter to show that $H^2(G,K^*)$ is an abelian group.

We now prove the following results on the schur multiplier of a finite group G. In what follows, K is an algebraically closed field.

<u>2.1.3. Lemma.</u> (see, e.g. [15, P. 251-252])

Every class $[\alpha]$ in $H^2(G,K^*)$ of order t contains a representative α' whose values are t^{th} roots of unity in K. Furthermore, α' is a nomalised factor set.

<u>Proof.</u>

Let the field K be of characteristic $P>0$. Since the order of $[\alpha]$ is t, then we may set $t=P^s q$, where $s\geq 0$ and q is such that $P\nmid q$. Then there exists a map $\mu:G\to K^*$ such that for

for all $x,y \in G$,

$$\alpha(x,y)^t = \mu(x)\mu(y)\mu(xy)^{-1} . \qquad (2.4)$$

Since the characteristic of K is p, then

$$\alpha(x,y)^t = (\alpha(x,y)^q)^{p^s} = \mu(x)\mu(y)\mu(xy)^{-1},$$

and

$$\alpha(x,y)^q = [\mu(x)\mu(y)\mu(xy)^{-1}]^{\frac{1}{p^s}}$$

$$= \mu(x)^{\frac{1}{p^s}} \mu(y)^{\frac{1}{p^s}} \mu(xy)^{\frac{-1}{p^s}} .$$

Therefore, $[\alpha]$ cannot be of order t unless $p^s=1$. Therefore $p \nmid t$.

Now, using (2.4) for each $x \in G$, we can find another map $\nu : G \to K^*$ such that

$$\nu(x)^t = \mu(x)^{-1};$$

and letting $\alpha'(x,y) = \nu(x)\nu(y)\nu(xy)^{-1}\alpha(x,y)$, we get

$$\alpha'(x,y)^t = 1,$$

So that the values of α' are t^{th} roots of unity in K.

Now to prove that α' is normalised, we set

$\nu(x) = [\alpha(x,x^{-1})]^{-\frac{1}{2}}$, for all $x \in G$. Then

$$\alpha'(x,x^{-1}) = \nu(x)\nu(x^{-1})\nu(x.x^{-1})\alpha(x,x^{-1})$$

$$= (\alpha(e,e^{-1}))^{\frac{1}{2}}[\alpha(x,x^{-1})\alpha(x^{-1},x)]^{-\frac{1}{2}}\alpha(x,x^{-1})$$

$$= [\alpha(x,x^{-1})\alpha^{-1}(x,x^{-1})]^{\frac{1}{2}}$$

$$= 1,$$

So that α' is a normalised factor set.

2.1.4. Theorem (see, e.g. [15, P. 251-252]).

The order of every element of $H^2(G,K^*)$ is a factor of $|G|$, the order of G, and $H^2(G,K^*)$ is a finite group.

Proof.

We note that $H^2(G,K^*)$ is an abelian group. Let $|G|$ be the order of G and α a factor set of G. Then, for fixed elements $x,y\varepsilon G$, we have

$$\alpha(x,zy)\alpha(z,y) = \alpha(xz,y)\alpha(x,z),$$

so that if we set

$$\mu(x) = \prod_{y\varepsilon G}\alpha(x,y),$$

where $\mu:G\to K^*$ has the same meaning as in (2.4), then $\mu(x)\mu(z)\mu(xz)^{-1} = \alpha(x,z)^{|G|} = 1$, and $[\alpha]$ is such that $[\alpha]^{|G|} = 1$, thus proving the first part of the theorem.

Now, by lemma 2.1.3, and since from above, t divides $|G|$, it follows that there is a finite number of elements in $H^2(G,K^*)$, and thus $|H^2(G,K^*)|$ is finite, which completes the proof of the theorem.

2.2. Central Extensions

A normal subgroup N of a group H determines the factor group H/N. We write G = H/N and call H an extension of G by N.

Given a finite group G, we now consider a natural way in which the projective representations of G may arise. We first consider the following.

2.2.1. Definition

A central extension of a group G is an exact sequence

$$1 \to N \to H \xrightarrow{\phi} G \to 1$$

such that $N \subseteq Z(H)$, the centre of H.

A central extension is thus characterised by a pair (H,ϕ) of a group H together with a homomorphism $\phi: H \to G$, such that $\ker \phi$ lies in $Z(H)$. That is, $H/N \cong G$ and $N \subseteq Z(H)$, where $N = \ker \phi$. For this reason, a central extension of a group G shall be denoted simply by (H, ϕ).

Let (H, ϕ) be a central extension of G and $N = \ker \phi$. Let $\{\gamma(x) \mid x \in G\}$ be a set of coset representatives for N in H which are in 1-1 correspondence with elements of G, and let $H = \bigcup_{x \in G} \gamma(x)N$ be a left coset decomposition of H mod N. We define an element $n(x,y) \in N$ for all $x, y \in G$ by

$$\gamma(x)\gamma(y) = n(x,y)\gamma(xy). \tag{2.5}$$

Then the associative law in H gives that, for $x, y, z \in G$.

$$(\gamma(x)\gamma(y))\gamma(z) = n(x,y)\gamma(xy)\gamma(z)$$
$$= n(x,y)n(xy,z)\gamma(xyz)$$

and

$$\gamma(x)(\gamma(y)\gamma(z)) = n(y,z)\gamma(x)\gamma(yz)$$
$$= n(y,z)n(x,yz)\gamma(xyz),$$

so that $n(x,y)n(xy,z) = n(y,z)n(x,yz)$. (2.6)

As a convention, we fix $\gamma(e) = 1$. We now state the following result proved by Haggarty and Humphreys [6, P. 177].

2.2.2. Proposition

With the above notation we may choose a set $\{\gamma(x)/x\in G\}$ of coset representatives for N in H with an isomorphism $\theta:G\to H/N$ defined by $\theta(x)=\gamma(x)N$, for all $x\in G$, and also such that $\gamma(x_1)$ is conjugate to $\gamma(x_2)$ in H whenever x_1 is conjugate to x_2 in G.

Let ψ denote a character of a 1-dimensional ordinary representation of N, and for all $x,y\in G$, set

$$\alpha(x,y) = \psi(n(x,y)). \tag{2.7}$$

Then (2.6) now implies that, for all $x,y\in G$,

$$\alpha(x,y)\alpha(xy,z) = \alpha(x,z)\alpha(x,yz) \tag{2.8}$$

so that α is a factor set of G.

2.2.3. Definition (see [6, p. 178])

A factor set α defined in (2.7) above is known as a special factor set of G.

Let T be a linear representation of H of degree d and let (H,ϕ) and N be as above. Since $nx = xn$ for all $n\in N$, $x\in G$, then Schur's lemma now implies that elements of N are mapped into scalar multiples of I_d. That is

$$T(n) = \alpha(n)I_d, \text{ for some } \alpha(n)\in \mathrm{Hom}(N,K^*).$$

In particular, we may now set $Tn(x,y) = \alpha(x,y)I_d$, for all $x,y\in G$. If we define $P(x) = T(\gamma(x))$, $x\in G$, then

$$\begin{aligned} P(x)P(y) &= T(\gamma(x))T(\gamma(y)) \\ &= T(\gamma(x)\gamma(y)) \\ &= T(n(x,y)\gamma(xy)) \\ &= T(n(x,y))T(\gamma(xy)) \text{ (since } T \text{ is linear)} \\ &= \alpha(x,y)P(xy), \end{aligned}$$

for all $x,y \in G$.

Thus, P is a projective representation of G with factor set α which arises from a linear representation T.

2.2.4. Definition

Let (H,ϕ) be a central extension of a group G. A projective representation P of G which arises from an ordinary irreducible representation T of H by $P(x) = T(\gamma(x))$, is said to be a linearizable representation of G or to be 'linearized' by the linear representation T of H.

We prove the following result.

2.2.5 Theorem

Let G be a finite group, H an arbitrary group (not necessarily finite) with a normal abelian subgroup N such that $H/N \cong G$. Let $\psi \in \mathrm{Hom}(N,K^*)$ and α' be a special factor set obtained by

$$\alpha'(x,y) = \psi(n(x,y)).$$

Then the map $\xi : \mathrm{Hom}(N,K^*) \to H^2(G,K^*)$ defined by $\xi(\psi) = \alpha' B^2(G,K^*)$, is a homomorphism with $\ker \xi = (N \cap H')^{\perp}$, where H' is the

derived group of H. In particular, φ is an isomorphism if and only if $N \subseteq H'$.

Proof.

Let $H = \bigcup_{x \in G} N\gamma(x)$ and define $\gamma(x)\gamma(y) = n(x,y)\gamma(xy)$.

The first question to settle is, when does the special factor set α' lie in $B^2(G,K^*)$?

Suppose that $\alpha' \in B^2(G,K^*)$. Then $\alpha'(x,y) = \mu(x)\mu(y)\mu(xy)^{-1}$, for some $\mu \in B^2(G,K^*)$ with $\mu(e) = 1$. Let $\tau : H \to K^*$ be defined by

$$\tau(n(\gamma(x)) = \psi(n)\mu(x).$$

Then

$$\begin{aligned}\tau(\gamma(x))\tau(\gamma(y)) &= \mu(x)\mu(y) \\ &= \alpha'(x,y)\mu(xy) \\ &= \psi(n(x,y))\mu(xy) \\ &= \tau(n(x,y))\gamma(xy)\end{aligned}$$

and

$\tau(\gamma(e)) = \mu(e) = 1$, so that τ is a homomorphism.

Since $\tau(x^{-1}y^{-1}xy) = 1$, the restriction of τ to H' is $\tau|H' = 1$. Also $1 = \tau(\gamma(e)\gamma(e)^{-1}) = \psi(n(e,e^{-1})\mu(e)$

$$= \psi(e)\mu(e).$$

Therefore $\tau(n\gamma(e)\gamma(e)^{-1}) = \tau(n) = \psi(n)$

and

$\tau|N = \psi(n)$, $n \in N$. Thus, if $n \in N \cap H'$, then $\tau(n) = 1 = \psi(n)$.

Now, let $(N\cap H')^{\perp}$ be defined as follows:

$(N\cap H')^{\perp} = \{\lambda\in \mathrm{Hom}(N,K^*) \mid \lambda(n) = 1,\ n\in N\cap H'\}$. Then $\psi\in(N\cap H')\}$.

Conversely, suppose that $\psi\in (N\cap H')^{\perp}$. Let $\ell : NH'\to K^*$ be defined by $\ell(nx) = \psi(n)$. Then ℓ is a homomorphism which can be extended to a homomorphism $\phi : H\to K^*$ defined by

$$\begin{aligned}\phi(\gamma(x))\phi(\gamma(y)) &= \phi(\gamma(x)\gamma(y))\\ &= \phi(n(x,y)\gamma(xy))\\ &= \psi(n(x,y))\phi(\gamma(xy)).\end{aligned}$$

Now, if we let $\mu(x) = \phi(\gamma(x))$, then

$$\begin{aligned}\phi(\gamma(x))\phi(\gamma(y)) &= \psi(n(x,y))\phi(\gamma(xy)\\ &= \alpha'(x,y)\mu(xy)\end{aligned}$$

or

$$\mu(x)\mu(y) = \alpha'(x,y)\mu(xy) \quad \text{so that} \quad \alpha'\in B^2(G,K^*).$$

Thus, given a linear character ψ of N, the special factor set α' determined by ψ lies in $B^2(G,K^*)$ if and only if $\psi\in (N\cap H')^{\perp}$

Now, if we define $\xi : \mathrm{Hom}(N,K^*)\to H^2(G,K^*)$ by $\xi(\psi) = \alpha' B^2(G,K^*)$, then ξ will be a homomorphism with $\ker\xi = \{\psi\in \mathrm{Hom}(N,K^*) \mid \psi(n) = 1\} = (N\cap H')^{\perp}$.

In particular, ξ is an isomorphism if and only if $N\subseteq H'$, that is $N = \ker\xi$ is trivial.

2.2.6. Definition.

Let ψ be a linear character of N such that for all $x,y\in G$,

$$\alpha(x,y) = \psi(n(x,y)).$$

Then $\xi: \mathrm{Hom}(N,K^*) \to H^2(G,K^*)$ given by $\xi(\psi) = \alpha B^2(G,K^*)$, is called the <u>standard map</u>.

This definition implies that given a factor set α, the standard map determines a unique special factor set contained in $[\alpha]$.

The following result gives a necessary and sufficient condition for a given projective representation of G to be linearized.

2.2.7. Theorem

Let (H,ϕ) be a finite central extension of G with $\ker \phi = N$ and ξ be the associated standard map. Then the projective representation P of G with factor set α is linearized by a representation T of H if and only if $[\alpha] \in \mathrm{Im}\ \xi$.

Proof.

Let $\{\gamma(x) \mid x \in G\}$ be a set of coset representatives of H/N, with $\phi(\gamma(x)) = x$ and $n(x,y) = \gamma(x)\gamma(y)\gamma(xy)^{-1}$. Let ψ be a linear character of N and suppose that

$$\xi(\psi) = [\alpha] \in H^2(G,K^*).$$

Then a factor set α' given by $\alpha'(x,y) = \psi(n(x,y))$, for all $x,y \in G$, is equivalent to α. That is $\alpha'(x,y) = \mu(x)\mu(y)\mu(xy)^{-1}\alpha(x,y)$ for some $\mu: G \to K^*$.

Now, define a linear representation $T: H \to GL(V)$ by

$$T(n(\gamma(x))) = \psi(n)P(x)\mu(x), \text{ for all } x \in G,\ n \in N.$$

Then $T(\gamma(x))T(\gamma(y)) = \mu(x)P(x)\mu(y)P(y)$

$$= \alpha(x,y)P(xy)\mu(x)\mu(y)$$

$$= \alpha(x,y)\mu(x)\mu(y)P(xy)$$

$$= \alpha'(x,y)\mu(xy)P(xy)$$

$$= \alpha'(x,y)T(\gamma(xy))$$

$$= T(\eta(x,y)\gamma(xy))$$

$$= T(\gamma(x)\gamma(y)),$$

so that T is a linear representation of H which linearizes P.

Conversely, suppose that P is a projective representation of G linearizable in H. That is $T(\gamma(x)) = P(x)\mu(x)$, where T is some linear representation of H and $\mu: H \to K^*$. Therefore $P(e) = T(e)P(e)^{-1}$ and

$$T(n) = \mu(n)P(e) = \mu(n)\mu(e)^{-1}T(e), \quad n \in N.$$

Thus $\psi(n) = \mu(n)$ is a linear character of N.

Now, set $\eta(x) = \mu(\gamma(x))$.

Then $\alpha'(x,y)T(\gamma(xy)) = T(\eta(x,y)\gamma(xy))$

$$= T(\gamma(x)\gamma(y))$$

$$= T(\gamma(x)T(\gamma(y))$$

$$= \eta(x)P(x)\eta(y)P(y)$$

$$= \eta(x)\eta(y)P(x)P(y).$$

Therefore $\alpha'(x,y)\eta(xy)P(xy) = \alpha(x,y)\eta(x)\eta(y)P(xy)$, since $T(\gamma(xy)) = \mu(\gamma(xy))P(xy)$ and $\mu(\gamma(xy)) = \eta(xy)$. That is $\alpha'(x,y) = \eta(x)\eta(y)\eta(xy)^{-1}\alpha'(x,y)$ so that the special factor set α' is equivalent to the factor set α of P.

Thus $\xi(\psi) = [a'] = [a] \in H^2(G,K^*)$. That is a projective representation of P with factor set a is linearizable if and only if $[a]$ lies in the image $\mathrm{Im}\,\xi$ of ξ.

2.2.8. Definition

A representation group H of a group G is a finite group H of lowest possible order, which is a central group extension of G, such that every projective representation P of G occurs as a representation linearized from a linear representation of H.

We prove the following:

2.2.9. Corollary

Let (H,ϕ) be a finite central extension of G with $\ker \phi = N$, and ξ be the standard map.

(i) If $N \subseteq H'$, then N is isomorphic to a subgroup of $H^2(G,K^*)$;

(ii) Assume $|N| = |H^2(G,K^*)|$. Then $N \subseteq H'$ if and only if every projective representation of G is linearized by a representation of H.

Proof.

Let $\xi: \mathrm{Hom}(N,K^*) \to H^2(G,K^*)$ be defined by $\xi(\psi) = a\, B^2(G,K^*)$. Then $\ker \xi = (N \cap H')^{\perp}$ and $\mathrm{Im}\ \xi \subseteq H^2(G,K^*)$ by theorem 2.2.5.

By the first isomorphism theorem,

$\mathrm{Hom}(N,K^*)/(N\cap H')^{\perp}$ is isomorphic to a subgroup of $H^2(G,K^*)$. Infact, $\mathrm{Hom}(N\cap H', K^*)=\mathrm{Hom}(N,K^*)/(N\cap H')^{\perp}$ since $N\subseteq H'$.

Also, $N\cap H'=\mathrm{Hom}(N\cap H',K^*)$, and therefore if $N\subseteq H'$, then N is isomorphic to a subgroup of $H^2(G,K^*)$, which proves (i).

Now, suppose that $N\cap H'\neq\{e\}$. Then $\mathrm{Im}\,\xi\neq\{e\}$, which gives rise to trivial factor sets, hence, the corresponding projective representations are linear representations of G.

If $N\cap H'\neq\{e\}$, then α is not a trivial factor set. In particular, $\mathrm{Im}\,\xi\neq\{e\}$ and so $N\subseteq H'$ implies that every projective representation of G is linearizable by theorem 2.2.7. Furthermore, this is the only time when projective representations of G are linearizable. In this case $N=H^2(G,K^*)$.

We now prove the following result.

2.2.10. Theorem

We maintain the above notation.

Let G be a finite group. Then G has at least one representation group H of order $|H^2(G,K^*)|\,|G|$.

Furthermore, the kernel, $\ker\phi$ of the homomorphism $\phi: H\to G$ is isomorphic to $H^2(G,K^*)$.

<u>Proof:</u> (see, e.g. [6, P. 182-183])

Being a finite and abelian group, we can express $H^2(G,K^*)$ as a direct product of cyclic groups;

$$H^2(G,K^*) \approx \langle [\alpha^{(1)}]\rangle \times \ldots \times \langle [\alpha^{(n)}]\rangle ,$$

where each generator $[\alpha^{(i)}]$ is of order t_i. Since the $\alpha^{(i)}(x,y)$ are roots of unity, we have $\alpha^{(i)}(x,y) = \xi_i^{a^{(i)}_{x,y}}$, where $0 \le a^{(i)}_{x,y} \le t_i - 1$ and ξ_i is a t_i^{th} root of unity. From the property

$$\alpha(x,y)\alpha(xy,z) = \alpha(x,yz)\alpha(y,z), \text{ we have}$$

$$a^{(i)}_{x,y} + a^{(i)}_{xy,z} \equiv a^{(i)}_{x,yz} + a^{(i)}_{y,z} \pmod{t_i}. \tag{2.9}$$

If $[\alpha] \in H^2(G,K^*)$, then α is equivalent to β where

$$\beta(x,y) = (\alpha^{(1)}(x,y))^{\ell_1} \times (\alpha^{(2)}(x,y))^{\ell_2} \times \ldots \times (\alpha^{(r)}(x,y))^{\ell_r}$$

$$= (\xi_1^{\ell_1})^{a^{(1)}_{x,y}} \times (\xi_2^{\ell_2})^{a^{(2)}_{x,y}} \times \ldots \times (\xi_r^{\ell_r})^{a^{(r)}_{x,y}} \tag{2.10}$$

$(0 \le \ell_i \le t_i - 1)$.

Now, let $\ker \phi = A$, and let $a_1, a_2, \ldots, a_n$ be the generators of A corresponding to the generators $[\alpha^{(i)}]$ of $H^2(G,K^*)$. For each $x,y \in G$, let $a(x,y) \in A$ be defined by

$$a(x,y) = \prod_{i=1}^{n} a_i^{a^{(i)}_{x,y}} .$$

$$a(x,y)a(xy,z) = a(x,yz)a(y,z)$$

Then $a(x,yz)a(y,z) = a(x,y)a(xy,z)$ (2.11)

by (2.9) above.

If for $\chi \in \mathrm{Hom}(A,K^*)$, we define

$$\psi_\chi(x,y) = \chi(a(x,y)),$$

then $\psi_\chi(x,y) = \prod_{i=1}^{n} \chi(a_i)^{a^{(i)}_{x,y}}$

That is $\psi_\chi \in H^2(G,K^*)$; that is as χ runs through all the linear characters of A, ψ_χ runs through the elements of $H^2(G,K^*)$.

Now, set $H = \{(x,a) \mid x \in G,\ a \in A\}$ with a composition on H being defined as follows:

for $x,y \in G$, and $a,b \in A$,

$$(x,a)(y,b) = (xy, a(x,y)ab).$$

Then H is a group with $\{(1,a) \mid a \in A\} \cong A$ contained in $Z(H)$, the centre of H.

Furthermore, if we let $\nu(x) = (x,1)$, for all $x \in G$, $1 \in A$, then $\{\nu(x) \mid x \in G\}$ is a set of coset representatives of H mod A.

Therefore $H/A \cong G$ and H is a central extension of G.

Now, to show that every projective representation of G can be linearized by a linear representation of H, let $P: G \to GL(V)$ be a projective representation of G with factor set α. Then, as in theorem 2.2.7, there exists a linear character ψ of A such that

$$\psi(a(x,y)) = \alpha(x,y), \text{ for all } x,y \in G.$$

Now, let $T:H\to GL(V)$ be defined by $T(\nu(x)a) = P(x)\psi(a)$. Then

$$P(x)P(y) = T(\nu(x))T(\nu(y)) = T(\nu(x)\nu(y))$$
$$= T(a(x,y)\nu(xy)) = \psi(a(x,y))P(xy)$$
$$= a(x,y)P(xy) .$$

so that P is a projective representation of G with factor set a, which is linearized by T.

That $|H| = |H^2(G,K^*)| \cdot |G|$ follows from the fact that $H/A \cong G$ and that $A \cong H^2(G,K^*)$. Thus, H is a representation group of G.

Let (H,ϕ), N and $n(x,y) \in N$ be as before. Then we have

2.2.11. Lemma

$$N = gp\{n(x,y) \mid x,y \in G\}.$$

Proof. (see Haggarty and Humphreys[6, P. 179])

2.2.12. Definition

A stem extension of a group G is a pair (H,ϕ) such that

$$1 \to \ker\phi = N \to H \xrightarrow{\phi} G \to 1$$

is an exact sequence of groups and $N \subseteq H' \cap Z(H)$

2.2.13. Lemma.

Let K be an algebraically closed field. Then G has atmost a finite number of inequivalent irreducible projective representations.

Proof

Proof

Suppose that P is an irreducible projective representation of G. Then P is linearizable by a linear representation T of H; that is

$P(x) = T(\gamma(x))$, for all $x \in G$.

Suppose that P' is another irreducible projective representation of G, then $P'(x) = T'(\gamma(x))$, for all $x \in G$. That is P' can also be linearized, and so on.

Thus, the number of inequivalent irreducible projective representations of G is less than or equal to the number of inequivalent irreducible linear representations of H.

But the number of inequivalent linear representations of H is known to be finite. Hence G has a finite number of inequivalent irreducible projective representations.

2.2.14. Theorem

The degrees of the irreducible projective representations of a finite group G divide $|G|$.

Proof.

Let P be a projective representation of G. Then P is linearizable in H.

Set $P(x) = T(\gamma(x))$.

Hence, degree of P equals the degree of T, where T is the linear representation of H.

But degree of T divides $|H:Z(H)|$ (result due to Huppert, B.). Since $N \subseteq Z(H)$ and $H/N \cong G$, then degree of T divides $|H/N| = |G|$.

Hence, degree of P divides $|G|$.

The following result due to Schur gives a characterization of a representation group H of G; and will be required in chapter 3 where we consider projective characters of finite groups.

2.2.15. Theorem (I. Schur)

Given a finite group G, there exists a group H, a representation group of G, such that H has a central subgroup N with

(i) N contained in the derived group of H,

(ii) $H/N \cong G$, and

(iii) $|N| = |H^2(G,K^*)|$.

CHAPTER THREE

PROJECTIVE CHARACTERS OF FINITE GROUPS

In this chapter, we consider projective characters of a finite group G in terms of characters of linear representations of its representation group H, and investigate those properties of projective characters which are analogues of properties of linear characters.

3.1 Projective characters

Let P be a projective representation of a group G with factor set α.

3.1.1. Definition.

The projective character χ of a projective representation P is defined by

$$\chi(x) = \text{trace } P(x), \quad x \in G.$$

Given that (H, ϕ) is a stem extension of G, we define the projective characters of G in three stages:

(i) Assume that α is a special factor set, and let $P_1, \ldots, P_m$ be representatives of the linear equivalence classes of irreducible projective representations of G with factor set α [6, P. 179]. Then, each of the P_i can be linearized by linear irreducible representations T_i of H such that for all $x \in G$,

$$P_i(x) = T_i(\gamma(x)) \quad (i=1,\ldots,m).$$

We define the <u>projective character</u> χ_i of P_i by

$$\chi_i(x) = \text{trace } T_i(\gamma(x)).$$

(ii) Given any irreducible projective representation P of G with factor set α, we know that there exists a unique special factor set α' in the class $[\alpha]$ containing α (see, e.g.[6,P. 179]). Thus P is projectively equivalent to a representation P' with factor set α'. We define the projective character of P to be that of P'.

(iii) Let P be any projective representation of G with factor set α, which is linearly equivalent to a direct sum of irreducible projective representations $P_1,\ldots,P_n$ each with factor set α. We define the projective character of P to be (see, e.g.[6, P.179]) the sum of the projective characters of $P_1,\ldots,P_n$.

We now prove the following result which is a consequence of proposition 2.2.2.

<u>3.1.2. Lemma.</u>

Let P be a projective representation of G with simple factor set α. Then the projective character χ is a class function on G.

<u>Proof:</u>

Let g be α-regular in G and x be any element of G. Then

$$P(x)P(g)P(x^{-1}) = \alpha(x,g)P(xg)P(x^{-1})$$

$$= \alpha(x,g)\,\alpha(xg,x^{-1})P(xgx^{-1})$$

$$= f_\alpha(x,g)P(xgx^{-1}) \text{ by } (1.1)$$

$$= P(xgx^{-1}), \text{ since } \alpha \text{ is simple.}$$

i.e. $P(g)$ and $P(xgx^{-1})$ are similar matrices. Taking traces, we have,

$$\text{trace } P(x)P(g)P(x^{-1}) = \text{trace } P(xgx^{-1}).$$

i.e. $$\chi(g) = \chi(xgx^{-1}).$$

The following result is proved in Haggarty and Humphreys [6, P.179-181].

3.1.3. Theorem

Let P_1 and P_2 be projective representations of G with special factor set α. Let $\chi_i (i=1,2)$ be their respective characters. Then P_1 and P_2 are projectively equivalent if and only if there exists a one-dimensional linear character λ of G such that for all $x \in G$

$$\chi_1(x) = \lambda(x)\chi_2(x).$$

3.2 Properties of Projective Characters

In this section, we consider projective representations defined over $\mathbb{C}$, the field of complex numbers. For any group G, let τ_1, τ_2 be two class functions on G. Define an inner product as follows:

$$\langle \tau_1, \tau_2 \rangle_G = |G|^{-1} \sum_{x \in G} \tau_1(x)\,\overline{\tau_2(x)}, \tag{3.1}$$

where $\bar{\tau}_2$ is the complex conjugate of τ_2.

Let χ_1, χ_2 be irreducible projective characters of G and let ξ_1, ξ_2 be the linear (ordinary) characters of H such that $\xi_i(\gamma(x)) = \chi_i(x)$ $(i=1,2)$. Suppose that ξ_i determines the linear character ψ_i of N. Since the $\chi_i (i=1,2)$ are class functions of G, and since

$$T_i(\gamma(x)^{-1}) = T_i(n(x^{-1},x))\gamma(x^{-1})$$
$$= \psi_i(n(x^{-1},x))T_i(\gamma(x^{-1})),$$

it follows that (see [6, P.188])

$$\bar{\chi}_2(x) = \psi_2(n(x^{-1},x))\chi_2(x^{-1}). \tag{3.2}$$

Therefore

$$\langle \chi_1, \chi_2 \rangle_G = |G|^{-1} \sum_{x \in G} \psi_2(n(x^{-1},x))\chi_1(x)\chi_2(x^{-1}). \tag{3.3}$$

<u>3.2.1 Lemma</u> (see, e.g. [6, P. 188]).

Maintaining the above notation, then

$$\langle \xi_1, \xi_2 \rangle_H = \langle \psi_1, \psi_2 \rangle_N \langle \chi_1, \chi_2 \rangle_G .$$

<u>Proof:</u>

Using the adoptation $\gamma(x) = (x,a)$ (see 2.2), the proof follows from

$$\xi_i(x,a) = \psi_i(a) \otimes \chi_i(x), \qquad x \in G, \quad a \in N.$$

ie. $$\langle \xi_1, \xi_2 \rangle_H = |N \times G|^{-1} \sum_{(x,a) \in H} \xi_1(x,a)\, \xi_2(x,a)^{-1}$$

$$= |N|^{-1} |G|^{-1} \sum_{a \in N} \sum_{x \in G} \psi_i(a) \chi_i(x) \psi_i(a^{-1}) \chi_i(x^{-1})$$

$$= [\, |N|^{-1} \sum_{a \in N} \psi_i(a)\, \psi_i(a^{-1})\,][\, |G|^{-1} \sum_{x \in G} \chi_i(x) \chi_i(x^{-1})\,]$$

$$= \langle \psi_1, \psi_2 \rangle_N \, \langle \chi_1, \chi_2 \rangle_G .$$

3.2.2. Corollary

Let P_1 and P_2 be irreducible projective representations of G with special factor set $\psi(n(x,y))$. If χ_i is the projective character of $P_i (i=1,2)$, then

$$\langle \chi_1, \chi_2 \rangle_G = \begin{cases} 1, & \text{if } P_1 \text{ is linearly equivalent to } P_2 \\ 0 & \text{otherwise} \end{cases}$$

Proof.

If P_1 is linearly equivalent to P_2, then $\chi_1 = \chi_2$ (see, corollary 1.4, [6, P.181]), so that $\xi_1 = \xi_2$ and hence $\langle \chi_1, \chi_2 \rangle_G = 1$ from the orthogonality relations in H, since $\xi_i(x,a) = \psi_i(a) \otimes \chi_i(x)$.

If P_1 and P_2 are inequivalent, then from the orthogonality relations in H, we have $\langle \chi_1, \chi_2 \rangle_G = 0$.

3.2.3. Corollary

Let P be a projective representation of G with factor set α and character χ. Then P is irreducible if and only if $\langle \chi, \chi \rangle_G = 1$.

Proof.

If P is irreducible, then $\langle \chi\chi \rangle_G = 1$, by 3.2.2.

Conversely, if $\langle \chi , \chi \rangle_G = 1$, we have P is equivalent to P' where

$$P'(x) = \begin{pmatrix} P_1(x) & & & 0 \\ & P_1(x) & & \\ & & \ddots & \\ 0 & & & P_n(x) \end{pmatrix},$$

where the $P_i (i=1,\ldots,n)$ are irreducible projective representations with factor set a, and the P_i's occur with multiplicity a_i in P.

Let χ_i be the character of P_i. Since P is equivalent to P', we have

$$\chi(x) = \sum_{i=1}^{n} a_i \chi_i(x).$$

Therefore

$$\langle \chi, \chi \rangle_G = |G|^{-1} \sum_{x \in G} \chi(x)\, \chi(x^{-1})$$

$$= \sum_{x \in G} |G|^{-1} [\sum_{1}^{n} a_i \chi_i(x)] [\sum_{1}^{n} a_i \chi_i(x^{-1})]$$

$$= \sum_{1}^{n} a_i^2 [|G|^{-1} \sum_{x \in G} \chi_i(x)\, \chi_i(x^{-1})]$$

$$= \sum_{1}^{n} a_i^2 = 1.$$

Since the a_i's are positive integers, then $a_s = 1$ for some $1 \leq s \leq n$ and $a_t = 0$ for all $t \neq s$. i.e P' has only one

irreducible component, vis, P_s. i.e P' is irreducible, and hence P is irreducible.

The following analogue of the second orthogonality relation may be deduced in a similar manner to the linear case.

3.2.4. Lemma (see [6, P. 188]).

Let $P_1, P_2, \ldots, P_n$ form a complete set of irreducible projecitve representations of G with factor set α and degrees $d_1, d_2, \ldots, d_n$, respectively. Let $\chi_i (i=1,\ldots,n)$ be the projective character of P_i and let $\ell_1, \ell_2, \ldots \ell_n$ be the α-regular classes of G, with x_i as the representative for $\ell_i (i=1,\ldots,n)$.

Then

$$\sum_{i=1}^{n} \chi_i(x_j)\overline{\chi}_i(x_k) = | C_G(x_j) | \ \delta_{jk},$$

where $C_G(x_j)$ is the centralizer of x_j in G.

3.2.5. Remark

The above result implies that if x is an α-regular element of G, then there exists an irreducible projective representation with character χ say, for which $\chi(x) \neq 0$.

We now state a necessary and sufficient condition for an element x of G to be α-regular.

The following is proved in [6, P. 189] .

3.2.6. Lemma

An element $x \in G$ is α-regular if and only if there exists an irreducible projective character χ of G with factor set α such that $\chi(x) \neq 0$.

3.3. Induced projective Characters

Let G be a finite group, M its subgroup and let K be an algebraically closed field of characteristic zero. Denote by W, a $(KG)_\alpha$-module. Then W may be restricted to $(KM)_\alpha$-module. Call this restriction W_M.

Let P be a projective representation of G with factor set α which W affords and let P_M be a projective representation of M afforded by W_M. Then (see [6]) α determines a factor set α_M of M by restriction, and

$$P_M(m) = P(m), \text{ for all } m \in M. \tag{3.4}$$

Let χ and χ_M be the characters of these representations. Then (see [6]) if χ is irreducible, χ_M is not irreducible in general.

Since an element g of M which is α_M-regular need not be α-regular in G (see section 5.3), the theory of projective characters is much more complicated than that of ordinary characters.

We now consider a construction which associates with each $(KM)_\alpha$-module W, a $(KG)_\alpha$-module W^G.

3.3.1. Definition

Let M be a subgroup of G and W be a left $(KM)_\alpha$-module. Then we may form a left $(KG)_\alpha$-module W^G:

$$W^G = (KG)_\alpha \otimes_{(KM)_\alpha} W,$$

which is said to be <u>induced</u> from M.

A representation of G afforded by W^G is called an <u>induced representation</u> of G, denoted P^G.

Let $P(x) = (S_{ij}(x))$ be a projective representation of M afforded by the left $(KM)_\alpha$-module W. Choose $[G:M] = n$ and let $\{v_1, \ldots, v_r\}$ be a K-basis for W. Let

$G = \bigcup_{i=1}^{n} x_i M$ be the right coset decomposition of G mod. M, where $\{x_i\}$ is the transversal and $|\{x_i\}| = n$. Then each element $x \in G$ has an expression of the form

$$x_i m, \quad 1 \leq i \leq n, \quad m \in M. \tag{3.5}$$

Therefore, each element in $(KG)_\alpha$-module W^G can be uniquely expressed in the form (see., e.g [6])

$$\sum_{i=1}^{n} \gamma(x_i) b_i \tag{3.6}$$

where $b_i \in (KM)_\alpha$. Therefore,

$$(KG)_\alpha = \gamma(x_1)(KM)_\alpha \oplus \ldots \oplus \gamma(x_n)(KM)_\alpha . \tag{3.7}$$

Thus $\{\gamma(x_1), \ldots, \gamma(x_n)\}$ is a basis for $(KG)_\alpha$ and hence

$$W^G = (\gamma(x_1)(KM)_\alpha \otimes_{(KM)\alpha} W) \oplus \ldots \oplus (\gamma(x_n)(KM)_\alpha \otimes_{(KM)\alpha} W) \qquad \ldots (3.8)$$

and so, $\{\gamma(x_i) \otimes v_j \mid i=1,\ldots,n;\ j=1,\ldots,r\}$ (3.9

forms a K-basis for W^G.

Also, from (3.8) and (3.9), we have

$$(W^G:K) = [G:M](W:K). \qquad (3.10)$$

Now, if we express $\gamma(x)(\gamma(x_i) \otimes v_j)$ as a K-linear combination of basis elements, we have

$$\begin{aligned}
\gamma(x)(\gamma(x_i) \otimes v_j) &= \alpha(x,x_i)\gamma(xx_i) \otimes v_j \\
&= \alpha(x,x_i)\gamma(x_k x_k^{-1} x x_i) \otimes v_j, \quad k \in \{1,2,\ldots,n\} \\
&= \alpha(x,x_i)\alpha^{-1}(x_k, k_k^{-1} x x_i)\gamma(x_k)\gamma(x_k^{-1} x x_i) \otimes v_j \\
&= \alpha(x,x_i)\alpha^{-1}(x_k, x_k^{-1} x x_i)\gamma(x_k) \otimes \gamma(x_k^{-1} x x_i) \otimes v_j \\
&= \alpha(x,x_i)\alpha^{-1}(x_k, x_k^{-1} x x_i)\gamma(x_k) \otimes \sum_{\ell=1}^{r} S_{\ell_j}(x_k^{-1} x x_i) v \\
&= \sum_{\ell=1}^{r} \alpha(x,x_i)\alpha^{-1}(x_k, x_k^{-1} x x_i) S_{\ell_j}(x_k^{-1} x x_i) \\
&\qquad \gamma(x_k) \otimes v_\ell
\end{aligned}$$

Re-arranging the basis elements in the form $\gamma(x_1) \otimes v_1$, $\gamma(x_1) \otimes v_2, \ldots, \gamma(x_1) \otimes v_r, \gamma(x_2) \otimes v_1, \ldots, \gamma(x_n) \otimes v_1, \ldots$ $\gamma(x_n) \otimes v_r$, the above equation now implies that for each $x \in G$, we have

$$P^G(x) = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{cc} & \begin{array}{ccc} (i,1) & & (i,r) \end{array} \end{array} \\ \left(\begin{array}{c|ccc} * & & & \\ \hline * & & a(x,x_i)a^{-1}(x_k,x_k^{-1}xx_i)P(x_k^{-1}xx_i) & \\ \hline * & & & * \end{array}\right) \end{array} \begin{array}{l} k,1 \\ \\ (k,r) \end{array}$$

where P is extended to the whole of G by setting $P(x) = 0$ if $x \notin M$.

Now, if χ_M is the character of P and χ that of P^G, then

$$\chi(x) = \sum_{i=1}^{n} a(x,x_i)a^{-1}(x_i,x_i^{-1}xx_i)\chi(x_i^{-1}xx_i),\ldots \qquad (3.11)$$

which is the <u>formular</u> for the <u>induced projective character</u>.

CHAPTER FOUR

SCHUR MULTIPLIERS OF ROTATION SUBGROUPS OF WELY GROUPS

In this chapter, we consider Schur Multipliers of rotation subgroups of certain finite groups of orthogonal transformations generated by reflections. Our work follows that of Maxwell [7], who determined the Schur Multipliers of the rotation subgroups of Coxeter groups.

4.1. Weyl groups and their Rotation Subgroups

Let V be an ℓ-dimensional Real-Euclidean space with a positive definite inner product $(,)$. For each non-zero vector $r \in V$, let τ_r be the reflection in the hyperplane $\langle r \rangle^{\perp}$ perpendicular to r. τ_r is the linear map defined by

$$\tau_r(s) = s - \frac{2(r,s)}{(r,r)}.r \qquad \text{for all } s \in V.$$

4.1.1. Definition

A subset Φ of V is said to be a root system in V if the following axioms are satisfied:

(i) Φ is a finite subset of non-zero vectors which generates V;

(ii) If $r, s \in \Phi$, then $\tau_r(s) \in \Phi$;

(iii) If $r, \lambda r \in \Phi$ where $\lambda \in \mathbb{R}$, then $\lambda = \pm 1$.

A root system Φ is said to satisfy the Crystallographic Condition if in addition, $\frac{2(r,s)}{(r,r)}$ is a rational integer, for any $r,s \in \Phi$.

4.1.2. Definition

Let Φ be a root system in V. A subset π of Φ is called a fundamental (or simple) system of vectors if

(i) Φ is linearly independent over $\mathbb{R}$; and

(ii) If $r \in \Phi$, then r is a linear combination of the elements in π in which all non-zero coefficients are either all positive or all negative. That is π is a basis for Φ.

We shall now define the Weyl groups and state some of their properties.

4.1.3 Definition

Let $W = W(\Phi) = \langle \tau_r / r \in \Phi \rangle$. Then W is a subgroup of the orthogonal group $O(\ell)$ called a Finite reflection group of Φ.

If, in particular, Φ satisfies the Crystallographic condition, then $W(\Phi)$ is called a Weyl group of type Φ.

The rank of $W(\Phi)$ is the dimension of V. The reflections τ_{r_i} corresponding to $r_i \in \pi$ are known as the fundamental (simple) reflections.

4.1.4. Definition

Let Φ be a root system. Then Φ is said to be irreducible if it cannot be decomposed into a union of two proper subsets, such that each root in one set is perpendicular to each root in

the other; otherwise, Φ is reducible. The group $W(\Phi)$ is said to be irreducible if its associated root system is, and reducible if otherwise.

In our discussions (also see chapter 5), we shall consider only irreducible Weyl groups, since any Weyl group of type Φ is isomorphic to the direct product of the respective Weyl groups of its irreducible components.

Let τ_i denote the reflection which corresponds to a root $r_i \varepsilon \pi$, and m_{ij} be the order of $\tau_i \tau_j$. Thus, in particular, $m_{ii} = 1$ for each $r_i \varepsilon \pi$ and $(\tau_i \tau_j)^{m_{ij}} = 1$ gives a set of relations in $W(\Phi)$.

4.1.5. Definition:

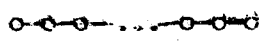
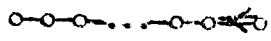
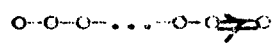
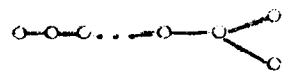
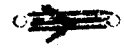
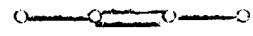
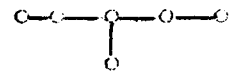
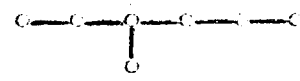
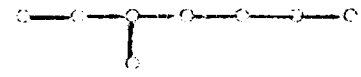
A coxeter group is a group G which has a subset C such that (i) every element in C is of order 2, and (ii) if for every pair τ_i, τ_j of elements in C, m_{ij} denotes the order of their product $\tau_i \tau_j$, then $(\tau_i \tau_j)^{m_{ij}} = 1$ are the defining relations of G.

Thus, every Weyl group is a Coxeter group. To each such Coxeter group, (see, e.g.[1]) there corresponds a graph, called a Coxeter graph, whose nodes are in 1-1 correspondence with the generators $\tau_i (r_i \varepsilon \pi)$, the bonds corresponding to τ_i and τ_j being joined by $m_{ij}-2$ bonds if $r_i \neq r_j$.

If the root system Φ is such that two lengths appear, the Coxeter graph fails to determine which of a pair of nodes should correspond to a short simple root, and which, to a long simple root. Whenever a double or a tripple bond occurs in the coxeter graph of Φ, an arrow is added which points to the shorter of

the two roots. The resulting diagram is known as the <u>Dynkin diagram</u> of Φ .

The irreducible Weyl groups have been classified [1] and correspond to the following types (with associated Dynkin diagrams):

Type	Diagram	Rank
A_ℓ		$\ell \geq 1$
B_ℓ		$\ell \geq 2$
C_ℓ		$\ell \geq 3$
D_ℓ		$\ell \geq 4$
G_2		2
F_4		4
E_5		6
E_7		7
E_8		8

The groups of types $A_\ell (\ell \geq 1)$, $B_\ell (\ell \geq 2)$, $C_\ell (\ell \geq 3)$ and $D_\ell (\ell \geq 4)$ are called the <u>classical Weyl groups</u>, and those of types $E_\ell (\ell = 6,7,8)$, F_4 and G_2 are called the exceptional <u>Weyl groups.</u>

The conjugacy classes of the individual Weyl groups are well known and Carter [3] has given a unified description of these in terms of certain admissible diagrams which are associated with the root system Φ (For details concerning these, we refer the reader to Carter's paper [3]). Alternatively, conjugacy classes

may be parametrized in terms of pairs of partitions

$(a = [1^{a_1} 2^{a_2} \ldots r^{a_r}]\ ;\ b[= \bar{1}^{-b_1} \bar{2}^{-b_2} \ldots \bar{s}^{-b_s}])$ of ℓ, where

$|a| + |b| = \ell$, r denotes a positive r-cycle and $\bar{s}$ denotes a negative s-cycle (see chapter 5). Thus, the conjugacy classes of $W(\Phi)$ are in 1-1 correspondence with pairs of partitions of ℓ.

Now let V be an ℓ-dimensional real Euclidean space, Φ be a root system in V with basis $\pi = \{r_1, \ldots, r_\ell\}$, and let $W(\Phi)$ be the Weyl group of Φ. If for each $r_i \epsilon \pi$ we let $\tau_i = \tau_{r_i}$, then the Weyl group $W(\Phi)$ has a presentation (see Bourbaki [1])

$$W(\Phi) = \langle \tau_i,\ i=1,\ldots,\ell / \tau_i^2 = 1,\ (\tau_i \tau_j)^{m_{ij}} = 1 \rangle, \qquad (4.1)$$

where m_{ij} are the integers defined in 4.1.5.

Each element $w \epsilon W(\Phi)$, can thus be expressed in the form

$$w = \tau_1 \tau_2 \ldots \tau_k (r_i \epsilon \pi). \qquad (4.2)$$

Let $\ell(w)$ denote the smallest value of k in any such expression for w. Then the above expression for w is said to be <u>reduced</u> if $\ell(w) = k$

<u>4.1.6. Definition</u>

We maintain the above notation. Let W^+ be the group defined as follows:

$$W^+ = \{w \epsilon W \mid \ell(w) \equiv 0 (\mathrm{mod}\ 2)\}.$$

Then following Maxwell [8], we call W^+ the <u>rotation sub group</u> of W, and W^+ is a subgroup of index 2 in W.

Now for $r_i, r_j \epsilon \pi$, let m_{ij} have the same meaning as in (4.1.5) and define an element g_i of $W^+(\Phi)$ by

$$g_i = \tau_i \tau_1 \quad (i \geq 2). \tag{4.3}$$

Then the rotation subgroup $W^+(\Phi)$ has the following presentation (see, e.g [1])

$$W^+(\Phi) = \langle g_i (i \geq 2) \mid g_i^{m_{i1}} = 1 (2 < i \leq \ell),\ (g_i g_j)^{m_{ij}} = 1 (2 \leq i < j \leq \ell) \rangle \tag{4.4}$$

4.2. Schur Multipliers of the rotation subgroups W^+ of W.

In this section, we state Maxwell's results on the Schur Multipliers of the rotation subgroups W^+ of W, for all cases of Φ, where Φ is a Crystallographic root system.

Let $\mathbb{C}$ be the field of complex numbers. Then the results of Maxwell are given by the following (c.f.[8]):

4.2.1. Theorem

If W is of type Φ then the schur multiplier of W^+ is as follows:

Type of Φ	$H^2(W^+, \mathbb{C}^*)$
$A_\ell (\ell \geq 3, \ell \neq 5,6)$	Z_2
A_5, A_6	$Z_2 \oplus Z_3$
$C_\ell (\ell \geq 4)$	$(Z_2)^2$
C_3	Z_2
$D_\ell (\ell \geq 5, \ell \neq 6,7)$	$(Z_2)^2$
D_4	$(Z_2)^3$
D_6, D_7	$(Z_2)^2 \oplus Z_3$
$E_\ell (\ell = 6,7,8)$	Z_2
F_4	$Z_2 \oplus Z_3$.

In the proof of this result, Maxwell [8] established that if α is a factor set in $Z^2(W^+,\mathbb{C}^*)$, and $[\alpha] \in H^2(W^+,\mathbb{C}^*)$, then there exists a solution in $GL(v)$ to the following equations:

$$T_2^3 = S_{22}I_v,\ T_i^2 = S_{ii}I_v(2<i\leq\ell),\ (T_iT_j)^{m_{ij}} = S_{ij}I_v(2\leq i<j\leq\ell) \quad (4.5)$$

where $T_i = T(g_i)(i\geq 2)$, and the S_{ij} are as given in Table 1 below. The T_i generate a projective representation $T:W^+\to GL(V)$ associated with the factor set α.

Table 1: Values to solutions to equations in (4.5)

Type of α	(S_{ij})
A_ℓ $(\ell\geq 3,\ \ell\neq 5,6)$	$S_{ij} = d$
A_5	$S_{ij} = d$(unless $i=2, j=4$), $S_{24} = de$
A_6	$S_{ij} = d$(unless $i=2,\ j=5$), $S_{25}=de$
C_ℓ $(\ell\geq 4)$	$S_{ij} = d$(unless $i=\ell-1, j=\ell$), $S_{\ell-1,\ell} = c$
C_3	$S_{22} = S_{33} = 1,\ S_{23} = m$
D_ℓ $(\ell\geq 5, \ell\neq 6,7)$	$S_{ij} = d$(unless $i=\ell-1, j=\ell$), $S_{\ell-1,\ell} = b$
D_4	$S_{22} = S_{23}=S_{24} = 1, S_{33}=m,\ S_{44}=n,$ $S_{34} = bmn$
D_6	$S_{ij} = d$(unless $i=2,\ j=4$ or $i=5,\ j=6$), $S_{24}=de$ $S_{56} = b$
D_7	$S_{ij} = d$(unless $i=2, j=5$ or $i=5, j=6$), $S_{25}=de$ $S_{56}=b$
E_ℓ $(\ell=6,7,8)$	$S_{ij}=d$
F_4	$S_{22} = S_{33} = S_{44} = 1,\ S_{23} = m,\ S_{24} = e.$

where the numbers b, c, d, m and n depend only on the class of factor set in $H^2(W^+, \mathbb{C}^*)$ and can assume the values ± 1, while e is a cube root of 1.

CHAPTER FIVE

THE PROJECTIVE CHARACTERS OF $W^+(D_6)$ AND $W^+(D_7)$

5.1. General Properties of the Weyl group $W(D_\ell)$ and its Rotation Subgroup $W^+(D_\ell)$

Before investigating the properties of the Weyl group $W(D_\ell)$ and its rotation subgroup $W^+(D_\ell)$, we first consider the Weyl group $W(C_\ell)$. The group $W(C_\ell)$ is generated by reflections $\{r_1,\ldots r_\ell\}$ (see e.g [4, p.117-129]) subject to relations [12]

$$r_i^2 = 1,\ i = 1,\ \ldots,\ell;$$

$$(r_i r_{i+1})^3 = 1,\ i=1,\ldots,\ell-2;$$

$$(r_{\ell-1} r_\ell)^4 = 1,$$

$$(r_i r_j)^2 = 1,\ |i-j|\geq 2. \qquad (5.1)$$

It is isomorphic to the hyperoctahedral group on ℓ elements which permutes the set $\{1,\ldots,\ell\}$ as well as changing the sign of any number of them.

The hyperoctahedral group is a semi-direct product $C_2^\ell \rtimes S_\ell$ of the normal subgroup C_2^ℓ and the symmetric group S_ℓ. Hence, it has order $2^\ell.\ell!$ (see e.g. [12]).

Let $r_i (i=1,2,\ldots,\ell-1)$ denote the transposition $(i,i+1)$ and $w_j (j=1,2,\ldots,\ell)$ denote the negative 1-cycle $(j-j)$ (see e.g [12, P. 131]), then $W(C_\ell)$ has the alternative presentation

$$W(C_\ell) = \langle r_i (1 \le i \le \ell-1), w_j (1 \le j \le \ell) \mid r_i^2 = (r_i r_{i+1})^3 = (r_i r_j)^2 = 1$$
$$|i-j| \ge 2,\ w_j^2 = 1,\ w_j w_k = w_k w_j,\ r_i w_i = w_{i+1} r_i,$$
$$r_i w_j = w_j r_i,\ j = i,\ i+1 \rangle \quad . \tag{5.2}$$

As in the case of the symmetric group, any element of $W(C_\ell)$ may be uniquely expressed as a product of disjoint cycles (see Read [12]). The k-cycle

$$\sigma = \begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ \pm a_2 & \pm a_3 & \cdots & \pm a_1 \end{pmatrix}$$

is said to be <u>positive</u> if $\sigma^k = 1$ and negative otherwise. An element of $W(C_\ell)$ which has a_1 positive 1-cycles, a_2 positive 2-cycles, ..., b_1 negative 1-cycles, b_2 negative 2-cycles,..., is said to have signed cycle-type $[1^{a_1} 2^{a_2} \ldots ; 1^{-b_1} 2^{-b_2} \ldots]$.

The Weyl group $W(C_\ell)$ has three subgroups of index 2 (see[16, P. 259-260]), and we discuss these below.

By ignoring sign changes, each element $w \in W(C_\ell)$ gives a uniquely defined element of $W(A_{\ell-1})$; the symmetric group of degree ℓ. If this map is defined by ρ, then this defines a map $\rho : W(C_\ell) \to W(A_{\ell-1})$ [12, P.132-133] such that $\rho[1^{a_1} 2^{a_2} \ldots ; 1^{-b_1} 2^{-b_2} \ldots] = [1^{a_1 + b_1} 2^{a_2 + b_2} \ldots]$. We say that w is <u>even</u> if $\rho(w)$ lies in $W^+(A_{\ell-1})$, the alternating group of degree ℓ and <u>odd</u> otherwise. The set

$$H = \{w \in W(C_\ell) / w \text{ is even}\} \tag{5.3}$$

is a subgroup of $W(C_\ell)$ of index 2 (see e.g. [16, P. 260]).

The group H may also be considered as a wreath product $S_2 \wr U_\ell$ (see [16, P.260]), where S_2 is the cyclic group of order 2. A class of conjugate elements of $W(C_\ell)$ splits in H if and only if it contains an element of type $[1^{a_1} 3^{a_2} \ldots ; 1^{-b_1} 3^{-b_2} \ldots]$, where $a_i, b_i = 0,1$.

The Rotation subgroup $W^+(C_\ell)$ of $W(C_\ell)$ consists of those even number of generators [16, P. 260]. That is, $W^+(C_\ell)$ consists of all elements of signed-cycle type of the form $[1^{a_1} 2^{a_2} \ldots ; 1^{-b_1} 2^{-b_2} \ldots]$ such that either

$$\text{(i)} \qquad \Sigma a_{2i} \equiv 0 (\text{mod } 2) \text{ and } \Sigma b_{2i-1} \equiv 0 (\text{mod } 2) \text{ or}$$

$$\text{(ii)} \qquad \Sigma a_{2i} \not\equiv 0 (\text{mod } 2) \text{ and } \Sigma b_{2i-1} \not\equiv 0 (\text{mod } 2) \tag{5.4}$$

The other subgroup of index 2 in $W(C_\ell)$ is the Weyl group $W(D_\ell)$ of type D_ℓ $(\ell \geq 4)$.

5.1.1. Definition

The group $W(D_\ell)$ is generated by the elements $r_1, \ldots, r_{\ell-1}, r_\ell r_{\ell-1} r_\ell$ of (5.1) (see [12]).

In terms of generators and relations, it has a presentation

$$W(D_\ell) = \langle \tau_1, \ldots \tau_\ell / \tau_i^2 = 1 (1 \leq i \leq \ell), (\tau_i \tau_{i+1})^3 = 1 (1 \leq i \leq \ell-2),$$
$$(\tau_{\ell-2} \tau_\ell)^3 = 1, \ (\tau_i \tau_j)^2 = 1 \ (i,j=1,\ldots,$$
$$\ell-2, |i-j| \geq 2) \rangle \tag{5.5}$$

An element of $W(C_\ell)$ belongs to $W(D_\ell)$ if and only if it changes the sign of an even number of basis elements (see e.g. [3, P. 26]). Thus, in terms of signed cycle-types, an element $w \in W(C_\ell)$ of type $[1^{a_1}2^{a_2}\ldots;\ 1^{-b_1}2^{-b_2}\ldots]$ lies in the subgroup $W(D_\ell)$ if and only if $\Sigma b_i \equiv 0 \pmod 2$.

The following result will be required.

5.1.2. Lemma [16]

Let H and $W^+(C_\ell)$ be defined as in (5.3) and (5.4) respectively. Then

$$H \cap W(D_\ell) = H \cap W^+(C_\ell) = W(D_\ell) \cap W^+(C_\ell) = W^+(D_\ell).$$

5.1.3. Definition

The group $W^+(D_\ell)$ defined above, is called the Rotation subgroup of $W(D_\ell)$ and is of index 2 in $W(D_\ell)$.

This group consists of all elements of signed cycle-type of the form $[1^{a_1}2^{a_2}\ldots;\ 1^{-b_1}2^{-b_2}\ldots]$, where $\Sigma b_i \equiv 0 \pmod 2$ and $\Sigma(a_{2i}+b_{2i}) \equiv 0 \pmod 2$, and has order $2^{\ell-2}\ell!$.

5.2. Conjugacy classes in $W^+(D_\ell)$

The following result gives the conjugacy classes of $W(D_\ell)$ (see e.g. [3, P. 26]):

5.2.1. Lemma

(i) Two elements of $W(D_\ell)$ are conjugate if and only if they have the same cycle-type, except in the case of classes of $W(C_\ell)$ of type $[2^{a_2} 4^{a_4} \ldots ; \phi]$ which split into two classes in $W(D_\ell)$.

(ii) If an element $w \in W(D_\ell)$ has cycle type $[1^{a_1} 2^{a_2} \ldots ; 1^{-b_1} 2^{-b_2} \ldots]$, then its centralizer if of order $2^{p-1}(1^{a_1} a_1! \, 2^{a_2} a_2! \ldots 1^{b_1} b_1! \, 2^{b_2} b_2! \ldots)$, where $p = a_1 + a_2 + \ldots + b_1 + b_2 + \ldots$, unless w is of type $[2^{a_2} 4^{a_4} \ldots ; \phi]$ when its centralizer is of order $2^p(2^{a_2} a_2! \, 4^{a_4} a_4! \ldots)$, where $P = a_2 + a_4 + \ldots$

The following gives a description of the classes of $W(D_\ell)$ in terms of the carter diagrams (see e.g. [3]).

5.2.2. Proposition

In $W(D_\ell)$, a positive i-cycle $[i]$ is represented by the admissible diagram A_{i-1} and the pair of negative cycles $[\bar{i}\,\bar{j}]$ with $i \geq j$ is represented by the admissible diagram D_{i+1} if $j=1$ and $D_{i+j}(a_{j-1})$ if $j > 1$. The admissible diagram representing any other class is obtained by splitting the signed cycle-type into positive cycles and pairs of negative cycles, and then taking the union of the admissible diagrams corresponding to these.

In what follows, we use both the signed cycle notation and carter diagrams (admissible diagrams) in describing

classes in $W(D_\ell)$.

The following was proved in [16, P261-206].

5.2.3. Theorem

A conjugacy class of $W(D_\ell)$ splits over its rotation subgroup $W^+(D_\ell)$ if and only if it has one of the following as its signed cycle-types:

(i) $[\phi ; 2^{-b_2} 4^{-b_4} \ldots]$

(ii) $[1^{a_1} 3^{a_3} \ldots ; 1^{-b_1} 3^{-b_3} \ldots]$; $a_i, b_i = 0,1$; $i=1,3,5,\ldots$

5.2.4. Remark

It is now clear from theorem 5.2.3 that the pairs of classes of $W(D_\ell)$ which have the same signed cycle-type $[2^{a_2} 4^{a_4} \ldots ; \phi]$, remain complete classes in $W^+(D_\ell)$.

5.3. α-regular classes in $W^+(D_\ell)$

We maintain the above notation. Let α be the factor set of $W^+(\Phi)$ associated with the basic (spin) projective representation of W^+ (see [17, P. 29]). Then the α-regular classes in $W^+(\Phi)$ have been determined in [16] in all cases. Here we use these results to determine the α-regular classes of $W^+(D_\ell)$.

Let α_{W^+} denote the restriction of the factor set α of $W(\Phi)$ to $W^+(\Phi)$. Then the α-regular classes in $W(\Phi)$ are α_{W^+}-regular in $W^+(\Phi)$ (see [16, P. 268]), though in general, the converse of

this is not true as an α_{W^+} - regular element in $W^+(\Phi)$ may not be α-regualr in $W(\Phi)$.

The α-regular classes in $W(D_\ell)$ have been determined by Read [12] and the results are now summarised in the following:

5.3.1. Lemma:

The α-regular classes in $W(D_\ell)$ are those with the following signed cycle-types (where $\alpha \in (-1,-1)$):

(i) $[\,1^{a_1}\,3^{a_3};\ldots;\ 2^{-b_2}\,4^{-b_4}\ldots]$;

(ii) $[\,\phi;\ 1^{-b_1}\,3^{-b_3}\,5^{-b_5}\ldots\,]$, where $b_i = 1,0; i=1,3,5,\ldots$

(iii) $[\,\phi;\ 1^{-b_1}\,2^{b_2}\ldots]$ (only when ℓ is odd).

Thus, all the conjugacy classes of $W(D_\ell)$, containing the elements of the subgroup $W^+(D_\ell)$, which appear in lemma 5.3.1 are α_{W^+} -regular in $W^+(D_\ell)$.

Let α be the factor set of $W(D_\ell)$ corresponding to the basic projective representation of $W(D_\ell)$ considered by Morris in [9], and let α_{W^+} be its restriction to $W^+(D_\ell)$. The following result gives the α-regular classes in $W^+(D_\ell)$ which are not α-regular in $W(D_\ell)$ (see [16, P.268-270]).

5.3.2. Theorem

Let α and α_{W^+} be as above. Then an element $w \in W^+(D_\ell)$ which is not α-regular in $W(D_\ell)$ is α_{W^+} -regular in $W^+(D_\ell)$ if and only if it has cycle-type $[\,\phi;\ 1^{-a_1}\,2^{-b_2}\,3^{-a_3}\,4^{-b_4}\ldots]$ in $W(D_\ell)$.

The a-regular classes in $W^+(D_\ell)$ are now easily determined form lemma 5.3.1 and Theorem 5.3.2.

5.4. The Basic Projective Characters of $W^+(D_\ell)$

The projective representations of the rotation subgroup $W^+(\Phi)$ have been determined from those of $W(\Phi)$ by Theo [17], for each root system Φ. This is done by constructing non-trivial central extensions of $W^+(\Phi)$ via the double converings of the rotation groups $SO(\ell)$. This adaptation gives a unified way of obtaining the basic projective representation of $W^+(\Phi)$ from those of $W(\Phi)$ determined by Morris in [10]. For the details, we refer the reader to [10] and [17].

We now consider how the basic characters of $W^+(D_\ell)$ may be obtained from those of $W(D_\ell), \ell \geq 4$, and our work follows that in [17].

The basic character of $W(D_\ell)$ and its values on the a-regular classes are given in [9]. The restriction $\chi \downarrow W^+(D_\ell)$ of the basic projective character χ of $W(D_\ell)$ to $W^+(D_\ell)$ is determined in [17] and the results are given by the following result (see e.g. [17 P. 31]).

5.4.1. Theorem

Let $k=s+t$ and χ be the basic projective character of $W(D_\ell)$. If $w \in W^+(D_\ell)$ has carter diagram

$$A_{i_1} + \ldots + A_{i_s} + D_{\lambda_1}(a_{k_1-1}) + \ldots + D_{\lambda_t}(a_{k_t-1}),$$

where $\sum_{r=1}^{s} (i_r+1) + \sum_{r=1}^{t} \lambda_r = \ell$, then

(i) If ℓ is odd, $\chi \downarrow W^+(D_\ell)$ is an irreducible basic projective character of $W^+(D_\ell)$ and

$$\chi(w) = \begin{cases} 2^{\frac{1}{2}(k+t-1)} & \text{if all the } i_r, \lambda_r, k_r \text{ are even,} \\ 0 & \text{for all other } w \in W^+(D_\ell) \end{cases}$$

(ii) If ℓ is even, $\chi \downarrow W^+(D_\ell)$ is a sum of two irreducible basic characters of $W^+(D_\ell)$ and for $j = 1,2$,

$$\chi^{(j)}(w) = \begin{cases} 2^{\frac{1}{2}(k+t-2)} & \text{if all the } i_r, \lambda_r, k_r \text{ are even,} \\ [1 \pm i^{\frac{1}{2}\ell}]\, 2^{t-1} & \text{if } i_r = 0, \text{ all the } \lambda_r, k_r \text{ are even,} \\ \pm i^{\frac{1}{2}\ell} 2^{t-1} & \text{if } i_r = 0 \text{ and an even number of the } \lambda_r \text{ are odd.} \\ 0 & \text{for all other } w \in W^+(D_\ell) \end{cases}$$

5.4.2. Remark

1. If ℓ is even and $w \in W^+(D_\ell)$ lies in an α-regular class in $W(D_\ell)$, then from 5.4.1, it is clear that $\chi^{(j)}$ $(j=1,2)$ take on different values at w if and only if its carter diagram is of the form

$$D_{\lambda_1}(a_{k_1-1}) + \ldots + D_{\lambda_t}(a_{k_t-1})$$

where (i) the λ_r and k_r are all even integers, or (ii) the λ_r are all even and the k_r are all odd. These classes belong to splitting classes of $W(D_\ell)$ by 5.2.3.

2. By 5.4.1 (ii), the classes with carter diagram $D_{\lambda 1}(a_{k_1-1}) + \ldots + D_{\lambda t}(a_{k_t-1})$, where an even number of the λ_r are odd; has a non-zero basic character in $W^+(D_\ell)$. This class is not α-regular in $W(D_\ell)$.

5.5. The root system $\Phi = D_6$

We now apply the foregoing to the root system of type D_6. The Weyl group $W(D_6)$ is of order 23040 and from the results of Read [12], it is easily seen that this group has 37 conjugacy classes. Of these, 20 contain the elements of its rotation subgroup $W^+(D_6)$, which is of order 11520.

It is now a consequence of Theorem 2.2. of [16, P.261] that the conjugacy classes in $W^+(D_6)$ are those given in Table II The group $W^+(D_6)$ contains the central inversion $-I_6$, which reverses every vector in the Euclidean 6-space (see e.g. [4, P. 127]), and we denote this element by $3D_2$ in Carter's notat or by T^6 in the class symbol notation.

The basic (spin) projective character of $W^+(D_6)$ is of degree 8 (see e.g. [17, P. 29]) and has non-zero values on 12 conjugacy classes. Thus, these are all α-regular classes in $W^+(D_6)$. The values of the basic projective character are now listed in Table II.

5.6. The root system $\Phi = D_7$

When Φ is the root system of type D_7, the Weyl group $W(D_7)$ is of order 322 560 and from Read [12], it is easily

seen that this group has 55 conjugacy classes. Of these classes, 29 contain the elements of $W^+(D_7)$; its rotation subgroup, which is of order 161 280.

By Theorem 2.2. of [16, P. 261], we see that $W(D_7)$ does not contain classes which split in its rotation subroup; hence $W^+(D_7)$ contains only the conjugacy classes given in Table III. Unlike $W^+(D_6)$, $W^+(D_7)$ does not contain the central inversion (see e.g. [4, P. 127]).

Its basic projective character is of degree 8 (see e.g. [17, P. 29]) and has non-zero values on 8 conjugacy classes, which are all α-regular classes in $W^+(D_7)$.

A list of the values of the basic projective character is given in Table III.

Table II. Basic projective characters of $W^+(D_6)$

Class		order	$\chi^{(1)}$	$\chi^{(2)}$
$[1^6;\phi]$	ϕ	1	4	4
$[1^3 3;\ \phi]$	A_2	160	2	2
$[1^2 2^2;\ \phi]$	$2A_1$	180	0	0
$[15;\phi]$ *	A_4*	2304	1	1
$[24;\ \phi]'$	$(A_1+A_3)'$	360	0	0
$[24;\phi]''$	$(A_1+A_3)''$	360	0	0
$[3^2;\ \phi]$	$2A_2$	640	1	1
$[1^4;1^{-2}]$	D_2	15	0	0
$[1^2;1^{-4}]$	$2D_2$	15	0	0
$[\phi;1^{-6}]$	$3D_2$	1	$-4i$	$4i$
$[\phi;1^{-3}\bar{3}]$	D_2+D_4	160	$-2i$	$2i$
$[\phi;1^{-2}2^{-2}]$	$D_2+D_4(a_1)$	180	0	0
$[\phi;\bar{1}\ \bar{5}]$*	D_6*	2304	$-i$	i
$[\phi;\bar{2}\ \bar{4}]$*	$D_6(a_1)$*	1440	$1-i$	$-(1-i)$
$[\phi;\bar{3}^2]$	$D_6(a_2)$	640	$-i$	i
$[1^2;\bar{1}\bar{3}]$	D_4	480	0	0
$[1^2;\bar{2}^2]$	$D_4(a_1)$	180	0	0
$[13;1^{-2}]$	A_2+D_2	480	0	0
$[12;\bar{1}\ \bar{2}]$	A_1+D_3	720	0	0
$[2^2;1^{-2}]$	$2A_1+D_2$	180	0	0

* denotes a class which splits in $W^+(D_6)$

Table III. Basic projective characters of $W^+(D_7)$

Class		Order	χ
$[1^7;\phi]$	ϕ	1	8
$[1^4 3;\phi]$	A_2	280	4
$[1^2 5;\phi]$	A_4	8064	2
$[2^2 3;\phi]$	$2A_1+A_2$	3360	0
$[124;\phi]$	A_1+A_3	10080	0
$[13^2;\phi]$	$2A_2$	4480	2
$[1^3 2^2;\phi]$	$2A_1$	420	0
$[7;\phi]$	A_6	46080	1
$[1^5;\bar{1}^2]$	D_2	21	0
$[1^3;\bar{1}^4]$	$2D_2$	35	0
$[1^3;\bar{1}\,\bar{3}]$	D_4	1120	0
$[1^3;\bar{2}^2]$	$D_4(a_1)$	420	$\sqrt{2}$
$[1;\bar{1}^6]$	$3D_2$	7	0
$[1;\bar{1}^3\bar{3}]$	D_2+D_4	1120	0
$[1;\bar{1}^2\bar{2}^2]$	$D_2+D_4(a_1)$	1260	0
$[1;\bar{1}\,\bar{5}]$	D_6	16128	0
$[1;\bar{2}\,\bar{4}]$	$D_6(a_1)$	10080	$\sqrt{2}$
$[1;\bar{3}^2]$	$D_6(a_2)$	4480	0
$[2;\bar{1}^3\bar{2}]$	$A_1+D_2+D_3$	840	0
$[2;\bar{2}\,\bar{3}]$	$A_1+D_5(a_1)$	6720	0
$[2;\bar{1}\,\bar{4}]$	A_1+D_5	10080	0
$[3;1^{-4}]$	A_2+2D_2	280	0
$[3,\bar{1}\,\bar{3}]$	A_2+D_4	8960	0
$[3;\bar{2}^2]$	$A_2+D_4(a_1)$	3360	2
$[4;\bar{1}\,\bar{2}]$	A_3+D_3	10080	0
$[1^2 2;\bar{1}\,\bar{2}]$	A_1+D_3	2520	0
$[1^2 3;\bar{1}^2]$	A_2+D_2	1680	0
$[5;\bar{1}^2]$	A_4+D_2	8064	0

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