THE GEOMETRY OF HOMOGENEOUS SPACES WITH APPLICATION TO HAMILTONIAN MECHANICS

By

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Abstract

In this thesis we study some of the properties of homogeneous spaces. We are more interested in homogeneous spaces which are also manifolds. We have shown that homogeneous spaces are basically quotient spaces. Working with quotient spaces, we pushed to symplectic quotient, the Marsden-Weinstein-Meyer quotient or the symplectic reduction using an equivariant momentum mapping. We have shown that the reduction can be performed using an affine action of a Lie group G on the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ using the momentum with cocycle σ. In this direction we also proved that a Riemannian structure on a symplectic manifold can be induced to the symplectic quotient through a Riemannian submersion. We have proved that if G is a compact, connected and semisimple Lie group, acting transitively on its Lie algebra $\mathfrak{g}$ by the adjoint representation, and acting transitively on the dual $\mathfrak{g}^*$ of its Lie algebra by the coadjoint representation, then there is a symplectic diffeomorphism between the homogeneous space $\mathfrak{g}/G$, the adjoint orbit of the adjoint action and the homogeneous space $\mathfrak{g}^*/G$, the coadjoint orbit of the coadjoint action. We have extended this result to the applications to Hamiltonian mechanics and have shown that Hamiltonian vector fields on symplectic manifold lift to Hamiltonian vector fields on the cotangent bundle of this manifold. On the way to this result we have written equations of Hamiltonian systems using the deformed Poisson bracket and have proved that many properties of Hamiltonian systems with canonical Poisson bracket also hold with a more general structure, the deformed Poisson bracket.

Preface

The research and the writing of this thesis was supervised by Professor Augustin Batubenge initially member of staff in the department of Mathematical Sciences at the University of South Africa, Pretoria, from March 2013 to December 2016, and later a member of staff in the department of Mathematics and Statistics at the University of Zambia, Lusaka, from November 2017 to March 2020.

The work in this thesis is the original work by the author and to the best of my knowledge has not been submitted in any form for any degree or diploma to any other university. Where other people's work has been made use of, acknowledgement has been duly given in the text.

Signed:

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Wallace M. Haziyu (Student)

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Dedication

I dedicate this work to my three daughters, Mwendalubi Mudenda, Kuliwa Sarah and Bwiiche Jessica.

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I want to thank my supervisor Professor Batubenge for his committed advisory and guidance role he played throughout this study. His contribution towards my study is difficult to quantify.

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Approval

This thesis of WALLACE MULEYA HAZIYU is a fulfilling requirement for the award of the degree of Doctor of Philosophy in Mathematics and Statistics of the University of Zambia.

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Introduction

1.1. Overview

The study of homogeneous spaces was initiated by Kostant and Souriau and recently developed by Chu ,(see[35, p. 113]). A homogeneous space is basically a manifold M on which a Lie group G acts in a transitive way. For this reason, Klein considered them (homogeneous spaces) to be the geometries in the sense that they are obtained from a manifold M and a transitive action of a Lie group G on M [3, p.xiii]. One of the advantages of homogeneous spaces is that if we know the value of a geometric quantity at a point, usually taken to be the identity coset $eH \in G/H$, then we can use some smooth maps, say translations, to calculate the value of this quantity at any other point of G/H. There are many examples of homogeneous spaces, however, in this thesis we have mainly used coadjoint orbits to obtain other results. Coadjoint orbits are obtained through the action of a Lie group G on the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G through Ad^* representation. It was shown by Kostant and Souriau that there is up to covering (see [11, p 61 theorem 2.26]), an isomorphism between a symplectic manifold (M, ω), homogeneous under the action of a Lie group G and a coadjoint orbit. (See [11]).

1.2. Main results

Following are the main results which we have stated and proved in different Sections in this thesis.

Theorem 1.2.1 *Let $\Psi : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ defined by $\Psi(g, \alpha) = Ad^*_g \alpha + \sigma(g)$ be the affine action of a Lie group G on its dual $\mathfrak{g}^*$ to its Lie algebra $\mathfrak{g}$. Let $\beta \in \mathfrak{g}^*$.*

Then, the orbit

$$G \cdot \beta = \{\Psi(g, \beta) : g \in G\}$$

is a symplectic manifold with the symplectic 2-form given by

$$\omega_\beta(\xi_{\mathfrak{g}^*}(v), \eta_{\mathfrak{g}^*}(v)) = -\beta[\xi, \eta] + \sum(\xi, \eta),$$

where $\xi, \eta \in \mathfrak{g}$, *and* $\xi_{\mathfrak{g}^*}$ *and* $\eta_{\mathfrak{g}^*}$ *are vector fields on* $\mathfrak{g}^*$.

Theorem 1.2.2 *Let* (M, ω) *be a symplectic manifold and* G *be a Lie group of isometries of* M *whose action on* M *is a Hamiltonian action. Let* $\mathfrak{g}$ *be the Lie algebra of* G, *and let*

$$\mu : M \to \mathfrak{g}^*$$

be the Ad^**-equivariant momentum mapping of the action, where* $\mathfrak{g}^*$ *is the dual of the Lie algebra of* G. *Let* $\beta \in \mathfrak{g}^*$ *be a regular value of* μ *and* G_β *the isotropy subgroup of* β *which acts freely and properly on* $\mu^{-1}(\beta)$. *Then, there exists a Riemannian metric* g_β *on the reduced space* $\mu^{-1}(\beta)/G_\beta$ *such that the projection map*

$$\pi_\beta : \mu^{-1}(\beta) \to \mu^{-1}(\beta)/G_\beta$$

is a Riemannian submersion. That is,

$$\pi_\beta^* g_\beta = i^* g_M,$$

where g_M *is a Riemannian metric on* M *and*

$$i : \mu^{-1}(\beta) \to M$$

is the inclusion map.

Theorem 1.2.3 *Let* (M, ω) *be a symplectic manifold and* G *a Lie group of isometries of* M. *Let* $\Phi : G \times M \to M$ *be a Hamiltonian action of* G *on* M *with* Ad^**-equivariant momentum mapping*

$$\mu : M \to \mathfrak{g}^*.$$

Let $\beta \in \mathfrak{g}^*$ be a regular value of μ and G_β be the isotropy subgroup of β acting freely and properly on $\mu^{-1}(\beta)$. Given a compatible almost complex structure J_M on M and a Riemannian metric g_M which satisfies the compatibility condition,

$$\omega(X, Y) = g_M(J_M X, Y)$$

for all $X, Y \in TM$, let ω_β be the reduced symplectic form on the reduced symplectic manifold $\mu^{-1}(\beta)/G_\beta$. Then there exists an almost complex structure J_β and a Riemannian metric g_β on the reduced space $\mu^{-1}(\beta)/G_\beta$ which make

$$\pi : \mu^{-1}(\beta) \to \mu^{-1}(\beta)/G_\beta$$

a Riemannian submersion and satisfies the condition

$$\omega_\beta([u], [v]) = g_\beta(J_\beta[u], [v])$$

for all $[u], [v] \in T(\mu^{-1}(\beta)/G_\beta)$ if and only if

$$\pi : \mu^{-1}(\beta) \to \mu^{-1}(\beta)/G_\beta$$

is an almost complex mapping.

Theorem 1.2.4 *Let G be a compact, connected semisimple Lie group. Let $\mathfrak{g}$ be its Lie algebra and $\mathfrak{g}^*$ the dual of $\mathfrak{g}$. Assume further that G acts transitively on $\mathfrak{g}$ by the adjoint action and transitively on $\mathfrak{g}^*$ by the coadjoint action. Let $B^\flat$ be as in theorem 5.5.1 and let $\hat{B}^\flat : \mathfrak{g}/G \to \mathfrak{g}^*/G$ be the map induced by passage to quotients between adjoint and coadjoint orbit spaces. Then, the map $\hat{B}^\flat$ is a symplectic diffeomorphism.*

Theorem 1.2.5 *Let $(M, \{\cdot, \cdot\}^r)$ be a Poisson manifold with a deformed Poisson structure. If X_f^r and X_g^r are Hamilotnian vector fields with corresponding Hamiltonian functions f and g respectively, then their bracket $[X_f^r, X_g^r]$ is a Hamiltonian vector field with the Hamiltonian function $\{f, g\}^r$. Thus,*

$$[X^r_f, X^r_g] = X^r_{\{f,g\}^r}.$$

Theorem 1.2.6 *Let (M, ω) be a symplectic manifold and X_H be a Hamiltonian vector field on M with the Hamiltonian function H. Then, X_H induces a Hamiltonian vector field X_{T^*M} on the cotangent bundle T^*M, whose flow is the lift of the flow of X_H.*

1.3. Organisation of the thesis

The thesis is organised in the following way.

Chapter 1 is basically an introduction in which we have given an overview of the research project as well as stating the main results.

Chapter 2 is a preliminary chapter of concepts. Here we give the notions of Lie groups and homogeneous spaces. These notions are important in many constructions and in the understanding of differential geometry of the underlying space. We have also shown that there is a diffeomorphism between a manifold M on which a Lie group G acts in a transitive way and the quotient manifold G/H, where H is some closed subgroup of G.

In chapter 3 we describe symplectic manifolds and how they come about as quotient manifolds of group actions. Central in this chapter is the description of the coadjoint orbits introduced by Kirillov in the 1960's, (see [24]). We show that a symplectic structure can be defined on a modified action of the Lie group G on $\mathfrak{g}^*$ through a one cocycle σ so that the orbit obtained is a symplectic manifold. We have described the momentum mapping with one cocycle which induces another action which makes the momentum mapping Ad^*-equivariant with respect to the new G-structure on $\mathfrak{g}^*$.

In chapter 4 we determine conditions for which the reduced space of a symplectic manifold inherits an induced Riemannian structure through a Riemannian submersion. We think that if we have a symplectic manifold with a compatible Riemann metric, it would be good to end up with a Marsden-Weinstein-Meyer quotient which is also a Riemannian space having a Riemannian structure (metric) inherited from the one on the original space. We have also put up a great deal of effort to describe the spaces of positive complex structure which is compatible with a given symplectic structure.

In chapter 5 we study the adjoint orbits under the action of a semi simple, connected and compact Lie group G. We have given an example of generalised flag manifolds as a special case of adjoint orbits. Generalised flag manifolds are a class of homogeneous spaces which admit a symplectic structure and other structures such as the complex structure ([3],[2]). The main result of the chapter is the following statement.

Theorem 1.3.1 *Let G be a compact, connected semisimple Lie group. Let $\mathfrak{g}$ be its Lie algebra and $\mathfrak{g}^*$ the dual of $\mathfrak{g}$. Assume further that G acts transitively on $\mathfrak{g}$ by the adjoint action and transitively on $\mathfrak{g}^*$ by the coadjoint action. Let $B^\flat$ be as in theorem 5.5.1 and let $\hat{B}^\flat : \mathfrak{g}/G \to \mathfrak{g}^*/G$ be the map induced by passage to quotients between adjoint and coadjoint orbit spaces. Then the map $\hat{B}^\flat$ is a symplectic diffeomorphism.*

This result also gives another proof that under certain conditions the adjoint orbit is a symplectic homogeneous space by showing that there is a smplectic diffeomorphism between an adjoint orbit and a coadjoint orbit.

In chapter 6 we study the Hamiltonian formalisms on symplectic manifolds. In our paper [10], we provided a way forward on a deformation of the standard Poisson bracket on the algebra of smooth functions. In this chapter we have discussed Hamiltonian mechanics with a deformed Poisson bracket and have shown that many properties of Hamiltonian systems which hold with canonical Poisson bracket also hold true with a deformed Poisson bracket. In the last section of this chapter we investigate the relationship between the Hamiltonian systems on the base space, the symplectic manifold $\mathfrak{g}/G$, taken as a single adjoint orbit of a transitive action and the Hamiltonian systems on the phase space $T^*(\mathfrak{g}/G)$, where the systems on the phase space is induced by the systems on the base space.

2

Lie group actions and homogeneous spaces

2.1. Lie groups and Lie algebras

Lie groups play a central role in the study of manifolds for a number of reasons. Lie groups are manifolds in their own right, as such, they provide another class of examples of manifolds. Perhaps another major reason is that Lie groups appear as symmetries of various geometric objects. The link between Lie groups and their Lie algebras provide a means of solving some of difficult problems in geometry by methods of linear algebra.

Definition 2.1.1 *A Lie group is a differentiable manifold such that the group operations are smooth with regard to its manifold smooth structure. That is, the operations*

(i) $G \times G \to G$, $(x, y) \mapsto xy$

(ii) $G \to G$, $x \mapsto x^{-1}$

are smooth.

Definition 2.1.2 *An algebra of vector fields, which we shall for now denote by* $\mathfrak{L}_A$*, is a vector space over* $\mathbb{R}$ *together with a bilinear operation* $[,] : \mathfrak{L}_A \times \mathfrak{L}_A \to \mathfrak{L}_A$*, called the bracket, which is skew symmetric and satisfies the Jacobi identity. That is, for all* $X, Y, Z \in \mathfrak{L}_A$*,* $a, b \in \mathbb{R}$

(a) $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ *and* $[Z, aX + bY] = a[Z, X] + b[Z, Y]$

(b) $[X, Y] = -[Y, X]$

(c) $[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]=0$*, the Jacobi identity.*

Definition 2.1.3 *Let $g \in G$ be an element of G. A left translation L_g is a map $L_g : G \to G$ defined by $L_g(x) = gx$ for all $x \in G$. Similarly, a right translation is a map $R_g : G \to G$ defined by $R_g(x) = xg$ for all $x \in G$.*

Since $L_{g^{-1}} \circ L_g = L_g \circ L_{g^{-1}} = id_G$ we have $L_{g^{-1}} = L_g^{-1}$. Similarly, $R_{g^{-1}} = R_g^{-1}$. Thus these maps L_g and R_g are diffeomorphisms of G.

Let $L_g : G \to G$ be the left translation on a Lie group G. Then the differential of L_g is a linear map

$$(dL_g)_h : T_hG \to T_{gh}G$$

Definition 2.1.4 *Let G be a Lie group. A vector field $X \in \mathfrak{X}(G)$ is called left invariant if*

$$(dL_g)_h(X(h)) = X(gh) = X_{gh} \tag{2.1}$$

for all $g, h \in G$.

That is, a vector field X is left invariant if $dL_g(X) = X$ for all $g \in G$. That is, X is L_g- related to itself.

If $h = e$, the identity element of the Lie group G, then equation (2.1) gives

$$(dL_g)_e : T_eG \to T_gG, \quad X_e \mapsto X_g.$$

That is, a left invariant vector field is determined by its value at the identity since if $X_e = Y_e$ then $X_g = (dL_g)(X_e) = (dL_g)(Y_e) = Y_g$ for all $g \in G$ implying that $X = Y$.

We denote by $L(G)$ the space of all left invariant vector fields on the Lie group G.

Proposition 2.1.1 *Let $L(G)$ be the space of all left invariant vector fields on a Lie group G. Then* *(i) $L(G)$ is a real vector space,*

(ii) $L(G)$ is closed under the bracket operation on vector fields.

Proof. Let $X, Y \in L(G)$, then

(i) for all $p, q \in \mathbb{R}$ and for all $g \in G$ we have,

$$\begin{aligned} dL_g(pX + qY) &= dL_g(pX) + dL_g(qY) \\ &= pdL_g(X) + qdL_g(Y) \\ &= pX + qY, \end{aligned}$$

which shows that $pX + qY \in L(G)$.

(ii) for all $g, h \in G$ and $f \in C^\infty(G)$, we have,

$$\begin{aligned} dL_g[X, Y]_h f &= [X, Y]_h(f \circ L_g) \\ &= X_h(Y(f \circ L_g)) - Y_h(X(f \circ L_g)) \\ &= X_h(dL_g Y)f - Y_h(dL_g X)f \\ &= X_h Y f - Y_h X f \\ &= (X_h Y - Y_h X)f \\ &= [X, Y]_h f. \end{aligned}$$

This shows that the bracket of two left invariant vector fields is also a left invariant vector field. □

By this proposition, the space of all left invariant vector fields of a Lie group G is an algebra called the Lie algebra of G. The Lie algebra of G is denoted by $\mathfrak{g}$.

Proposition 2.1.2 *Let $\mathfrak{g}$ be the Lie algebra of left invariant vector fields of a Lie group G. Then the map*

$$\mathfrak{g} \to T_e G, \quad X \mapsto X_e$$

is an isomorphism of vector spaces.

Proof. Proposition 2.1.1 implies that this map is linear. To see that it is injective let $X_e = 0$. Then for all $g \in G$ we have $X_g = (dL_g)_e(X_e) = 0$. Thus only a zero left invariant vector field maps to a zero tangent vector at the identity proving the map is injective. To see that the map is surjective, let $v \in T_e G$ and define $X_g = (dL_g)_e(v)$ for every $g \in G$. Then X_g is left invariant since, for all $g, h \in G$ we have $X(gh) = (dL_{gh})_e(v) = (dL_g)_h(dL_h)_e(v) = (dL_g)_h(X_h)$. That is, $(dL_g)_h(X_h) = X_{gh}$. Thus X_g is left invariant. $\square$

Proposition 2.1.3 *Left invariant vector fields of a Lie group G are smooth vector fields.*

For the proof of this proposition see [17, Prop5.1.19] or [39, Prop 3.7b].

Because of the isomorphism in proposition 2.1.2 we shall from now identify the Lie algebra $\mathfrak{g}$ of G with the tangent space T_eG of G at the identity,

$$\mathfrak{g} \cong T_eG$$

Definition 2.1.5 *Let G be a Lie group. The Lie algebra of G is a vector space $\mathfrak{g} = T_eG$ which is equipped with the bracket operation.*

Definition 2.1.6 *Let G_1 and G_2 be two Lie groups. A smooth map $\Phi : G_1 \to G_2$ is called a Lie group homomorphism if Φ is a homomorphism of abstract groups G_1 and G_2. If Φ is a diffeomorphism, then it is called an isomorphism.*

Definition 2.1.7 *Let $\Phi : G \to G$ be an isomorphism of a Lie group G into itself, then Φ is called an automorphism of G. The set of all the automorphisms of G is a group under composition of maps and is denoted by $Aut(G)$.*

Definition 2.1.8 *Let G be a Lie group and V a finite dimensional vector space. Then the map $\Phi : G \to Aut(V)$ is called the representation of G.*

Theorem 2.1.1 *Let $\Phi : G_1 \to G_2$ be a Lie group homomorphism. Let $\mathfrak{g} = T_eG$ be a Lie algebra of a Lie group of G and $\bar{\mathfrak{g}} = T_{\bar{e}}G_2$ be the Lie algebra of a Lie group G_2. Then the tangent map*

$$d\Phi : \mathfrak{g} \to \bar{\mathfrak{g}}$$

is a Lie algebra homomorphism.

First note that Φ maps the identity of G_1 to the identity of G_2. Therefore, the differential $d\Phi$ of Φ is a linear transformation $d\Phi : \mathfrak{g} \to \bar{\mathfrak{g}}$. Thus, it takes left invariant vector fields of G_1 to the left invariant vector fields of G_2.

Lemma 2.1.1 *If $F : G \to G'$ is a homomorphism of Lie groups, then for each left invariant vector field $X \in \mathfrak{g}$ there is a left invariant vector field $X' \in \mathfrak{g}'$ such that $X'(e') = dF(X(e))$.*

For the proof of this lemma, (see [12, Corollary 7.10]).

Proof of theorem 2.1.1. Let $X \in \mathfrak{g}$ and let $\bar{X} \in \bar{\mathfrak{g}}$ be the unique left invariant vector field in lemma 2.1.1 such that $\bar{X} = d\Phi(X)$. We must first show that $\bar{X}$ and X are Φ-related. But since Φ is a homomorphism we have $L_{\Phi(a)}\Phi(b) = \Phi(a)\Phi(b) = \Phi(ab) = \Phi(L_a(b))$ so that

$$\Phi \circ L_a = L_{\Phi(a)} \circ \Phi$$

Now

$$\begin{aligned}
\bar{X}(\Phi(a)) &= (dL_{\Phi(a)})_{\bar{e}}(\bar{X}(\bar{e})) \\
&= (dL_{\Phi(a)})_{\bar{e}}(d\Phi)_e(X(e)), \qquad \bar{e} = \Phi(e) \\
&= d(L_{\Phi(a)} \circ \Phi)_e(X(e)) \\
&= d(\Phi \circ L_{\Phi(a)})_e(X(e)) \\
&= (d\Phi)_{L_a(e)}(dL_a)_e(X(e)) \\
&= (d\Phi)_a(X(a)), \qquad \text{for all } a \in G.
\end{aligned}$$

Thus, $\bar{X}$ and X are Φ-related.

We remain to show that if $\bar{X}$ is Φ-related to X and $\bar{Y}$ is Φ-related to Y then $[\bar{X}, \bar{Y}] = \overline{[X, Y]}$. But $[X, Y]$ is Φ-related to the left invariant vector field $[\bar{X}, \bar{Y}]$ ([12, Thm 7.9 p150]). So

$$[\bar{X}, \bar{Y}](\bar{e}) = d\Phi([X, Y](e))$$

But also lemma 2.1.1 implies that $\overline{[X, Y]}$ is the unique left invariant vector field on G_2 whose value at the identity is $d\Phi([X, Y](e))$. Therefore, we must have

$$\overline{[X, Y]} = [\bar{X}, \bar{Y}].$$

This completes the proof of the theorem. □

2.1.1 Exponential map

We now turn to a very important map in the study of Lie groups and their Lie algebras, the exponential map

$$\exp : \mathfrak{g} \to G.$$

We shall use the notion of one-parameter subgroup to define it because in general, multiplication in $\mathfrak{g}$ is not defined and so it would not be possible to use power series except in the case that $\mathfrak{g}$ is a Lie algebra of matrices.

Definition 2.1.9 *Let G be a Lie group. A smooth map $\sigma : \mathbb{R} \to G$ is called a one-parameter subgroup of G if*

(i) $\sigma(0) = e$, the identity element of G,

(ii) $\sigma(t + \tau) = \sigma(t)\sigma(\tau), \quad$ for all $t, \tau \in \mathbb{R}$.

Note that this is a homomorphism of Lie groups since $\mathbb{R}$ is a Lie group under addition operation.

The following proposition gives the existence and uniqueness of one-parameter subgroups. (See also [17, Prop 5.1.23]).

Proposition 2.1.4 *Let G be a Lie group and $X \in \mathfrak{g} = T_eG$. Then, there exists a unique one-parameter subgroup $\sigma_X : \mathbb{R} \to G$ such that $\dot{\sigma}_X(0) = X(e)$.*

Proof. We shall first prove uniqueness assuming existence.

Suppose that $\sigma : \mathbb{R} \to G$ is a one-parameter subgroup with $\dot{\sigma}(0) = X(e)$, then we have, for all $t_1, t_2 \in \mathbb{R}$, $\sigma(t + \tau) = \sigma(t)\sigma(\tau) = L_{\sigma(t)}\sigma(\tau)$.

Differentiating with respect to τ using chain rule and setting $\tau = 0$ we have

$$\dot{\sigma}(t + \tau) = L_{\sigma(t)*}\dot{\sigma}(\tau),$$

and putting $\tau = 0$ yields

$$\dot{\sigma}(t) = L_{\sigma(t)*}\dot{\sigma}(0) = L_{\sigma(t)*}X(e).$$

That is, $\dot{\sigma}(t) = X(\sigma(t))$.

The existence and uniqueness of integral curve ([12, Theorem 4.1]) now implies that σ is the unique integral curve of X through e. This proves uniqueness.

To prove existence, first note that a left invariant vector field on a Lie group G is complete ([12, Cor 5.8, p138]). So, let $\Phi_t^X : G \to G$ be the flow of X. Define a map $\sigma_X : \mathbb{R} \to G$ by

$$\sigma_X(t) = \Phi_t^X(e). \tag{2.2}$$

We must show that equation (2.2) defines a one-parameter subgroup of G. Now, if $X \in \mathfrak{g}$ and Φ_t^X is the flow of X, then we have for all $a \in G$, the identity

$$L_a \circ \Phi_t^X \circ L_{a^{-1}} = \Phi_t^X. \tag{2.3}$$

See ([17, Pro 5.1.23, p165]).

Now from equation (2.2) we have $\sigma_X(0) = \Phi_0^X(e) = e$ since Φ_t^X is a flow. We also have

$$\begin{aligned}
\sigma_X(t+\tau) &= \Phi_{t+\tau}^X(e) \\
&= \Phi_\tau^X(\Phi_t^X(e)) \\
&= \Phi_\tau^X(\sigma_X(t)) \\
&= \sigma_X(t)\sigma_X(t)^{-1}\Phi_\tau^X(\sigma_X(t)e) \\
&= \sigma_X(t)\Phi_\tau^X(e) \qquad \text{by equation(2.3)} \\
&= \sigma_X(t)\sigma_X(\tau).
\end{aligned}$$

Thus, $\sigma_X(t) = \Phi_t^X(e)$ is a one-parameter subgroup of G. We also have

$$\dot{\sigma}_X(0) = \frac{d}{dt}\Phi_t^X(e)|_{t=0} = X(e).$$

This proves the existence part and completes the proof of the proposition. □

Definition 2.1.10 *Let G be any Lie group with $\mathfrak{g} = T_eG$ its Lie algebra. Then for any $X \in \mathfrak{g}$ we define the exponential map*

$$\exp : \mathfrak{g} \to G$$

by $\exp(X) = \sigma_X(1)$.

Proposition 2.1.5 *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. If $X \in \mathfrak{g}$, then* $\sigma_X(t) = \exp(tX)$.

See ([17, Lemma 5.1.26]).

Because of this proposition, for any $X \in \mathfrak{g}$ we shall express its one-parameter subgroup by $\exp(tX)$.

Proposition 2.1.6 *The exponential map* $\exp : T_eG \to G$ *is a smooth map and carries some neighbourhood of* $0 \in T_eG$ *diffeomorphically onto a neighbourhood of* $e \in G$.

See ([17, Prop 5.1.27, p167]).

One of the basic properties of the exponential map is given in the following proposition.

Proposition 2.1.7 *Let* $f : G_1 \to G_2$ *be a smooth homomorphism of Lie groups. Then, for any* $\eta \in \mathfrak{g}_1$*, the Lie algebra of* G_1*, we have*

$$f(\exp_{G_1} \eta) = \exp_{G_2} (df)_e \cdot \eta. \tag{2.4}$$

Proof. Let $\eta \in \mathfrak{g}_1$. Then $\Phi_1 : \mathbb{R} \to G_1$, $t \mapsto \exp t\eta$ is the one-parameter subgroup of G_1 generated by η. We then have that $f \circ \Phi_1 : \mathbb{R} \to G_2$, $t \mapsto f(\exp t\eta)$ is the one-parameter subgroup of G_2. Let $\Phi_2(t) = f(\exp_{G_1} t\eta)$. There is $\xi \in \mathfrak{g}_2$, the Lie algebra of G_2, such that $\Phi_2(t) = \exp_{G_2} t\xi$. Differentiating the relation $\exp_{G_2} t\xi = \Phi_2(t) = f(\exp_{G_1} t\eta)$, gives

$$\begin{aligned} \frac{d}{dt} \exp_{G_2} t\xi|_{t=0} &= \frac{d}{dt}\Phi_2|_{t=0} \\ &= df_{\exp_{g_1} t\eta} \frac{d}{dt} \exp_{G_1} t\eta|_{t=0}, \end{aligned}$$

which yields $\xi(e_2) = df_{e_1} \cdot \eta$, where e_1 and e_2 are the identities of G_1 and G_2 respectively. Thus, $\Phi_2(1) = \exp_{G_2} \xi = \exp_{G_2} df_{e_1} \cdot \eta$. $\square$

Definition 2.1.11 *Let* G *be a Lie group and* H *an algebraic subgroup of* G*. Then* H *is called a Lie subgroup if the inclusion map* $i : H \hookrightarrow G$ *is an immersion. That is,* H *is a Lie subgroup if it is a submanifold with its smooth structure as an immersed submanifold of* G.

The following theorem is called the Cartan's Theorem on closed subgroups. For the proof of the theorem is given in a number of text books. We refer the reader to ([1, Prop 4.1.12, p259]) or ([33, Thm 5.1, p14]).

Theorem 2.1.2 *Let G be a Lie group and H a closed subgroup of G. Then H is an embedded submanifold and hence a Lie subgroup of G.*

2.1.2 Group actions on manifolds

As we have stated at the beginning of this chapter, Lie groups form an integral part of the study of manifolds mainly because they appear as symmetries of various geometric objects. We shall now study the actions of Lie groups on smooth manifolds.

Definition 2.1.12 *Let M be a smooth manifold and G a Lie group. A smooth map $\Phi : G \times M \to M,\ (g, m) \mapsto \Phi(g, m)$ is called an action of G on M if*

(i) $\Phi(e, m) = m$ for all $m \in M$, where, $e \in G$ is the identity of G,

(ii) $\Phi(gh, m) = \Phi(g, \Phi(h, m))$ for all $g, h \in G$ and for all $m \in M$.

If we now fix $g \in G$ in the definition then we get a map $\Phi_g : M \to M$ on M. By property (ii) in the definition we have, for each $g \in G$, $\Phi_g \circ \Phi_{g^{-1}} = \Phi_{g^{-1}} \circ \Phi_g = id_M$. Thus, $(\Phi_g)^{-1} = \Phi_{g^{-1}}$. Clearly if Φ_g is smooth then its inverse $\Phi_{g^{-1}}$ is also smooth. This implies that Φ_g is a diffeomorphism of M for each $g \in G$. Then the map $g \mapsto \Phi_g$ is a smooth homomorphism of G into the group of diffeomorphisms of M

$$G \to \mathrm{Diff}(M).$$

Suppose that M is a vector space, then for each $g \in G$, $\Phi_g : M \to M$ is a linear transformation and note that the map $G \to \mathrm{Diff}(M)$ is a map of G into the automorphisms of M. In this case, the action of G on M is called a representation of G on M, (see definition 2.1.8).

We have already seen that a Lie group G can act on itself by left (or right) translation. Another way a Lie group G acts on itself is by conjugation, $I : G \times G \to G$, $I_g(a) = gag^{-1}$. For each $g \in G$, the map $g \mapsto I_g$ is a map into the group of diffeomorphisms of G. Since $I_g = R_{g^{-1}} \circ L_g$ is a composition of diffeomprphisms, I_g is a diffeomorphism of G into itself. Note that $I_g(e) = geg^{-1} = gg^{-1} = e$, so that this map fixes the identity element in G. Thus, $I_{g*e} : T_eG \to T_eG$.

Definition 2.1.13 *(Adjoint representation)*

Let G be a Lie group. The adjoint representation of G is a homomorphism $Ad : G \to Aut(\mathfrak{g})$ defined by $Ad(g) = (dI_a)_e$ for all $g \in G$, where we have identified the tangent space of G at the identity T_eG with its Lie algebra $\mathfrak{g}$.

Proposition 2.1.8 *If G is a matrix group then $Ad_g X = gXg^{-1}$ for all $g \in G$ and for all $X \in \mathfrak{g}$. The multiplication is the ordinary multiplication of matrices.*

(See [3, Prop 2.9 p29]).

Corollary 2.1.1 *Let G be a Lie group of matrices, then we have;*

(i) $Ad_g \circ Ad_h = Ad_{gh}$,

(ii) $\dfrac{d}{dt} Ad_{\exp tX}(Y) \mid_{t=0} = [X, Y]$ *for all* $X, Y \in \mathfrak{g} = T_eG$.

Proof. (i) let $X \in \mathfrak{g}$, then from the above proposition

$$Ad_g \circ Ad_h(X) = Ad_g(Ad_h(X)) = Ad_g(hXh^{-1}) = g(hXh^{-1})g^{-1} = ghX(gh)^{-1} = Ad_{gh}(X).$$

(ii) let $X, Y \in \mathfrak{g}$ and let $t \mapsto \exp tX$ and $s \mapsto \exp sY$ be the one-parameter subgroups associated with X and Y respectively. Then

$$\begin{aligned}
\frac{d}{dt} Ad_{\exp tX}(Y) \mid_{t=0} &= \frac{d}{dt}\Big\{(dI_{\exp tX})Y\Big\}\mid_{t=0} \\
&= \frac{d}{dt}\Big\{\frac{d}{ds} I_{\exp tX}(\exp sY) \mid_{s=0}\Big\}\mid_{t=0} \\
&= \frac{d}{dt}\Big\{\frac{d}{ds} \exp tX \cdot \exp sY \cdot \exp(-tX) \mid_{s=0}\Big\}\mid_{t=0} \\
&= \frac{d}{dt}\Big\{\exp tX \frac{d}{ds} \exp sY \mid_{s=0} \exp(-tX)\Big\}\mid_{t=0} \\
&= \frac{d}{dt}\Big\{\exp tX(Y)\exp(-tX)\Big\}\mid_{t=0} \\
&= XY - YX \\
&= [X, Y].
\end{aligned}$$

□

Definition 2.1.14 . *The adjoint representation of the Lie algebra* $\mathfrak{g}$, *is the homomorphism* $ad : \mathfrak{g} \to End(\mathfrak{g})$ *defined by* $ad(X) = (dAd)_e(X)$, *where* $End(\mathfrak{g})$ *is the group of endomorphisms of* $\mathfrak{g}$, *([3, p 28]).*

Definition 2.1.15 *(Coadjoint Representation)*

Let G *be a Lie group with* $\mathfrak{g}^*$ *the dual of its Lie algebra* $\mathfrak{g}$. *The coadjoint representation of a Lie group* G *is the map* $Ad^* : G \times \mathfrak{g}^* \to \mathfrak{g}^*, \quad (g, \beta) \mapsto Ad^*_g\beta,$

given by $\langle Ad^*_g\beta, X\rangle = \langle \beta, Ad_{g^{-1}}X\rangle$ *for all* $\beta \in \mathfrak{g}^*$, $X \in \mathfrak{g}$ *and* $g \in G$.

By using g^{-1} in the definition of $Ad^*_g\beta$ we obtain a group homomorphism $Ad^*_g \circ Ad^*_h = Ad^*_{gh}$, as can be seen from the following calculations; for any $X \in \mathfrak{g}$,

$$\begin{aligned}
\langle Ad^*_g \circ Ad^*_h\beta, X\rangle &= \langle Ad^*_h\beta, Ad_{g^{-1}}(X)\rangle \\
&= \langle \beta, Ad_{h^{-1}} \circ Ad_{g^{-1}}(X)\rangle \\
&= \langle \beta, Ad_{h^{-1}g^{-1}}(X)\rangle \\
&= \langle \beta, Ad_{(gh)^{-1}}(X)\rangle \\
&= \langle Ad^*_{(gh)}\beta, X\rangle.
\end{aligned}$$

Definition 2.1.16 *Let* $\Phi : G \times M \to M$ *be an action of a Lie group* G *on a manifold* M. *If* $m \in M$, *then the orbit of* m *under the action of* Φ, *or the* Φ*-orbit of* m *is defined by*

$$G \cdot m = \{\Phi_g(m) : g \in G\}.$$

The isotropy group or the stabilizer group of $m \in M$ *is given by*

$$G_m = \{g \in G : \Phi_g(m) = m\}.$$

The action is called transitive if there is only one orbit. That is, if for each pair $m_1, m_2 \in M$ *there is a* $g \in G$ *such that* $m_2 = \Phi_g(m_1)$. *The action is effective or faithful if* $G_m = \{e\}$. *That is, if the assignment* $g \mapsto \Phi_g$ *is one-to-one. The action is called free if for each* $m \in M$, $g \mapsto \Phi_g(m)$ *is one-to-one. That is, if* $\Phi_g(m) = m$ *for some* $g \in G$, *then* $g = e$.

Definition 2.1.17 *Let* $\Phi : G \times M \to M$ *be an action of a Lie group* G *on a manifold* M. *Then the action* Φ *is said to be a proper action if the map* $\tilde{\Phi}$:

$G \times M \to M \times M$, defined by, $\tilde{\Phi}(g, x) = (x, \Phi(g, x))$ is a proper mapping. This means that Φ is proper if whenever $K \subset M \times M$ is a compact subset, then the inverse image $\tilde{\Phi}^{-1}(K)$ is compact.

The alternative way to state the property of a proper action is to say that Φ is a proper action if whenever x_n converges in M, and $\Phi(g_n, x_n)$ converges in M, then g_n has a convergent subsequence in G. This condition is automatically satisfied if G is a compact Lie group.

An action $\Phi : G \times M \to M$ of a Lie group G on a manifold M partitions M into equivalence classes. That is, each orbit is an equivalence class. Then the relation of belonging to the same orbit is an equivalence relation. We denote the set of all the equivalence classes by M/G. The map which takes an element to its orbit is $\pi : M \to M/G$, $x \mapsto [x]$, where $[x]$ is the orbit containing x.

The topology on M/G is the quotient topology, that is, $U \subset M/G$ is open if and only if $\pi^{-1}(U)$ is open in M, (see [1, p 261]).

Proposition 2.1.9 *Let G be a compact Lie group acting on a smooth manifold M. Then M/G is a Hausdorff space and it is second countable.*

To prove the proposition first we have the following claims:

Claim 1: Distinct orbits are disjoint.

Proof. (Of claim). Let $[x]$ and $[y]$ be distinct orbits through x and y respectively. If $[x] \cap [y] \neq \emptyset$, let $z \in [x] \cap [y]$. Then $z = g_1 x = g_2 y$ for some $g_1, g_2 \in G$. This gives $x = g_1^{-1} g_2 y$ so that $x \in [y]$. Then, for any $w \in [x]$ such that $w = gx$ for some $g \in G$, we have $w = g(g_1^{-1} g_2 y) = (g g_1^{-1} g_2) y$. This implies that $w \in [y]$. Thus $[x] \subset [y]$. Reversing the argument gives $[y] \subset [x]$. This gives $[x] = [y]$.

Claim 2: Any orbit $[x]$ is a closed subset of M.

Proof. (Of claim). We shall show that the complement $M \setminus [x]$ is open. Let $y \in M \setminus [x]$, since M is Hausdorff, choose disjoint open sets U_x and V_x with $y \in U_x$ and $x \in V_x$. The collection $\{V_{x_i} : x_i \in [x]\}$ is a covering for $[x]$ by open sets in M. But $[x] = Gx$ is the image by a smooth map $\Phi_x : G \to M$ of a compact set, G, hence $[x]$ is compact. Let $V_{x_1}, V_{x_2}, \ldots, V_{x_n}$ be the finite sub-collection which also covers $[x]$, let $V = V_{x_1} \cup V_{x_2} \cup \ldots \cup V_{x_n}$ and $U = U_{x_1} \cap U_{x_2} \cap \ldots \cap U_{x_n}$. If $z \in V$ then $z \in V_{x_i}$ for some x_i and so $z \notin U_{x_i}$. Then $z \notin U$. Thus V is an open set containing $[x]$ which is disjoint from U, an open set containing y. Since this is true for each $y \in M \setminus [x]$, we conclude that $M \setminus [x]$ is an open set so that its complement $[x]$ is closed.

We now prove the proposition 2.1.9.

To prove Hausdorff property, suppose that the orbits $[x]$ and $[y]$ of x and y respectively, cannot be separated in M/G. For each positive integer n let U_n and V_n be open balls of radius $\frac{1}{n}$ around x and y respectively. Then since for each n, $GU_n \cap GV_n \neq \emptyset$, there is $g_n, h_n \in G$, $x_n \in U_n$ and $y_n \in V_n$ such that

$$\Phi(g_n, x_n) = \Phi(h_n, y_n), \quad \text{that is,}$$

$$x_n = \Phi(g_n^{-1}, \Phi(h_n, y_n)).$$

Taking limit as $n \to \infty$, we see that $x = \Phi(g^{-1}, \Phi(h, y))$ so that $x \in [y]$. But x is in the closure of its orbit $[x]$ which is a closed set. So we must have $[x] = [y]$ a contradiction. Thus, M/G must be Hausdorff. (See also [1, p 261, prop 4.1.19]).

To show that it is second countable, let $\{U_i\}$ be the countable basis for the topology of M, then $\{\pi U_i\}$ is a countable collection of open subsets of M/G. We need to show that $\{\pi U_i\}$ is a basis for the topology of M/G. First note that if U is a subset of M/G then $\pi^{-1}U$ is the union of sets of elements whose orbits belong to U, so if U is an open subset of M/G, then U is a collection of orbits whose union is an open subset $\pi^{-1}U$ of M. This means that for each element x of $\pi^{-1}U$ there is a basis element U_i containing x. But then πU_i is an element of $\{\pi U_i\}$ which contains an orbit of x an element of U. Since this is true for each element of U, $\{\pi U_i\}$ is a countable basis for the topology of M/G. $\square$

Theorem 2.1.3 *Let $\Phi : G \times M \to M$ be an action of a Lie group G on a smooth manifold M. If Φ is a proper action, then M/G has a smooth manifold structure such that the map $\pi : M \to M/G$ is a submersion.*

See ([1, Thm 4.1.20, p262]).

If G is a Lie group and $\mathfrak{g} = T_eG$ is its Lie algebra, then for each $\xi \in \mathfrak{g}$, the action $\Phi : G \times M \to M$ induces on M, a smooth vector field called the infinitesimal generator of the action corresponding to ξ and is defined by

$$\xi_M(m) = \frac{d}{dt}\Phi(\exp -t\xi, m)|_{t=0}. \tag{2.5}$$

We shall see in the next section an extension of this theorem to Lie groups and their closed subgroups.

2.2. Homogeneous spaces

We now come to the special kind of spaces, the homogeneous spaces on which Lie groups acts in a transitive way. We shall be more interested in homogeneous spaces which are also manifolds.

Definition 2.2.1 *Let G be a Lie group and M a smooth space.*
Let $\Phi : G \times M \to M$ define an action of G on M. Then the space M is called a homogeneous space if whenever $x, y \in M$, then there is a $g \in G$ such that

$$\Phi_g(x) = \Phi(g, x) = y.$$

Such an action, as we have already seen, is called a transitive action

Example 2.2.0.1 *Every Lie group G is a homogeneous space under the left translation*

$$L : G \times G \to G; \;\; (g, x) \mapsto L_g(x),$$

or indeed under the right translation

$$R_g(x) = xg, \;\; \textit{for all} \;\; g, x \in G.$$

Example 2.2.0.2 *Consider a Lie group G and any subgroup H of G. Let*

$$G/H = \{xH : x \in G\},$$

be the set of left cosets of H in G. Define a left action

$$\theta : G \times G/H \to G/H \;\; \textit{by} \;\; \theta(g, xH) = gxH.$$

This is an action since $\theta(e, xH) = exH = xH$, *and*

$$\begin{aligned}
\theta(g_1g_2, xH) &= g_1g_2xH \\
&= g_1(g_2xH) \quad \textit{by associativity in } G \\
&= \theta(g_1, g_2xH) \\
&= \theta(g_1, \theta(g_2, xH)).
\end{aligned}$$

This action is transitive since if xH *and* yH *are any two points of* G/H, *with* $x, y \in G$, *then*

$$\theta(xy^{-1}, yH) = xH.$$

Special type of example 2.2.0.2 is the following:

Theorem 2.2.1 *Suppose that* H *is a closed Lie subgroup of a Lie group* G. *Let*

$$G/H = \{xH : x \in G\},$$

be the set of the left cosets of H *in* G. *Then there exists a unique smooth manifold structure on* G/H *such that*

(i) $\pi : G \to G/H$ *is smooth,*

(ii) each point $g \in G$ *is the image* $\sigma(V)$ *of a* C^∞ *section* (V, σ) *on* G/H,

(iii) the natural action

$$\theta : G \times G/H \to G/H, \quad \textit{defined by} \quad \theta(g, xH) = gxH,$$

is a C^∞ *action of* G *on* G/H *with respect to this structure.*

The dimension of G/H *is given by* $\dim G - \dim H$.

For the proof of this important theorem, see [39, Thm 3.58, p120] and [12, Thm 9.2, p161].

Lemma 2.2.1 *Let* G *be a Lie group with Lie algebra* $\mathfrak{g}$ *and* H *its Lie subgroup with Lie subalgebra* $\mathfrak{h}$, *then* $T_{\mathfrak{o}}(G/H) \cong \mathfrak{g}/\mathfrak{h}$, *where* $\mathfrak{o} = \pi(e) = eH$.

Proof. We compute the differential of the projection map

$$\pi : G \to G/H,$$

at the identity,

$$d\pi_e : T_e(G) \to T_{\mathfrak{o}}(G/H).$$

Let $X \in \mathfrak{g} \cong T_e(G)$, and let $\exp tX$ be the one-parameter subgroup corresponding to X. Then

$$\begin{aligned} d\pi_e(X) &= d\pi_e \circ \frac{d}{dt} \exp tX|_{t=0} \\ &= \frac{d}{dt}(\pi \circ \exp tX)|_{t=0} \\ &= \frac{d}{dt}\left((\exp tX)H\right)|_{t=0}. \end{aligned}$$

The last equality is because $\pi(a) = aH$ for all $a \in G$. But now if $x \in H$ then $xH = H$. Thus, if $X \in \mathfrak{h}$ then

$$d\pi_e(X) = 0, \text{ the zero vector.}$$

That is to say

$$\ker d\pi_e = \mathfrak{h},$$

the Lie algebra of H. But $d\pi_e$ is onto, hence

$$\mathfrak{g}/\mathfrak{h} \cong T_{\mathfrak{o}}(G/H)$$

as required. □

Theorem 2.2.2 *Let $\Phi : G \times M \to M$ be a transitive action of a Lie group G on M, so that M is a homogeneous space with respect to the action Φ. Then there is a closed subgroup H of G such that the map $F : G/H \to M$ is a diffeomorphism.*

Proof. We want to show that the homogeneous space G/H is naturally diffeomorphic to the homogeneous space M.

To begin the proof, let M be a smooth manifold and G a compact Lie group. Let G act transitively on M by the rule $\Phi : G \times M \to M$ defined by $\Phi(g, x) = gx$ for g in G and x in M. Let $x_0 \in M$ be arbitrary and let

$$H = \{g \in G : \Phi_g(x_0) = x_0\}$$

be the isotropy group. H is a closed subgroup of G since if $\{g_n\}$ is a sequence in H converging to $g \in G$ then $gx_0 = \lim g_n x_0 = \lim x_0 = x_0$. Therefore, as we have seen before, G/H is a smooth manifold and G acts naturally on G/H by the rule $\theta : G \times G/H \to G/H$, $\theta(g, xH) = gxH$. This action is smooth since it is a composition of the left translation L_g with the projection $\pi : G \to G/H$. The action is transitive by Example 2.2.0.2.

Let $\tilde{F} : G \to M$ be a map defined by $\tilde{F}(g) = \Phi(g, x_0) = gx_0$.

Define a map $F : G/H \to M$ by $F(gH) = gx_0$. That is, $F(gH) = \tilde{F}(g)$. Then, F is well-defined since if $aH = bH$ then $a^{-1}b \in H$ implying that $x_0 = a^{-1}bx_0$ which implies that $ax_0 = bx_0$.

We now show that the map $F : G/H \to M$ is a diffeomorphism. To do this we shall show that F is injective, it is surjective, it is C^∞ and that the map $F_* : T(G/H) \to T(M)$ is an isomorphism.

The map F is injective since $F(g_1H) = F(g_2H)$ implies that $\tilde{F}(g_1) = \tilde{F}(g_2)$. That is, $\Phi_{g_1}(x_0) = \Phi_{g_2}(x_0)$, which gives $g_1x_0 = g_2x_0$, implying that $g_2^{-1}g_1x_0 = x_0$, so that $g_2^{-1}g_1 \in H$ and $g_1H = g_2H$.

To see that F is surjective, we note first that

$$\begin{aligned}
\tilde{F}(gh) &= \Phi_{gh}(x_0) \\
&= \Phi(gh, x_0) \\
&= \Phi(g, \Phi(h, x_0)) \\
&= \Phi(g, x_0) \\
&= \Phi_g(x_0) \\
&= \tilde{F}(g), \quad \text{for all} \quad g \in G \text{ and all } h \in H.
\end{aligned}$$

Thus, if $x \in M$ then $\tilde{F}^{-1}(x) = gH$ where g is such that $\tilde{F}(g) = x$.
This gives $F(gH) = \tilde{F}(g) = x$. Note that we could also have used the fact that the action Φ is transitive to show that F is surjective.

To see that F is smooth, let $y \in G/H$ and let (V, σ) be the smooth section defined on a neighborhood V of y, then $F|V = \tilde{F} \circ \sigma$, a composition of smooth maps. Thus, F is C^∞ in a neighborhood of every point, hence on G/H.

It remains to show that $F_* : T(G/H) \to T(M)$ is an isomorphism.
Let $s_X(t)$ be a 1 - parameter subgroup of G. By identification $L(G) = T_e(G)$, define exponential map $exp : T_e(G) \to G, exp(X) = s_X(1)$ for $X \in T_e(G)$. We know that $exp : T_e(G) \to G$ is a smooth map and carries some neighborhood of 0 in $T_e(G)$ diffeomorphically onto a neighborhood of e in G. ([17, Prop 5.1.27, p167]).
Let $\mathfrak{h} = L(H)$ and let $\mathfrak{m}$ be any complementary subspace of $L(G)$ such that $L(G) = \mathfrak{m} \oplus \mathfrak{h}$. Let C_0 be an open neighborhood of 0 in $\mathfrak{m}$ and U_e the corresponding neighborhood of $e \in G$ such that $exp : exp(C_0) \to U_e$ is a diffeomorphism. We now show that $F_{*eH} : T_{eH}(G/H) \to T_{x_0}(M)$ is an isomorphism by showing that $\tilde{F}_{*e} : T_e(exp(C_0)) \to T_{x_0}(M)$ is an isomorphism. Let $v \in T_e(exp(C_0)) = \mathfrak{m}$ and consider the curve $s(t) = exp(tv)x_0$. Then,

$$\tilde{F}_{*e}(v) = \dot{s}(0) = \frac{d}{dt}(exp(tv)x_0) \mid_{t=0} = vx_0.$$

We only need to show that $\dot{s}(0) = 0$ implies that $v = 0$.
Let $a = exp(t_0 v)$ so that $L_a(s(t)) = s(t + t_0)$. Then, $L_{*a}(\dot{s}(0)) = \dot{s}(t_0)$. Since $t_0 \in \mathbb{R}$ is arbitrary, this gives $\dot{s}(0) = 0$ implies that $\dot{s}(t) = 0$ for all $t \in \mathbb{R}$. Thus, $exp(tv)x_0 = x_0$ for all $t \in \mathbb{R}$. This implies that $exp(tv) \subset H$, which shows that $v \in \mathfrak{h}$. But then this means that $v \in \mathfrak{m} \cap \mathfrak{h} = \{0\}$ so that $v = 0$ as required. Thus $\tilde{F}_{*e}$ is injective, and since it is surjective, $\tilde{F}_{*e}$ and hence

$$F_{*eH} : T_{eH}(G/H) \to T_{x_0}(M)$$

is an isomorphism at the identity eH. The Inverse Function Theorem then implies that F is locally a diffeomorphism at eH. But then if $aH \in G/H$, we have by equation 2.3,

$$F_{*aH} = (L_a)_{*x_0} \circ F_{*eH} \circ (L_{a^{-1}})_{*aH}$$

is an isomorphism of $T_{aH}(G/H) \to T_{ax_0}(M)$. Hence F_* is an isomorphism of $T(G/H)$ onto $T(M)$. This completes the proof that F is a diffeomorphism. $\square$

We give below an example of homogeneous spaces called the Grassmann manifold.

Example 2.2.0.3 *A Grassmann manifold of k-planes in $\mathbb{R}^n$, denoted by $G_k(\mathbb{R}^n)$, is by definition a set of all k-dimensional subspaces of $\mathbb{R}^n$. (A typical k-dimensional subspace of $\mathbb{R}^n$ is called a k-plane). Let $U_k(\mathbb{R}^n)$ be the set of all k-bases of $\mathbb{R}^n$. We shall assume that all the elements of $U_k(\mathbb{R}^n)$ are normalised. That is, $U_k(\mathbb{R}^n)$ consists of orthonormal k-bases of $\mathbb{R}^n$. For each element $(u_1, \cdots, u_k) \in U_k(\mathbb{R}^n)$, consider the map*

$$U_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n), \quad (u_1, \cdots, u_k) \mapsto \langle u_1, \cdots, u_k \rangle,$$

which takes each k-basis of $\mathbb{R}^n$ to a k-plane it generates. This map is surjective since for any given k-plane, we can choose, using the methods of linear algebra, an orthonormal k-basis which spans it. Let $O(n)$ be the orthogonal group of matrices. Consider the map

$$\Psi : O(n) \times G_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n), \quad (A, \langle u_1, \cdots, u_k \rangle) \mapsto \langle Au_1, \cdots, Au_k \rangle, \quad A \in O(n).$$

We see that Ψ is an action since if $A = I_n$, the identity, then

$$\Psi(I_n, \langle u_1, \cdots, u_k \rangle) = \langle u_1, \cdots, u_k \rangle.$$

Also if $A, B \in O(n)$ then

$$\begin{aligned}
\Psi(AB, \langle u_1, \cdots, u_k \rangle) &= \langle ABu_1, \cdots, ABu_k \rangle \\
&= \langle A(Bu_1) \cdots, A(Bu_k) \rangle \quad \textit{matrix multiplication is associative,} \\
&= \Psi(A, \langle Bu_1, \cdots, Bu_k \rangle) \\
&= \Psi(A, \Psi(B, \langle u_1, \cdots, u_k \rangle)).
\end{aligned}$$

The action Ψ is transitive since given any two k-planes in $\mathbb{R}^n$, choose in each one of them a k-basis which spans it. Then both bases can be completed to an orthonormal basis of $\mathbb{R}^n$. But given any two orthonormal bases of $\mathbb{R}^n$, the transitional matrix from one basis to the other basis is orthogonal. Thus, there is

$P \in O(n)$ which transforms one k-plane generated by one k-basis to the other k-plane generated by the other k-basis.

To determine the isotropy group consider a point $x = a_1e_1 + \cdots + a_ku_k$, where, $(e_1, \cdots, e_k)$ are such that $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ with a 1 in the i^{th} position. The element of $O(n)$ which leaves x invariant is the matrix of the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

where $A \in O(k)$ and $B \in O(n-k)$. Now, for each integer $n \geq 1$ define a map

$$f : GL(n, \mathbb{R}) \to M_n(\mathbb{R}); \quad A \mapsto A - A^T.$$

Then $O(n) = f^{-1}(0)$ (see [3, p 10]).

Thus, $O(n)$ is a closed set. Therefore, the matrix $O(k) \times O(n-k)$ is closed being the product of two closed sets. Consequently, we have

$$G_k(\mathbb{R}^n) \cong O(n)/O(k) \times O(n-k)$$

is a homogeneous space.

As a special case of Example 2.2.0.3 is when $k = 1$. In this case, $G_1(\mathbb{R}^n)$ is the set of all 1-dimensional planes. These are straight lines in $\mathbb{R}^n$ passing through the origin. We call this space the projective space, and is denoted by $\mathbb{R}P^{n-1}$. Thus, the projective space is a homogeneous space

$$\mathbb{R}P^{n-1} \cong O(n)/O(1) \times O(n-1).$$

We shall see other interesting examples of homogeneous spaces such as the flag manifolds when we discus the adjoint orbits. Flag manifolds will be of special interest because they are known to hold a symplectic structure as well.

3

Symplectic manifolds

3.1. Symplectic algebra

Definition 3.1.1 *Let V and W be finite dimensional vector spaces. A pairing is a bilinear map*

$$\langle \cdot, \cdot \rangle : V \times W \to \mathbb{R}$$

It is non degenerate if $\langle v_0, w \rangle = 0$ for all $w \in W \Rightarrow v_0 = 0$, and $\langle v, w_0 \rangle = 0$ for all $v \in V \Rightarrow w_0 = 0$.

Example 3.1.0.4 *Let V be a finite dimensional vector space and V^* be its dual, then*

$$\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}$$

given by

$$\langle \alpha, \xi \rangle = \alpha(\xi),$$

is a non degenerate pairing.

Proposition 3.1.1 *Let V and W be finite dimensional vector spaces.*
If $b : V \times W \to \mathbb{R}$ is a non degenerate pairing, then $V \cong W^*$ and $W \cong V^*$.

Proof. Let $v \in V$ and $w \in W$. Consider the map $b^\flat : V \to W^*$ defined by

$$(b^\flat(v))(w) = b(v, w).$$

We have that $b^\flat$ is a linear map since b is a linear map. The kernel of $b^\flat$ is given by

$$\begin{aligned} \ker b^\flat &= \{v_0 \in V : b^\flat(v_0) = 0\} \\ &= \{v_0 \in V : b(v_0, w) = 0, \quad \text{for all } w \in W\} \\ &= \{0\}, \text{ since we assumed non degeneracy.} \end{aligned}$$

Thus $b^\flat$ is injective so that $\dim V \leq \dim W^* = \dim W$.

Similarly we have $\dim W \leq \dim V^* = \dim V$. Combining the two inequalities we get
$\dim V = \dim W$. Hence $b^\flat$ is an isomorphism. $\square$

Definition 3.1.2 *Let V be a finite dimensional vector space over $\mathbb{R}$ and V^* its dual space. Then the space $\bigwedge^2 V^*$ is identified with the space of skew symmetric bilinear forms*

$$\omega : V \times V \to \mathbb{R},$$

where $\omega(u, v) = -\omega(v, u)$ for all $u, v \in V$.

The form ω is called a symplectic form if it is non degenerate, that is if $\omega(u, v) = 0$ for all $v \in V$ implies that $u = 0$.

Definition 3.1.3 *A vector space V equipped with a symplectic form ω, is called a symplectic vector space. That is, a symplectic vector space is a pair (V, ω), where V is a finite dimensional real vector space and ω a non degenerate skew symmetric bilinear form.*

Example 3.1.0.5 *Let $V = \mathbb{R}^{2n}$. If $x = (x_1, \cdots, x_{2n})$ and $y = (y_1, \cdots, y_{2n})$ are vectors in V, define $\omega(x, y)$ by:*

$$\omega(x, y) = \sum_{i=1}^{n} x_{i+n}y_i - x_iy_{i+n}$$

Then $(\mathbb{R}^{2n}, \omega)$ is a symplectic vector space.

Clearly ω is bilinear. We must show that it is skew symmetric and non degenerate. To see that it is skew symmetric, we have from the definition:

$$\begin{aligned}
\omega(x, y) &= \sum_{i=1}^{n} x_{i+n}y_i - x_iy_{i+n} \\
&= \sum_{i=1}^{n} -(x_iy_{i+n} - x_{i+n}y_i) \\
&= \sum_{i=1}^{n} -(y_{i+n}x_i - y_ix_{i+n}) \\
&= -\sum_{i=1}^{n} y_{i+n}x_i - y_ix_{i+n} \\
&= -\omega(y, x).
\end{aligned}$$

To prove non degeneracy, suppose that $\omega(x,y)=0$ for all $y\in\mathbb{R}^{2n}$, then it is zero on all the basis elements. Thus, for example

$$0=\omega(x,e_1)=x_{1+n}\cdot 1-x_1\cdot 0=x_{1+m}.$$

Thus, $x_{1+n}=0$. Trying each basis element gives $x_i=0$ for $i=1,\cdots,2n$. Hence $x=0$ proving that ω is non degenerate.

The skew symmetric condition implies that $\omega(u,u)=0$ for all $u\in V$.

If $(e_1,\cdots,e_n)$ is the given basis for V, then the bilinear form ω on V can be expressed, relative to this basis, in matrix form

$$\omega\mapsto(\omega_{ij})\in M_n(\mathbb{R}),$$

where $\omega_{ij}=\omega(e_i,e_j)$.

Definition 3.1.4 *Let (V,ω) be a symplectic vector space, and let W be a linear subspace of V. Then the symplectic complement (or symplectic orthogonal) of W in V, denoted by W^ω, is defined by*

$$W^\omega=\{v\in V:\omega(v,w)=0,\ \textit{for all } w\in W\}.$$

Lemma 3.1.1 *Let (V,ω) be any symplectic vector space and let $W\subset V$ be any linear subspace. Then*

(i) $\dim W+\dim W^\omega=\dim V$,

(ii) $(W^\omega)^\omega=W$.

Proof. Define a map

$$\omega^\flat:V\to V^*;\ v\mapsto\omega^\flat(v):V\to\mathbb{R},\ \text{such that } w\mapsto\omega(v,w),$$

for all $v,w\in V$, where V^* is the dual space of V. Since ω is non degenerate, we have

$$\ker\omega^\flat=\{v\in V:\omega(v,w)=0\text{ for all }w\in V\}=\{0\}.$$

Thus $\omega^\flat$ is an isomorphism. Note that if $W\subset V$ then $\omega^\flat(W^\omega)=W^\perp\subset V^*$, is the orthogonal complement of W. That is,
$W^\perp=\{\alpha\in V^*:\alpha(w)=0,\ \text{for all } w\in W\}$.

Part (i) now follows from the fact that $\dim W + \dim W^{\perp} = \dim V$.

To prove (ii), note that $W \subset (W^{\omega})^{\omega}$ since if $w \in W$ and $v \in W^{\omega}$ then $\omega(v, w) = 0$ which implies that $w \in (W^{\omega})^{\omega}$. But from Part (i) above we have

$$\dim W = \dim V - \dim W^{\omega} = \dim (W^{\omega})^{\omega}.$$

Combining the two we conclude that $W = (W^{\omega})^{\omega}$. $\square$

Theorem 3.1.1 *Let (V, ω) be any symplectic vector space, then there exists a basis $u_1, \cdots, u_n, v_1, \cdots, v_n$ of V such that*

$$\omega(u_j, u_k) = \omega(v_j, v_k) = 0, \quad \omega(u_j, v_k) = \delta_{jk}.$$

In particular, $\dim V = 2n$ *for some positive integer n.*

Proof. We prove by induction on the dimension of V, $\dim V$.
Note that $\dim V \geq 2$ since $\omega \neq 0$.

When $\dim V = 2$, since ω is non degenerate, there exists non zero vectors $u, v \in V$ such that $\omega(u, v) \neq 0$. This implies that u and v are linearly independent so that they form a basis for V. After multiplying v by a scalar, it can be assumed that $\omega(u, v) = 1$ and the condition is satisfied for $\dim V = 2$ and the theorem is true for this case.

Now suppose that the theorem is true when $\dim V \leq m - 1$. We prove that it is also true when $\dim V = m$. Again the non degeneracy condition of ω implies that there exists $u_1, v_1 \in V$ such that u_1 and v_1 are linearly independent and $\omega(u_1, v_1) = 1$. Set $W =$ span(u_1, v_1) and consider the space $(W^{\omega}, \omega|_{W^{\omega}})$. To see that this space is a symplectic vector space we must show that $\omega|_{W^{\omega}}$ is non degenerate. Let $w \in W^{\omega}$ be such that $\omega(w, z) = 0$ for all $z \in W^{\omega}$. We need to show that $w = 0$. From lemma 3.1.1 part (i) we note that $W \cap W^{\omega} = \{0\}$, so that $V = W \oplus W^{\omega}$. Now for any $z \in V$ we can write $z = z_1 + z_2$ where $z_1 \in W$ and $z_2 \in W^{\omega}$. Thus $\omega(w, z_1) = 0$ because $w \in W^{\omega}$ and $\omega(w, z_2) = 0$ by assumption on w. Hence $\omega(w, z) = 0$ and therefore $w = 0$ since ω is non degenerate on V. Therefore, $(W^{\omega}, \omega|_{W^{\omega}})$ is a symplectic vector space. Since $\dim W^{\omega} = \dim V - 2 \leq m - 1$, the inductive hypothesis implies that there is a symplectic basis $u_2, \cdots, u_n, v_2, \cdots, v_n$ of $(W^{\omega}, \omega|_{W^{\omega}})$. Therefore, the basis $u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n$ is a symplectic basis for (V, ω) and the theorem is proved. $\square$

With respect to symplectic basis, the form ω is represented by the matrix $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, where $I_n \in M_n(\mathbb{R})$ is the identity matrix.

Remark 3.1.1 *Note that for $V = \mathbb{R}^{2n}$ with the standard Euclidean inner product $\langle \cdot, \cdot \rangle$, the form defined in example 3.1.0.5 is the form given by $\omega(x, y) = \langle Jx, y \rangle$.*

Definition 3.1.5 *A subspace $W \subseteq V$ of a symplectic vector space is called*

(a) Isotropic if $W \subseteq W^\omega$;

(b) Co-isotropic if $W^\omega \subseteq W$;

(c) Lagrangian if $W = W^\omega$;

(d) Symplectic if $W \cap W^\omega = \{0\}$.

Example 3.1.0.6 *(a) $W =$ Span$\{u_1, u_2\}$ is isotropic.*

(b) $W =$ Span$\{u_1, \cdots, u_n, v_1\}$ is co-isotropic.

(c) $W =$ Span$\{u_1, u_2, \cdots, u_n\}$ is Lagrangian.

(d) $W =$ Span$\{u_1, u_2, \cdots, u_k, v_1, v_2, \cdots, v_k\}$ for some $k \leq n$ is symplectic.

Corollary 3.1.1 *Let V be a symplectic vector space and let ω_1 and ω_2 be two symplectic forms on V, then there exists an invertible linear map $A : V \to V$ such that*
$\omega_1(Au, Av) = \omega_2(u, v)$ for all $u, v \in V$. That is, $A^*\omega_1 = \omega_2$,
where $A^*\omega_1(u, v) = \omega_1(Au, Av)$.

Proof. Let $e_1, \cdots, e_n, f_1, \cdots, f_n$ be a basis for V such that $\omega_1 = \sum e_i^* \wedge f_i^*$, where $e_1^*, \cdots, e_n^*, f_1^*, \cdots, f_n^*$ is its dual basis. There also exists a basis $u_1, \cdots, u_n, v_1, \cdots, v_n$ for V such that $\omega_2 = \sum u_i^* \wedge v_i^*$, where $u_1^*, \cdots u_n^*, v_1^*, \cdots, v_n^*$ is the dual basis relative to the basis $u_1, \cdots, u_n, v_1, \cdots, v_n$. Define a map $A : V \to V$ by $A(u_i) = e_i$ and $A(v_j) = f_j$, $i, j = 1, \cdots, n$. Then $A^*(e_i^*) = u_i^*, A^*(f_j^*) = v_j^*$. Therefore,

$$\begin{aligned} A^*\omega_1 &= A^*(\textstyle\sum e_i^* \wedge f_i^*) \\ &= \textstyle\sum A^*(e_i^* \wedge f_i^*) \\ &= \textstyle\sum A^*(e_i^*) \wedge A^*(f_i^*) \\ &= \textstyle\sum u_i^* \wedge v_i^* \\ &= \omega_2. \qquad \square \end{aligned}$$

Corollary 3.1.2 *Any even dimensional vector space V admits a symplectic form.*

Proof. Let $n = \frac{1}{2}\dim V$ and choose a basis $e_1, \cdots, e_n, f_1, \cdots, f_n$ for V. Let $e_1^*, \cdots, e_n^*, f_1^*, \cdots, f_n^*$ be the dual basis. Let $\omega = \sum e_i^* \wedge f_i^*$. We must show that ω is non degenerate.

Let $u = \sum_{i=1}^{n} a_i e_i + b_i f_i \in V$ be such that $\omega(u, v) = 0$ for all $v \in V$. Then we have $\omega(u, e_i) = \omega(u, f_i)$ for each basis element. Thus

$$\begin{aligned} 0 &= \sum_{i=1}^{n} e_i^* \wedge f_i^*(u, e_j) \\ &= \sum_{i=1}^{n} (e_i^*(u) f_i^*(e_j) - e_i^*(e_j) f_i^*(u)) \\ &= 0 - b_j = -b_j. \end{aligned}$$

Thus $b_j = 0$ for all $j = 1, 2, \cdots, n$.

Similarly,

$$\begin{aligned} 0 &= \sum_{i=1}^{n} e_i^* \wedge f_i^*(u, f_k) \\ &= \sum_{i=1}^{n} (e_i^*(u) f_i^*(f_k) - e_i^*(f_k) f_i^*(u)) \\ &= a_k - 0 = a_j. \end{aligned}$$

Giving $a_k = 0$ for all $k = 1, 2, \cdots, n$.

Hence $u = 0$ and ω is non degenerate. $\square$

Remark 3.1.2 *Let (V, ω) be a symplectic vector space. A subspace $U \subset V$ is symplectic if the restriction of the symplectic form ω to U is non degenerate.*

Proof. We must show that U is even dimensional if $\omega|_U$ is non degenerate. Let $u_1, \cdots, u_k, v_1, \cdots, v_l$ be the basis for U. We must show that $l = k$. Note that this basis can be extended to the symplectic basis for V, so that $\omega(u_i, u_j) = \omega(v_i, v_j) = 0$ and $\omega(u_i, v_j) = \delta_{ij}$. Suppose that $l \neq k$, we first assume that $l > k$. We pair up $u_1 v_1, u_2 v_2, \cdots, u_k v_k$ such that $\omega(u_i, v_i) = 1$. Let p be such that $k < p < l$ and $\omega(u_i, v_p) \neq 0$. We scale v_p so that $\omega(u_i, v_p) = 1$. Thus $\omega(u_i, v_i) = \omega(u_i, v_p)$ which implies that $\omega(u_i, v_i - v_p) = 0$. But since $u_i \neq 0$ and ω is non degenerate, we must have $v_i - v_p = 0$ so that $v_i = v_p$. Thus $l > k$ is not possible. A similar argument shows that $k > l$ is also not possible. Therefore, we must have $l = k$ and $\dim U = 2n$ for some positive integer n. □

Remark 3.1.3 *Let (V, ω) be a symplectic vector space. A subspace $U \subset V$ is symplectic if and only if $U \cap U^\omega = \{0\}$.*

Proof. Suppose U is symplectic, let $u_1, \cdots, u_k, v_1, \cdots, v_k$ be its symplectic basis. Now if $w = \sum_{i=1}^{n}(a_i u_i + b_i v_i) \in U \cap U^\omega$, then $w \in U^\omega$ and it follows that $0 = \omega(w, u_i) = -b_i$. Thus, $b_i = 0$ for $i = 1, 2, \cdots, n$. Similarly, $0 = \omega(w, vj) = a_j$ showing that $a_j = 0$ for $j = 1, 2, \cdots, n$. Therefore, $w = 0$ and $U \cap U^\omega = \{0\}$. Suppose now that $U \cap U^\omega = \{0\}$. Consider the restriction $\omega|_U$. The condition that $U \cap U^\omega = \{0\}$ implies that ω restricted to U is non degenerate. By remark 3.1.2, U is symplectic. □

Example 3.1.0.7 *Let (V, ω) be a symplectic vector space and let $e_1, \cdots, e_n, f_1, \cdots, f_n$ be its symplectic basis. Then, $U = Span\{e_1, \cdots, e_k, f_1, \cdots, f_k\}$ for some $k \leq n$ is a symplectic subspace.*

Remark 3.1.4 *From lemma 3.1.1 we have $(U^\omega)^\omega = U$, it follows that $U \cap U^\omega = \{0\}$ if and only if $U^\omega \cap (U^\omega)^\omega = \{0\}$. Thus, a subspace U is symplectic if and only if its symplectic orthogonal U^ω is symplectic.*

3.1.1 Lagrangian subspaces

Lemma 3.1.2 *Let (V, ω) be a symplectic vector space, and let U, W be subspaces of V. Then*

(i) $U \subset W \Rightarrow W^\omega \subset U^\omega$;

(ii) $(U + W)^\omega = U^\omega \cap W^\omega$;

(iii) $(U \cap W)^\omega = U^\omega + W^\omega$.

Proof. To prove (1) we have $v \in W^\omega \Rightarrow \omega(u, v) = 0$ for all $u \in W$, and in particular, $\omega(u, v) = 0$ for all $u \in U$ so that $v \in U^\omega$. Thus $v \in W^\omega \Rightarrow v \in U^\omega$. Hence $W^\omega \subset U^\omega$.

To prove (2) and (3) first note that

$$U \cap W \subseteq U \Rightarrow U^\omega \subseteq (U \cap W)^\omega, \text{ and}$$
$$U \cap W \subseteq W \Rightarrow W^\omega \subseteq (U \cap W)^\omega. \text{ Then}$$

$$U^\omega + W^\omega \subset (U \cap W)^\omega. \tag{3.1}$$

We also have,

$$U \subset U + W \Rightarrow (U + W)^\omega \subseteq U^\omega, \text{ and}$$
$$W \subseteq U + W \Rightarrow (U + W)^\omega \subseteq W^\omega. \text{ Thus}$$
$$(U + W)^\omega \subseteq U^\omega \cap W^\omega. \tag{3.2}$$

We already have $(U^\omega)^\omega = U$. So from inclusion (3.1) and inclusion (3.2) we have,

$$U \cap W = ((U \cap W)^\omega)^\omega \subseteq (U^\omega + W^\omega)^\omega \subseteq (U^\omega)^\omega \cap (W^\omega)^\omega = U \cap W.$$

It follows that both inclusions are equalities. Therefore, we have

$$(U + W)^\omega = U^\omega \cap W^\omega$$

proving 2, and

$$(U \cap W)^\omega = U^\omega + W^\omega$$

proving 3. □

Let (V, ω) be a symplectic vector space and let $\{U_k\}$ be a strictly increasing sequence of isotropic subspaces of V. If V is finite dimensional then this sequence terminates at a maximal isotropic subspace.

Lemma 3.1.3 *Any maximal isotropic subspace L of a finite dimensional symplectic vector space (V, ω) is a Lagrangian subspace.*

Proof. We must show that $L = L^\omega$. But since if k is a scalar, $\omega(v, kv) = k\omega(v, v) = 0$ we note that a one dimensional subspace is necessarily isotropic. Suppose $L \subset L^\omega$ and $L \neq L^\omega$, let $v \in L^\omega \setminus L$ and consider $L' = L + kv$ for some scalar k. From $(L + kv)^\omega = L^\omega \cap (kv)^\omega$, we have $L \subset (L + kv)^\omega$ since $L \subset L^\omega$ and $kv \subset L^\omega$. This last inclusion implies that $L \subset (kv)^\omega$. We also have $v \in (L + kv)^\omega$ since $v \in L^\omega$ and $v \in (kv)^\omega$. It follows that $L + kv \subset (L + kv)^\omega$ which implies that $L' = L + kv$ is isotropic and $\dim L' = \dim L + 1$. Therefore, L

is a maximal isotropic linear subspace if and only if $L = L^\omega$. Hence L is maximal isotropic subspace if and only if L is a Lagrangian subspace. $\square$

This lemma also shows that for any symplectic vector space (V, ω) there exists a Lagrangian subspace.

Lemma 3.1.4 *Let (V, ω) be a symplectic vector space with $\dim V = 2n$. For any Lagrangian subspace L of V there exists another Lagrangian subspace M of V such that $L \cap M = \{0\}$ and $V = L \oplus M$.*

Proof. Let M be isotropic subspace such that $M \neq M^\omega$. Then, there exists $v \in M^\omega \setminus M$ such that if $M' = M + kv$ then $L \cap M' = \{0\}$, for if $L \cap M' \neq \{0\}$, then there exists $w \in M$ and a scalar b such that $u = w + bv$ is a non zero element of L. Then $v \in L + M$. If this is the case for every $v \in M^\omega \setminus M$, then $M^\omega \setminus M = L + M$ which implies that $L \cap M^\omega = L^\omega \cap M^\omega = (L + M)^\omega \subset (M^\omega)^\omega = M$. But $L \cap M = \{0\}$, this implies that $L \cap M^\omega = \{0\}$. On the other hand, we have,

$$\dim M^\omega = \dim V - \dim M > \dim V - n = \dim V - \dim L$$

This gives $\dim M^\omega + \dim L > \dim V$ and
$\dim L \cap M^\omega = \dim L + \dim M^\omega - \dim V > 0$ which is a contradiction since $L \cap M^\omega = \{0\}$. Hence $M = M^\omega$ and $V = L \oplus M$. $\square$

3.1.2 Symplectic maps

Definition 3.1.6 *Let (V_1, ω_1) and (V_2, ω_2) be two symplectic vector spaces. A linear map $\Phi : V_1 \to V_2$ is called symplectic if*

$$\omega_2(\Phi u, \Phi v) = \omega_1(u, v) \ \textit{for} \ \ \textit{all} \ u, v \in V_1.$$

(See [11, p 35]).

Note that if $\Phi v \neq 0$, then by the non degeneracy of ω_2 we have
$0 = \omega_2(\Phi u, \Phi v) \Rightarrow \Phi u = 0$. But $\Phi v \neq 0 \Rightarrow v \neq 0$ since Φ is linear. Therefore,

$$0 = \omega_2(\Phi u, \Phi v) = \omega_1(u, v) \Rightarrow u = 0.$$

Thus, $\Phi u = 0 \Rightarrow u = 0$ shows that Φ is injective.

Let M be a C^∞ manifold and $\omega \in \Omega^2(M)$. Then ω is non degenerate if and only if for all $m \in M$, $\omega_m \in \wedge^2(T_mM)$ is non degenerate.
That is, $\omega_m(X_m, Y_m) = 0$ for all $Y_m \in T_mM \Rightarrow X_m = 0$, or equivalently, $\omega(X,Y) = 0$ for all $Y \in \mathfrak{X}(M) \Rightarrow X = 0$.

Let M be a C^∞ manifold and ω a 2-form on M. Define a map

$$\omega^\flat : \mathfrak{X}(M) \to \Omega^1(M);\ X \mapsto \omega^\flat(X) = i_X\omega = \alpha$$

such that

$$Y \mapsto i_X\omega(Y) := \omega(X,Y),$$

for all $X, Y \in \mathfrak{X}(M)$.

Proposition 3.2.1 *$\omega \in \Omega^2(M)$ is non degenerate if and only if $\omega^\flat$ defined above is an isomorphism of modules $\mathfrak{X}(M)$ and $\Omega^1(M)$ on $C^\infty(M)$.*

Proof. If $\omega^\flat$ is an isomorphism then $\ker \omega^\flat = \{0\}$. If $\omega(X,Y) = 0$ for all $Y \in \mathfrak{X}(M)$ then $\omega(X,Y) = i_X\omega(Y) = \omega^\flat(X)(Y) = 0$ for all $Y \in \mathfrak{X}(M)$. This gives $X = 0$ and ω is non degenerate. On the other hand, if ω is non degenerate, then $\omega^\flat(X) = 0$ implies that $\omega^\flat(X)(Y) = \omega(X,Y) = 0$ for all $Y \in \mathfrak{X}(M)$. But ω is non degenerate, implying that $X = 0$. Thus we get that $\omega^\flat$ is injective. $\omega^\flat$ is also surjective since for any $\alpha \in \Omega^1(M)$ we can find $X \in \mathfrak{X}(M)$ such that $\alpha = i_X\omega = \omega^\flat(X)$. Thus $\omega^\flat$ is an isomorphism. $\square$

Theorem 3.2.1 *(Darboux) Let $\omega \in \Omega^2(M)$ be non degenerate with $\dim M = 2n$ for some integer n. Then ω is closed if and only if for each $m \in M$ there is a chart (U, φ) containing m such that $\varphi(m) = 0 \in \mathbb{R}^{2n}$ and for all $u \in U$, $\varphi(u) = (x^1(u), \cdots, x^n(u), y^1(u), \cdots, y^n(u))$ with $\omega|_U = \sum_{i=1}^{n} dx^i \wedge dy^i$.*

(See [1, Thm 3.2.2 p175]).

Definition 3.2.1 *A symplectic structure on a manifold M of dimension n is a given 2-form $\omega \in \Omega^2(M)$ which is;*

(i) closed. That is, $d\omega = 0$,

(ii) non degenerate. That is, $\omega(X, Y) = 0$ for all Y implies that $X = 0$ for $X, Y \in \mathfrak{X}(M)$. That is to say, for each $x \in M$, $\omega_x(X_x, Y_x) = 0$ for all Y_x implies that $X_x = 0$, $X_x, Y_x \in T_xM$.

For $x \in M$, ω_x is a non degenerate bilinear form on the tangent space T_xM. Also ω_x is skew symmetric. From Linear algebra, the condition ω_x is skew symmetric implies that the dimension of T_xM is even. That is, $\dim T_xM = 2n(= m)$, and ω_x has maximal rank. Therefore, M is an even dimensional manifold. The form $\Omega_\omega = \frac{(-1)^{[\frac{n}{2}]}}{n!}\omega^n$ denote the standard volume form, where $\omega^n = \omega \wedge \omega \wedge \cdots \wedge \omega$ is the volume on M. The rank of ω is $2n$ which is the dimension of M. (See also [1, p 166]).

Definition 3.2.2 *If ω is a symplectic structure on a manifold M, then we call (M, ω) a symplectic manifold.*

Note that a symplectic manifold is always even dimensional.

Definition 3.2.3 *Let (M, ω) be a symplectic manifold and (U, φ) a chart on M such that for each $u \in U$, $\varphi(u) = (x^1(u), \cdots, x^n(u), y^1(u), \cdots, y^n(u))$, then the coordinates (x^i, y^i) are called symplectic coordinates about $u \in U$ and the chart (U, φ) is called a symplectic chart about $u \in U$.*

Definition 3.2.4 *Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds. A smooth map $f : M_1 \to M_2$ is called a symplectic map if $f^*\omega_2 = \omega_1$.*

Proposition 3.2.2 *(a) Let $f : M_1 \to M_2$ be a symplectic map and $\dim M_1 = \dim M_2$, then*

(i) f perserves a volume form

(ii) f is a local diffeomorphism

(b) Let (M, ω) be a symplectic manifold. If $\Phi : M \to M'$ is a diffeomorphism onto a manifold M', then $(M', (\Phi^{-1})^*\omega)$ is a symplectic manifold.

Proof. For (a) see ([1, Prop 3.2.2 p177]).

We prove (b) of the proposition. We need to show that $(\Phi^{-1})^*\omega$ is non degenerate and also closed. Closedness is straight forward since pullbacks and exterior differentiation commute. That is, $df^* = f^*d$. So we have
$d((\Phi^{-1})^*\omega) = (\Phi^{-1})^* d\omega = 0$. Thus, $(\Phi^{-1})^*\omega$ is closed. To show that it is non degenerate suppose $(\Phi^{-1})^*\omega(Y, Z) = 0$ for all $Z \in (M')$. Then
$\omega(\Phi_*^{-1}Y, \Phi_*^{-1}Z) = 0$ for all $Z \in \mathfrak{X}(M')$ and note that $\Phi_*^{-1}Y, \Phi_*^{-1}Z \in \mathfrak{X}(M)$ since Φ and Φ^{-1} are diffeomorphisms. But ω is non degenerate so that $\Phi_*^{-1}Y = 0$. Since Φ_*^{-1} is an isomorphism, we must have that $Y = 0$.

To see that $\Phi : (M, \omega) \to (M, (\Phi^{-1})^*\omega)$ is symplectic, we need to show that the pullback Φ^*, takes back $(\Phi^{-1})^*\omega$ to ω on M. But this is straight forward since

$$\Phi^*(\Phi^{-1})^*\omega = (\Phi^{-1} \circ \Phi)^*\omega = \omega.$$

$\square$

We shall show that under some conditions, the coadjoint orbit is a symplectic manifold.

Definition 3.2.5 *Given a finite Lie group G with its Lie algebra $\mathfrak{g}$, let $Ad^* : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ be the coadjoint action of G on $\mathfrak{g}^*$, the dual of its Lie algebra, we define the coadjoint orbit of $\beta \in \mathfrak{g}^*$ to be*

$$O_\beta = \{Ad_g^*\beta : g \in G\} \subset \mathfrak{g}^*.$$

The isotropy subgroup of β is given by

$$G_\beta = \{g \in G : Ad_g^*\beta = \beta\}.$$

We show that $O_\beta \cong G/G_\beta$. That is, the coadjoint orbit is a homogeneous space. Define a map $\varphi : O_\beta \to G/G_\beta$ as follows. If $\eta = Ad_g^*\beta$ for some $g \in G$, then $\varphi(\eta) = gG_\beta$. The map φ is well-defined (single-valued) because if $\varphi(\eta) = hG_\beta$ also, then $Ad_g^*\beta = Ad_h^*\beta$ so that $Ad_{h^{-1}}^*(Ad_g^*\beta) = \beta$. This implies that $Ad_{h^{-1}g}^*\beta = \beta$ so that $h^{-1}g \in G_\beta$ and $gG_\beta = hG_\beta$.

The map φ is injective because if $\eta = Ad_g^*\beta$, $\mu = Ad_h^*\beta$ and $gG_\beta = hG_\beta$, then $h^{-1}g \in G_\beta$ so that $Ad_{h^{-1}g}^*\beta = \beta$. This implies that $Ad_{h^{-1}}^* \circ Ad_g^*\beta = \beta$. It follows then that

$$\eta = Ad_g^*\beta = Ad_h^*\beta = \mu.$$

The map is surjective since if $gG_\beta \in G/G_\beta$, then since $g \in G$, $\eta = Ad_g^*\beta \in O_\beta$ gives $\varphi(\eta) = gG_\beta$ by construction.

Definition 3.2.6 *Let G be a Lie group and $\mathfrak{g}$ the Lie algebra of G. Let G_β be the isotropy subgroup of β. We shall denote the Lie algebra of G_β by $LieG_\beta$.*

Proposition 3.2.3 *Let $\eta \in O_\beta$ be related to β by the equation $\eta = Ad_h^*\beta$ for some $h \in G$, then the isotropy subgroups G_β and G_η are conjugates.*

Proof. Define a map $\psi : G/G_\beta \to G/G_\eta$ by $[g]_\beta \mapsto [hgh^{-1}]_\eta$. To see that ψ is a well-defined isomorphism, let $x \in G_\beta$ so that $Ad_x^*\beta = \beta$. Since $\eta = Ad_h^*\beta$, we have

$$
\begin{aligned}
Ad_h^*(Ad_x^*)Ad_{h^{-1}}^*\eta &= Ad_h^*(Ad_x^*)Ad_{h^{-1}}^*(Ad_h^*\eta) \\
&= Ad_h^*Ad_x^*(Ad_{h^{-1}}^*Ad_h^*\beta) \\
&= Ad_h^*Ad_x^*\beta \\
&= Ad_h^*\beta \\
&= \eta.
\end{aligned}
$$

Since $x \in G_\beta$ was arbitrary, it follows that $Ad_h^*G_\beta Ad_{h^{-1}}^*$ is a subgroup of G_η. Taking $\beta = Ad_{h^{-1}}^*\eta$ gives the reverse inclusion. Therefore, $G_\eta = Ad_h^*G_\beta Ad_{h^{-1}}^*$. This concludes the proof. It follows that $\psi : G/G_\beta \to G/G_\eta$, $[g]_\beta \mapsto [hgh^{-1}]_\eta$ is an isomorphism. □

We have shown that if G is a finite Lie group acting on the dual $\mathfrak{g}^*$ of its Lie algebra, $\mathfrak{g}$ and if η and β are in the same coadjoint orbit, then the map $\gamma_{Ad_h^*} : G_\beta \to G_\eta$, where $\gamma_{Ad_h^*}$ is conjugation by Ad_h^*, is an isomorphism.

The above discussion also implies that if $M_\beta = G/G_\beta \cong O_\beta$ is the coadjoint orbit through β, then for all $g \in G$ we have a diffeomorphism $G/G_\beta \cong G/G_{Ad_g^*\beta}$,induced by the map $g \mapsto hgh^{-1}$. Thus the definition of M_β does not depend on the choice of the element β in its orbit.

We want to define a symplectic structure on the coadjoint orbit.

Let $X \in \mathfrak{g}$. The infinitesimal generator of the action corresponding to X is given by

$$X_{\mathfrak{g}^*}(\beta) = \frac{d}{dt}(Ad_{\exp tX}^*\beta)|_{t=0}$$

Now let $Y \in \mathfrak{g}$, then we have that

$$\begin{aligned}
(X_{\mathfrak{g}^*}(\beta))Y &= \frac{d}{dt}(Ad^*_{\exp tX}\beta)Y|_{t=0} \\
&= \frac{d}{dt}(\beta Ad_{\exp(-tX)}(Y))|_{t=0} \\
&= \beta(\frac{d}{dt}Ad_{\exp(-tX)}(Y))|_{t=0} \\
&= \beta(-[X,Y]).
\end{aligned}$$

Define $\beta([X,Y]) := \langle \beta, [X,Y]\rangle$, where $\langle \cdot, \cdot \rangle$ is the natural pairing.

Then we have

$$\begin{aligned}
\langle \frac{d}{dt}Ad^*_{\exp tX}(\beta), Y\rangle|_{t=0} &= \frac{d}{dt}\langle Ad^*_{\exp tX}(\beta), Y\rangle|_{t=0} \\
&= \frac{d}{dt}\langle \beta, Ad_{\exp -tX}(Y)\rangle|_{t=0} \\
&= \langle \beta, \frac{d}{dt}Ad_{\exp -tX}(Y)\rangle|_{t=0} \\
&= \langle \beta, -[X,Y]\rangle \\
&= \langle \beta, -ad_X(Y)\rangle \\
&= \langle ad^*_X\beta, Y\rangle.
\end{aligned}$$

where, $d(Ad^*) = ad^*$ and $ad^*_X = (-ad_X)^*$.

Let $X \in \mathfrak{g}$, denote by $X^\sharp$ the vector field on $\mathfrak{g}^*$ generated by X. That is;

$$X^\sharp_\beta = X^\sharp(\beta) = \frac{d}{dt}(Ad^*_{\exp tX}\beta)|_{t=0}.$$

To compute the tangent space of O_β at β, let $x(t) = \exp tX$ be a curve in G which is tangent to X at $t = 0$. Then

$$\beta(t) = Ad^*_{x(t)}\beta = Ad^*_{\exp tX}\beta$$

is a curve in O_β such that $\beta(0) = \beta$. If $Y \in \mathfrak{g}$ then,

$$\langle \beta(t), Y\rangle = \langle Ad^*_{\exp tX}\beta, Y\rangle = \langle \beta, Ad_{\exp(-tX)}(Y)\rangle.$$

Differentiating with respect to t at $t = 0$ yields,

$$\langle \beta'(0), Y\rangle = \langle \beta, -ad_X(Y)\rangle = \langle ad^*_X\beta, Y\rangle.$$

This shows that

$$\beta'(0) = ad^*_X\beta. \tag{3.3}$$

Therefore, the tangent space of the orbit O_β at β is given by

$$T_\beta O_\beta = \{ad^*_X\beta : X \in \mathfrak{g}\}. \tag{3.4}$$

Proposition 3.2.4 *Let $\omega_\beta : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be defined by*

$$\omega_\beta(X, Y) = \beta([X, Y]) = \langle \beta, [X, Y]\rangle,$$

for all $X, Y \in \mathfrak{g}$ and $\beta \in \mathfrak{g}^*$. Then

(i) ω_β is a skew-symmetric bilinear form on $\mathfrak{g}$.

(ii) $ker\omega_\beta = LieG_\beta$ where $G_\beta = \{g \in G : Ad^*_g\beta = \beta\}$.

(iii) ω_β is G-invariant. That is, given any $h \in G$ we have

$$\omega_{Ad^*_h\beta}(Ad_hX, Ad_hY) = \omega_\beta(X, Y).$$

Proof.

(i) The fact that ω_β is skew-symmetric and bilinear follows directly since the Lie bracket is symmetric and bilinear.

(ii) We have

$$
\begin{aligned}
\ker\omega_\beta &= \{X \in \mathfrak{g} : \omega_\beta(X,Y) = 0, \text{ for all } Y \in \mathfrak{g}\} \\
&= \{X \in \mathfrak{g} : \langle \beta, [X,Y] \rangle = 0, \text{ for all } Y \in \mathfrak{g}\} \\
&= \{X \in \mathfrak{g} : \langle \beta, ad_X(Y) \rangle = 0, \text{ for all } Y \in \mathfrak{g}\} \\
&= \{X \in \mathfrak{g} : \langle -ad_X^*\beta, Y \rangle = 0, \text{ for all } Y \in \mathfrak{g}\} \\
&= \{X \in \mathfrak{g} : -ad_X^*\beta = 0\} \\
&= \mathrm{Lie}G_\beta.
\end{aligned}
$$

(iii) Let $h \in G$. Then

$$
\begin{aligned}
\omega_{Ad_h^*\beta}(Ad_h X, Ad_h Y) &= Ad_h^*\beta([Ad_h X, Ad_h Y] \\
&= Ad_h^*\beta([hXh^{-1}, hYh^{-1}]) \\
&= Ad_h^*\{hXh^{-1}hYh^{-1} - hYh^{-1}hXh^{-1}\} \\
&= Ad_h^*\beta\{hXYh^{-1} - hYXh^{-1}\} \\
&= Ad_h^*\beta(h[X,Y]h^{-1}) \\
&= \langle Ad_h^*\beta, h[X,Y]h^{-1} \rangle \\
&= \langle \beta, Ad_{h^{-1}}(h[X,Y])h^{-1} \rangle \\
&= \langle \beta, h^{-1}h[X,Y]h^{-1}h \rangle \\
&= \langle \beta, [X,Y] \rangle \\
&= \omega_\beta(X,Y).
\end{aligned}
$$

The proof of (iii) shows that ω_β is G-inveriant and hence it is smooth. $\square$

For $\beta \in \mathfrak{g}^*$ define a map

$$
\Omega_\beta : T_\beta O_\beta \times T_\beta O_\beta \to \mathbb{R} \text{ by } \Omega_\beta(X^\sharp, Y^\sharp) = \omega_\beta(X,Y) \text{ for all } X, Y \in \mathfrak{g}. \tag{3.5}
$$

Proposition 3.2.5 *Let G be a Lie group and $\mathfrak{g}^*$ the dual of its Lie algebra. Then, Ω_β defined above is a well-defined G-invariant differential 2-form on the coadjoint orbit O_β, through β, of the action of G on $\mathfrak{g}^*$.*

Proof. To see that Ω_β is well defined we must show that the definition of Ω_β does not depend on the choice of $X, Y \in \mathfrak{g}$. To this effect first note that if $Z \in \mathrm{Lie}G_\beta$ then

$$\begin{aligned}\Omega_\beta(Z^\sharp, Y^\sharp) &= \omega_\beta(Z, Y) \\ &= \beta([Z, Y]) \\ &= \langle \beta, [Z, Y] \rangle \\ &= \langle \beta, ad_Z(Y) \rangle \\ &= \langle -ad^*_Z \beta, Y \rangle \\ &= 0.\end{aligned}$$

for all $Y \in \mathfrak{g}$. But now it follows that if $Z \in \mathrm{Lie}G_\beta$ and $X, Y \in \mathfrak{g}$ then

$$\begin{aligned}\Omega_\beta(X^\sharp + Z^\sharp, Y^\sharp) &= \omega_\beta(X + Z, Y) = \beta([X + Z, Y]) \\ &= \langle \beta, [X + Z, Y] \rangle \\ &= \langle \beta, [X, Y] \rangle + \langle \beta, [Z, Y] \rangle \\ &= \langle \beta, [X, Y] \rangle = \beta([X, Y]) \\ &= \omega_\beta(X, Y) \\ &= \Omega_\beta(X^\sharp, Y^\sharp).\end{aligned}$$

Hence Ω_β is well-defined.

We have already seen that, locally, ω_β is skew-symmetric, bilinear, G-invariant form on the tangent space T_eG. Therefore, it follows that Ω_β is skew symmetric on the tangent space $T_\beta O_\beta$. To show that Ω_β defines a differential 2-form on the coadjoint orbit through β, we must show that it is non-degenerate and closed.

To prove non-degeneracy we must show that if $X \notin \mathrm{Lie}G_\beta$, that is, if $-ad_X\beta \neq 0$ then there exists a $Y \in \mathfrak{g}$ such that $\Omega_\beta(X^\sharp, Y^\sharp) \neq 0$. Now pick any $Y \in \mathfrak{g}$ and any $X \notin \mathrm{Lie}G_\beta$, then

$$\begin{aligned}\Omega_\beta(X^\sharp, Y^\sharp) &= \omega_\beta(X, Y) \\ &= \beta([X, Y]) \\ &= \langle \beta, [X, Y] \rangle \\ &= \langle \beta, ad_X(Y) \rangle \\ &= \langle -ad^*_X \beta, Y \rangle.\end{aligned}$$

But then $ad^*_X\beta \neq 0$ if and only if $X \notin \mathrm{Lie}G_\beta$. Therefore, $\langle -ad^*_X\beta, Y \rangle \neq 0$ as required. Hence Ω_β is non-degenerate.

It remains to show that Ω_β is closed. To prove closure we shall use the formula

$$
\begin{aligned}
d\omega(X,Y,Z) &= (L_X\omega)(Y,Z) - (L_Y\omega)(X,Z) \\
&+ (L_Z\omega)(X,Y) + \omega(X,[Y,Z]) \\
&- \omega(Y,[X,Z]) + \omega(Z,[X,Y])
\end{aligned}
$$

whose proof can be found in [11, p 53]. Therefore,

$$
\begin{aligned}
d\Omega_\beta(X^\sharp,Y^\sharp,Z^\sharp) &= d\omega_\beta(X,Y,Z) \\
&= [(L_X\omega_\beta)(Y,Z) - (L_Y\omega_\beta)(X,Z) + (L_Z\omega_\beta)(X,Y)] \\
&+ [\omega_\beta(X,[Y,Z]) - \omega_\beta(Y,[X,Z]) + \omega_\beta(Z,[X,Y])]\,.
\end{aligned}
$$

Dealing with the second square bracket first, we have

$$
\begin{aligned}
\omega_\beta(X,[Y,Z]) - \omega_\beta(Y,[X,Z]) + \omega_\beta(Z,[X,Y]) &= \langle\beta,[X,[Y,Z]]\rangle - \langle\beta,[Y,[X,Z]]\rangle \\
&+ \langle\beta,[Z,[X,Y]]\rangle \\
&= \langle\beta,[X,[Y,Z]] - [Y,[X,Z]] \\
&+ [Z,[X,Y]]\rangle \\
&= 0
\end{aligned}
$$

by Jacobi identity.

To deal with the first square bracket, note that $(L_X\omega)(Y,Z) = \omega(Z,[X,Y]) - \omega(Y,[X,Z])$. Therefore

$$
\begin{aligned}
(L_X\omega_\beta)(Y,Z) - (L_Y\omega_\beta)(X,Z) + (L_Z\omega_\beta)(X,Y) &= (\omega_\beta(Z,[X,Y]) - \omega_\beta(Y,[X,Z])) \\
&- (\omega_\beta(Z,[Y,X]) - \omega_\beta(X,[Y,Z])) \\
&+ (\omega_\beta(Y,[Z,X]) - \omega_\beta(X,[Z,Y])) \\
&= \langle\beta,[Z,[X,Y]]\rangle - \langle\beta,[Y,[X,Z]]\rangle \\
&- \langle\beta,[Z,[Y,X]]\rangle + \langle\beta,[X,[Z,Y]]\rangle \\
&+ \langle\beta,[Y,[Z,X]]\rangle - \langle\beta,[X,[Z,Y]]\rangle \\
&= 2\langle\beta,[Z,[X,Y]]\rangle - 2\langle\beta,[Y,[X,Z]]\rangle \\
&+ 2\langle\beta,[X,[Y,Z]]\rangle \\
&= 2\langle\beta,[X,[Y,Z]] - [Y,[X,Z]] \\
&+ [Z,[X,Y]]\rangle \\
&= 0
\end{aligned}
$$

again by Jacobi identity.

Thus $d\Omega_\beta = 0$. Hence Ω_β is closed.

We therefore conclude that Ω_β defines a symplectic structure on the coadjoint orbit through β for action of the Lie group G on the dual of its Lie algebra. $\square$

We have proved that the coadjoint orbit is a symplectic homogeneous space.

3.2.1 The momentum map

Definition 3.2.7 *Let G be a Lie group and (M, ω) a symplectic manifold. Let*

$$\begin{aligned} \Phi \ &: \ G \times M \to M \\ &\quad (g, m) \mapsto \Phi_g(m) = g \cdot m \end{aligned}$$

be an action of G on M. The action Φ is called symplectic if the diffeomorphisms

$$\begin{aligned} \Phi_g \ &: \ M \to M \\ &\quad m \mapsto \Phi_g(m) \end{aligned}$$

are symplectic. That is, if $\Phi_g^* \omega = \omega$ for each $g \in G$.

Let $\mathfrak{g}$ be the Lie algebra of the Lie group G. For $X \in \mathfrak{g}$ let

$$X_M(m) = \frac{d}{dt} \Phi_{\exp tX} m|_{t=0} \tag{3.6}$$

be the infinitesimal generator of the action. If F_t is the corresponding flow of X_M, then

$$L_{X_M} \omega = \frac{d}{dt} (F_t^* \omega)|_{t=0}. \tag{3.7}$$

(See [11, p72]).

Definition 3.2.8 *Let (M, ω) be a symplectic manifold. A vector field X on M is said to be symplectic if it preserves the two form ω. That is, X is symplectic if $L_X \omega = 0$.*

If the vector field X on a symplectic manifold M is symplectic, then the flow F_t corresponding to X also preserves ω. That is, $F_t^*\omega = \omega$ for all t. (See [14, p106]).

Suppose now that the vector field X_M in equation (3.6) is symplectic, then $L_{X_M}\omega = 0$. But by Cartan's formula

$$L_{X_M}\omega = di_{X_M}\omega + i_{X_M}d\omega = di_{X_M}\omega,$$

we have $0 = L_{X_M}\omega = di_{X_M}\omega$. This implies that the 1-form $i_{X_M}\omega$ is closed. Poincare Lemma (see [17, p261]), now states that $i_{X_M}\omega$ is locally exact. That is, a function

$$\hat{\mu}(X) : M \to \mathbb{R}$$

can be defined on M such that locally,

$$i_{X_M}\omega = d\hat{\mu}(X). \tag{3.8}$$

Definition 3.2.9 *Let (M, ω) be a symplectic manifold. A vector field $X \in \mathfrak{X}(M)$ is called locally Hamiltonian if for each point $m \in M$ there is a neighbourhood U and a function $F \in C^\infty(U)$ such that on U,*

$$i_X\omega = dF$$

In particular, the vector field X_M defined by equation 3.8 is locally Hamiltonian.

Suppose now that for every $X \in \mathfrak{g}$ the vector field X_M defined by equation (3.8) is globally Hamiltonian on M, then the functions μ_X's are globally defined on M. Let $\mathfrak{g}^*$ be the dual of the Lie algebra of G, $\mathfrak{g}$, it follows that we can then define a map

$$\begin{aligned} \mu \quad : \quad & M \to \mathfrak{g}^* \\ & m \mapsto \mu(m), \end{aligned}$$

such that for every $X \in \mathfrak{g}$ we have

$$\langle \mu(m), X \rangle = \hat{\mu}(X)(m) \text{ or } \mu(m) \cdot X = \hat{\mu}(X)(m) \tag{3.9}$$

The map μ defined by equation (3.9) is called the momentum map (or the moment map).

Definition 3.2.10 *Let* $\Phi : G \times M \to M$, $(g, m) \mapsto \Phi_g(m)$ *be an action of a Lie group* G *on a symplectic manifold* (M, ω) *such that* $\Phi_g^* \omega = \omega$ *for all* $g \in G$. *Let* $\mathfrak{g}^*$ *be the dual of the Lie algebra* $\mathfrak{g}$ *of* G. *Then the map*

$$\mu : M \to \mathfrak{g}^*$$

is called the momentum map (or the moment map) for the action if for each $X \in \mathfrak{g}$ *there is a function*

$$\mu_X : M \to \mathbb{R} \text{ with } d\mu_X = i_{X_M}\omega$$

such that equation (3.9) holds, where X_M *is the infinitesimal generator of the action corresponding to* $X \in \mathfrak{g}$.

The equation $d\mu_X = i_{X_M}\omega$ implies that

$$X_{\mu_X} = X_M \text{ for all } X \in \mathfrak{g}. \tag{3.10}$$

The space (M, ω, Φ, μ) is called a Hamiltonian G-space.

Definition 3.2.11 *Let* G *act on a symplectic manifold* (M, ω) *by a symplectic action* Φ. *If the action admits a momentum map, then the action is called a Hamiltonian action.*

Note that not every locally Hamiltonian vector field is globally Hamiltonian (see [11, p78]). Therefore, it follows that not every symplectic action has a momentum map. However, if the vector field X_M is globally Hamiltonian, then there is a momentum map.

Definition 3.2.12 *Let* $\Phi : G \times M \to M$ *be the symplectic action of a Lie group* G *on a symplectic manifold* (M, ω) *which admits a momentum map. Let* $\mathfrak{g}$ *be the Lie algebra og* G *and let* $\mathfrak{g}^*$ *be its dual. Then the momentum map* $\mu : M \to \mathfrak{g}^*$ *is called equivariant if it is equivariant with respect to the coadjoint action* $Ad^* : G \times \mathfrak{g}^* \to \mathfrak{g}^*$. *That is, if for every* $g \in G$, *the following equation holds*

$$\mu \circ \Phi_g = Ad_g^* \circ \mu. \tag{3.11}$$

Since $Ad^* : G \times \mathfrak{g}^* \to \mathfrak{g}^*$, then for each $g \in G$, we have $Ad^*_g : \mathfrak{g}^* \to \mathfrak{g}^*$ is an automorphism on the dual of its Lie algebra. So, Ad^* maps G into automorphims of $\mathfrak{g}^*$. We can express this as a map $Ad^* : G \to Aut\ \mathfrak{g}^*$.

Now if $\rho : G' \to G$ is a homomorphism of Lie groups, then ρ defines a representation $\rho : G' \to Aut\ \mathfrak{g}^*$ of G' into the automorphisms of the dual of the Lie algebra of G by the composition

$$G' \to G \to Aut\ \mathfrak{g}^*,$$

since the map $G' \times \mathfrak{g}^* \to G \times \mathfrak{g}^* \to \mathfrak{g}^*$ defined by

$$(g', \alpha) \mapsto (\rho(g'), \alpha) \mapsto Ad^*_{\rho(g')}(\alpha),$$

is smooth.

Taking $\rho = Id_G$, the identity map on G, then the momentum map is equivariant with respect to the coadjoint action if the following diagram commute:

$$\begin{tikzcd}
G \times M \arrow[r, "(Id_G \times \ \mu)"] \arrow[d, "\Phi_g"'] & G \times \mathfrak{g}^* \arrow[d, "Ad^*"] \\
M \arrow[r, "\mu"] & \mathfrak{g}^*
\end{tikzcd}$$

Equivariant momentum maps play an important role in many constructions in symplectic geometry. One such area is the constructions in symplectic reduction theory. However, there are cases when the momentum mapping is not equivariant with respect to the coadjoint action of the Lie group G. In such cases we can define an action of G on $\mathfrak{g}^*$ such that the momentum mapping is equivariant with respect to this action.

Much of the material which now follows in the remainder of this chapter is contained in our first paper. (See [10]).

3.2.2 Momentum with cocycle

Let (M, ω, Φ, μ) be a Hamiltonian G-space. For $g \in G$ and $\xi \in \mathfrak{g}$, define a function

$$\Psi_{g,\xi} : M \to \mathbb{R}$$

by $\Psi_{g,\xi}(x) = \hat{\mu}(\xi)(\Phi_g(x)) - \hat{\mu}(Ad_{g^{-1}}\xi)(x),$ for all $x \in M.$

We shall show that Ψ is constant on M. Differentiating at $x \in M$ gives

$$\begin{aligned} d\Psi_{g,\xi}(x) &= d(\hat{\mu}(\xi)(\Phi_g(x))) - d(\hat{\mu}(Ad_{g^{-1}}\xi)(x)) \\ &= d(\hat{\mu}(\xi)(\Phi_g(x)))T_x\Phi_g(x) - d\hat{\mu}(Ad_{g^{-1}}\xi)(x) \\ &= i_{\xi_M}\omega(\Phi_g(x)) \cdot T_x\Phi_g(x) - i_{(Ad_{g^{-1}}\xi)_M}\omega(x) \end{aligned}$$

by definition of momentum mapping.

This gives $d\Psi_{g,\xi}(x) = \Phi_g^*(i_{\xi_M}\omega)(x) - i_{(Ad_{g^{-1}}\xi)_M}\omega(x)$.

Now, using the identities:

(a) $(Ad_{g^{-1}}\xi)_M = \Phi_g^*\xi_M$ and

(b) $\Phi_g^* i_{\xi_M}\omega = i_{\Phi_g^*\xi_M}\Phi_g^*\omega$,

we get $d\Psi_{g,\xi}(x) = 0$.

If M is connected then Ψ is constant on M, otherwise it is constant on connected components.

Now define a function

$$\sigma : G \to \mathfrak{g}^*,\ g \mapsto \mu(\Phi_g(m)) - Ad_g^*\mu(m),$$

for all $m \in M$ so that

$$\sigma(g) \cdot \xi = \Psi_{g,\xi}(m),$$

for all $g \in G$, $\xi \in \mathfrak{g}$ and all $m \in M$.

The map σ is called a coadjoint cocycle on G. It satisfies the cocycle identity

$$\sigma(gh) = \sigma(g) + Ad_g^*\sigma(h), \tag{3.12}$$

for all $g, h \in G$.

Proposition 3.2.6 *Let Φ be a symplectic action of a Lie group G on a symplectic manifold (M, ω) which admits a momentum mapping μ. Let σ be a one-cocycle. Define a map*

$$\Psi : G \times \mathfrak{g}^* \to \mathfrak{g}^*,$$

by

$$\Psi(g, \alpha) = Ad_g^* \alpha + \sigma(g).$$

Then the map Ψ is an action and the momentum map is equivariant with respect to this action.

For the proof see ([10, Prop 3.4]).

In order to discuss commutation relations associated with a given momentum map, we first state the following proposition.

Proposition 3.2.7 *Let $\Phi : G \times M \to M$ be a smooth action of a Lie group G on a smooth manifold M. For $\xi \in \mathfrak{g}$ let*

$$\xi_M(m) = \tfrac{d}{dt}\Phi(\exp t\xi, m) \mid_{t=0},$$

be the infinitesimal generator of the action. Then for $\xi, \eta \in \mathfrak{g}$, we have:

$$[\xi_M, \eta_M] = -[\xi, \eta]_M.$$

For the proof see ([1, p 269, Prop 4.1.26]).

Theorem 3.2.2 *Let $\Phi : G \times M \to M$ be a symplectic action of G on M which admits a momentum mapping*

$$\mu : M \to \mathfrak{g}^*,$$

and let

$$\sigma : G \to \mathfrak{g}^*,$$

be the cocycle of the momentum map μ. Let the function

$$\hat{\sigma_\eta} : G \to \mathbb{R},$$

be defined by

$$\hat{\sigma_\eta}(g) = \sigma(g) \cdot \eta.$$

Define also a function

$$\Sigma : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R},$$

by

$$\Sigma(\xi, \eta) = d\hat{\sigma_\eta}(e) \cdot \xi,$$

for all $\xi, \eta \in \mathfrak{g}$. Then,

(i) Σ is skew symmetric bilinear form on $\mathfrak{g}$ and satisfies the Jacobi's identity
$0 = \Sigma(\xi, [\eta, \zeta]) + \Sigma(\eta, [\zeta, \xi]) + \Sigma(\zeta, [\xi, \eta])$.

(ii) $\{\hat{\mu}(\xi), \hat{\mu}(\eta)\} = \hat{\mu}([\xi, \eta]) - \Sigma(\xi, \eta)$, and since $\Sigma(\xi, \eta)$ is a constant, we have
$X_{\{\hat{\mu}(\xi), \hat{\mu}(\eta)\}} = X_{\hat{\mu}([\xi, \eta])}$.

Proof. We first obtain an expression for $\Sigma(\xi, \eta)$. From the expression

$$\begin{aligned} \hat{\sigma_\eta}(g) &= \mu(\Phi_g(x)) \cdot \eta - Ad^*_g \mu(x) \cdot \eta \\ &= \hat{\mu}_\eta(\Phi_g(x)) - \hat{\mu}_{Ad_{g^{-1}}\eta}(x), \end{aligned}$$

differentiating in g at $g = e$ in the direction of $\xi \in \mathfrak{g}$ we get;

$$\begin{aligned}
d\hat{\sigma}_\eta(e)\cdot\xi &= d(\hat{\mu}_\eta(\Phi_g(x))\cdot\xi - \hat{\mu}_{Ad_{g^{-1}}\eta}(x)\cdot\xi) \\
&= \tfrac{d}{dt}\hat{\mu}_\eta(\Phi_{\exp t\xi}(x)) \mid_{t=0} - \tfrac{d}{dt}\hat{\mu}_{Ad_{\exp(-t\xi)}\eta}(x) \mid_{t=0} \\
&= (i_{\eta_M}\omega)\tfrac{d}{dt}\Phi_{\exp t\xi}(x) \mid_{t=0} - \tfrac{d}{dt}\langle Ad_{\exp(-t\xi)}\eta, \mu(x)\rangle \mid_{t=0} \\
&= (i_{\eta_M}\omega)(\xi_M(x)) - \langle \tfrac{d}{dt}Ad_{\exp(-t\xi)}\eta \mid_{t=0}, \mu(x)\rangle \\
&= (i_{\xi_M} i_{\eta_M}\omega)(x) - \langle[\eta,\xi],\mu(x)\rangle \\
&= -\{\hat{\mu}_\xi,\hat{\mu}_\eta\}(x) - \hat{\mu}_{[\eta,\xi]}(x) \\
&= -\{\hat{\mu}_\xi,\hat{\mu}_\eta\}(x) + \hat{\mu}_{[\xi,\eta]}(x).
\end{aligned}$$

Thus,

$$\Sigma(\xi,\eta) = -\{\hat{\mu}_\xi,\hat{\mu}_\eta\} + \hat{\mu}_{[\xi,\eta]}. \tag{3.13}$$

But both the Poisson bracket $\{\hat{\mu}_\xi,\hat{\mu}_\eta\}$ and the Lie bracket $[\xi,\eta]$ are skew symmetric bilinear. This implies that the right side of equation (3.13) is skew symmetric and bilinear. Therefore, $\Sigma(\xi,\eta)$ is skew symmetric and bilinear form on $\mathfrak{g}$. The right side also satisfies Jacobi's identity which implies that $\Sigma(\xi,\eta)$ also satisfies the Jacobi's identity. This proves the first part.

To prove the second part first we show that

$$-\{\hat{\mu}_\xi,\hat{\mu}_\eta\} + \hat{\mu}_{[\xi,\eta]}$$

is a constant. We shall show that

$$d\{\hat{\mu}_\xi,\hat{\mu}_\eta\} = d\hat{\mu}_{[\xi,\eta]}. \tag{3.14}$$

Evaluating the right hand side of equation (3.14), we have

$$\begin{aligned}
d\hat{\mu}_{[\xi,\eta]} &= i_{[\xi,\eta]_M}\omega \\
&= -i_{[\xi_M,\eta_M]}\omega \text{ by the proposition 3.2.7} \\
&= -(L_{\xi_M} i_{\eta_M}\omega - i_{\eta_M} L_{\xi_M}\omega), \text{ an identity,} \\
&= -L_{\xi_M} i_{\eta_M}\omega \text{ (since } \xi_M = X_{\hat{\mu}_\xi} \text{ so that } L_{\xi_M}\omega = 0) \\
&= -L_{\xi_M} d\hat{\mu}_\eta \\
&= -dL_{\xi_M}\hat{\mu}_\eta \\
&= -d(L_{X_{\hat{\mu}_\xi}}\hat{\mu}_\eta) \\
&= -d(-\{\hat{\mu}_\xi, \hat{\mu}_\eta\}) \\
&= d(\{\hat{\mu}_\xi, \hat{\mu}_\eta\}).
\end{aligned}$$

Thus, $d\{\hat{\mu}_\xi, \hat{\mu}_\eta\} = d\hat{\mu}_{[\xi,\eta]}$. This shows that

$$\Sigma(\xi, \eta) = -\{\hat{\mu}_\xi, \hat{\mu}_\eta\} + \hat{\mu}_{[\xi,\eta]}$$

is a constant and from equation (3.13) we have

$$\{\hat{\mu}_\xi, \hat{\mu}_\eta\} = \hat{\mu}_{[\xi,\eta]} - \Sigma(\xi, \eta).\Box$$

The main result of this section is the following:

Theorem 3.2.3 *Let $\Psi : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ defined by $\Psi(g, \alpha) = Ad_g^*\alpha + \sigma(g)$ be the affine action of a Lie group G on its dual $\mathfrak{g}^*$ to its Lie algebra $\mathfrak{g}$. Let $\beta \in \mathfrak{g}^*$. Then, the orbit*

$$G \cdot \beta = \{\Psi(g, \beta) : g \in G\}$$

is a symplectic manifold with the symplectic 2-form given by

$$\omega_\beta(\xi_{\mathfrak{g}^*}(v), \eta_{\mathfrak{g}^*}(v)) = -\beta[\xi, \eta] + \sum(\xi, \eta),$$

where $\xi, \eta \in \mathfrak{g}$, and $\xi_{\mathfrak{g}^*}$ and $\eta_{\mathfrak{g}^*}$ are vector fields on $\mathfrak{g}^*$.

Proof. We shall first show that the orbit $O_\beta = \{\Psi(g, \beta) : g \in G\}$ is a manifold. Thereafter we shall define a symplectic structure on it.

Define the orbit of $\beta \in \mathfrak{g}^*$ by $O_\beta = \{\Psi(g, \beta) : g \in G\} \subset \mathfrak{g}^*$. The isotropy group of β is given by

$$G_\beta = \{g \in G : \Psi(g, \beta) = \beta\}.$$

This is a closed subgroup of G since if g_n is a sequence in G_β which converges to $g \in G$ then we have:

$$\begin{aligned}\beta &= \lim_{n\to\infty} \Psi(g_n, \beta)\\ &= \Psi(\lim_{n\to\infty} g_n, \beta)\\ &= \Psi(g, \beta).\end{aligned}$$

The second equality is because Ψ is an action and so it is smooth. This shows that $g \in G_\beta$.

We now show that $O_\beta \cong G/G_\beta$. Define a map

$$\varphi : O_\beta \to G/G_\beta$$

by

$$\varphi(\eta) = gG_\beta$$

for $\eta \in O_\beta$, where $\eta = \Psi(g, \beta)$ for some $g \in G$.

The map φ is well-defined since if $\varphi(\eta) = hG_\beta$ also, then we have

$$\Psi(g, \beta) = \Psi(h, \beta)$$

so that

$$\Psi(h^{-1}, \Psi(g, \beta)) = \beta \Rightarrow \Psi(h^{-1}g, \beta) = \beta,$$

which implies that

$$h^{-1}g \in G_\beta,$$

and consequently

$$gG_\beta = hG_\beta.$$

The map φ is injective. To see this let $\eta = \Psi(g, \beta)$, $\zeta = \Psi(h, \beta)$ and $gG_\beta = hG_\beta$ for $h, g \in G$ and $\eta, \zeta \in O_\beta$. Then $h^{-1}g \in G_\beta$ so that $\Psi(h^{-1}g, \beta) = \beta$. This implies that $\Psi(h^{-1}, \Psi(g, \beta)) = \beta$. It follows that $\Psi(g, \beta) = \Psi(h, \beta)$ so that $\eta = \zeta$.

The map is surjective since if $gG_\beta \in G/G_\beta$, then $\eta = \Psi(g, \beta) \in O_\beta$ gives $\varphi(\eta) = gG_\beta$ by construction.

Hence φ is an isomorphism. $\square$

Lemma 3.2.1 *Suppose that $\eta \in O_\beta$ so that $\eta = \Psi(h, \beta)$ for some $h \in G$, then the isotropy groups G_β and G_η are conjugates.*

Proof. We shall change the notation a bit and write $\Psi_g(\beta)$ for $\Psi(g, \beta)$. We have already seen that Ψ is an action and so, it is a homomorphism

$$\Psi(gh, \beta) = \Psi(g, \Psi(h, \beta)).$$

Define a map

$$\gamma : G/G_\beta \to G/G_\eta,$$

by

$$[g]_\beta \mapsto [hgh^{-1}]_\eta.$$

Then γ is a well-defined isomorphism. To see this, let $x \in G_\beta$ so that

$$\Psi_x(\beta) = \beta.$$

Since $\eta = \Psi_h(\beta)$, we have

$$\begin{aligned}
\Psi_h \circ \Psi_x \circ \Psi_{h^{-1}}(\eta) &= \Psi_h \circ \Psi_x \circ \Psi_{h^{-1}}(\Psi(h, \beta)) \\
&= \Psi_h \circ \Psi_x(\Psi(hh^{-1}, \beta)) \\
&= \Psi_h \circ \Psi_x(\beta) \\
&= \Psi(h, \Psi(x, \beta)) \\
&= \Psi(h, \beta) \\
&= \eta.
\end{aligned}$$

Since $x \in G_\beta$ was arbitrary, it follows that $\Psi_h G_\beta \Psi_{h^{-1}}$ is a subgroup of G_η. Taking $\beta = \Psi(h^{-1}, \eta)$ gives the reverse inclusion. Thus $G_\eta = \Psi_h G_\beta \Psi_{h^{-1}}$.

Hence γ is an isomorphism. $\square$

We now write the orbit of Ψ through β as $G \cdot \beta = G/G_\beta \cong O_\beta$. From the discussion above, it is clear that the orbit $G \cdot \beta$ does not depend on the choice of the element β in its orbit. We already have that G_β is a closed subgroup of G. Thus

$$G \cdot \beta = G/G_\beta$$

is a manifold.

We shall now define a symplectic structure on the orbit of the action Ψ through β.

Let $\xi \in \mathfrak{g}$. We define the vector field on $\mathfrak{g}^*$, called the infinitesimal generator of the action to be:

$$\begin{aligned}
\xi_{\mathfrak{g}^*}(\beta) &= \tfrac{d}{dt}\Psi(\exp t\xi, \beta) \mid_{t=0} \\
&= \tfrac{d}{dt}[Ad^*_{\exp t\xi}\beta + \sigma(\exp t\xi)] \mid_{t=0} \\
&= \tfrac{d}{dt}Ad^*_{\exp t\xi}\beta \mid_{t=0} + \tfrac{d}{dt}\sigma(\exp t\xi) \mid_{t=0} \\
&= \tfrac{d}{dt}Ad^*_{\exp t\xi}\beta \mid_{t=0} + d\sigma(e) \cdot \xi \\
&= \tfrac{d}{dt}Ad^*_{\exp t\xi}\beta \mid_{t=0} + d\hat{\sigma}_\xi(e).
\end{aligned}$$

If now $\eta \in \mathfrak{g}$, then we have:

$$\begin{aligned}
(\xi_{\mathfrak{g}^*}(\beta))\eta &= \tfrac{d}{dt}(Ad^*_{\exp t\xi}\beta)\eta \mid_{t=0} + d\hat{\sigma}_\xi(e) \cdot \eta \\
&= \beta(\tfrac{d}{dt}Ad_{\exp -t\xi}(\eta)) \mid_{t=0} + \textstyle\sum(\eta, \xi) \\
&= \beta(-[\xi, \eta]) + \textstyle\sum(\eta, \xi).
\end{aligned}$$

To compute the tangent space to the orbit $G \cdot \beta$ at β, for $\xi \in \mathfrak{g}$ let $x(t) = \exp t\xi$ be a curve in G which is tangent to ξ at $t = 0$, then $\beta(t) = \Psi(x(t), \beta)$ is the curve in $G \cdot \beta$ such that $\beta(0) = \beta$ since $\sigma(e) = 0$.

If $\eta \in \mathfrak{g}$, then

$$\begin{aligned}
\langle \beta(t), \eta \rangle &= \langle \Psi(x(t), \beta), \eta \rangle \\
&= \langle Ad^*_{x(t)}\beta + \sigma(x(t)), \eta \rangle \\
&= \langle Ad^*_{\exp t\xi}\beta, \eta \rangle + \langle \sigma(\exp t\xi), \eta \rangle,
\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the natural pairing of $\mathfrak{g}$ and its dual $\mathfrak{g}^*$.

Differentiating with respect to t at $t = 0$ gives

$$\langle \beta'(0), \eta \rangle = \langle ad^*_\xi \beta, \eta \rangle + \sum(\eta, \xi).$$

This implies that

$$\beta'(0) = ad^*_\xi \beta + \sum(\cdot, \xi).$$

Therefore, the tangent space to $G \cdot \beta$ at β is given by;

$$T_\beta G \cdot \beta = \{ad^*_\xi \beta + \sum(\cdot, \xi) : \xi \in \mathfrak{g}\}.$$

Consider now the function $\omega_\beta : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ defined by:

$$\omega_\beta(\xi, \eta) = \beta(-[\xi, \eta]) + \sum(\eta, \xi).$$

Clearly ω_β is skew symmetric and bilinear on $\mathfrak{g}$ since both the Lie bracket $[\cdot, \cdot]$ and the form $\sum(\cdot, \cdot)$ are skew symmetric bilinear.

$$\begin{aligned} \ker \omega_\beta &= \{\xi \in \mathfrak{g} : \omega_\beta(\xi, \eta) = 0, \forall \eta \in \mathfrak{g}\} \\ &= \{\xi \in \mathfrak{g} : \beta(-[\xi, \eta]) + \sum(\eta, \xi) = 0, \forall \eta \in \mathfrak{g}\} \\ &= LieG_\beta. \end{aligned}$$

Now, for $\xi \in \mathfrak{g}$ let $\tilde{\xi}$ denote the vector field on $\mathfrak{g}^*$ generated by ξ. That is,

$$\tilde{\xi}_\beta = \tilde{\xi}(\beta) = \tfrac{d}{dt} \Psi(\exp t\xi, \beta) \mid_{t=0} .$$

Then for $\beta \in \mathfrak{g}^*$, define the function $\Omega_\beta : T_\beta G \cdot \beta \times T_\beta G \cdot \beta \to \mathbb{R}$ by

$$\Omega_\beta(\tilde{\xi}, \tilde{\eta}) = \omega_\beta(\xi, \eta), \tag{3.15}$$

for all $\xi, \eta \in \mathfrak{g}$.

Now to complete the proof of theorem 3.2.3, we have the following proposition.

Proposition 3.2.8 *Ω_β defined by equation (3.15) above is a well-defined 2-form on $G \cdot \beta$, the orbit of the affine action $\Psi : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ through β.*

Proof. First note that if $\zeta \in LieG_\beta$, then $\Omega_\beta(\tilde{\zeta}, \tilde{\xi}) = \omega_\beta(\zeta, \xi) = 0$ for all $\xi \in \mathfrak{g}$. Now let $\xi, \eta \in \mathfrak{g}$. If $\zeta \in LieG_\beta$ then

$$\begin{aligned} \Omega_\beta(\tilde{\xi} + \tilde{\zeta}, \tilde{\eta}) &= \omega_\beta(\xi + \zeta, \eta) \\ &= \omega_\beta(\xi, \eta) + \omega_\beta(\zeta, \eta) \text{ since } \omega_\beta \text{ is bilinear} \\ &= \omega_\beta(\xi, \eta) \text{ since } \omega_\beta(\zeta, \eta) = 0 \\ &= \Omega_\beta(\tilde{\xi}, \tilde{\eta}). \end{aligned}$$

Thus Ω_β does not depend on the choice of $\xi, \eta \in \mathfrak{g}$. Hence Ω_β is well-defined. Since locally ω_β is skew symmetric, bilinear on the tangent space T_eG, it follows that Ω_β is skew symmetric bilinear on the tangent space $T_\beta G \cdot \beta$. It remains to show that Ω_β is non-degenerate and closed on $G \cdot \beta$.

To prove non-degeneracy let $\xi \in \mathfrak{g}$ be such that $\xi \notin LieG_\beta$, we must show that there exists $\eta \in \mathfrak{g}$ such that $\Omega_\beta(\tilde{\xi}, \tilde{\eta}) \neq 0$.

But now if $\eta \in \mathfrak{g}$ and $\xi \notin LieG_\beta$ then $\Omega_\beta(\tilde{\xi}, \tilde{\eta}) = \omega_\beta(\xi, \eta) \neq 0$ if and only if $\xi \notin \ker \omega_\beta = LieG_\beta$. This shows that if $\xi \notin LieG_\beta$, there exists $\eta \in \mathfrak{g}$ such that $\Omega_\beta(\tilde{\xi}, \tilde{\eta}) \neq 0$. Hence Ω_β is non-degenerate.

To show that Ω_β is closed, let $\xi, \eta, \zeta \in \mathfrak{g}$, then

$$\begin{aligned} d\Omega_\beta(\tilde{\xi}, \tilde{\eta}, \tilde{\zeta}) &= d\omega_\beta(\xi, \eta, \zeta) \\ &= (L_\xi \omega_\beta)(\eta, \zeta) - (L_\eta \omega_\beta)(\xi, \zeta) \\ &+ (L_\zeta \omega_\beta)(\xi, \eta) + \omega_\beta(\xi, [\eta, \zeta]) \\ &- \omega_\beta(\eta, [\xi, \zeta]) + \omega_\beta(\zeta, [\xi, \eta]). \end{aligned}$$

Repeated application of Jacobi identity then shows that $d\Omega_\beta = 0$ which implies that Ω_β is closed.

We have therefore shown that if the affine action $\Psi : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ defined by $\Psi(g, \alpha) = Ad_g^* \alpha + \sigma(g)$ is used in place of the coadjoint action, then the orbit $G \cdot \alpha$ is a symplectic manifold with the 2-form given by:

$$\omega_\alpha(\xi_{\mathfrak{g}^*}(v), \eta_{\mathfrak{g}^*}(v)) = -\alpha[\xi, \eta] + \textstyle\sum(\eta, \xi).$$

This completes the proof of the theorem. $\square$

4

Riemannian structure on homogeneous spaces

We now study the inheritance of a Riemannian metric of a symplectic manifold on its symplectic quotient. Starting with a symplectic manifold having a Riemann metric, we would like to end up with a Marsden-Weinstein-Meyer quotient which is also a Riemannian space with a Riemannian metric inherited from the one on the original space.

4.1. Riemannian manifolds

A Riemannian structure (or Riemannian metric) on a smooth manifold M, which is usually denoted by g is a smooth positive definite, symmetric bilinear form such that for each $p \in M$ we have

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

It is a smooth assignment of an inner product $\langle \cdot, \cdot \rangle$, to each tangent space T_pM of M. We denote by (M, g) a manifold on which the Riemannian structure g is defined and call it the Riemannian manifold.

For notational convenience, we shall denote the inner product at $p \in M$ by $g_M(p)$ if there is need to emphasize that g_M is the Riemannian metric on M. That is, we shall either write g_p or $g_M(p)$ whichever is suitable.

We recall that if (M, g) is a Riemannian manifold and $f : N \to M$ an immersion, then f^*g is a Riemannian metric on N called the induced metric.

Let (M, g) and (N, h) be two Riemannian manifolds, a diffeomorphism

$f : M \to N$ is called an isometry if

$$g_p(X, Y) = h_{f(p)}(T_p f \cdot X, T_p f \cdot Y)$$

for all $X, Y \in T_p M$, where $p \in M$ and, where $T_p f \cdot X$ is the image of the tangent vector X by the differential mapping associated with f at p. We also say that $f : M \to M$ is an isometry on M if for all $u, v \in T_p M$, $p \in M$, we have

$$g_p(u, v) = g_{f(p)}(T_p f \cdot u, T_p f \cdot v).$$

It is easily checked that if f is an isometry on M, then its inverse is also an isometry on M. Clearly the identity map on M is an isometry on M and if f, g are isometries on M then their composition is also an isometry on M. Thus the set of isometries on M is a group under the composition of maps.

Proposition 4.1.1 *(Myers-Steenrod). A group of isometries on a Riemannian manifold M is a Lie group.*

(See [3, p 67, Theorem 4.3]).

Theorem 4.1.1 *Let G be a Lie group of isometries of a Riemannian manifold (M, g) acting transitively on M, then G is compact if and only if M is compact.*

For the proof of this theorem (see [20, p 63, Theorem 2.35]).

Definition 4.1.1 *Let $\Phi : G \times M \to M$ be an action of a Lie group G on a smooth manifold M. Then a Riemannian metric $g(\cdot, \cdot)$ on M is called invariant if for each $m \in M$ we have*

$$g_m(u, v) = g_{\Phi_a(m)}(T_m \Phi_a \cdot u, T_m \Phi_a \cdot v)$$

for all $u, v \in T_m M$ and $a \in G$.

Theorem 4.1.2 *(See [5, p. 56]).*

Let G be a Lie group acting on a smooth manifold M. If G is compact then there exists an invariant Riemannian metric on M.

4.2. Riemannian submersions

Definition 4.2.1 *Let $f : M \to N$ be a smooth map. An element $x \in N$ is called a regular value of f if $f^{-1}(x)$ is a submanifold of M, and if whenever $m \in f^{-1}(x)$ then*

$$T_m f : T_m M \to T_{f(m)} N,$$

is surjective. A point $m \in M$ is called a regular point of f if $T_m f$ is surjective.

Definition 4.2.2 *Let M and N be smooth manifolds. A smooth map $\Phi : M \to N$ is called a submersion if all points of M are regular points of Φ. That is, Φ is a submersion if*

$$(d\Phi)_x : T_x M \to T_{\Phi(x)} N,$$

is surjective for all $x \in M$

Let $V(M)_p = T_p \Phi^{-1}(b) = \ker d\Phi_p$, for $p \in \Phi^{-1}(b), b \in N$.

Since M is a Riemannian manifold, it is appropriate to talk about the orthogonal complement of $V(M)_p$. We denote by $H(M)_p$ the orthogonal complement of $V(M)_p$.

Definition 4.2.3 *Let (M, g) and (B, h) be Riemannian manifolds, a smooth map*

$$\pi : M \to B,$$

is called a Riemannian submersion if:

(i) π has maximum rank at each point $p \in M$. That is to say

$$(d\pi)_p : T_p M \to T_{\pi(p)} B,$$

is surjective, and

(ii) $(d\pi)_p$ is an isometry between $H(M)_p$ and $T_{\pi(p)} B$. That is, if $X_p, Y_p \in H(M)_p$, then

$$g_p(X_p, Y_p) = h_{\pi(p)}((d\pi)_p X_p, (d\pi)_p Y_p).$$

The set $V(M)_p$ is the set of vertical vectors, and $H(M)_p$ is the set of horizontal vectors.

The tangent space T_pM decomposes into an orthogonal direct sum

$$T_pM = H(M)_p \oplus V(M)_p,$$

where

$$H(M)_p \cap V(M)_p = \{0\}.$$

Proposition 4.2.1 *Let G be a Lie group of isometries acting properly and freely on a Riemannian manifold (M, g) and let $p : M \to M/G$ be the canonical projection map (note that $N = M/G$ is a manifold). Then there exists a unique metric on $N = M/G$ such that the projection map p is a Riemannian submersion.*

For the proof see ([20, p. 61 Prop 2.28]).

We remark the following:

(a) If $b \in N = M/G$, and if $m_1, m_2 \in p^{-1}(b)$ then there is $h \in G$ such that

$$\Phi_h(m_1) = m_2,$$

([20, Proposition 2.28]), where Φ is the action of G on M. Thus the isometry group G acts transitively on each fibre so that the action of G preserves the fibres.

(b) Let $x \in p^{-1}(b)$. For each $\xi \in \mathfrak{g} = T_eG$, let

$$F(t) = \exp t\xi,$$

be its flow of ξ, then

$$\xi_M(x) = \frac{d}{dt}\Phi(\exp t\xi, x)\mid_{t=0}$$

is a tangent vector to the fibre through x. If $\xi \neq 0$ then $\xi_M(x) \neq 0$. Thus there is a one-to-one correspondence between $\mathfrak{g} = T_eG$ and the tangent space to the fibre at each point x in the fibre.

4.3. Almost complex structure

Let $\mathbb{C}^n$ denote n-dimensional space of complex numbers $(z^1, z^2, \cdots, z^n)$. We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by the correspondence $(z^1, \cdots, z^n) \to (x^1, y^1, \cdots, x^n, y^n)$, with $z^k = x^k + iy^k$, where $i = \sqrt{-1}$. By this identification we can consider $\mathbb{C}^n$ as a $2n-$dimensional Euclidean space. Similarly, if M is an $n-$ dimensional complex manifold with local coordinates $(z^1, \cdots, z^n)$, by identifying these coordinates with $(x^1, y^1, \cdots, x^n, y^n)$, where $z^k = x^k + iy^k$, $i = \sqrt{-1}$, $k = 1, \cdots, n$, we can regard M to be a $2n-$ dimensional differentiable manifold. Then for $p \in M$, the tangent space T_pM has the basis $\{(\frac{\partial}{\partial x^1})_p, (\frac{\partial}{\partial y^1})_p, \cdots, (\frac{\partial}{\partial x^n})_p, (\frac{\partial}{\partial y^n})_p\}$.

Definition 4.3.1 *Let M be a smooth manifold. A map*

$$J : TM \to TM$$

is called an almost complex structure on M if for each $p \in M$, J assigns a linear transformation

$$J_p : T_pM \to T_pM$$

such that

$$\begin{aligned} J_p \left(\frac{\partial}{\partial x^i}\right)_p &= \left(\frac{\partial}{\partial y^i}\right)_p \\ J_p \left(\frac{\partial}{\partial y^i}\right)_p &= -\left(\frac{\partial}{\partial x^i}\right)_p \end{aligned}$$

$i = 1, 2, \cdots, n.$

Clearly $J_p^2 = -Id_{T_pM}$.

The definition of J_p does not depend on the choice of local coordinates $(z^1, \cdots, z^n)$, (see [30, p. 107]).

The pair (M, J) is called an almost complex manifold.

Recall that if $F : M \to N$ is a smooth map and let $\varphi = (x^1, \cdots, x^n)$ be local coordinates about $p \in M$ and $\psi = (y^1, \cdots, y^m)$ local coordinates about $F(p) \in N$. Then

$$
\begin{aligned}
F_*\left(\tfrac{\partial}{\partial x^i}\right)_p &= \sum_{j=1}^m \left(F_*\left(\frac{\partial}{\partial x^i}\right)_p y^j\right)\frac{\partial}{\partial y^j}\,|_{F(p)} \\
&= \sum_{j=1}^m \left(\frac{\partial}{\partial x^i}\right)_p (y^j \circ F)\frac{\partial}{\partial y^j}\,|_{F(p)} \\
&= \sum_{j=1}^m \frac{\partial(y^j \circ F)}{\partial x^i}(p)\frac{\partial}{\partial y^j}\,|_{F(p)}
\end{aligned}
$$

If f is a smooth function on N, then the pull back of f under F is a smooth function on M given by

$$F^* f = f \circ F.$$

Proposition 4.3.1 *A differentiable map $\phi : M_1 \to M_2$, between two almost complex manifolds M_1 and M_2 with almost complex structures J_1 and J_2 respectively is holomorphic if and only if*

$$\phi_* \circ J_1 = J_2 \circ \phi_*, \text{ where } \phi_*$$

is the differential of the map ϕ.

Proof. Let $p \in M_1$ and let $(z^1, \cdots, z^n)$ be the complex local coordinates in the neighborhood of p and identify these coordinates with $(x^1, y^1, \cdots, x^n, y^n)$ of $\mathbb{R}^{2n}$. Let $(w^1, \cdots, w^m)$ be the local coordinates of the neighborhood of $\phi(p)$ in M_2 identified with $(u^1, v^1, \cdots, u^m, v^m)$ of $\mathbb{R}^{2m}$, where

$$
\begin{aligned}
z^k &= x^k + iy^k \qquad k = 1, 2, \cdots, n \\
w^j &= u^j + iv^j \qquad j = 1, 2, \cdots, m
\end{aligned}
$$

Set

$$
\begin{aligned}
\phi^* u^j &= a_j(x^1, y^1, \cdots, x^n, y^n) \text{ and} \\
\phi^* v^j &= b_j(x^1, y^1, \cdots, x^n, y^n), j = 1, \cdots, m
\end{aligned}
$$

Then by the above comments we have

$$
\begin{aligned}
\phi_*\left(\tfrac{\partial}{\partial x^i}\right)_p &= \sum_{j=1}^m \frac{\partial(u^j \circ \phi)}{\partial x^i}(p)\frac{\partial}{\partial u^j}|_{\phi(p)} + \sum_{j=1}^m \frac{\partial(v^j \circ \phi)}{\partial x^i}(p)\frac{\partial}{\partial v^j}|_{\phi(p)} \\
&= \sum_{j=1}^m \frac{\partial a_j}{\partial x^i}(p)\frac{\partial}{\partial u^j}|_{\phi(p)} + \sum_{j=1}^m \frac{\partial b_j}{\partial x^i}(p)\frac{\partial}{\partial v^j}|_{\phi(p)}.
\end{aligned}
$$

Similarly

$$\phi_*\left(\tfrac{\partial}{\partial y^i}\right)_p \;=\; \sum_{j=1}^m \frac{\partial a_j}{\partial y^i}(p)\frac{\partial}{\partial u^j}|_{\phi(p)} + \sum_{j=1}^m \frac{\partial b_j}{\partial y^i}(p)\frac{\partial}{\partial v^j}|_{\phi(p)}.$$

Now, from

$$\phi_*\left(J_1\tfrac{\partial}{\partial x^i}\right)_p = \phi_*\left(\tfrac{\partial}{\partial y^i}\right)_p,$$

we have,

$$\phi_*\left(J_1\frac{\partial}{\partial x^i}\right)_p = \sum_{j=1}^m \frac{\partial a_j}{\partial y^i}(p)\frac{\partial}{\partial u^j}|_{\phi(p)} + \sum_{j=1}^m \frac{\partial b_j}{\partial y^i}(p)\frac{\partial}{\partial v^j}|_{\phi(p)}. \tag{4.1}$$

Also

$$\phi_*\left(J_1\tfrac{\partial}{\partial y^i}\right)_p = -\phi_*\left(\tfrac{\partial}{\partial x^i}\right)_p,$$

gives

$$\phi_*\left(J_1\frac{\partial}{\partial y^i}\right)_p = -\sum_{j=1}^m \frac{\partial a_j}{\partial x^i}(p)\frac{\partial}{\partial u^j}|_{\phi(p)} - \sum_{j=1}^m \frac{\partial b_j}{\partial x^i}(p)\frac{\partial}{\partial v^j}|_{\phi(p)}. \tag{4.2}$$

On the other hand, from

$$J_2\circ\phi_*\left(\tfrac{\partial}{\partial x^i}\right)_p = \sum_{j=1}^m \frac{\partial a_j}{\partial x^i}(p)J_2\left(\frac{\partial}{\partial u^j}\right)_{\phi(p)} + \sum_{j=1}^m \frac{\partial b_j}{\partial x^i}(p)J_2\left(\frac{\partial}{\partial v^j}\right)_{\phi(p)},$$

we get

$$J_2\circ\phi_*\left(\frac{\partial}{\partial x^i}\right)_p = \sum_{j=1}^m \frac{\partial a_j}{\partial x^i}(p)\frac{\partial}{\partial v^j}|_{\phi(p)} - \sum_{j=1}^m \frac{\partial b_j}{\partial x^i}(p)\frac{\partial}{\partial u^j}|_{\phi(p)}. \tag{4.3}$$

and

$$J_2\circ\phi_*\left(\tfrac{\partial}{\partial y^i}\right)_p = \sum_{j=1}^m \frac{\partial a_j}{\partial y^i}(p)J_2\left(\frac{\partial}{\partial u^j}\right)_{\phi(p)} + \sum_{j=1}^m \frac{\partial b_j}{\partial y^i}(p)J_2\left(\frac{\partial}{\partial v^j}\right)_{\phi(p)},$$

gives

$$J_2 \circ \phi_* \left(\frac{\partial}{\partial y^i} \right)_p = \sum_{j=1}^{m} \frac{\partial a_j}{\partial y^i}(p) \frac{\partial}{\partial v^j}|_{\phi(p)} - \sum_{j=1}^{m} \frac{\partial b_j}{\partial y^i}(p) \frac{\partial}{\partial u^j}|_{\phi(p)}. \tag{4.4}$$

Now equation (4.2) = equation (4.4) if and only if

$$\frac{\partial a_j}{\partial x^i} = \frac{\partial b_j}{\partial y^i},$$

that is, if and only if

$$\frac{\partial u^j}{\partial x^i} = \frac{\partial v^j}{\partial y^i}.$$

and equation (4.1) = equation (4.3) if and only if

$$\frac{\partial a_j}{\partial y^i} = -\frac{\partial b_j}{\partial x^i},$$

that is, if and only if

$$\frac{\partial v^j}{\partial x^i} = -\frac{\partial u^j}{\partial y^i}.$$

which are Cauchy-Riemann equations. Thus ϕ is holomorphic if and only if

$$\phi_* \circ J_1 = J_2 \circ \phi_*$$

as required. This completes the proof of the theorem. $\square$

If (M, ω) is a symplectic manifold, an almost complex structure J on M is said to be compatible if whenever $m \in M$ and

$$g_m : T_m M \times T_m M \to \mathbb{R},$$

then

$$g_m(u, v) := \omega_m(u, Jv)$$

defines a Riemannian metric on M, for all $u, v \in T_m M$.

Proposition 4.3.2 *For every symplectic manifold (M, ω), there exists an almost complex structure J and a Riemannian metric $g(\cdot, \cdot)$ on M such that for each $m \in M$ we have*

$$\omega_m(u, Jv) = g_m(u, v)$$

for all $u, v \in T_m M$.

For the proof see ([23, p 14, Prop 5]).

Note that we can also write the compatibility condition in the form

$$\omega_m(u, v) = g_m(Ju, v), \quad u, v \in T_m M$$

Proposition 4.3.3 *Let G be a compact Lie group and $\Phi : G \times M \to M$ a symplectic action of G on the symplectic manifold (M, ω). Let $g(\cdot, \cdot)$ be an invariant metric on M and A a field of endomorphisms of TM, that is, $A : TM \to TM$ such that for each $m \in M$ we have $\omega_m(X, Y) = g_m(A_m X, Y)$, $X, Y \in T_m M$, then A is $G-$invariant.*

Proof. Let $a \in G$, $m \in M$. Suppose further that X, Y are vectors such that $X \in T_m M$ $Y \in T_{\Phi_a(m)} M$. Then we have:

$$\begin{aligned} g_{\Phi_a(m)}(T_m \Phi_a \circ A_m X, Y) &= g_m(A_m X, (T_m \Phi_a)^{-1} Y) \\ &= \omega_m(X, (T_m \Phi_a)^{-1} Y) \\ &= \omega_{\Phi_a(m)}(T_m \Phi_a \cdot X, Y) \\ &= g_{\Phi_a(m)}(A_{\Phi_a(m)} \circ (T_m \Phi_a) X, Y) \end{aligned}$$

Thus $T_m \Phi_a \circ A_m = A_{\Phi_a(m)} \circ T_m \Phi_a$.

This proves the proposition. □

Proposition 4.3.4 *Let (M, ω) be a symplectic manifold with a compatible almost complex structure J. If G is a group of isometries of M acting in a symplectic way, then the compatible almost complex structure J is G-invariant.*

Proof. Let g be a Riemannian metric on M such that for each $x \in M$, we have

$$g_x(Ju, v) = \omega_x(u, v)$$

for all $u, v \in T_xM$. Let $\Phi : G \times M \to M$ be the action of G on M. Then, for all $x \in M$ we have:

$$\begin{aligned} g_x(Ju, v) &= \omega_x(u, v) = \Phi_a^*\omega_x(u, v) \\ &= \omega_{\Phi_a(x)}(T_x\Phi_a u, T_x\Phi_a v) \\ &= g_{\Phi_a(x)}(JT_x\Phi_a u, T_x\Phi_a v) \\ &= g_{\Phi_a^{-1}\circ\Phi_a(x)}(T_x\Phi_a^{-1} \circ J \circ T_x\Phi_a u, v) \\ &= g_x(T_x\Phi_a^{-1} \circ J \circ T_x\Phi_a u, v) \end{aligned}$$

for all $u, v \in T_xM$.

Thus

$$Ju = T_x\Phi_a^{-1} \circ J \circ T_x\Phi_a u,$$

which gives

$$T_x\Phi_a \circ J = J \circ T_x\Phi_a.$$

This completes the proof. □

4.4. Riemannian structure on a reduced space

Definition 4.4.1 *Let (M, ω) be a symplectic manifold and G a Lie group. Let $\Phi : G \times M \to M$ be a Hamiltonian action of G on M. Let $\mu : M \to \mathfrak{g}^*$ be the Ad^*-equivariant momentum mapping of the action and $\beta \in \mathfrak{g}^*$ a regular value of μ. We define the symplectic reduced space of the G-action on M to be*

$$M_\beta := \mu^{-1}(\beta)/G_\beta,$$

where G_β is the isotropy subgroup of β.

(i) Since $\beta \in \mathfrak{g}^*$ is a regular value of μ, the inverse image $\mu^{-1}(\beta)$ is a submanifold of M of dimension $\dim M - \dim G$.

(ii) If the action of G_β on $\mu^{-1}(\beta)$ is free and proper then the reduced space $M_\beta = \mu^{-1}(\beta)/G_\beta$ is a manifold of dimension $\dim M - \dim G - \dim G_\beta$. (See [29, p. 124]).

In this case, the projection map

$$\pi_\beta : \mu^{-1}(\beta) \to \mu^{-1}(\beta)/G_\beta$$

is a smooth submersion. (See [1, pp. 298-299]). By the Marsden-Weinstein-Meyer reduction theorem there is a unique symplectic form ω_β on the reduced space M_β which is characterized by the equation:

$$\pi_\beta^* \omega_\beta = i_\beta^* \omega, \tag{4.5}$$

where

$$i_\beta : \mu^{-1}(\beta) \to M$$

is the inclusion map and

$$\pi_\beta : \mu^{-1}(\beta) \to \mu^{-1}(\beta)/G_\beta$$

is the quotient map. That is, if x is a point in $\mu^{-1}(\beta)$ so that

$$\pi_\beta(x) = [x]$$

is a point on the quotient space $\mu^{-1}(\beta)/G_\beta$ and $u \in T_x(\mu^{-1}(\beta))$ is a tangent vector so that

$$[u] \in T_{[x]}(\mu^{-1}(\beta)/G_\beta) \cong T_x(\mu^{-1}(\beta))/T_x(G_\beta \cdot x),$$

then the equation (4.5) is equivalent to the following:

$$\omega_\beta([x])([u],[v]) = \omega(x)(u,v),$$

for all $u, v \in T_x(\mu^{-1}(\beta))$. (See [27, p. 15]).

Let g_M be a Riemannian metric on the symplectic manifold (M, ω) and let J_M be an almost complex structure such that

$$\omega(\cdot, \cdot) = g_M(J_M \cdot, \cdot),$$

then for $u, v \in T_x(\mu^{-1}(\beta))$, we have:

$$\begin{aligned} i^*\omega(x)(u,v) &= \omega(x)(i_*u, i_*v) \\ &= g_M(x)(J_M(i_*u), i_*v) \\ &= g_M(x)(J_M u, v) \\ &= g_M(x)(i_*(J_M u), i_*v) \\ &= i^*g_M(x)(J_M u, v), \end{aligned}$$

where

$$i : \mu^{-1}(\beta) \to M,$$

is the inclusion map. That is,

$$i^*\omega(\cdot, \cdot) = i^*g_M(J_M \cdot, \cdot).$$

Theorem 4.4.1 *Let (M, ω) be a symplectic manifold having a compatible Riamannian metric g_M, and G a Lie group of isometries of M whose action on M is a Hamiltonian action. Let $\mathfrak{g}$ be the Lie algebra of G, and let*

$$\mu : M \to \mathfrak{g}^*$$

be the Ad^*-equivariant momentum mapping of the action, where $\mathfrak{g}^*$ is the dual of the Lie algebra of G. Let $\beta \in \mathfrak{g}^*$ be a regular value of μ and G_β the isotropy subgroup of β which acts freely and properly on $\mu^{-1}(\beta)$. Then, there exists a Riemannian metric g_β on the reduced space $\mu^{-1}(\beta)/G_\beta$ such that the projection map

$$\pi_\beta : \mu^{-1}(\beta) \to \mu^{-1}(\beta)/G_\beta$$

is a Riemannian submersion. That is,

$$\pi_\beta^* g_\beta = i^* g_M,$$

where g_M is a Riemannian metric on M and $i : \mu^{-1}(\beta) \to M$, is the inclusion map.

Proof. Let $\pi : \mu^{-1}(\beta) \to \mu^{-1}(\beta)/G_\beta$ be the projection onto the reduced space. For convenience we shall write M_β for $\mu^{-1}(\beta)$ and B_β for $\mu^{-1}(\beta)/G_\beta$. If $x \in B_\beta$ then $\pi^{-1}(x)$ is called the fibre over x. If $m \in \pi^{-1}(x)$, then

$$\pi^{-1}(x) = \{gm : g \in G \text{ and } \pi(gm) = x\},$$

is the fibre through m. Since G_β acts freely and properly on $\mu^{-1}(\beta)$, the projection $\pi : M_\beta \to B_\beta$, is a submersion. (See [1, pp. 298-299]). But π is constant on $\pi^{-1}(x)$, for each $x \in B_\beta$, that is, $\pi(\pi^{-1}(x)) = \{x\}$, so, if $u \in T_m\pi^{-1}(x)$, for $m \in \pi^{-1}(x)$, then $d\pi_m(u) = 0$. That is, $T_m\pi^{-1}(x) = \ker d\pi_m = V(M_\beta)_m$ is the set of vertical vectors. Let $H(M_\beta)_m$ be the orthogonal complement of $V(M_\beta)_m$. Then, T_mM_β decomposes into a direct sum

$$T_mM_\beta = H(M_\beta)_m \oplus V(M_\beta)_m,$$

with $H(M_\beta)_m \cap V(M_\beta)_m = \{0\}$. Thus, if $X \in T_mM_\beta$, then $X = Y + Z$, with $Y \in H(M_\beta)_m$ and $Z \in V(M_\beta)_m$. It follows that $d\pi_m(X) = d\pi_m Y$.
So, if $X \notin V(M_\beta)_m$, then $d\pi_m(X) \neq 0$, and $d\pi_m(X) \in T_{[m]}B_\beta$, where $[m] = \pi(m)$. Thus, for each $X \in H(M_\beta)_m$, we have

$$d\pi_m(X) \in T_{[m]}B_\beta.$$

Let $d\pi_m|_{H_m}$ be the restriction of $d\pi_m$ to $H(M_\beta)_m$, the space of horizontal vectors. Since π and $d\pi$ are surjective ([1, p. 299]), then $d\pi_m|_{H_m}$ is surjective and it is linear. But

$$\ker d\pi_m|_{H_m} = \{0\},$$

so if $[u] \in T_{[m]}B_\beta$, there is a unique $u \in H(M_\beta)_m$ such that

$$d\pi_m|_{H_m}(u) = [u].$$

That is the map $d\pi_m|_{H_m}$ is also injective. It follows therefore that the map

$$d\pi_m|_{H_m} : H(M_\beta)_m \to T_{[m]}B_\beta$$

is an isomorphism of vector spaces. Because of this isomorphism, we shall write the tangent vectors of $T_{[m]}B_\beta$ as say w instead of $[w]$, when we refer to the restriction map $d\pi_m|_{H_m}$.

Let $v, w \in T_x B_\beta$. Then there exists unique vectors $\tilde{v}, \tilde{w} \in H(M_\beta)_m$, $m \in \pi^{-1}(x)$ such that

$$d\pi_m|_{H_m}(\tilde{v}) = v,$$

and

$$d\pi_m|_{H_m}(\tilde{w}) = w.$$

Define a metric h on $T_x B_\beta$ by

$$h_x(v, w) = i^* g_M(\tilde{v}, \tilde{w}).$$

We shall show that the assignment $x \mapsto h_x$ smoothly depends on x. First note that, if $m_1, m_2 \in \pi^{-1}(x)$, then there is an isometry $f \in G$ with $f(m_1) = m_2$, and $\pi \circ f = \pi$, (see [20, proposition 2.20]). We then have,

$$T_{f(m_1)}\pi \circ T_{m_1} f = T_{m_1}\pi.$$

Thus, $T_{m_1} f$ is an isometry between $H(M_\beta)_{m_1}$ and $H(M_\beta)_{m_2}$. This shows that h_x does not depend on the choice of m in the fibre $\pi^{-1}(x)$.

Let $m \mapsto p_m$ be a smooth assignment of the orthogonal projection

$$p_m : T_m M_\beta \to H(M_\beta)_m$$

of $T_m M_\beta$ onto $H(M_\beta)_m$. Since $d\pi_m|_{H_m}$ is an isomorphism, π is a local diffeomorphism. Let σ be the local section of π. If U is an open subset of B_β and $x \in U$, let $v', w' \in T_{\sigma(x)} M_\beta$, then

$$h_x(v,w) = i^* g_M(\sigma(x))(p_{\sigma(x)}v', p_{\sigma(x)}w'),$$

where

$$p_{\sigma(x)}v' = \tilde{v} \in H(M_\beta)_{\sigma(x)}$$

and

$$p_{\sigma(x)}w' = \tilde{w} \in H(M_\beta)_{\sigma(x)}.$$

As the right side is the composition of smooth maps we conclude that $x \mapsto h_x$ is smooth and

$$d\pi_m|_{H_m} : H(M_\beta)_m \to T_{\pi(m)}B_\beta$$

is an isometry. By this construction we have shown that

$$\pi : \mu^{-1}(\beta) \to \mu^{-1}(\beta)/G_\beta$$

is a Riemannian submersion. □

Definition 4.4.2 *An almost Hermitian manifold is an almost complex manifold (M, J) with a chosen Riemannian structure g_M such that*

$$g_M(JX, JY) = g_M(X, Y),$$

for all $X, Y \in TM$.

Definition 4.4.3 *Let (M, J_M) and (N, J_N) be almost Hermitian manifolds, a map*

$$\Phi : M \to N$$

is called almost complex if it commutes with almost complex structures, that is, if

$$\Phi_* \circ J_M = J_N \circ \Phi_*.$$

An almost complex mapping between almost Hermitian manifolds which is also a Riemannian submersion is called a almost Hermitian submersion.

Proposition 4.4.1 *Let $\Phi : M \to N$ be an almost Hermitian submersion, then the horizontal and the vertical distributions determined by Φ are J_M-invariant. That is*

$$J_M\{V(M)\} = V(M)$$
$$J_M\{H(M)\} = H(M).$$

Proof. Let (M, J_M, g_M) and (N, J_N, g_N) be two almost Hermitian manifolds and $\Phi : M \to N$ an almost Hermitian submersion. Then Φ is an almost complex mapping and we have

$$\Phi_* \circ J_M = J_N \circ \Phi_*.$$

Let V be a vertical vector, then $\Phi_* V = 0$ since $V \in \ker \Phi_*$. We now have

$$\Phi_*(J_M V) = J_N(\Phi_* V) = 0.$$

Thus $\Phi_*(J_M V) = 0$ which shows that $J_M V$ is a vertical vector. If now X is a horizontal vector, then for any vertical vector V, we have

$$g_M(X, V) = 0,$$

since they belong to orthogonal complement subspaces. We then have,

$$\begin{aligned} g_M(J_M X, V) &= g_M(J_M^2 X, J_M V) \\ &= -g_M(X, J_M V) \\ &= 0. \end{aligned}$$

Thus, $J_M X$ is horizontal vector.() See [40, p. 151]). □

Definition 4.4.4 *Let $\Phi : M \to N$ be an almost Hermitian submersion. A horizontal vector field X on M is called a basic vector field if there is a smooth vector field denoted by X_* on N such that X and X_* are Φ-related.*

We shall first state the difficulties that may arise with regard to transformation of the almost complex structure to the reduced space by the projection map.

Let the symplectic manifold (M, ω) be a real manifold and g_M a Riemannian structure on M such that

$$\omega(\cdot, \cdot) = g_M(J\cdot, \cdot).$$

Let X be a vector field on M. Then

$$0 = \omega(X, X) = g_M(JX, X).$$

That is, $g_M(JX, X) = 0$, and since g_M is positive definite we conclude that JX is orthogonal to X. Thus, if X is a horizontal vector field then JX belong to the orthogonal complement which in this case is the vertical space. Therefore, even if X is a basic vector field there is no guarantee that JX will be a basic vector field.

Another difficulty arises from the push-forward of the almost complex structure. Even when the kernel of the differential of π_β is preserved by J, there need not be an almost complex structure on the image $\pi_\beta(M_\beta)$ which make $d\pi_\beta$ complex linear as the following example shows.

Consider the twistor fibration (see [13]).

$$\begin{aligned} \pi \quad : \quad & \mathbb{C}P^3 \to \mathbb{H}P^1 = S^4 \\ & \mathbb{C} \cdot v \mapsto \mathbb{H} \cdot v, \qquad v \in \mathbb{C}^4 \end{aligned}$$

which sends a complex line through the origin in $\mathbb{C}^4$ to its quaternionic span in $\mathbb{H}^2$. For each point $x \in \mathbb{H}P^1$, the inverse image $\pi^{-1}(x)$ are complex lines in $\mathbb{C}P^3$. Thus the fibers of π are holomorphic submanifolds of $\mathbb{C}P^3$ which are compact and connected. However, it has been proved that $\mathbb{H}P^1$ does not admit any almost complex structure. This shows that the push-forward of an almost complex structure by a submersion does not necessarily yield an almost complex structure on its image for which the differential of the map is complex linear. (See [6, p. 8]) for the details of this example.

Another example of this phenomenon is found among covering maps of smooth manifolds $\pi : E \to B$ where E has a complex structure. The immediate example is the covering map $\mathbb{C}P^1 \to \mathbb{R}P^2$. It is immediate that $\mathbb{R}P^2$ does not admit any complex structure since it is not orientable. For the next theorem see also [9].

Theorem 4.4.2 *Let (M, ω) be a symplectic manifold and G a Lie group of isometries of M. Let $\Phi : G \times M \to M$ be a hamiltonian action of G on M with Ad^*-equivariant momentum mapping*

$$\mu : M \to \mathfrak{g}^*.$$

Let $\beta \in \mathfrak{g}^*$ be a regular value of μ and G_β be the isotropy subgroup of β acting freely and properly on $\mu^{-1}(\beta)$. Given a compatible almost complex structure J_M on M and a Riemannian metric g_M which satisfies the compatibility condition,

$$\omega(X, Y) = g_M(J_M X, Y),$$

for all $X, Y \in TM$, let ω_β be the reduced symplectic form on the reduced symplectic manifold $\mu^{-1}(\beta)/G_\beta$. Then there exists an almost complex structure J_β and a Riemannian metric g_β on the reduced space $\mu^{-1}(\beta)/G_\beta$ which make

$$\pi : \mu^{-1}(\beta) \to \mu^{-1}(\beta)/G_\beta$$

a Riemannian submersion and satisfies the condition

$$\omega_\beta([u], [v]) = g_\beta(J_\beta[u], [v])$$

for all $[u], [v] \in T(\mu^{-1}(\beta)/G_\beta)$ if and only if

$$\pi : \mu^{-1}(\beta) \to \mu^{-1}(\beta)/G_\beta$$

is an almost complex mapping.

Proof. Let h_β be the Riemannan metric on $\mu^{-1}(\beta)/G_\beta$ as in theorem 4.4.1. Since $\mu^{-1}(\beta)/G_\beta$ is a symplectic manifold, there is a almost complex structure J_β and a Riemannian metric g_β such that if $[u], [v] \in T(\mu^{-1}(\beta)/G_\beta)$ then

$$\omega_\beta([u],[v]) = g_\beta(J_\beta[u],[v]), \text{ see Proposition 4.3.2.}$$

It is sufficient to find a condition for which

$$h_\beta = g_\beta.$$

Let $x \in \mu^{-1}(\beta)/G_\beta$, we have seen from theorem 4.4.1 that if $m \in \pi^{-1}(x)$, then

$$\pi^{-1}(x) = \{gm : g \in G\}$$

is the fibre through m . The tangent space to the fibre $T_m(\pi^{-1}(x))$ is the kernel of the differential of π at m. That is,

$$\ker d\pi_m = T_m(\pi^{-1}(x)).$$

We have classified this tangent space as the set of vertical vectors of the Riemannian submersion π. We also have by the Symplectic Reduction Theorem (see [27, p. 15]) that

$$(T_m(\mu^{-1}(\beta)))^\omega = T_m(G\cdot m).$$

But $G\cdot m = \{gm : g \in G\} = \pi^{-1}(x)$ is the fibre through m. So if $X \notin T_m(\pi^{-1}(x))$ then there is a $Y \in T_m(\mu^{-1}(\beta))$ such that $\omega(X,Y) \neq 0$. That is,

$$\omega(m)(X,Y) = \omega_\beta([m])([X],[Y]) = g_\beta([m])(J_\beta[X],[Y]) \neq 0. \tag{4.6}$$

But we also have that

$$\omega(m)(X,Y) = g_M(m)(J_M X, Y) = h_\beta(\pi(m))(\pi_*(J_M X), \pi_* Y), \tag{4.7}$$

by Theorem 4.4.1. In particular, if X and Y are basic vector fields, then equation (4.6) and (4.7) imply that

$$\begin{aligned} g_\beta(\pi(m))(J_\beta(\pi_* X), \pi_* Y) &= h_\beta(\pi(m))((J_M X)_*, Y_*)\circ\pi \\ &= \pi^* h_\beta(m)(JX, Y) \\ &= h_\beta(\pi(m)(\pi_*(J_M X), \pi_* Y). \end{aligned}$$

But this relation holds if and only if

$$J_\beta(\pi_* X) = \pi_*(J_M X),$$

if and only if

$$\pi_* \circ J_M = J_\beta \circ \pi_*,$$

if and only if π is an almost complex mapping. This completes the proof of the theorem. $\square$

5

Adjoint orbits and coadjoint Orbits

This chapter involves background ideas from representation theory. That is, the adjoint and coadjoint representations as well as the actions of Lie groups giving orbit spaces. We are more interested in those quotient spaces that result from transitive actions, the homogeneous spaces. We mention flag and generalised flag manifolds. These are an important class of homogeneous spaces which admit a complex structure, a Kähler structure and a symplectic structure as mentioned in [3]

5.1. Adjoint action

Definition 5.1.1 *Let G be a Lie group and $\mathfrak{g} \cong T_eG$ be its Lie algebra where e is the identity element in G. Then the smooth action*

$$\Phi : G \times \mathfrak{g} \to \mathfrak{g};\ \ (g, \xi) \mapsto Ad(g)\xi$$

is called the adjoint action of G on its Lie algebra $\mathfrak{g}$, which we denote by $Ad : G \times \mathfrak{g} \to \mathfrak{g}$.

Definition 5.1.2 *Let $Ad : G \times \mathfrak{g} \to \mathfrak{g}$ be the adjoint action of a Lie group G on its Lie algebra $\mathfrak{g}$ and let $\xi \in \mathfrak{g}$. We define the adjoint orbit of ξ to be*

$$O_\xi = \{Ad(g)\xi : g \in G\} \subset \mathfrak{g}.$$

The stability group also called the isotropy group of ξ is given by

$$G_\xi = \{g \in G : Ad(g)\xi = \xi\}.$$

This is a closed subgroup of G (see [18, p 16]). If $\eta \in O_\xi$ then there is some $g \in G$ such that $\eta = Ad(g)\xi$.

We shall now show that an adjoint orbit can be represented as homogeneous space. For a similar construction (see [10, pp 127-129]). Define a map

$$\rho : O_\xi \to G/G_\xi$$

by

$$\rho(\eta) = gG_\xi,$$

for all $\eta \in O_\xi$ and $g \in G$ such that

$$\eta = Ad(g)\xi.$$

The map ρ is well defined since if also $\rho(\eta) = hG_\xi$ for some $h \in G$ then

$$Ad(g)\xi = Ad(h)\xi,$$

which implies that

$$Ad(h^{-1}) \circ Ad(g)\xi = \xi.$$

This gives $h^{-1}g \in G_\xi$ and

$$gG_\xi = hG_\xi.$$

The map ρ is injective. Let $\eta = Ad(g)\xi$, $\mu = Ad(h)\xi$ and suppose that $gG_\xi = hG_\xi$. Then

$$h^{-1}g \in G_\xi,$$

so that

$$Ad(h^{-1}g)\xi = Ad(h^{-1}) \circ Ad(g)\xi = \xi.$$

This implies then that

$$\eta = Ad(g)\xi = Ad(h)\xi = \mu.$$

Clearly ρ is surjective, since for $g \in G$ and $\eta = Ad(g)\xi \in O_\xi$ gives $\rho(\eta) = gG_\xi$ by construction.

If $\eta = Ad(h)\xi$, for some $h \in G$, then $G_\eta = Ad(h)G_\xi Ad(h^{-1})$. Thus, for all $g \in G$ we have

$$G/G_\xi \cong G/G_{Ad(g)\xi}.$$

This shows that the definition of G/G_ξ does not depend on the choice of the element ξ in its adjoint orbit. Thus,

$$G/G_\xi \cong O_\xi.$$

Now let $M = G/G_\xi \cong O_\xi$. Then the Lie group G acts transitively on M so that M is a homogeneous space. (See Example 2.2.0.2).

Let $X \in \mathfrak{g}$. The vector field on $\mathfrak{g}$ corresponding to X, called the infinitesimal generator of the action, is defined by

$$X_{\mathfrak{g}}(\xi) = \frac{d}{dt}(Ad(\exp tX)\xi) \mid_{t=0} .$$

To determine the tangent space to the adjoint orbit O_ξ at ξ, let $X \in \mathfrak{g}$ and let $x(t) = \exp tX$ be the curve in G which is tangent to X at $t = 0$, then

$$\xi(t) = Ad(\exp tX)\xi$$

is the curve on O_ξ such that $\xi(0) = \xi$. Let $Y \in \mathfrak{g}$, then

$$\langle \xi(t), Y\rangle = \langle Ad(\exp tX)\xi, Y\rangle,$$

where $\langle \cdot, \cdot\rangle$ is the natural pairing on $\mathfrak{g}$. Differentiating with respect to t at $t = 0$ we get

$$\begin{aligned}\langle \xi'(0), Y\rangle &= \frac{d}{dt}\langle Ad(\exp tX)\xi, Y\rangle \mid_{t=0} \\ &= \langle \frac{d}{dt}(Ad(\exp tX)\xi) \mid_{t=0}, Y\rangle \\ &= \langle ad(X)\xi, Y\rangle.\end{aligned}$$

Thus $\xi'(0) = ad(X)\xi$. Therefore, the tangent space to the orbit O_ξ at ξ is given by

$$T_\xi O_\xi = \{ad(X)\xi : X \in \mathfrak{g}\}$$

5.2. An example of adjoint orbits as flag manifolds

The examples of adjoint orbits that will be of interest to us are the generalized flag manifolds. These orbits are known to hold a symplectic structure. Generalized flag manifolds are homogeneous spaces which can be expressed in the form $G/C(S)$, where G is a compact Lie group and

$$C(S) = \{g \in G : gx = xg, \text{ for all } x \in S\}$$

is the centraliser of a torus S in G.

Definition 5.2.1 *Let $\mathbb{C}^n$ be an $n-$dimensional complex space. A flag is an increasing sequence of complex subspaces ordered by inclusion*

$$W = V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n$$

in the sense that each V_i is a proper subset of V_{i+1} for $i = 1, \cdots, n-1$ and such that $\dim V_k = k$ *for $k = 1, \cdots, n$.*

Remark 5.2.1 *This definition holds for subspaces of a finite dimensional vector space.*

Since $V_n \simeq \mathbb{K}^n$, the example below deal with the flags in the canonical vector space or field $\mathbb{K}^n$.

Example 5.2.0.1 *Let $\{e_1, e_2, \cdots, e_n\}$ be the canonical basis for the complex vector space $\mathbb{C}^n$. Then the standard flag is given by*

$$W_0 = Span_{\mathbb{C}}\{e_1\} \subset Span_{\mathbb{C}}\{e_1, e_2\} \subset \cdots \subset Span_{\mathbb{C}}\{e_1, \cdots e_n\} = \mathbb{C}^n.$$

We shall now show that flag manifolds are homogeneous spaces.

Let F_n be the set of all flags in $\mathbb{C}^n$ and let W_0 be the standard flag above. The Lie group

$$U(n) = \{A \in Gl(n, \mathbb{C}) : \bar{A}^T A = I\},$$

where $\bar{A}^T$ denotes the transpose of the conjugate of A, will play a key role here. First the action of the Lie group $U(n)$ on F_n is transitive. To see this consider an arbitrary flag

$$W = V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n.$$

Then $U(n)$ acts on F_n by left multiplication. That is, if $S \in U(n)$ then

$$SW = SV_1 \subset SV_2 \subset \cdots \subset SV_n = \mathbb{C}^n.$$

Let v_1 be a unit vector in V_1 such that

$$V_1 = Span_{\mathbb{C}}\{v_1\}.$$

Next choose a unit vector v_2 in V_2 orthogonal to V_1 such that

$$V_2 = Span_{\mathbb{C}}\{v_1, v_2\}.$$

Having chosen unit vectors $\{v_1, \cdots, v_k\}$ with

$$V_k = Span_{\mathbb{C}}\{v_1, \cdots, v_k\},$$

choose a unit vector v_{k+1} in V_{k+1} orthogonal to V_k such that

$$V_{k+1} = Span_{\mathbb{C}}\{v_1, \cdots, v_{k+1}\}.$$

Continuing this construction we obtain a set of orthonomal unit vectors $\{v_1, \cdots, v_{n-1}\}$ such that

$$V_j = Span_{\mathbb{C}}\{v_1, \cdots, v_j\}.$$

Let v_n be a unit vector in V_n orthogonal to V_{n-1}. The set $\{v_1, v_2, \cdots, v_n\}$ is another orthonormal basis for $\mathbb{C}^n$. It is now a result of linear algebra that there is $n \times n$ matrix $S = (a_{ij})$ such that

$$v_i = \sum_{j=1}^{n} a_{ij} e_j.$$

Then $S \in U(n)$ and $SW_0 = W$. Thus $U(n)$ acts transitively on F_n as earlier claimed.

The isotropy subgroup of W is

$$\{A \in U(n) : AV_j = V_j\}.$$

In particular, this is a set of matrices $A \in U(n)$ such that

$$Av_k = \lambda_k v_k,$$

for some complex number λ_k with $\mid \lambda_k \mid = 1$ since $A \in U(n)$. Thus

$$\lambda_k = e^{i\theta_k} \in U(1).$$

Since this must be true for each v_j, $j = 1, 2, \cdots, n$, the matrix A must be of the form

$$A = diag(e^{i\theta_1}, \cdots, e^{i\theta_n}).$$

Thus

$$F_n = U(n)/U(1) \times \cdots \times U(1).$$

The generalized flag manifold can now be constructed as follows:

Let $\{n_1, \cdots, n_k\}$ be a set of positive integers such that $n_1 + n_2 + \cdots + n_k = n$. A partial flag is an element

$$W = V_1 \subset \cdots \subset V_k,$$

with

$$\dim V_k = n_1 + \cdots + n_k.$$

We can visualize this as a sum of vector spaces. For example, let $Q_1, Q_2, \cdots, Q_n$ be a set of subspaces of $\mathbb{C}^n$ with $\dim Q_1 = n_1$, $\dim Q_2 = n_2 \cdots \dim Q_{n-1} = n-1$. Set

$$\begin{aligned} V_1 &= Q_1 \\ V_2 &= Q_1 \oplus Q_2 \\ &\cdots \\ V_{n-1} &= Q_1 \oplus Q_2 \oplus \cdots \oplus Q_{n-1}. \end{aligned}$$

Then $V_1 \subset \cdots \subset V_{n-1}$ and $\dim V_j = n_1 + \cdots + n_j$. The flag

$$W = V_1 \subset \cdots \subset V_k,$$

with

$$\dim V_k = n_1 + \cdots + n_k, \ k \leq n,$$

is called a partial flag.

A generalized flag manifold is a set $F(n_1, \cdots, n_k)$ of all partial flags with $n_1+n_2+\cdots+n_k = n$. Generalized flag manifolds just like flag manifolds are homogeneous spaces (see[3, p 70]).

Throughout, the discussion that follow, the Lie group G will be compact and connected.

Let
$$U(n) = \{A \in GL(n, \mathbb{C}) : \bar{A}^T A = I\},$$
be the unitary group, where $\bar{A}^T$ denotes the transpose of the conjugate of A. Then

(i) $U(n)$ is compact:

First notice that $U(n)$ is a closed subgroup of $GL(n, \mathbb{C})$ since $U(n) = \det^{-1}(S^1) = \det^{-1}(U(1)$. Also, $U(n)$ is bounded. For let $A = (\alpha_{ij}) \in U(n)$. One has $\sum \alpha_{ij} \cdot \beta_{jk} = \delta_{ik}$, the Kronecker delta, with $\beta_{jk} = \bar{\alpha}_{kj}$. Hence, if $k = i$ one has
$$\sum \alpha_{ij} \cdot \bar{\alpha}_{ji} = 1$$
which implies that
$$\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |\alpha_{ij}|^2 \right) = n.$$
Now,
$$\|A\| = \left(\sum_{i,j=1}^{n} |\alpha_{ij}|^2 \right)^{\frac{1}{2}} = \sqrt{n} < \sqrt{n+1}.$$
Thus,
$$A \in B(0, \sqrt{n+1}), \text{ where } r = \sqrt{n+1}.$$
One concludes that $A \in U(n)$ implies that $A \in B(0, r)$ and $U(n) \subset B(0, r)$, where $r = \sqrt{n+1}$. Since this is true for each $A \in U(n)$, then $U(n)$ is also bounded. Thus, $U(n)$ is compact. (See [7]).

(ii) $U(n)$ is connected since if we consider the action of $U(n)$ on $\mathbb{C}^n$ given by
$$(A, X) \mapsto AX,$$
for all $A \in U(n)$ and $X \in \mathbb{C}^n$, then

$$\begin{aligned}\|AX\|^2 &= (\overline{AX})^T(AX)\\ &= \bar{X}^T\bar{A}^TAX\\ &= \bar{X}^TX\\ &= \|X\|^2.\end{aligned}$$

Thus, this action takes sets of the form

$$\{(z_1,\cdots,z_n) :| z_1 |^2 + | z_2 |^2 + \cdots + | z_n |^2 = 1\}$$

into sets of the same kind. In particular, the orbit of e_1 under this action is the unit sphere S^{2n-1}. The stabilizer of the same element e_1 are matrices of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & A_1 \end{pmatrix},$$

where $A_1 \in U(n-1)$. Thus

$$S^{2n-1} = U(n)/U(n-1).$$

But S^{2n-1} is connected which implies that $U(n)$ is connected if and only if $U(n-1)$ is connected. Since $U(1) = S^1$ is connected, we conclude by induction on n that $U(n)$ is connected.

The Lie algebra of $U(n)$ is the space of all skew-Hermitian matrices

$$\mathfrak{u}(n) = \{A \in Mat_{n\times n}(\mathbb{C}) : A + \bar{A}^T = 0\}.$$

We now want to determine the orbits of adjoint representation of the Lie group $G = U(n)$ on its Lie algebra $\mathfrak{g} = \mathfrak{u}(n)$.

Let $Ad : G \times \mathfrak{g} \to \mathfrak{g}$ be the action of G on its Lie algebra $\mathfrak{g}$. Let $X \in \mathfrak{g}$, then the orbit of X is given by:

$$\begin{aligned}O_X &= \{Ad_gX : g \in G\}\\ &= \{Y \in \mathfrak{g} : Y = gXg^{-1}, \text{ for some } g \in G\}\end{aligned}$$

This is a set of similar matrices since the action is by conjugation. Recall that every skew Hermitian matrix is diagonalizable and that all the eigenvalues of a skew Hermitian matrix are purely imaginary. This means that X is $U(n)-$ conjugate to a matrix of the form $X_\lambda = \operatorname{diag}(i\lambda_1, i\lambda_2, \cdots, i\lambda_n)$ for $\lambda_j \in \mathbb{R}, \quad j = 1, \cdots, n$. Since similar matrices have same eigenvalues, without loss of generality we can describe the adjoint orbit of X to be the set of all skew Hermitian matrices with eigenvalues $i\lambda_1, i\lambda_2, \cdots, i\lambda_n$. Denote this set of eigenvalues by λ and the orbit determined by the corresponding eigenspaces by $H(\lambda)$. Note that $H(\lambda)$ is a vector space since it is a closed subgroup of a linear group $GL(n, \mathbb{C})$.

Case 1 : All the n eigenvalues are distinct.

Let x_j be the eigenvector corresponding to the eigenvalue $i\lambda_j$, then we have $gx_j = i\lambda_j x_j$. This gives a 1-dimensional subspace P_j of $\mathbb{C}^n$ which is a line in the complex plane passing through the origin.

Assuming $\lambda_1 < \lambda_2 < \cdots < \lambda_n$. Note that the eigenvectors corresponding to distinct eigenvalues are orthogonal. Now each element in $H(\lambda)$ has same eigenvalues $i\lambda_1, \cdots, i\lambda_n$, however, it is only distinguished by its corresponding eigenspaces $P_1, \cdots, P_n$. Thus for each $n-$tuple $(P_1, P_2, \cdots, P_n)$ of complex lines in $\mathbb{C}^n$ which are pairwise orthogonal, there will be an associated element $h \in H(\lambda)$ and each element $h \in H(\lambda)$ determines a family of eigenspaces $(P_1, P_2, \cdots, P_n)$.

Let $(P_1, \cdots, P_n) \mapsto P_1 \subset P_1 \oplus P_2 \subset \cdots \subset P_1 \oplus P_2 \oplus \cdots \oplus P_n = \mathbb{C}^n$ and define the vector space V_j by $V_j = P_1 \oplus \cdots \oplus P_j$. Then $W = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n$ is a flag we have already seen and the totality of such flags $F_n = U(n)/U(1) \times \cdots \times U(n)$ is the flag manifold described earlier. There is a bijection from $H(\lambda)$ to F_n which associates to each element $h \in H(\lambda)$ the subspaces $V_j = P_1 \oplus \cdots \oplus P_j$ where P_j is the eigenspace of h corresponding to the eigenvalue $i\lambda_j$. This shows that the adjoint orbits are diffeomorphic to flag manifolds.

Case 2: There are $k < n$ distinct eigenvalues.

We again order the eigenvalues $\lambda_1 < \cdots < \lambda_k$. Let $n_1, n_2, \cdots, n_k$ be their multiplicities respectively. Let Q_j be the eigenspace corresponding to the eigenvalue $i\lambda_j$. We assume that $\dim Q_i = n_i, \quad i = 1, \cdots, k$. Then the orbit of X is again determined by the eigenspaces $Q_1, \cdots, Q_k$. We form an increasing sequence ordered by inclusion as before

$(Q_1, Q_2, \cdots, Q_k) \mapsto Q_1 \subset Q_1 \oplus Q_2 \subset \cdots \subset Q_1 \oplus \cdots \oplus Q_k = \mathbb{C}^n.$

Let $F(n_1, n_2, \cdots, n_k)$ be the set of all such sequences. Then the orbit of X is diffeomorphic to the homogeneous space
$F(n_1, \cdots, n_k) = U(n)/(U(n_1) \times \cdots \times U(n_k))$ which as we have already seen is a generalized flag manifold. (See also [5, proposition II.1.15]).

5.2.1 Killing form

Definition 5.2.2 *Given any Lie algebra $\mathfrak{g}$, the Killing form of $\mathfrak{g}$ denoted by B, is a symmetric bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ given by*

$$B(X, Y) = tr(ad(X) \circ ad(Y)), \text{ for all } X, Y \in \mathfrak{g}$$

where tr is the trace of the composition.

We call B the Killing form of the Lie group G provided $\mathfrak{g}$ is the Lie algebra of G.

Remark 5.2.2 *If $\mathfrak{g}$ is the Lie algebra of the Lie group G, then the Killing form B is Ad-invariant. That is,*

$$B(X, Y) = B(Ad(g)X, Ad(g)Y)$$

for all $g \in G$ and $X, Y \in \mathfrak{g}$.

(See [3, proposition 2.10]).

By Cartan's criterion for semisimplicity, a finite dimensional Lie group G is said to be semisimple if its Killing form is non-degenerate. (See [3, p. 34]).

Proposition 5.2.1 *Let G be an n-dimensional semisimple Lie group. Then the center of its Lie algebra is trivial, that is $Z(\mathfrak{g}) = 0$.*

Proof. Let $X \in Z(\mathfrak{g})$, then for all $Y \in \mathfrak{g}$ we have $[X, Y] = 0$ since X commutes with every element of $\mathfrak{g}$. Thus

$$[X, Y] = ad_X(Y) = 0.$$

This shows that ad_X is a zero operator. But then we have

$$B(X, X) = tr(ad(X) \circ ad(X)) = 0.$$

Since B is non-degenerate we must have $X = 0$. This completes the proof. $\square$

For the following two theorems see ([3, pp. 34-35]).

Theorem 5.2.1 *Let G be an n-dimensional semisimple Lie group. If G is compact then its Killing form is negative definite.*

Theorem 5.2.2 *Let G be an n-dimensional connected Lie group. If the Killing form of G is negative definite on $\mathfrak{g}$, then G is compact and semisimple.*

5.3. Adjoint orbits as symplectic manifolds

We have seen that the adjoint orbits of flag manifolds are determined by the eigenspaces corresponding to a set of eigenvalues $i\lambda_1, \cdots, i\lambda_k$. Denote this set of eigenvalues by λ and the orbit determined by the corresponding eigenspaces by $H(\lambda)$. Let $G = U(n)$ be a Lie group and $\mathfrak{g} = \mathfrak{u}(n)$ its Lie algebra. First note that the dimension of orbit $H(\lambda)$ is $n^2 - n$ which is even.

For $X \in \mathfrak{g}$ we have seen that if $x(t) = \exp tX$ is a curve in G tangent to X at $t = 0$, then $\xi(t) = Ad_{x(t)}\xi = Ad_{\exp tX}\xi$ is a curve in $H(\lambda)$ passing through $\xi \in \mathfrak{u}(n)$. Then the tangent vector to this curve at $t = 0$ is given by

$$\xi'(t) = \tfrac{d}{dt} Ad_{\exp tX}\xi \,|_{t=0},$$

or

$$\xi'(0) = ad(X)\xi = [\xi, X].$$

We shall now construct a symplectic 2-form on the orbit $H(\lambda)$. Let h be an element of $\mathfrak{u}(n)$. Define a map

$$\omega_h : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$$

by

$$\omega_h(X, Y) = B(h, [X, Y]),$$

where B is the Killing form of $\mathfrak{g}$, the Lie algebra of G.

Proposition 5.3.1 *Let ω_h be as defined above. Then*

(i) ω_h *is skew symmetric bilinear form on* $\mathfrak{g} = \mathfrak{u}(n)$;

(ii) $\ker \omega_h = \{X \in \mathfrak{u}(n) : [h, X] = 0\}$;

(iii) ω_h *is G-invariant. That is, for each $g \in G$ we have*

$$\omega_{Ad(g)(h)}(Ad_g X, Ad_g Y) = \omega_h(X, Y).$$

Proof. Part (i) follows from the properties of the Lie bracket.

For part (ii) (see [2, p 19]). We prove part (iii).

$$\begin{aligned}
\omega_{Ad(g)(h)}(Ad_g X, Ad_g Y) &= B(Ad_g h, [Ad_g X, Ad_g Y]) \\
&= B(Ad_g h, [gXg^{-1}, gYg^{-1}]) \\
&= B(Ad_g h, \{gXYg^{-1} - gYXg^{-1}\}) \\
&= B(Ad_g h, g[X, Y]g^{-1}) \\
&= B(Ad_g h, Ad_g[X, Y]) \\
&= B(h, [X, Y]) \\
&= \omega_h(X, Y). \qquad \square
\end{aligned}$$

Now for $h \in \mathfrak{u}(n)$, we consider the orbit map

$$\begin{aligned}
\Phi_h \;:\; & U(n) \to \mathfrak{u}(n) \\
& g \mapsto ghg^{-1}.
\end{aligned}$$

That is

$$\Phi_h : U(n) \to H(\lambda) \subset \mathfrak{u}(n).$$

Then we have:

$$T_I\Phi_h : \mathfrak{u}(n) \to T_h H(\lambda).$$

But the tangent space on the orbit is generated by the vector field

$$ad(X)\xi = [X, \xi],$$

where $X, \xi \in \mathfrak{g}$. Define a 2-form Ω_h on $T_h H(\lambda)$ by the formula

$$\Omega_h([h, X], [h, Y]) = \omega_h(X, Y), \quad \text{for } X, Y \in \mathfrak{u}(n)$$

Proposition 5.3.2 *The Ω_h defined above is a closed and nondegenerate 2-form on the orbit $H(\lambda)$.*

Proof. First note that Ω_h does not depend on the choice of $X, Y \in \mathfrak{u}(n)$ since if $Z \in \ker \omega_h$ then we have:

$$\begin{aligned}
\Omega_h([h, X+Z], [h, Y+Z]) &= \omega_h(X+Z, Y+Z) = B(h, [X+Z, Y+Z]) \\
&= B(h, [X, Y] + [X, Z] + [Z, (Y+Z)]) \\
&= B(h, [X, Y]) + B(h, [X, Z]) + B(h, [Z, (Y+Z)]) \\
&= \omega_h(X, Y) + \omega_h(X, Z) + \omega_h(Z, (Y+Z)) \\
&= \omega_h(X, Y) \\
&= \Omega_h([h, X], [h, Y]).
\end{aligned}$$

Thus Ω_h is well defined. It is skew-symmetric bilinear form and $G-$invariant by the construction so it is smooth. Since the Killing form B is non-degenerate, Ω_h is non-degenerate. We only have to show that it is closed.

Let $X, Y, Z \in \mathfrak{u}(n)$. Then,

$$\begin{aligned}
d\Omega_h([h, X], [h, Y], [h, Z]) &= d\omega_h(X, Y, Z) \\
&= \{L_X\omega_h(Y, Z) - L_Y\omega_h(X, Z) + L_Z\omega_h(X, Y)\} \\
&+ \{\omega_h(X, [Y, Z]) - \omega_h(Y, [X, Z]) + \omega_h(Z, [X, Y])\}.
\end{aligned}$$

We now apply the Jacobi identity to each bracket given by the braces. The second bracket gives:

$$\begin{aligned} \omega_h(X,[Y,Z]) &- \omega_h(Y,[X,Z]) + \omega_h(Z,[X,Y]) \\ &= B(h,[X,[Y,Z]]) - B(h,[Y,[X,Z]]) + B(h,[Z,[X,Y]]) \\ &= B(h,[X,[Y,Z]] - [Y,[X,Z]] + [Z,[X,Y]]), \end{aligned}$$

and the term in the bracket is zero by the Jacobi identity since $\mathfrak{u}(n)$ is the Lie algebra of $U(n)$. To deal with the first bracket we have:

$$\begin{aligned} L_X\omega_h(Y,Z) &= \omega_h(Z,[X,Y]) - \omega_h(Y,[X,Z]) \\ L_Y\omega_h(X,Z) &= \omega_h(Z,[Y,X]) - \omega_h(X,[Y,Z]) \\ L_Z\omega_h(X,Y) &= \omega_h(Y,[Z,X]) - \omega_h(X,[Z,Y]). \end{aligned}$$

Substituting into the first bracket and simplifying gives:

$$\begin{aligned} L_X\omega_h(Y,Z) &- L_Y\omega_h(X,Z) + L_Z\omega_h(X,Y) \\ &= 2\left(\omega_h(X,[Y,Z]) + \omega_h(Y,[Z,X]) + \omega_h(Z,[X,Y])\right), \end{aligned}$$

which again vanishes by Jacobi identity.

Thus, $d\Omega_h = 0$ proving that Ω_h is indeed closed on the orbits of the adjoint action of the Lie group G on its Lie algebra $\mathfrak{g}$. □

5.4. Coadjoint orbits

We now describe briefly the orbits of the coadjoint action of a Lie group G on the dual of its Lie algebra. There are many references to this section such as [1] as well as [38].

Consider the Lie group G acting on itself by left translation $L_g : G \to G$, $h \mapsto gh$, for $g \in G$. This map is a diffeomorphism so by lifting of diffeomorphisms induces a symplectic action on its cotangent bundle

$$\begin{aligned} \Phi \;:\; & G \times T^*G \to T^*G \\ & (g,\alpha_h) \mapsto \Phi(g,\alpha_h) = L^*_{g^{-1}}(\alpha_h). \end{aligned}$$

This action has a momentum mapping which is equivariant with the coadjoint action. The momentum mapping of this action is given by

$$\begin{aligned}\mu \quad : \quad & T^*G \to \mathfrak{g}^* \\ & \mu(\alpha_g)\xi = \alpha_g(\xi_G(g)) = \alpha_g(R_g)_{*e}\xi = (R_g^*\alpha_g)\xi,\end{aligned}$$

for all $\xi \in \mathfrak{g}$.

That is, $\mu(\alpha_g) = R_g^*\alpha_g$. Every point $\beta \in \mathfrak{g}^*$ is a regular value of the momentum mapping μ (see [38, p 282]). So we have for each $\beta \in \mathfrak{g}^*$

$$\begin{aligned}\mu^{-1}(\beta) &= \{\alpha_g \in T^*G : \mu(\alpha_g) = \beta\} \\ &= \{\alpha_g \in T^*G : R_g^*\alpha_g\xi = \beta \cdot \xi \text{ for all } \xi \in \mathfrak{g}\}.\end{aligned}$$

In particular, $R_e^*\alpha_e\xi = \beta \cdot \xi$ implying that $\alpha_e = \beta$. Denote this 1-form by α_β so that

$$\alpha_\beta(e) = \beta. \tag{5.1}$$

For $g \in G$, applying the right translation $R_{g^{-1}}^*$ to Equation (5.1) gives a right-invariant 1-form on G

$$\alpha_\beta(g) = R_{g^{-1}}^*\beta. \tag{5.2}$$

But now for all $g \in G$ we have

$$\begin{aligned}\mu(\alpha_\beta(g)) &= \mu(\alpha_g) \\ &= R_g^*R_{g^{-1}}^*\beta = \beta.\end{aligned}$$

Thus, Equation (5.2) defines all and only points of $\mu^{-1}(\beta)$. Since the action is defined by

$$\Phi(g, \alpha_h) = L_{g^{-1}}^*(\alpha_h),$$

the isotropy subgroup of β is

$$G_\beta = \{g \in G : L_{g^{-1}}^*(\alpha_\beta) = \beta\}.$$

From the map

$$L_{g^{-1}}^* : (h, \alpha_\beta(h)) \longrightarrow (gh, \alpha_\beta(gh))$$

we see that G_β acts on $\mu^{-1}(\beta)$ by left translation on the base points. This action is proper (see [38, p 283]). Since β is also a regular value of the momentum mapping μ, then $\mu^{-1}(\beta)/G_\beta$ is a symplectic manifold. There is a diffeomorphism

$$\mu^{-1}(\beta)/G_\beta \simeq G\cdot\beta = \{Ad^*_{g^{-1}}\beta : g \in G\} \subset \mathfrak{g}^* \quad \text{(see [38, p 284])},$$

of the reduced space $\mu^{-1}(\beta)/G_\beta$ onto the coadjoint orbit of $\beta \in \mathfrak{g}^*$. Thus the coadjoint orbit $G\cdot\beta$ is a symplectic manifold. The symplectic 2-form is given by the Kirillov-Kostant-Souriau form

$$\omega_\beta(\nu)(\xi_{\mathfrak{g}^*}(\nu), \eta_{\mathfrak{g}^*}(\nu)) = -\nu\cdot[\xi,\eta] \text{ (see [1, pp 302-303])},$$

where $\xi, \eta \in \mathfrak{g}$ and $\nu \in \mathfrak{g}^*$.

If G is semisimple, it is known that in this case, $H^1(\mathfrak{g}, \mathbb{R}) = 0$. (see [2, p 19]). Thus if ω is closed then it is exact. So, there is a 1-form $\alpha \in \mathfrak{g}^*$ such that $d\alpha = \omega$ where $\mathfrak{g}^*$ is the dual to the Lie algebra of G. The 1-form α satisfies $d\alpha(X,Y) = \alpha([X,Y])$.

Thus if the Lie group G is semisimple, compact and connected, then we have the relation:

$$\alpha([X,Y]) = d\alpha(X,Y) = \omega(X,Y) = B([\xi,X],Y) = B(\xi,[X,Y]), \tag{5.3}$$

where $\alpha \in \mathfrak{g}^*$,$\omega$ a 2-form on the homogeneous space G/H, B the Killing form on G/H and $\xi, X, Y \in \mathfrak{g}$, the Lie algebra of G. Note that the first term in equation (5.3), is the 2-form on the coadjoint orbit while the last term is the 2-form on the adjoint orbit.

5.5. Adjoint and coadjoint orbits are symplectomorphic homogeneous spaces

We shall now show that the adjoint orbit is diffeomorphic to the coadjoint orbit and that the diffeomorphim between them is actually a symplectic morphism. As such, we can use this map to pull back the symplectic structure on the coadjoint orbit to the adjoint orbit. This will provide another proof that the adjoint orbit is a symplectic homogeneous space. (See also [8]).

In this section we make a general assumption that the Lie group G acts transitively on its Lie algebra $\mathfrak{g}$ by the adjoint action, and also acts transitively on the dual $\mathfrak{g}^*$ by the coadjoint action.

Theorem 5.5.1 *Let $Ad : G \times \mathfrak{g} \to \mathfrak{g}$ be an adjoint action of an n-dimensional semisimple, compact, connected Lie group G on its Lie algebra $\mathfrak{g} \cong T_eG$. Let $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$.Then there is an Ad^*-equivariant isomorphism $B^\flat : \mathfrak{g} \to \mathfrak{g}^*$.*

Proof. Let

$$B^\flat : \mathfrak{g} \to \mathfrak{g}^*; (X \mapsto B^\flat(X) : \mathfrak{g} \to \mathbb{R}) \text{ such that } Y \mapsto B^\flat(X)Y := B(X,Y),$$

where B is the Killing form. Then $B^\flat$ is linear since of for all $X, Y, Z \in \mathfrak{g}$ and using the fact that the Killing form B is bilinear, we have:

$$\begin{aligned} B^\flat(aX + bY)Z &= B(aX + bY, Z) \\ &= aB(X,Z) + bB(Y,Z) \\ &= aB^\flat(X)Z + bB^\flat(Y)Z \\ &= (aB^\flat(X) + bB^\flat(Y))Z. \end{aligned}$$

Thus

$$B^\flat(aX + bY) = aB^\flat(X) + bB^\flat(Y).$$

To see that $B^\flat$ is injective let $B^\flat(X) = B^\flat(Y)$. Then for all $Z \in \mathfrak{g}$,

$$B^\flat(X)Z = B^\flat(Y)Z \Rightarrow B(X,Z) = B(Y,Z) \Rightarrow B(X - Y, Z) = 0,$$

and since the Killing form is non degenerate, we get

$$X = Y.$$

To see that the map is surjective first note that G is finite dimensional Lie group, we have $\dim \mathfrak{g}^* \leq \dim G = \dim \mathfrak{g}$. But $B^\flat$ is injective so that
$\dim \mathfrak{g} \leq \dim \mathfrak{g}^*$. We have $\ker B^\flat = \{0\}$ implying that $\dim \ker B^\flat = 0$. But $\dim \ker B^\flat + \mathrm{Rank} B^\flat = \dim \mathfrak{g}$, so we must have
$\dim \mathfrak{g}^* = \dim \mathrm{Im} B^\flat = \mathrm{Rank} B^\flat = \dim \mathfrak{g}$. This shows that the map $B^\flat$ is surjective.

To show that $B^\flat : \mathfrak{g} \to \mathfrak{g}^*$ is equivariant with respect to the adjoint action of G on $\mathfrak{g}$ and the coadjoint action of G on $\mathfrak{g}^*$ define a map

$$u \ : \ G \times \mathfrak{g} \to G \times \mathfrak{g}^*; \ (g, X) \mapsto (g, B^\flat X), \quad \text{for all } X \in \mathfrak{g} \text{ and } g \in G,$$

where the map u is defined by

$$u := Id_G \times B^\flat.$$

We must then show that the following diagram commutes:

$$\begin{tikzcd} G \times \mathfrak{g} \arrow[r, "u"] \arrow[d, "Ad_g"'] & G \times \mathfrak{g}^* \arrow[d, "Ad^*"] \\ \mathfrak{g} \arrow[r, "B^\flat"] & \mathfrak{g}^* \end{tikzcd}$$

In effect let $(g, X) \in G \times \mathfrak{g}$, then for all $Y \in \mathfrak{g}$, and using the natural pairing we have

$$\begin{aligned} B^\flat(Ad_g X)Y &= B(Ad_g X, Y) \\ &= B(Ad_{g^{-1}} \circ Ad_g X, Ad_{g^{-1}} Y) \\ &= B(X, Ad_{g^{-1}} Y) \\ &= B^\flat(X)(Ad_{g^{-1}} Y) \\ &= Ad_g^* B^\flat(X)(Y). \end{aligned}$$

The second and the third equalities is because the Killing form B is Ad-invariant. The last equality is by definition of Ad^*, see definition 2.1.15.

That is,

$$B^\flat(Ad_g X) = Ad_g^* B^\flat X,$$

and the above diagram commute as we required.

This gives the equivariance relation

$$B^\flat \circ Ad_g = Ad_g^* \circ B^\flat. \tag{5.4}$$

This completes the proof of the theorem. $\square$

Let

$$\pi_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}/G$$

and

$$\pi_{\mathfrak{g}^*} : \mathfrak{g}^* \to \mathfrak{g}^*/G,$$

be the projection maps into the respective orbit spaces. Then, (see [32, p 10]) and theorem 2.1.3, there is at most one manifold structure on $\mathfrak{g}/G$ respectively on $(\mathfrak{g}^*/G)$ such that $\pi_{\mathfrak{g}}$ respectively $(\pi_{\mathfrak{g}^*})$ are submersions. In fact note for example that the rank of $d\pi_{\mathfrak{g}}$ is equal to the dimension of its image and since $\dim \mathfrak{g}/G \leq \dim \mathfrak{g}$ then $\pi_{\mathfrak{g}}$ is a submersion. Since

$$B^\flat : \mathfrak{g} \to \mathfrak{g}^*$$

is equivariant, and $\pi_{\mathfrak{g}}$ and $\pi_{\mathfrak{g}^*}$ are submersions, the criterion of passage to quotients (see [1, p 264]) implies that there is an induced unique map

$$\begin{aligned} \hat{B^\flat} \quad : \quad & \mathfrak{g}/G \to \mathfrak{g}^*/G, \\ & \hat{B^\flat}[X] = [\alpha] := [B^\flat(X)], \end{aligned}$$

where $[X]$ is adjoint orbit through X and $[\alpha] := [B^\flat(X)]$ the corresponding coadjoint orbit through $B^\flat(X) = \alpha$. This gives the following commutative diagram:

$$\begin{tikzcd} G \times \mathfrak{g} \arrow[r, "u"] \arrow[d, "Ad_g"'] & G \times \mathfrak{g}^* \arrow[d, "Ad^*"] \\ \mathfrak{g} \arrow[r, "B^\flat"] \arrow[d, "\pi_{\mathfrak{g}}"'] & \mathfrak{g}^* \arrow[d, "\pi_{\mathfrak{g}^*}"] \\ \mathfrak{g}/G \arrow[r, "\hat{B^\flat}"] & \mathfrak{g}^*/G \end{tikzcd}$$

$$\mu := (Id_G \times B^\flat).$$

Theorem 5.5.2 *Let G be a compact, connected semisimple Lie group. Let $\mathfrak{g}$ be its Lie algebra and $\mathfrak{g}^*$ the dual of $\mathfrak{g}$. Assume further that G acts transitively on $\mathfrak{g}$ by the adjoint action and transitively on $\mathfrak{g}^*$ by the coadjoint action. Let $B^\flat$ be as in theorem 5.5.1 and let $\hat{B^\flat} : \mathfrak{g}/G \to \mathfrak{g}^*/G$ be the map induced by passage to quotients as described above between adjoint and coadjoint orbit spaces. Then, the map $\hat{B^\flat}$ is a symplectic diffeomorphism.*

Proof. The map $\hat{B^\flat}$ is well defined because of [37, proposition 1.3.5]. To show that $\hat{B^\flat}$ is injective note first that the following diagram commute:

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "B^\flat"] \arrow[d, "\pi_{\mathfrak{g}}"'] & \mathfrak{g}^* \arrow[d, "\pi_{\mathfrak{g}^*}"] \\
\mathfrak{g}/G \arrow[r, "\hat{B^\flat}"] & \mathfrak{g}^*/G
\end{tikzcd}$$

The commuting of this diagram is now a consequence of the fact that $B^\flat$ is both an isomorphism and is equivariant with respect to the adjoint action and the coadjoint action. That is,

$$B^\flat \circ Ad_g(X) = Ad_g^* \circ B^\flat(X),$$

for all $X \in \mathfrak{g}$ and for all $g \in G$. If we fix $X \in \mathfrak{g}$ and let g run through all the elements of G then on the left we get all the elements in the orbit through X while on the right we get all the elements in the orbit through $B^\flat(X) = \alpha$. Consequently, we must have

$$\hat{B^\flat} \circ \pi_{\mathfrak{g}}(X) = \pi_{\mathfrak{g}^*} \circ B^\flat(X),$$

for all $X \in \mathfrak{g}$.

We can now show that $\hat{B^\flat}$ is injective. The commuting of the above diagram says that

$$\hat{B^\flat} \circ \pi_{\mathfrak{g}} = \pi_{\mathfrak{g}^*} \circ B^\flat. \tag{5.5}$$

Suppose

$$\hat{B^\flat}([X]) = \hat{B^\flat}([Y]),$$

then

$$\pi_{\mathfrak{g}^*} \circ B^\flat(X) = \pi_{\mathfrak{g}^*} \circ B^\flat(Y),$$

so that

$$[B^\flat(X)] = [B^\flat(Y)].$$

This implies that $B^\flat(Y) \in [B^\flat(X)]$. Thus

$$B^\flat(Y) = Ad_g^* B^\flat(X),$$

for some $g \in G$, so that

$$B^\flat(Y) = B^\flat(Ad_g(X)),$$

by equivariance of $B^\flat$. But $B^\flat$ is an isomorphism, so we must have

$$Y = Ad_g(X),$$

and it follows that

$$Y \in [X].$$

But this means that

$$Y \in [X] \cap [Y].$$

Since equivalence classes are disjoint, we must have

$$[X] = [Y],$$

and $\hat{B^\flat}$ is injective. From the relation $\hat{B^\flat} \circ \pi_{\mathfrak{g}} = \pi_{\mathfrak{g}^*} \circ B^\flat$, the right hand side is a composition of smooth map and on the left $\pi_{\mathfrak{g}}$ is smooth, this then implies that $\hat{B^\flat}$ must be a smooth map.

To show that $\hat{B^\flat}$ is a surjective map consider the following commutative diagram:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "B^\flat"] \arrow[d, "\pi_{\mathfrak{g}}"'] \arrow[dr, "\varphi"] & \mathfrak{g}^* \arrow[d, "\pi_{\mathfrak{g}^*}"] \\ \mathfrak{g}/G \arrow[r, "\hat{B^\flat}"] & \mathfrak{g}^*/G \end{tikzcd}$$

We have $\varphi = \pi_{\mathfrak{g}^*} \circ B^\flat$. But the right hand side is surjective since $B^\flat$ is an isomorphism hence bijective and $\pi_{\mathfrak{g}^*}$ is the projection which is surjective, this shows that

$$\varphi : \mathfrak{g} \to \mathfrak{g}^*/G,\, X \mapsto [B^\flat(X)],$$

is surjective. But $\hat{B^\flat}$ is the factorization of φ through $\mathfrak{g}/G$,(see also [37, pp 15-16]), that is, $\varphi = \hat{B^\flat} \circ \pi_{\mathfrak{g}}$. Therefore, for any $[B^\flat(X)] \in \mathfrak{g}^*/G$ there is $X \in \mathfrak{g}$ such that

$$\varphi(X) = [B^\flat(X)].$$

This gives

$$\varphi(X) = \hat{B^\flat}(\pi_{\mathfrak{g}}(X)) = \hat{B^\flat}([X]) = [B^\flat(X)].$$

Thus, for each $[B^\flat(X)] \in \mathfrak{g}^*/G$ there is $[X] \in \mathfrak{g}/G$ such that

$$\hat{B^\flat}([X]) = [B^\flat(X)],$$

which shows that $\hat{B^\flat}$ is bijective so that its inverse $(\hat{B^\flat})^{-1}$ exists. We must show that the inverse is smooth. But now

$$(\hat{B^\flat})^{-1} \circ \pi_{\mathfrak{g}^*} \circ B^\flat = \pi_{\mathfrak{g}},$$

and since $\pi_{\mathfrak{g}}$ is smooth and the other two maps on the left are smooth, this forces $(\hat{B^\flat})^{-1}$ to be smooth. Therefore, $\hat{B^\flat}$ is a diffeomorphism. Denote by O_X the orbit $[X]$ and by $O_{B^\flat(X)}$ the orbit $[B^\flat(X)]$.

Let O_X be the adjoint orbit through $X \in \mathfrak{g}$. First notice that each element in O_X is of the form gX for some $g \in G$. Now, for any two points $y = hX$ and $z = gX$ in O_X, define a set map f_X on O_X, as follows:

$$f_X : O_X \to O_X;\ y \mapsto f_X(y) = (gh^{-1})y = z.$$

Then, f_X maps all points of O_X into points of O_X. Since G is a Lie group and gh^{-1} is smooth for all $g, h \in G$, the map f_X is smooth with smooth inverse

$$f_X^{-1} = hg^{-1}.$$

In a similar way, define a set map k_α on the coadjoint orbit $O_{B^\flat(X)} = O_\alpha$ corresponding to the adjoint orbit O_X. That is,

$$k_\alpha : O_\alpha \to O_\alpha;\ \beta \mapsto k_\alpha(\beta) = (rs^{-1})\beta = \gamma,$$

where $\alpha = B^\flat(X), \beta = s\alpha, \gamma = r\alpha$ and $r, s \in G$. Let $\hat{B^\flat}_X$ be the restriction of $\hat{B^\flat}$ to a small neighborhood of the point O_X. Then,

$$k_\alpha \circ \hat{B^\flat_X} \circ f_X^{-1} : O_X \to O_{B^\flat(X)} = O_\alpha, \tag{5.6}$$

maps points of O_X into points of $O_{B^\flat(X)} = O_\alpha$ and it is smooth since it is a composition of smooth maps. It is known that coadjoint orbits are symplectic manifolds with the two form $\hat{\omega}$, called the Kirillov-Kostant-Souriau (KKS) form. Notice that the orbit $O_{B^\flat(X)} = O_\alpha$ is symplectic since it is a coadjoint orbit. Let the KKS form be the two form on $O_{B^\flat(X)} = O_\alpha$, then for all $Y, Z \in \mathfrak{g}$ and $r, s \in G$ we have:

$$\begin{aligned} k_\alpha^*\hat{\omega}(Y,Z) &= \hat{\omega}(k_{\alpha*}Y, k_{\alpha*}Z) \\ &= \hat{\omega}\left((rs^{-1})_*Y, (rs^{-1})_*Z\right) \\ &= \hat{\omega}\left(r_*(s_*^{-1}Y), r_*(s_*^{-1}Z)\right) \\ &= \hat{\omega}(r_*Y, r_*Z) \\ &= \hat{\omega}(Y,Z), \end{aligned}$$

since $Y, Z \in \mathfrak{g}$ are left invariant. Thus, $k_\alpha^*\hat{\omega} = \hat{\omega}$. By similar calculations, for any 2-form $\hat{\Omega}$ on the adjoint orbit O_X we must have $f_X^*\hat{\Omega} = \hat{\Omega}$.

Consider now the pull back of the form $\hat{\omega}$ by the map in equation (5.6),

$$\left(k_\alpha \circ \hat{B^\flat_X} \circ f_X^{-1}\right)^* \hat{\omega}.$$

We have

$$\begin{aligned}\left(k_\alpha \circ \hat{B^\flat_X} \circ f_X^{-1}\right)^* \hat{\omega} &= (f_X^{-1})^* \circ (\hat{B^\flat_X})^* \circ k_\alpha^* \hat{\omega} \\ &= (f_X^{-1})^* \circ (\hat{B^\flat_X})^* \hat{\omega}.\end{aligned}$$

We now consider the 2-form $(\hat{B^\flat_X})^*\hat{\omega}$ induced by the map $\hat{B^\flat_X}$. We check if the form $(\hat{B^\flat_X})^*\hat{\omega}$ is symplectic. First we have

$$d\hat{B^{\flat *}_X}\hat{\omega} = (\hat{B^\flat_X})^* d\hat{\omega} = 0,$$

since $\hat{\omega}$ is closed. Thus, the 2-form $(\hat{B^\flat_X})^*\hat{\omega}$ is closed. To show non degeneracy, let

$$(\hat{B^\flat_X})^*\hat{\omega}(Y, Z) = 0, \quad \text{for all } Z \in \mathfrak{g},$$

then

$$\hat{\omega}(d\hat{B^\flat}_X(Y), d\hat{B^\flat}_X(Z)) = 0, \quad \text{for all } Z \in \mathfrak{g}.$$

But now since $\hat{\omega}$ is symplectic, $\hat{\omega}(d\hat{B^\flat}_X(Y), d\hat{B^\flat}_X(Z)) = 0$, for all $Z \in \mathfrak{g}$ implies that

$$d\hat{B^\flat}_X(Y) = 0.$$

Thus, since $d\hat{B^\flat}$ is a linear isomorphism,

$$d\hat{B^\flat}_X(Y) = 0 \Rightarrow Y \in \ker d\hat{B^\flat} = \{0\},$$

which gives

$$Y = 0.$$

Thus, $(\hat{B^\flat_X})^*\hat{\omega}(Y,Z) = 0$, for all $Z \in \mathfrak{g}$ implies that $Y = 0$ and $(\hat{B^\flat})^*_X\hat{\omega}$ is non degenerate.

But now the orbit space $\mathfrak{g}/G$ is a single orbit O_X since the action of G on $\mathfrak{g}$ is transitive, and the orbit space $\mathfrak{g}^*/G$ consists of a single orbit O_α since the action of G on $\mathfrak{g}^*$ is transitive. Thus $\hat{B^\flat_X}^*\hat{\omega} = \hat{B^\flat}^*\hat{\omega}$ is a symplectic form on O_X. This proves that $\hat{B^\flat}$ is a symplectic map. $\square$

The existence of adjoint orbits that support a symplectic structure is now not in question. For a connected compact and semi simple Lie group G and a stabilizer of an element $h_0 \in \mathfrak{g}$, the Lie algebra of G, Alekseevsky have described homogeneous spaces G/K which admit an invariant symplectic structure ω. (See [2, p 19]).

6
Applications to Hamiltonian mechanics

We now turn to the discussion of Hamiltonian mechanics which centers mainly around a real valued function, usually denoted by H and called the Hamiltonian function, or the energy function. The smooth real valued function H can be used to define a Hamiltonian system on a symplectic manifold (M, ω). Because the Hamiltonian functions play a fundamental role in Hamiltonian mechanics, we shall introduce a more general structure than the symplectic structure, the Poisson structure. The Poisson structure gives a Lie algebra structure to vector space of smooth functions on the manifold.

6.1. Poisson algebra on a symplectic manifold

There are several ways to introduce a Poisson structure. However, we will be more concerned about the Poisson structure which is induced by the symplectic structure. For this reason, we will fix a symplectic manifold (M, ω) and then introduce the Poisson bracket of 1-forms first before we introduce the Poisson bracket of smooth functions. We will extend the discussion of Hamiltonian systems using a deformed Poisson bracket.

Let (M, ω) be a symplectic manifold, $X \in \mathfrak{X}(M)$ and $\omega \in \Omega^2(M)$. We define the inner (interior) product of X and ω by $i_X\omega(Y) = \omega(X, Y)$ for all $Y \in \mathfrak{X}(M)$. Other notation for $i_X\omega$ is $\omega^\flat(X)$. That is, $i_X\omega(Y) = \omega^\flat(X)Y = \omega(X, Y)$. Since $i_X\omega : \mathfrak{X}(M) \to \mathbb{R}$, we see that $i_X\omega = \omega^\flat(X) \in \Omega^1(M)$. First note that $\omega^\flat$ is linear. That is, if $X, Y, Z \in \mathfrak{X}(M)$, then

$$\begin{aligned} i_{X+Y}\omega(Z) &= \omega(X+Y, Z) \\ &= \omega(X, Z) + \omega(Y, Z) \\ &= i_X\omega(Z) + i_Y\omega(Z). \end{aligned}$$

We also have that if (M, ω) is a symplectic manifold with ω given in symplectic coordinates by $\omega = dx \wedge dy$, then for a vector field $X \in \mathfrak{X}(M)$, the interior product of a 2-form ω by X is given by

$$i_X\omega = (i_X dx) \wedge dy - dx \wedge (i_X dy).$$

Secondly, since ω is non degenerate, the map $\omega^\flat : \mathfrak{X}(M) \to \Omega^1(M)$ is injective and it is also surjective by Proposition (3.2.1), hence

$$\begin{aligned} \omega^\flat : \mathfrak{X}(M) &\to \Omega^1(M) \\ X &\mapsto i_X\omega = \alpha, \end{aligned}$$

is an isomorphism [1, p 162], with the inverse

$$\begin{aligned} (\omega^\flat)^{-1} : \Omega^1(M) &\to \mathfrak{X}(M) \\ \alpha &\mapsto (\omega^\flat)^{-1}(\alpha) = X_\alpha. \end{aligned}$$

6.1.1 Poisson algebra of 1-forms

Definition 6.1.1 *Let $\alpha, \beta \in \Omega^1(M)$. Then, the Poisson bracket of α and β is a 1-form on M given by*

$$\{\alpha, \beta\} = -i_{[X_\alpha, Y_\beta]}\omega,$$

where $[X_\alpha, Y_\beta] = \lim_{t\to 0} \frac{1}{t}(Y_\beta - d\Phi_t(Y_\beta))$ is the Lie bracket, and $\{\Phi_t\}$ is the flow of X_α satisfying the property;

$$L_{[X,Y]} = L_X L_Y - L_Y L_X.$$

So the module of forms $\Omega^1(M)$ provided with the Poisson bracket $\{\cdot, \cdot\}$, is a Lie algebra on $\mathbb{R}$ by the structure induced by that of $(\mathfrak{X}(M), [,])$, the Lie algebra of vector fields. It is denoted by $(\Omega^1(M), \{\cdot, \cdot\})$.

Proposition 6.1.1 *Let $\alpha, \beta \in \Omega^1(M)$. Then*

(i) $\{\alpha, \beta\} = -L_{X_\alpha}(\beta) + L_{X_\beta}(\alpha) + d(i_{X_\alpha} \circ i_{X_\beta}\omega)$

(ii) If α and β are closed, then $\{\alpha, \beta\}$ is exact. This leads to the conclusion that the set of all closed 1-forms on M is a Lie subalgebra of the Lie algebra $\Omega^1(M)$.

Proof. See [11, Theorem 3.11 p 79].

Proposition 6.1.2 $\omega^\flat : (\mathfrak{X}(M), [,]) \to (\Omega^1(M), \{\cdot, \cdot\})$ *is*

antimorphism of Lie algebras. That is, $\omega^\flat([X, Y]) = -\{\omega^\flat(X), \omega^\flat(Y)\}$

Proof. See [11, Definition 3.10 p 79].

6.1.2 Poisson algebra of smooth functions

Let $f, g \in C^\infty(M) \equiv \Omega^0(M)$, then $df, dg \in \Omega^1(M)$. It follows from $(\omega^\flat)^{-1}$ above that there are vector fields X_f and X_g such that $df \mapsto X_f$ and $dg \mapsto X_g$. Then by $\omega^\flat : \mathfrak{X}(M) \to \Omega^1(M)$, $X_f \mapsto df = i_{X_f}\omega$ and $X_g \mapsto dg = i_{X_g}\omega$.

Definition 6.1.2 *Let f and g be smooth functions on a Poisson manifold M, then, the Poisson bracket of the functions f and g is defined by*

$$\{f, g\} = -i_{X_f} \circ i_{X_g}\omega \in \Omega^0(M).$$

Note that we have $\{f, g\} = -i_{X_f} \circ i_{X_g}\omega = i_{X_g} \circ i_{X_f}\omega$.

Proposition 6.1.3 *(i)* $\{f, g\} = -\mathfrak{L}_{X_f}(g) = \mathfrak{L}_{X_g}(f)$;

(ii) $d\{f, g\} = \{df, dg\}$;

(iii) $X_{\{f,g\}} = -[X_f, X_g]$.

Proof. See [11, pp 80-83].

Proposition 6.1.4 *In a local symplectic chart (U,φ) with local coordinates $(x_1,\cdots,x_n,y_1,\cdots,y_n)$, the Poisson bracket of functions f and g is given by*

$$\{f,g\} = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial y^i} - \frac{\partial f}{\partial y^i}\frac{\partial g}{\partial x^i}\right)$$

Proof. We have $\{f,g\} = -i_{X_f}\circ i_{X_g}\omega$ where $\omega|_U = \sum_{i=1}^{n} dx^i \wedge dy^i$ by Darboux theorem, see theorem 3.2.1.

We also have that $X_f = \sum_{i=1}^{n}\left(X^i\frac{\partial}{\partial x^i} + Y^i\frac{\partial}{\partial y^i}\right)$. We shall find the X^i and Y^i according to f which distinguishes it from X_g.

Now $i_{X_f}\omega = \omega^\flat(X_f) = df$ so that first we have

$$i_{X_f}\omega = i_{\sum_{i=1}^{n}\left(X^i\frac{\partial}{\partial x^i} + Y^i\frac{\partial}{\partial y^i}\right)} \sum_{i=1}^{n} dx^i \wedge dy^i, \tag{6.1}$$

but we also have

$$df = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x^i}dx^i + \frac{\partial f}{\partial y^i}dy^i\right). \tag{6.2}$$

A typical term in equation (6.1) is given by

$$i_{X^i\frac{\partial}{\partial x^i} + Y^i\frac{\partial}{\partial y^i}} dx^i \wedge dy^i. \tag{6.3}$$

Simplifying expression (6.3) we get

$$\begin{aligned}
i_{X^i\frac{\partial}{\partial x^i} + Y^i\frac{\partial}{\partial y^i}} dx^i \wedge dy^i &= i_{X^i\frac{\partial}{\partial x^i}} dx^i \wedge dy^i + i_{Y^i\frac{\partial}{\partial y^i}} dx^i \wedge dy^i \\
&= X^i dx^i\left(\tfrac{\partial}{\partial x^i}\right) \wedge dy^i - dx^i \wedge X^i dy^i\left(\tfrac{\partial}{\partial x^i}\right) \\
&\quad + Y^i dx^i\left(\tfrac{\partial}{\partial y^i}\right) \wedge dy^i - dx^i \wedge Y^i dy^i\left(\tfrac{\partial}{\partial y^i}\right) \\
&= X^i dy^i - Y^i dx^i.
\end{aligned}$$

Thus

$$i_{X_f}\omega = \sum_{i=1}^{n} X^i dy^i - Y^i dx^i. \tag{6.4}$$

Equating equation(6.2) to equation(6.4) we get

$$X^i = \frac{\partial f}{\partial y^i} \quad \text{and} \quad Y^i = -\frac{\partial f}{\partial x^i}.$$

This gives the vector field X_f in terms of f as

$$X_f = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial y^i} \frac{\partial}{\partial x^i} - \frac{\partial f}{\partial x^i} \frac{\partial}{\partial y^i} \right). \tag{6.5}$$

Similarly,

$$X_g = \sum_{i=1}^{n} \left(\frac{\partial g}{\partial y^i} \frac{\partial}{\partial x^i} - \frac{\partial g}{\partial x^i} \frac{\partial}{\partial y^i} \right). \tag{6.6}$$

But $\{f, g\} = -i_{X_f} \circ i_{X_g} \omega$. Computing the right hand side and using equation 6.5 and equation 6.6 we have;

$$\begin{aligned}
-i_{X_f} i_{X_g} \omega &= -i_{X_f} \left(i_{\frac{\partial g}{\partial y^i} \frac{\partial}{\partial x^i} - \frac{\partial g}{\partial x^i} \frac{\partial}{\partial y^i}} dx^i \wedge dy^i \right) \\
&= -i_{X_f} \left(i_{\frac{\partial g}{\partial y^i} \frac{\partial}{\partial x^i}} dx^i \wedge dy^i - i_{\frac{\partial g}{\partial x^i} \frac{\partial}{\partial y^i}} dx^i \wedge dy^i \right) \\
&= -i_{X_f} \left(\frac{\partial g}{\partial y^i} dy^i + \frac{\partial g}{\partial x^i} dx^i \right) \\
&= -i_{\frac{\partial f}{\partial y^i} \frac{\partial}{\partial x^i} - \frac{\partial f}{\partial x^i} \frac{\partial}{\partial y^i}} \left(\frac{\partial g}{\partial y^i} dy^i + \frac{\partial g}{\partial x^i} dx^i \right) \\
&= -\left(i_{\frac{\partial f}{\partial y^i} \frac{\partial}{\partial x^i}} \left(\frac{\partial g}{\partial y^i} dy^i + \frac{\partial g}{\partial x^i} dx^i \right) - i_{\frac{\partial f}{\partial x^i} \frac{\partial}{\partial y^i}} \left(\frac{\partial g}{\partial y^i} dy^i + \frac{\partial g}{\partial x^i} dx^i \right) \right) \\
&= -\left(\frac{\partial f}{\partial y^i} \frac{\partial g}{\partial x^i} - \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial y^i} \right) \\
&= \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial y^i} - \frac{\partial f}{\partial y^i} \frac{\partial g}{\partial x^i},
\end{aligned}$$

where summation is understood in the above calculations.

Therefore, the Poisson bracket is given by

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x^i} \frac{\partial g}{\partial y^i} - \frac{\partial f}{\partial y^i} \frac{\partial g}{\partial x^i} \right). \tag{6.7}$$

This proves the theorem. $\square$

Definition 6.1.3 *Let M be a finite dimensional C^∞-manifold. We define a C^∞ Poisson structure on M to be an $\mathbb{R}$-bilinear skew-symmetric map*

$$C^\infty(M) \times C^\infty(M) \to C^\infty(M),\ (f,g) \mapsto \{f,g\}$$

on the space of smooth functions on M which satisfies the following two identities:

(i) $\{fg,h\} = \{f,h\}g + f\{g,h\}$, *the Leibniz identity;*

(ii) $\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0$, *the Jacobi identity,*

for all $f,g,h \in C^\infty(M)$.

The bracket $\{\cdot,\cdot\}$ is called the Poisson bracket. The space of smooth functions on M, $C^\infty(M)$ equipped with the Poisson bracket $\{\cdot,\cdot\}$, is a Lie algebra which satisfies the Leibniz identity. The manifold M equipped with the Poisson bracket $\{\cdot,\cdot\}$ is called a Poisson manifold, and is denoted by $(M,\{\cdot,\cdot\})$. For the reasons that will appear in the next section, this bracket will be called the canonical or standard Poisson bracket.

6.2. Hamiltonian systems with deformed Poisson bracket

In our discussion of results on conservation laws, we would like to extend such results using a deformed Poisson bracket which looks more general than the canonical Poisson bracket.

The philosophy of deformations on algebras of functions over a Poisson manifold with Poisson bracket is largely attributed to Flato in the 70's ([21, p3]). Motivated by the potential of having a large number of applications in mathematical physics, he looked at deformations of infinite-dimensional Lie algebras of functions with Poisson bracket on symplectic manifolds. The theory of deformations inspired deformation quantization in physics. It was suggested that quantization should be understood as a deformation of the structure of algebra of classical observables rather than a radical change in the nature of observables ([36, p3]).

According to Flato, a formal deformation of the Lie algebra of smooth functions on a Poisson manifold M, is a new Lie algebra law

$$[f,g]_\lambda = \sum_{r=0}^{\infty} \lambda^r C_r(f,g)$$

where $C_r(f,g)$ are 2-cochains on $C^\infty(M,\mathbb{R})$ with $C_0(f,g) = \{f,g\}$ ([19]).

For more readings on the genesis of deformation theory, (see [21] and [19]).

Deformation of Poisson bracket has also come about as classical limit of deformed Heisenberg algebras.(See [16]). It is used in performing transition from the phase space of classical observables, such as functions depending on positions and momentums to the Hilbert space of physically well-defined Hermitian operator, ([22]). For applications to a Hamiltonian operator for the harmonic oscillator system see ([22]).

We have taken a general form of a deformed Poisson bracket on a symplectic manifold with canonical coordinates q and p to give the mathematical formalism of the deformed Poisson bracket.

Definition 6.2.1 *Let f and g be smooth functions on a $2n$-dimensional symplectic manifold M. Let (q_j, p_j), $j = 1, \cdots, n$ be the canonical coordinates and let $k(p)$ be smooth a function of the momentum variable p. We define a generalized deformed bracket of f and g by*

$$\{f,g\}_p = \{f,g\} + k(p)\{f,g\}, \tag{6.8}$$

where $f, g \in C^\infty(M)$ and $\{f,g\}$ is the canonical Poisson bracket defined by

$$\{f,g\} = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j} \right). \tag{6.9}$$

We shall write this bracket in short by

$$\{f,g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q},$$

since operations on the right of equation (6.9) can be done term by term. We shall then write the bracket of the deformed Poisson bracket as

$$\{f,g\}_p = \left(\frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} \right)_p. \tag{6.10}$$

Proposition 6.2.1 *Let M be a smooth manifold with the Poisson bracket defined by the equation 6.10, then for all $f, g, h \in C^\infty(M)$ we have;*

(a) $\{f, g\}_p$ *is bilinear in* f *and* g;

(b) $\{f, g\}_p$ = - $\{g, f\}_p$;

(c) $\{fg, h\}_p = f\{g, h\}_p + \{f, h\}_p g$;

(d) $\{\{f, g\}_p, h\}_p + \{\{h, f\}_p, g\}_p + \{\{g, h\}_p, f\}_p = 0.$

Proof. The proof of (a) and (b) is straight forward from the definition (6.2.1). We shall prove (c), the Leibniz identity and (d) the Jacobi identity.

(c) To show that $\{fg, h\}_p = f\{g, h\}_p + \{f, h\}_p g$. We have

$$\{fg, h\}_p = \{fg, h\} + k(p)\{fg, h\}.$$

The right hand side consists of canonical Poisson brackets which satisfies the Leibniz rule. This proves (c).

(d) We now prove the Jacobi identity

$\{\{f, g\}_p, h\}_p + \{\{h, f\}_p, g\}_p + \{\{g, h\}_p, f\}_p = 0.$

First note that

$$\begin{aligned}
\{f, \{g, h\}_p\}_p &= \{f, \{g, h\}_p\} + k(p)\{f, \{g, h\}_p\} \\
&= \{f, (\{g, h\} + k(p)\{g, h\})\} + k(p)\{f, (\{g, h\} + k(p)\{g, h\})\} \\
&= \{f, \{g, h\}\} + \{f, k(p)\{g, h\}\} + k(p)[\{f, \{g, h\}\} + \{f, k(p)\{g, h\}\}].
\end{aligned}$$

Similarly,

$\{g, \{h, f\}_p\}_p = \{g, \{h, f\}\} + \{g, k(p)\{h, f\}\} + k(p)[\{g, \{h, f\}\} + \{g, k(p)\{h, f\}\}],$

and

$\{h, \{f, g\}_p\}_p = \{h, \{f, g\}\} + \{h, k(p)\{f, g\}\} + k(p)[\{h, \{f, g\}\} + \{h, k(p)\{f, g\}\}].$

Now, since the canonical Poisson bracket satisfies the Jacobi identity as proved by Hounkonnou M. N. in ([4, pp 7-9]), we only need to show that

$$\{f, k(p)\{g, h\}\} + \{g, k(p)\{h, f\}\} + \{h, k(p)\{f, g\}\} = 0.$$

Now,

$$\begin{aligned}
\{f,k(p)\{g,h\}\} &= \tfrac{\partial f}{\partial q}\tfrac{\partial(k(p)\{g,h\})}{\partial p}-\tfrac{\partial f}{\partial p}\tfrac{\partial(k(p)\{g,h\})}{\partial q}\\
&= \tfrac{\partial f}{\partial q}[k'(p)\{g,h\}+k(p)\tfrac{\partial\{g,h\}}{\partial p}]-k(p)\tfrac{\partial f}{\partial p}\tfrac{\partial\{g,h\}}{\partial q}\\
&= \tfrac{\partial f}{\partial q}(k'(p)\{g,h\})+k(p)\tfrac{\partial f}{\partial q}\tfrac{\partial\{g,h\}}{\partial p}-k(p)\tfrac{\partial f}{\partial p}\tfrac{\partial\{g,h\}}{\partial q}\\
&= \tfrac{\partial f}{\partial q}(k'(p)\{g,h\})+k(p)\{f,\{g,h\}\}.
\end{aligned}$$

Thus,

$$\{f,k(p)\{g,h\}\}=\frac{\partial f}{\partial q}(k'(p)\{g,h\})+k(p)\{f,\{g,h\}\}. \tag{6.11}$$

Similarly,

$$\{g,k(p)\{h,f\}\}=\frac{\partial g}{\partial q}(k'(p)\{h,f\})+k(p)\{g,\{h,f\},\} \tag{6.12}$$

and

$$\{h,k(p)\{f,g\}\}=\frac{\partial h}{\partial q}(k'(p)\{f,g\})+k(p)\{h,\{f,g\}\}. \tag{6.13}$$

Adding equation (6.11), equation (6.12) and equation (6.13), we get

$$\begin{gathered}
\{f,k(p)\{g,h\}\}+\{g,k(p)\{h,f\}\}+\{h,k(p)\{f,g\}\}\\
=\tfrac{\partial f}{\partial q}(k'(p)\{g,h\})+\tfrac{\partial g}{\partial q}(k'(p)\{h,f\})+\tfrac{\partial h}{\partial q}(k'(p)\{f,g\})\\
+k(p)[\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\}].
\end{gathered}$$

The last sum in square brackets disappears since it is the Jacobi identity of the canonical Poisson bracket. So, we have

$$\begin{aligned}
\{f,k(p)\{g,h\}\}+\{g,k(p)\{h,f\}\}\\
&+ \{h,k(p)\{f,g\}\}\\
&= k'(p)[\tfrac{\partial f}{\partial q}\{g,h\}+\tfrac{\partial g}{\partial q}\{h,f\}+\tfrac{\partial h}{\partial q}\{f,g\}]\\
&= k'(p)[\tfrac{\partial f}{\partial q}(\tfrac{\partial g}{\partial q}\tfrac{\partial h}{\partial p}-\tfrac{\partial g}{\partial p}\tfrac{\partial h}{\partial q})\\
&+ \tfrac{\partial g}{\partial q}(\tfrac{\partial h}{\partial q}\tfrac{\partial f}{\partial p}-\tfrac{\partial h}{\partial p}\tfrac{\partial f}{\partial q})\\
&+ \tfrac{\partial h}{\partial q}(\tfrac{\partial f}{\partial q}\tfrac{\partial g}{\partial p}-\tfrac{\partial f}{\partial p}\tfrac{\partial g}{\partial q})].
\end{aligned}$$

This term is zero by equality of mixed partials.

This proves the Jacobi identity for the deformed Poisson bracket

$$\{f, g\}_p = \{f, g\} + k(p)\{f, g\}.$$

Proposition 6.2.2 *Let M be a smooth manifold. Then M endowed with the bracket $\{\cdot,\cdot\}_p$ on $C^\infty(M)$ is a Poisson manifold, and is denoted by $(M, \{\cdot,\cdot\}_p)$.*

Given a function $H \in C^\infty(M)$ on a symplectic manifold (M, ω), a vector field X_H such that $i_{X_H}\omega = dH$ is called a Hamiltonian vector field. This vector field is given in local coordinates by $X_H = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial y_i}\frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i}\frac{\partial}{\partial y_i} \right)$.

In terms of the deformed Poisson bracket, and using the canonical coordinates, we shall write this vector field as

$$X_H^p = \left(\frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p} \right)_p. \tag{6.14}$$

Thus, given any smooth function $f \in C^\infty(M)$ on M, we have

$$\begin{aligned} X_H^p(f) &= \left(\tfrac{\partial H}{\partial p}\tfrac{\partial}{\partial q} - \tfrac{\partial H}{\partial q}\tfrac{\partial}{\partial p} \right)_p (f) \\ &= \left(\tfrac{\partial H}{\partial p}\tfrac{\partial f}{\partial q} - \tfrac{\partial H}{\partial q}\tfrac{\partial f}{\partial p} \right)_p \\ &= \{H, f\}_p. \end{aligned}$$

We have proved the following proposition.

Proposition 6.2.3 *Let $(M, \{\cdot,\cdot\}_p)$ be a Poisson manifold and $H \in C^\infty(M)$. Then there is a unique vector field X_H on M such that*

$$X_H^p(f) = \{H, f\}_p,$$

for all $f \in C^\infty(M)$.

We call $X_H^p(f)$ the derivative of f in the direction of X_H^p.

Clearly $X_g^p(f) = \{g, f\}_p = -\{f, g\}_p = -X_f^p(g)$.

Theorem 6.2.1 *Let $(M, \{\cdot,\cdot\}_p)$ be a Poisson manifold with a deformed Poisson structure. If X_f^p and X_g^p are Hamilotnian vector fields with corresponding Hamiltonian functions f and g respectively, then their bracket $[X_f^p, X_g^p]$ is a Hamiltonian vector field with the Hamiltonian function $\{f,g\}_p$. That is,*

$$[X_f^p, X_g^p] = X_{\{f,g\}_p}^p.$$

Proof. Let $h \in C^\infty(M)$ be an arbitrary function, then we have:

$$\begin{aligned}
[X_f^p, X_g^p](h) &= X_f^p(X_g^p(h)) - X_g^p(X_f^p(h)) \\
&= X_f^p(\{g,h\}_p) - X_g^p(\{f,h\}_p) \\
&= \{f,\{g,h\}_p\}_p - \{g,\{f,h\}_p\}_p \\
&= \{f,\{g,h\}_p\}_p + \{g,\{h,f\}_p\}_p \\
&= -\{h,\{f,g\}_p\}_p \\
&= \{\{f,g\}_p,h\}_p \\
&= X_{\{f,g\}}^p(h). \qquad \square
\end{aligned}$$

Definition 6.2.2 *Let $(M, \{\cdot,\cdot\}_p)$ be a Poisson manifold. Let $X \in \mathfrak{X}(M)$ be a Hamiltonian vector field corresponding to a Hamiltonian function H, a function $f \in C^\infty(M)$ is called a first integral of X_H^p if*

$$X_H^p(f) = \{H,f\}_p = 0.$$

The following theorem is called the Law of conservation of energy.

Theorem 6.2.2 *Let $(M, \{\cdot,\cdot\}_p)$ be a Poisson manifold and $X_H^p \in \mathfrak{X}(M)$ a Hamiltonian vector field corresponding to a Hamiltonian function H, then H is a first integral of the flow of X_H^p.*

Proof. We have

$$\begin{aligned}
X_H^p(H) &= \left(\frac{\partial H}{\partial p}\frac{\partial}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial}{\partial p}\right)_p (H) \\
&= \left(\frac{\partial H}{\partial p}\frac{\partial H}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial H}{\partial p}\right)_p \\
&= 0.
\end{aligned}$$

Thus, $X_H^p(H) = \{H,H\}_p = 0.$ $\square$

Theorem 6.2.3 *Let $(M, \{\cdot,\cdot\}_p)$ be a Poisson manifold and $X_H^p \in \mathfrak{X}(M)$ a Hamiltonian vector field corresponding to a Hamiltonian function H. If $f, g \in C^\infty(M)$ are first integrals of X_H^p, then their bracket, $\{f,g\}_p$ is also a first integral.*

Proof. We have

$$\{H, f\}_p = 0,$$

and

$$\{H, g\}_p = 0.$$

We must show that $\{H, \{f,g\}_p\}_p = 0$. But Jacobi identity gives

$$\{H, \{f,g\}_p\}_p + \{f, \{H,g\}_p\}_p + \{g, \{H,f\}_p\}_p = 0.$$

The second term and the third term are zero since f and g are first integrals. This then gives

$$\{H, \{f,g\}_p\}_p = 0$$

as required. □

In the next section we shall write the Hamiltonian equations of mechanics using the deformed Poisson bracket.

6.3. Hamiltonian systems on a symplectic manifold

Definition 6.3.1 *Let (M, ω) be a symplectic manifold and let $H : M \to \mathbb{R}$ be a smooth function on M. A vector field $X \in \mathfrak{X}(M)$ such that*

$$i_X\omega = dH,$$

is called a Hamiltonian vector field. The corresponding function H is called the Hamiltonian function or the energy function. We denote this vector field by X_H. That is, X_H is a Hamiltonian vector field if

$$i_{X_H}\omega = dH, \tag{6.15}$$

for some function $H \in C^\infty(M)$.

We call (M, ω, X_H) a Hamiltonian system on M.

Definition 6.3.2 *Let M be a smooth manifold and $X \in \mathfrak{X}(M)$ a smooth vector field on M. Let I be an open and connected subset of $\mathbb{R}$ such that $0 \in I$. An integral curve of X through a point $p \in M$ is a curve*

$$\gamma_p : I \to M$$

such that

(i) $\gamma_p(0) = p$,

(ii) $\dot{\gamma}_p(t) = \frac{d}{ds}|_{s=t}\gamma_p(s) = X_{\gamma_p(t)}$.

Let $H : M \to \mathbb{R}$ be a smooth function on a symplectic manifold M. Then on a local symplectic chart $(U, (q^1, \cdots, q^n, p^1, \cdots, p^n))$ we have $H = H(q^1, \cdots, q^n, p^1, \cdots, p^n)$, so that

$$dH = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial p^i} dp^i \right).$$

Let $X_H = \sum_{i=1}^{n} \left(A_i \frac{\partial}{\partial q^i} + B_i \frac{\partial}{\partial p^i} \right)$ be a Hamiltonian vector field corresponding to the function H. From equation (6.4), we have

$$i_{X_H}\omega = \sum_{i=1}^{n} \left(q^i dp^i - p^i dq^i \right).$$

Then, $i_{X_H}\omega = dH$ gives

$$\begin{aligned} A_i &= \frac{\partial H}{\partial p^i} \\ B_i &= -\frac{\partial H}{\partial q^i}. \end{aligned}$$

Thus,

$$X_H = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p^i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p^i} \right). \tag{6.16}$$

Now let $\gamma(t) = (q(t), p(t)), \quad t \in I$ be an integral curve of X_H. Then, we have

$$\dot{\gamma}(t) = (\dot{q}(t), \dot{p}(t)) = \sum_{i=1}^{n} \left(\dot{q}_i(t) \frac{\partial}{\partial q^i} + \dot{p}_i(t) \frac{\partial}{\partial p^i} \right).$$

But, we also have

$$(X_H)_{\gamma(t)} = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p^i} \frac{\partial}{\partial q^i}|_{\gamma(t)} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p^i}|_{\gamma(t)} \right).$$

Thus, the equation $\dot{\gamma}(t) = (X_H)_{\gamma(t)}$ gives the Hamiltonian equations of the mechanics with the Hamiltonian function H as follows;

$$\begin{aligned} \frac{\partial H}{\partial p^i} &= \dot{q}_i(t) \\ \frac{\partial H}{\partial q^i} &= -\dot{p}_i(t). \end{aligned}$$

We shall now write these equations of Hamiltonian mechanics using the deformed Poisson bracket.

From equation (6.14), we have, $\{H, q^i\}_p = X_H^p(q^i) = \dfrac{\partial H}{\partial p^i}$ and $\{H, p^i\}_p = X_H^p(p^i) = -\dfrac{\partial H}{\partial q^i}$

Thus,

$$\begin{aligned} \dot{q}_i(t) &= \{H, q^i\}(1 + k(p)) \\ \dot{p}_i(t) &= \{H, p^i\}(1 + k(p)). \end{aligned}$$

Clearly, these equations are not independent of the deformation factor.

Remark 6.3.1 *Let $h : M \to N$ be a diffeomorphism of manifolds, X a C^∞ vector field on M. Then the map h maps integral curves of X into integral curves of $h_* X$*

Proof. Let $\gamma_p : (-\epsilon, \epsilon) \to M$ be an integral curve of X through $p \in M$. Then we have $\gamma_p(0) = p$ and

$$(\gamma_p)_* \tfrac{d}{dt}|_{t_0} = X_{\gamma_p(t_0)}, \text{ for all } t_0 \in (-\epsilon, \epsilon).$$

The map $h \circ \gamma_p : (-\epsilon.\epsilon) \to N$ is smooth as a composition of smooth maps and $(h \circ \gamma_p)(0) = h(p)$. Then we have

$$\begin{aligned} (h \circ \gamma_p)_*(\tfrac{d}{dt}|_{t_0}) &= (h_*)_{\gamma_p(t_0)} \circ (\gamma_p)_*(t_0))(\tfrac{d}{dt}|_{t_0}) \\ &= (h_*)_{\gamma_p(t_0)}((\gamma_p)_*(t_0)\tfrac{d}{dt}|_{t_0}) \\ &= h_*(X_{\gamma_p(t_0)}). \end{aligned}$$

Thus, $h \circ \gamma_p$ is the integral curve of $h_*(X)$ passing through $h(p)$. □

Recall that if $F : N \to M$ is a diffeomorphism of smooth manifolds, then for each $n \in N$, the map

$$T_nF : T_nN \to T_{F(n)}M$$

is an isomorphism. Thus, if $Y \in \mathfrak{X}(N)$ there is $X \in \mathfrak{X}(M)$ such that

$$(TF)^{-1}X = Y. \tag{6.17}$$

Proposition 6.3.1 *Let $F : N \to M$ be a diffeomorphism of smooth manifolds and $\omega \in \Omega^p(M)$. If $X \in \mathfrak{X}(M)$, then*

$$i_{F^*X}F^*\omega = F^*i_X\omega.$$

Proof. Let $X \in \mathfrak{X}(M)$, $n \in N$ and $m = F(n)$. Given the vector
$u_1, \cdots, u_{p-1} \in T_nN$, we have

$$\begin{aligned} i_{F^*X}F^*\omega(n)(u_1, \cdots, u_{p-1}) &= F^*\omega(n)(F^*X(n), u_1, \cdots, u_{p-1}) \\ &= F^*\omega(n)(Y(n), u_1, \cdots, u_{p-1}), \text{ where } (TF)Y(n) = X(m) \\ &= F^*\omega(n)((TF)^{-1}X(m), u_1, \cdots, u_{p-1}) \\ &= \omega(F(n))(TF \circ (TF)^{-1}X(m), (TF)u_1, \cdots, (TF)u_{p-1}) \\ &= \omega(m)(X(m), (TF)u_1, \cdots, (TF)u_{p-1}) \\ &= i_X\omega(m)((TF)u_1, \cdots, (TF)u_{p-1}) \\ &= F^*i_X\omega(n)(u_1, \cdots, u_{p-1}). \end{aligned}$$

Thus, $i_{F^*X}F^*\omega = F^*i_X\omega$ as required. $\square$

Example 6.3.0.1 *We have shown in the previous chapter in theorem 5.5.2 that the map $\hat{B}^\flat : \mathfrak{g}/G \to \mathfrak{g}^*/G$ is symplectic. Further we assume that the action of G on its Lie algebra $\mathfrak{g}$, and on the dual $\mathfrak{g}^*$ are both transitive actions. Since the coadjoint orbit $\mathfrak{g}^*/G$ is a symplectic manifold , let $\omega_{\mathfrak{g}^*}$ be the symplectic 2-form on $\mathfrak{g}^*/G$. This allows us to define Hamiltonian systems on the adjoint orbit $\mathfrak{g}/G$ as follows:*

Let X_h be a Hamiltonian vector field on $\mathfrak{g}^*/G$ with the corresponding energy function $h : \mathfrak{g}^*/G \to \mathbb{R}$. Then

$$h \circ \hat{B}^\flat : \mathfrak{g}/G \to \mathbb{R}$$

is a smooth function on $\mathfrak{g}/G$. We then have

$$\begin{aligned} d(h \circ \hat{B}^\flat) &= d((\hat{B}^\flat)^* h) \\ &= (\hat{B}^\flat)^* dh \\ &= (\hat{B}^\flat)^* i_{X_h}\omega_{\mathfrak{g}^*} \\ &= i_{(\hat{B}^\flat)^* X_h}(\hat{B}^\flat)^*\omega_{\mathfrak{g}^*} \\ &= i_{X_{h\circ\hat{B}^\flat}}(\hat{B}^\flat)^*\omega_{\mathfrak{g}^*} \text{ by Proposition 6.3.1.} \end{aligned}$$

But $\hat{B}^\flat$ is symplectic so that $(\hat{B}^\flat)^*\omega_{\mathfrak{g}^*}$ is a 2-form on $\mathfrak{g}/G$. This gives that $X_{h\circ\hat{B}^\flat}$ is a Hamiltonian vector field with the energy function $h\circ\hat{B}^\flat$. Thus $(\mathfrak{g}/G, (\hat{B}^\flat)^*\omega_{\mathfrak{g}^*}, X_{h\circ\hat{B}^\flat})$ is a Hamiltonian system on the space $\mathfrak{g}/G$.

In fact we have the following:

Remark 6.3.2 *The vector fields $X_{h\circ\hat{B}^\flat} \in \mathfrak{X}(\mathfrak{g}/G)$ and $X_h \in \mathfrak{X}(\mathfrak{g}^*/G)$ are $\hat{B}^\flat$ related.*

Proof. Let $v \in \mathfrak{X}(\mathfrak{g}/G)$ be a vector field on $\mathfrak{g}/G$, then we have

$$\begin{aligned} i_{X_{h\circ\hat{B}^\flat}}(\hat{B}^\flat)^*\omega_{\mathfrak{g}^*}(v) &= d(h \circ \hat{B}^\flat)(v) \\ &= dh \circ (d\hat{B}^\flat \cdot v) \\ &= i_{X_h}\omega_{\mathfrak{g}^*}(d\hat{B}^\flat \cdot v). \end{aligned}$$

Thus

$$(\hat{B}^\flat)^*\omega_{\mathfrak{g}^*}(X_{h\circ\hat{B}^\flat}, v) = \omega_{\mathfrak{g}^*}(X_h, d\hat{B}^\flat \cdot v),$$

or

$$\omega_{\mathfrak{g}^*}(d\hat{B}^\flat \cdot X_{h\circ\hat{B}^\flat}, d\hat{B}^\flat \cdot v) = \omega_{\mathfrak{g}^*}(X_h, d\hat{B}^\flat \cdot v).$$

Since $v \in \mathfrak{X}(\mathfrak{g}/G)$ was arbitrary, we have that

$$d\hat{B}^\flat \cdot X_{h\circ\hat{B}^\flat} = X_h.$$

Theorem 6.3.1 *Let (M, ω) be a symplectic manifold and X_H be a Hamiltonian vector field on M with the Hamiltonian function H. Then, X_H induces a Hamiltonian vector field X_{T^*M} on the cotangent bundle T^*M, whose flow is the lift of the flow of X_H.*

Proof. Let $X = X_H$ be the Hamiltonian vector field on M. Assume that M is compact, or X_H has compact support, then X_H is complete. It generates a one-parameter group of diffeomorphisms on M. Denote this group by $\tilde{G}$. Then $\tilde{G}$ is a Lie group (see [20, p 63]), and its Lie algebra is as a vector space, the space of vector fields on M. (see [1, p 274 Exercise 4.1G]).

Let $\Phi : \tilde{G} \times M \to M$, $(h, q) \mapsto \Phi_h(q)$, $h \in \tilde{G}, q \in M$, be the action of the group $\tilde{G}$ on M. We lift this action to the action of $\tilde{G}$ on the cotangent bundle T^*M, $\Phi^{T^*} : \tilde{G} \times T^*M \to T^*M$, $(h, \alpha_q) \mapsto \Phi_h^{T^*}(\alpha_q) = T^*\Phi_{h^{-1}}(\alpha_q)$, $h \in \tilde{G}, \alpha_q \in T_q^*M$. The infinitesimal generator of $X \in \mathfrak{X}(M)$ on T^*M is given by

$$X_{T^*M}(\alpha_q) = \frac{d}{dt} T^*\Phi_{\exp -tX}(\alpha_q)|_{t=0}, \tag{6.18}$$

where $\exp tX$ is the flow of X.

Let $\sigma_M : T^*M \to M$ be the canonical projection. Then σ_M is equivariant wth respect to the action on T^*M and the action on M. That is, the following diagram,

$$\begin{tikzcd}
T^*M \arrow[r, "T^*\Phi_{h-1}"] \arrow[d, "\sigma_M"'] & T^*M \arrow[d, "\sigma_M"] \\
M \arrow[r, "\Phi_h"] & M
\end{tikzcd}$$

commutes. We have

$$\sigma_M \circ T^*\Phi_{h^{-1}} = \Phi_h \circ \sigma_M.$$

Differentiating this relation with respect to t at $t = 0$ we get

$$\frac{d}{dt}\sigma_M \circ T^*\Phi_{\exp -tX}|_{t=0} = \frac{d}{dt}\Phi_{\exp tX} \circ \sigma_M|_{t=0}.$$

This gives

$$d\sigma_M \cdot X_{T^*M} = X_M \cdot \sigma_M. \tag{6.19}$$

Since $T^*\Phi_{h^{-1}}$ is symplectic, it preserves the canonical 1-form θ on T^*M, so that $L_{X_{T^*M}}\theta = 0$.(See [25, Proposition 13.18 p 343]). Thus, by Cartan's identity we have

$$0 = L_{X_{T^*M}}\theta = di_{X_{T^*M}}\theta + i_{X_{T^*M}}d\theta.$$

This gives

$$i_{X_{T^*M}}d\theta = -di_{X_{T^*M}}\theta. \tag{6.20}$$

But now using the definition of canonical one-form we have the following

$$\begin{aligned}
i_{X_{T^*M}}\theta(\alpha_q) &= \theta_{\alpha_q}(X_{T^*M}(\alpha_q)) \\
&= \alpha_q d\sigma_M(X_{T^*M}(\alpha_q)) \\
&= \alpha_q(X_M \circ \sigma_M(\alpha_q)) \\
&= \alpha_q(X_M(q)) \\
&= F(X)(\alpha_q).
\end{aligned}$$

For some function $F(X) : T^*M \to \mathbb{R}$.

Let $\tilde{\omega}$ be the canonical 2-form on T^*M defined by $\tilde{\omega} = -d\theta$. Then since $i_{X_{T^*M}}\theta = F(X)$, differentiating both sides gives $di_{X_{T^*M}}\theta = dF(X)$. Equation(6.20) now gives

$$-i_{X_{T^*M}}d\theta = dF(X),$$

or

$$i_{X_{T^*M}}\tilde{\omega} = dF(X).$$

This implies that

$$X_{T^*M} = X_{F(X)}. \tag{6.21}$$

This shows that X_{T^*M} is a Hamiltonian vector field whose Hamiltonian function is

$$F(X) = i_{X_{T^*M}}\theta,$$

where θ is the canonical 1-form on the cotangent bundle T^*M. □

To continue with the Example 6.3.0.1 above, we shall use this proposition to study the Hamiltonian dynamics on the cotangent bundles by the lifting $T^*\hat{B}^\flat : T^*(\mathfrak{g}^*/G) \to T^*(\mathfrak{g}/G)$ of the map $\hat{B}^\flat : \mathfrak{g}/G \to \mathfrak{g}^*/G$. It is known that the map $T^*\hat{B}^\flat$ is symplectic and each cotangent bundle has a natural symplectic 2-form arising from the canonical one-form, (see [28, Def 6.3.1 and Prop 6.3.2, p 170]).

Let

$$\sigma^{\mathfrak{g}^*} : T^*(\mathfrak{g}^*/G) \to \mathfrak{g}^*/G$$

be the projection from the cotangent bundle $T^*(\mathfrak{g}^*/G)$ onto the coadjoint orbit $\mathfrak{g}^*/G$, and let

$$\sigma^{\mathfrak{g}} : T^*(\mathfrak{g}/G) \to \mathfrak{g}/G,$$

be the projection from the cotangent bundle $T^*(\mathfrak{g}/G)$ onto the adjoint orbit $\mathfrak{g}/G$, then the following diagram commutes:

$$\begin{tikzcd} A \arrow[d, "\sigma^{\mathfrak{g}}"'] & C \arrow[l, "T^*\hat{B}^\flat"'] \arrow[d, "\sigma^{\mathfrak{g}^*}"] \\ B \arrow[r, "\hat{B}^\flat"'] & D \end{tikzcd}$$

where,

$$\begin{aligned} A &= T^*(\mathfrak{g}/G); \\ B &= \mathfrak{g}/G; \\ C &= T^*(\mathfrak{g}^*/G); \\ D &= \mathfrak{g}^*/G. \end{aligned}$$

We have that

$$\hat{B}^\flat \circ \sigma^{\mathfrak{g}} \circ T^*\hat{B}^\flat = \sigma^{\mathfrak{g}^*}. \tag{6.22}$$

Now let X_h be the Hamiltonian vector field on $\mathfrak{g}^*/G$ and $X_{h\circ\hat{B}^\flat}$ the corresponding Hamiltonian vector field on $\mathfrak{g}/G$ as in example 6.3.0.1. Let $X_h^{T^*(\mathfrak{g}^*/G)}$ be the induced Hamiltonian vector field on $T^*(\mathfrak{g}^*/G)$ of proposition 6.3.1. Then equation (6.22) gives

$$\begin{aligned} d(\hat{B}^\flat \circ \sigma^{\mathfrak{g}} \circ T^*\hat{B}^\flat)(X_h^{T^*(\mathfrak{g}^*/G)}) &= d\sigma^{\mathfrak{g}^*}(X_h^{T^*(\mathfrak{g}^*/G)}) \\ &= X_h \cdot \sigma^{\mathfrak{g}^*}, \quad \text{by equation (6.19).} \end{aligned}$$

Thus,

$$d\hat{B}^\flat \circ d(\sigma^{\mathfrak{g}} \circ T^*\hat{B}^\flat)(X_h^{T^*(\mathfrak{g}^*/G)}) = X_h \cdot \sigma^{\mathfrak{g}^*}.$$

The Remark 6.3.2 now implies that

$$d(\sigma^{\mathfrak{g}} \circ T^*\hat{B}^\flat)(X_h^{T^*(\mathfrak{g}^*/G)}) = X_{h\circ\hat{B}^\flat},$$

or

$$d\sigma^{\mathfrak{g}} \circ dT^*\hat{B}^\flat \cdot X_h^{T^*(\mathfrak{g}^*/G)} = X_{h\circ\hat{B}^\flat} \cdot \sigma^{\mathfrak{g}},$$

by the Proposition 6.3.1. The same proposition now implies that

$$dT^*\hat{B}^\flat \cdot X_h^{T^*(\mathfrak{g}^*/G)} = X_{h\circ\hat{B}^\flat}^{T^*(\mathfrak{g}/G)}$$

is the induced Hamiltonian vector field by $X_{h\circ\hat{B}^\flat}$ on the cotangent bundle $T^*(\mathfrak{g}/G)$.

We have shown that while the symplectic diffeomorphism $\hat{B}^\flat : \mathfrak{g}/G \to \mathfrak{g}^*/G$ pushes forward Hamiltonian vector fields from $\mathfrak{g}/G$ to $\mathfrak{g}^*/G$, the lift of this diffeomorphism pushes Hamiltonian vector fields from the cotangent bundle $T^*(\mathfrak{g}^*/G)$ to the cotangent bundle $T^*(\mathfrak{g}/G)$.

7

Conclusion

Our contribution in this thesis has been mainly to symplectic geometry. We have investigated the effects of a Lie group G acting transitively on a smooth manifold M and another parallel action Ad^* of G on the dual $\mathfrak{g}^*$ of its Lie algebra $\mathfrak{g}$, called the coadjoint representation. In a number of cases, when the action Φ of G on M is Hamiltonian, it turns out that the momentum mapping $\mu : M \to \mathfrak{g}^*$ is equivariant with respect to the action Ad^*. Comparisons can then be made between the manifold M and the coadjoint orbits of Ad^* action of G on $\mathfrak{g}^*$. However, there are cases when the momentum mapping fails to be equivariant. We have shown that in this case it is possible to redefine the action of G on $\mathfrak{g}^*$ through a one-coycle σ in such a way that, with this action, the momentum mapping becomes equivariant with respect to the new affine action and the result is that investigations that can be done with an equivariant momentum mapping can now be done with the momentum with one-cocycle.

Actions of Lie groups on smooth manifolds have led to constructions of new spaces. Some of the new spaces are the quotient spaces. In particular, when the action of a Lie group G on a symplectic manifold (M, ω) is Hamiltonian, a new space called the reduced space can be constructed. However, this space may not be symplectic itself due to a number of reasons such as, the dimension of the new space may not even be even. We have worked with the known reduced space, the Marsden-Mayer-Weinstein reduced space, which is known to be symplectic, to investigate the transfer of Riemannian structure from the original manifold M to the reduced space. Our investigations have shown that when the Lie group G is compact, there are some Riemannian submersions that make, under suitable conditions, the reduced space inherit an induced Riemannian structure from the original manifold. Our investigations in this case have involved another structure, the almost complex structure.

Finally, we have substituted the action Φ of a Lie group G on an arbitrary symplectic manifold M with the action Ad of G on its Lie algebra $\mathfrak{g}$, called the adjoint representation while maintaining the parallel action Ad^* of G on the dual $\mathfrak{g}^*$, of its Lie algebra $\mathfrak{g}$. When the Lie group G is semi-simple, compact and connected, we have used the Killing form to show that in this case there is a symplectic diffeomorphism between $\mathfrak{g}/G$ and $\mathfrak{g}^*/G$ if and only if the actions of G on its Lie algebra $\mathfrak{g}$ and on the dual $\mathfrak{g}^*$ of its Lie algebra, are both transitive actions. Note that these findings cannot immediately be extended to a case when the action of G is not transitive. The difficult here is that the spaces $\mathfrak{g}/G$ and $\mathfrak{g}^*/G$ are both disjoint unions of orbits and it is not yet known if in this case these spaces are manifolds. It would therefore be interesting to investigate further the case when the action of G is not transitive.

In the last chapter we have extended the application of Hamiltonian mechanics to deformed Poisson bracket. We have noted that many properties of Hamiltonian systems which hold with the standard Poisson bracket also hold with the deformed Poisson bracket. We have used the spaces $\mathfrak{g}/G$ and $\mathfrak{g}^*/G$ as homogeneous spaces, to investigate some Hamiltonian formalisms on the cotangent bundle through the lifting of the integral curves of Hamiltonian vector fields on these spaces. The result is that Hamiltonian vector fields are lifted to Hamiltonian vector fields on the cotangent bundle. However, more investigations on the lifting of Hamiltonian systems to the cotangent space are needed in the future to generalise the findings.

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