PERIODIC SOLUTIONS OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

by

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CONTENTS

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Notation

The following notation has been used throughout.

$\mathbb{R}$ is a real line.

I is an interval and $t \in I \subseteq \mathbb{R}$.

$\mathbb{R}^n$ is an n-dimensional real-valued vector space.

D is a connected open subset of $\mathbb{R}^n$ and $x \in D$.

$\phi(t)$, $\psi(t)$, etc. are real-valued n-dimensional vector functions.

$\|\cdot\|$ is a norm defined over $\mathbb{R}^n$ and defined as follows:

$$\|x\| = \max_{i=1,\dots,n} |x_i| .$$

General Introduction

Many physical problems are studied through mathematical equations especially differential equations. For example, problems in mechanics, electricity, aerodynamics, to mention just a few, use differential equations. While it is true to say that physical sciences and technology are the two main sources of problems which require the use of differential equations, biological and social sciences are increasingly being realised as other sources. For example population study is one area where differential equations are applied.

Much of the literature on differential equations is on linear differential equations. Methods of solving a variety of linear differential equations are known but most of these methods cannot be effectively extended to nonlinear differential equations. This makes the solving of nonlinear equations a difficult task. What has been done to ease this problem is to abandon the idea of solving an equation and instead get as much information as is possible about a class of solutions of the nonlinear differential equation by examining the equation itself. After extracting enough information, then one can find ways of approximating a particular solution as the exact one is almost impossible to get.

This work is a brief survey of the literature available concerning periodic solutions of nonlinear ordinary differential equations.

Chapter 1 is on the standard existence theory of differential equations. Chapter 2 is a brief account of critical points and Chapter 3 is on stability theory. The last chapter looks at some of the existence theorems for periodic solutions of nonlinear differential equations. References are given at the end.

CHAPTER 1

EXISTENCE AND UNIQUENESS OF SOLUTIONS

1.1 Introduction

Before the theory of differential equations can unfold in more interesting directions there are three basic questions to settle. Given a system of differential equations and some conditions, does a solution exist which satisfies the conditions? If solutions do exist, can there be more than one solution satisfying the same conditions? If the conditions are slightly varied, does a solution respond accordingly?

A system of differential equations whose solutions are unique and continuous in initial conditions constitutes what is called a well-posed problem.

We give here an account of the standard theory in existence and uniqueness.

1.2 Types of Ordinary Differential Equations

Let $f_i : I \times D \to \mathbb{R}$, $i=1,2,\ldots,n$ be real-valued functions defined on $I \times D$. Put $f = (f_1, f_2, \ldots, f_n)$. Then f maps $I \times D$ into $\mathbb{R}^n$.

Consider a system of differential equations

$$\frac{dx_i}{dt} = f_i(t,x), \quad i=1,2,\ldots,n. \tag{1.2.1}$$

An ordinary differential equation of order n

$$G(t,x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \ldots, \frac{d^nx}{dt^n}) = 0 \tag{1.2.2}$$

can be reduced to system (1.2.1) by first solving for $\frac{d^nx}{dt^n}$ to get

$$\frac{d^n x}{dt^n} = F(t, x, \frac{dx}{dt}, \ldots, \frac{d^{n-1}x}{dt^{n-1}}). \qquad (1.2.3)$$

Then, putting

$$x_1 = x,\ x_2 = \frac{dx}{dt},\ x_3 = \frac{d^2x}{dt^2}, \ldots, x_n = \frac{d^{n-1}x}{dt^{n-1}}$$

and

$$x = (x_1, x_2, \ldots, x_n).$$

Equation (1.2.3) then becomes

$$\dot{x}_i = x_i \qquad i=1,\ldots,n-1$$

$$\dot{x}_n = F(t,x) \qquad (\dot{x}_i = \frac{dx_i}{dt}),$$

which is clearly a system of type (1.2.1).

For brevity system (1.2.1) will be written as

$$\dot{x} = f(t,x). \qquad (1.2.4)$$

When f is independent of t (1.2.4) has the form

$$\dot{x} = f(x), \qquad (1.2.5)$$

known as an autonomous system as opposed to a nonautonomous system (1.2.4). Special cases of (1.2.4) are the linear equation

$$\dot{x} = A(t)x + b(t), \qquad (1.2.6)$$

where $A(t)$ is an $n \times n$ matrix of continuous functions $a_{ij}(t)$, $i,j=1,\ldots,n$ and $b(t)$ is an n-vector function; and the periodic equation

$$\dot{x} = f(t,x), \quad f(t+w,x) = f(t,x). \qquad (1.2.7)$$

1.3 Existence and Uniqueness

A function

$$x = \phi(t), \quad \phi(t) = (\phi_1(t),\ldots,\phi_n(t)) \in \mathbb{R}^n$$

is called a ***solution*** of system (1.2.4) if $\phi(t)$ is defined on some interval $I_o \subseteq I$, $\phi(t) \in D$ for every $t \in I_o$ and

$$\dot{\phi} = f(t,\phi(t)).$$

The problem as far as existence is concerned is to find a differentiable function which will satisfy the given system (1.2.4). When in addition uniqueness is sought, then the problem becomes that of finding a solution through a given point. This is the so-called *initial value problem* and has the form

(p): Find a function ϕ such that

$$\dot{\phi} = f(t,\phi(t)) \tag{1.3.0}$$

and

$$\phi(t_o) = \xi$$

for

$$(t_o,\xi) \text{ in } I \times D.$$

It is easily seen that $\phi(t)$ is a solution to the problem (p) if and only if

$$\phi(t) = \xi + \int_{t_o}^{t} f(s,\phi(s))ds. \tag{1.3.1}$$

For, differentiating with respect to t, (1.3.1) is seen to satisfy the problem (p). Conversely, integrating (1.3.0) from t_0 to t immediately gives (1.3.1). The problem (p) is therefore equivalent to finding a function ϕ which satisfies the integral equation (1.3.1).

With (1.3.1) at hand, conditions on f have been established which are sufficient for the existence of at least one solution for (p). One such condition is the continuity of f (see e.g. [3; page 2]). This is the weakest practical condition available. Examples exist which show that continuity alone fails to guarantee uniqueness of a solution. For instance, the equation

$$\frac{dx}{dt} = x^{\frac{1}{2}}, \qquad x \geq 0,$$

with $$x(0) = 0$$

has two solutions satisfying the same condition, namely

$$x(t) \equiv 0$$

and

$$x(t) = \frac{t^2}{4}.$$

One well known condition which in addition to continuity ensures uniqueness of a solution is the *Lipschitz condition* defined as follows:

A function f is said to satisfy a Lipschitz condition in $I \times D$ if there exists a constant K such that for every x and y in D

$$\|f(t,x) - f(t,y)\| \leq K\|x-y\|, \qquad (1.3.2)$$

where K is independent of t. It is however, always sufficient that f be Lipschitz in a neighbourhood of a given point $x = \xi$. In this case f is said to be *locally Lipschitz* in $I \times D$.

A function which satisfies a Lipschitz condition is clearly continuous in x. For if (1.3.2) is satisfied, choose $\varepsilon > 0$ such that

$$K\|x-y\| < \varepsilon.$$

Then $\delta > 0$ can be taken to be $\frac{\varepsilon}{K}$ so that

$$\|f(t,x) - f(t,y)\| < \varepsilon$$

whenever

$$\|x-y\| < \delta.$$

When a function is continuously differentiable in x it is also seen to satisfy a Lipschitz condition locally. For suppose that all the partial derivatives $\frac{\partial f_i}{\partial x_j}$; $i,j=1,\ldots,n$ are continuous. Let $S_\varepsilon(x_0)$ be a closed neighbourhood of $x_0 \in D$, $\varepsilon > 0$. Let t_0 correspond to x_0 and let I_0 be an interval $|t-t_0| \leq \varepsilon$ contained in I. Then since $\frac{\partial f_i}{\partial x_j}$ are continuous in $I \times D$ there exists a constant K such that

$$\left|\frac{\partial f_i}{\partial x_j}(t,x)\right| \leq K, \qquad i,j=1,\ldots,n$$

for $(t,x) \in I_0 \times S_\varepsilon(x_0)$. If (t,x) and (t,y) are any two points in $I_0 \times S_\varepsilon(x_0)$, the application of the mean value theorem for functions of n variables gives

$$f_i(t,x) - f_i(t,y) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(t,\xi)(x_j-y_j)$$

where $x_j < \xi_j < y_j$, $\xi = (\xi_1,\ldots,\xi_n)$.

Hence

$$|f_i(t,x) - f_i(t,y)| \leq \sum_{j=1}^{n} \left|\frac{\partial f_i}{\partial x_j}(t,\xi)\right| |x_i - y_j|$$

so that

$$\|f(t,x) - f(t,y)\| \leq Kn\|x-y\|$$

which is clearly a Lipschitz condition with constant Kn.

It follows therefore that if f is continuously differentiable in x, then f is locally Lipschitz in every compact subset of D.

Theorem 1.3.1 (Picard)

Suppose that f is continuous and satisfies a Lipschitz condition with a constant K in the region

$$R = \{(t,x) : |t-t_0| \leq a, \quad \|x-\xi\| \leq b\}$$

which is contained in $I \times D$. Then the system (1.2.4) has a unique solution $\phi(t)$ for which

$$\phi(t_0) = \xi$$

and it is defined for $|t-t_0| < \alpha$, where

$$\alpha = \min\{a, \frac{b}{M}\}$$

and

$$M = \max_{(t,x)\in R} \|f(t,x)\|.$$

Proof

The proof of this theorem proceeds through showing that a sequence of functions

$$\phi_k(t) = (\phi_{1k}(t),\dots,\phi_{nk}(t))$$

defined as

$$\phi_0(t) = \xi$$

$$\phi_k(t) = \xi + \int_{t_0}^{t} f(s,\phi_{k-1}(s))ds \qquad (1.3.3)$$

converges uniformly to a function $\phi(t)$, which is the required solution on $|t-t_0| < \alpha$.

We first show that for each k, $\phi_k(t)$ is defined and continuous on $[t_0,t_0+\alpha]$ and

$$\|\phi_k(t)-\xi\| \le M|t-t_0|. \qquad (1.3.4)$$

Clearly the statement holds for $k=0$ since

$$\phi_0(t) = \xi \quad \text{for all } t \in [t_0,t_0+\alpha].$$

Suppose the statement is true for $k=r$: that is

$$\phi_r(t) = \xi + \int_{t_0}^{t} f(s,\phi_{r-1}(s))ds$$

is defined and continuous, and (1.3.4) holds on $[t_0,t_0+\alpha]$. Since $\phi_r(t)$ is continuous in t, $f(t,\phi_r(t))$ is continuous in t. So

$$\phi_{r+1}(t) = \xi + \int_{t_0}^{t} f(s,\phi_r(s))ds$$

is defined and continuous in t. Also

$$\begin{aligned}\|\phi_{r+1}(t)-\xi\| &= \left\| \int_{t_0}^{t} f(s,\phi_r(s))ds \right\| \\ &\le \int_{t_0}^{t} \|f(s,\phi_r(s))\|ds \\ &\le M|t-t_0| \le M\alpha \le M\frac{b}{M} \le b.\end{aligned}$$

By induction $\phi_k(t)$ are defined and continuous and satisfy (1.3.4). Hence $\phi_k(t) \in R$ for k.

Next it has to be shown that (ϕ_k) converge uniformly on $[t_o, t_o+\alpha]$.

Let $d_k(t) = \|\phi_{k+1}(t) - \phi_k(t)\|$, $k=0,1,\ldots$.

By (1.3.3)

$$\phi_{k+1}(t)-\phi_k(t) = \int_{t_o}^{t} [f(s,\phi_k(s))-f(s,\phi_{k-1}(s))]ds.$$

Hence

$$\|\phi_{k+1}(t)-\phi_k(t)\| \leq K \int_{t_o}^{t} \|\phi_k(s)-\phi_{k-1}(s)\|ds.$$

This implies that

$$d_k(t) \leq K \int_{t_o}^{t} d_{k-1}(s)ds. \tag{1.3.5}$$

But

$$d_o(t) \leq M|t-t_o|.$$

Also

$$\begin{aligned}
d_1(t) &\leq KM \int_{t_o}^{t} |s-t_o|ds \\
&= KM\frac{|t-t_o|^2}{2!} \\
d_2(t) &\leq K \int_{t_o}^{t} d_1(s)ds \\
&\leq \frac{K^2M}{2} \int_{t_o}^{t} |s-t_o|^2 ds \\
&= \frac{K^2M}{2} \frac{|t-t_o|^3}{3} \\
&= M K^2 \frac{|t-t_o|^3}{3!} .
\end{aligned}$$

By induction

$$d_n(t) \le \frac{M}{K} \frac{K^n|t-t_o|^n}{n!} .$$

It follows then that $\sum_{n=0}^{\infty} d_n(t)$ converges uniformly on $[t_o, t_o+\alpha]$ since

$$\sum_{n=0}^{p} d_n(t) \le \sum_{n=0}^{p} \frac{M}{K} \frac{K^n|t-t_o|^n}{n!}$$

$$= \frac{M}{K} \sum_{n=0}^{p} \frac{K^n|t-t_o|^n}{n!} \to \frac{M}{K} e^{K|t-t_o|} .$$

Since

$$\left\| \sum_{k=0}^{\infty} \{\phi_{k+1}(t)-\phi_k(t)\} \right\| \le \sum_{k=0}^{\infty} \|\phi_{k+1}(t)-\phi_k(t)\|,$$

the series

$$\sum_{k=0}^{\infty} \{\phi_{k+1}(t)-\phi_k(t)\}$$

converges uniformly on $[t_o, t_o+\alpha]$. But

$$\sum_{k=0}^{n-1} \{\phi_{k+1}(t)-\phi_k(t)\} = \phi_n(t)-\phi_o(t),$$

so

$$\phi_n(t) = \boldsymbol{\xi} + \int_{t_o}^{t} f(s,\phi_{n-1}(s))ds$$

converges uniformly to a function $\phi(t)$.

To show that

$$\phi(t) = \xi + \int_{t_o}^{t} f(s,\phi(s))ds,$$

first note that for an arbitrary $\varepsilon > 0$

$$\|\phi(t)-\xi\| \leq \|\phi(t)-\phi_n(t)\| + \|\phi_n(t)-\xi\|$$

$$< \quad \varepsilon \quad + M|t-t_o| \ .$$

But $\|\phi(t)-\xi\|$ is independent of ε, so

$$\|\phi(t)-\xi\| \leq M|t-t_o| \leq M\alpha \leq M\frac{b}{M} = b.$$

Hence $\phi(t) \in R$ for $t \in [t_o,t_o+\alpha]$.

Now

$$\left\| \int_{t_o}^{t} [f(s,\phi(s))-f(s,\phi_n(s))]\,ds \right\|$$

$$\leq K \int_{t_o}^{t} \|\phi(s)-\phi_n(s)\|\,ds$$

$$\leq K\,\varepsilon\,|t-t_o|$$

for large n.

Hence

$$\lim_{n\to\infty} \int_{t_o}^{t} f(s,\phi_n(s))ds = \int_{t_o}^{t} f(s,\phi(s))ds.$$

So $\phi(t)$ is a solution of (1.2.4). To show that $\phi(t)$ is unique, suppose that $\psi(t)$ is another solution such that

$$\psi(t_o) = \phi(t_o) = \xi.$$

Since ψ and ϕ are solutions, there exist sequences ψ_n and ϕ_n such that $\psi_n \to \psi$ and $\phi_n \to \phi$. Also, since $\psi(t_o) = \phi(t_o)$, there exists a neighbourhood of t_o, $|t-t_o| < \delta < \alpha$, $\delta > 0$ such that for sufficientl large n

$$\|\phi_n(t)-\psi_n(t)\| < \frac{\varepsilon}{3}$$

for any $\varepsilon > 0$ and $t \in (t_o-\delta,t_o+\delta)$.

Therefore

$$\|\phi(t)-\psi(t)\| \le \|\phi(t)-\phi_n(t)\| + \|\phi_n(t)-\psi_n(t)\| + \|\phi_n(t)-\psi(t)\|$$

$$\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$

$$= \varepsilon.$$

Since ε is arbitrary, it follows that

$$\phi(t) = \psi(t)$$

for $|t-t_o| < \delta$. This completes the proof.

1.4 Maximal Interval of Existence

The interval on which a solution was proved to exist in Theorem 1.3.1 may be very small indeed. However, a solution may be defined over a larger interval. This raises the question of the extent to which a solution can be continued.

Let $\phi(t)$ be a solution of (1.2.4) defined on $I_1 = (t_1,t_2)$. If $\psi(t)$ is another solution defined on $I_2 = (t_3,t_4)$ and $I_1 \subset I_2$, then $\psi(t)$ is a *continuation* of $\phi(t)$ if $\phi(t) = \psi(t)$ for all $t \in I_1$.

The *maximal interval of existence* of $\phi(t)$ is (α,β) if $\phi(t)$ has no continuation on this interval. Naturally, it is desirable to know when a solution ϕ can be continued. Of course, if ϕ is defined *on (α,β) and say β is an end-point of I on which f in (1.2.4) is defined, ϕ cannot be continued beyond β. Thus the question of continuation may be considered when β (or α) is an interior point of I. Also if $\lim_{t\to\beta} \phi(t)$ does not exist, where $t \in (\alpha,\beta)$, then $\phi(t)$*

cannot be continued. This means therefore that continuation of a solution ϕ is possible only when either α or β is an interior point of I where f is defined on I $\times$ D and limit as $t \to \beta$ (or $t \to \alpha$) of $\phi(t)$ exists and remains in a compact subset A of D. For in this subset and $t \in (\alpha,\beta)$

$$\|f(t,x)\| \leq M.$$

Now if

$$\phi(t) = \xi + \int_{t_0}^{t} f(s,\phi(s))ds,$$

then for t_1 and t_2 such that $\alpha < t_1 < t_2 < \beta$

$$\begin{aligned}\|\phi(t_1)-\phi(t_2)\| &\leq \int_{t_1}^{t_2} \|f(s,\phi(s))\| ds \\ &\leq M|t_1-t_2|.\end{aligned}$$

By the General Principle of Convergence

$$\lim_{t \to \beta} \phi(t) = \eta$$

exists and $\eta \in A$.

So a solution $\psi(t)$ which at $t = \beta$ is such that $\psi(\beta) = \eta$ exists and is defined on some interval $(\beta-\delta,\beta+\delta)$, $\delta > 0$.

If now χ is defined by

$$\chi(t) = \begin{cases}\phi(t), & \alpha < t < \beta \\ \psi(t), & \beta \leq t < \beta + \delta\end{cases}$$

then $\chi(t)$ is a solution of (1.2.4) on $(\alpha,\beta+\delta)$. For if $\alpha < t < \beta$ then

$$\chi(t) = \xi + \int_{t_0}^{t} f(s,\phi(s))ds.$$

Taking the limit as $t \to \beta$, we have

$$\lim_{t \to \beta} \chi(t) = \eta$$

$$= \xi + \int_{t_0}^{t} f(s,\chi(s))ds.$$

On the other hand, if $\beta \leq t < \beta + \delta$ then

$$\chi(t) = \eta + \int_{\beta}^{t} f(s,\psi(s))ds$$

$$= \eta + \int_{\beta}^{t} f(s,\chi(s))ds.$$

From the discussion above it follows that if (α,β) is the maximal interval of existence of a solution ϕ, and $\beta < \infty$ (or $-\infty < \alpha$) then the solution leaves every compact subset A of D as t approaches β (or α

1.5 Dependence of Solutions on Initial Points and Parameters

Generally, a solution ϕ of the problem (p) is written in the form $\phi(t,t_0,\xi)$ to imply that it is a solution through $t = t_0$ and its value at t_0 is ξ. Since $\phi(t,t_0,\xi)$ is a solution to the problem (p), it is continuously differentiable in t. The question which may be asked is, how does a solution behave as a function of t_0 or ξ? In particular, conditions may be sought under which $\phi(t,t_0,\xi)$ is continuous in t_0 and ξ as well.

<u>Theorem 1.5.1</u>

<u>Let</u> f <u>be continuous and satisfy a Lipschitz condition in</u> $I \times D$. <u>Suppose that</u> $\phi(t,t_0,\xi)$ <u>is a solution of (1.2.4) defined over</u> (α,β). <u>Let</u> [c,d] <u>be a closed interval contained in</u> (α,β) <u>with</u> t_0 <u>as an</u>

interior point. If $\|\eta-\xi\|$ is sufficiently small, then $\phi(t,t_0,\eta)$ is defined for $t \in [c,d]$; moreover $\phi(t,t_0,\eta)$ tends to $\phi(t,t_0,\xi)$ uniformly for $t \in [c,d]$ as $\eta \to \xi$.

To prove this theorem, two lemmas will be proved first, as they will be required. One lemma will require the concept of an approximate solution defined as follows.

For $\delta > 0$, a differentiable function $\chi(t)$ is called a *δ-solution* of the system (1.2.4) in the interval $I_0 \subset I$ if $\psi(t) \in D$ for every $t \in I_0$ and

$$\|\dot{\psi}(t)-f(t,\psi(t))\| \leq \delta.$$

Lemma 1.5.2 (Gronwall's Inequality)

Let h and g be continuous real valued functions of t defined for $\alpha \leq t \leq \beta$ with $g(t) \geq 0$. Suppose that $K(t)$ is differentiable on (α,β), nondecreasing and such that

$$h(t) \leq K(t) + \int_\alpha^t h(s)g(s)ds, \quad (\alpha \leq t \leq \beta). \qquad (1.5.1)$$

Then

$$h(t) \leq K(t)\exp[\int_\alpha^t g(s)ds].$$

Proof

Take

$$U(t) = K(t) + \int_\alpha^t h(s)g(s)ds, \quad (\alpha \leq t \leq \beta)$$

so that

$$\dot{U}(t) = \dot{K}(t) + h(t)g(t).$$

Since $g(t) \geq 0$ and $U(t) \geq h(t)$ by (2.5.1), then

$$\dot{U}(t) \leq \dot{K}(t) + U(t)g(t).$$

or

$$\dot{U}(t) - g(t)U(t) \leq \dot{K}(t).$$

Using the integrating factor

$$\exp\{-\int_{\alpha}^{t} g(s)ds\}$$

we get

$$\frac{d}{dt}[U(t)\exp\{-\int_{\alpha}^{t} g(s)ds\}] \leq \dot{K}(t)\exp\{-\int_{\alpha}^{t} g(s)ds\}.$$

Since $g(s) \geq 0$, $\exp\{-\int_{\alpha}^{t} g(s)ds\} \leq 1$.

So $$\frac{d}{dt}[U(t)\exp\{-\int_{\alpha}^{t} g(s)ds\}] \leq \dot{K}(t).$$

Hence

$$U(t)\exp\{-\int_{\alpha}^{t} g(s)ds\} \leq K(t) + U(\alpha) - K(\alpha).$$

But $U(\alpha) = K(\alpha)$. So

$$U(t)\exp\{-\int_{\alpha}^{t} g(s)ds\} \leq K(t)$$

or

$$U(t) \leq K(t)\exp\int_{\alpha}^{t} g(s)ds.$$

Since $h(t) \leq U(t)$, the result follows.

Lemma 1.5.3

Let f satisfy a Lipschitz condition with a constant K in $I \times D$ and let Δ be an open subset of $I \times D$. Let $\phi(t)$ and $\psi(t)$ be two δ-solutions of (1.2.4) in $I_1 = (\alpha,\beta)$ such that $(t,\phi(t))$ and $(t,\psi(t))$ are in Δ for $t \in I_1$. If

$$\|\phi(t_o)-\psi(t_o)\| \leq \delta_1,$$

then

$$\|\phi(t)-\psi(t)\| \leq [\delta_1+2\delta|t-t_o|]\exp\{K|t-t_o|\}.$$

Proof

Let $\xi(t) = \phi(t)-\psi(t)$. Then

$$\begin{aligned}\xi(t) &= \xi(t_o) + \int_{t_o}^{t} [\dot{\phi}(s)-f(s,\phi(s))+f(s,\phi(s)) \\ &\qquad - \dot{\psi}(s)-f(s,\psi(s))+f(s,\psi(s))]ds \\ &= \xi(t_o) + \int_{t_o}^{t} \{[\dot{\phi}(s)-f(s,\phi(s))] - [\dot{\psi}(s)-f(s,\psi(s))] \\ &\qquad + [f(s,\phi(s))-f(s,\psi(s))]\}ds.\end{aligned}$$

Hence

$$\begin{aligned}\|\xi(t)\| &\leq \delta_1 + 2\delta|t-t_o| + K\int_{t_o}^{t}\|\phi(s)-\psi(s)\|ds \\ &= \delta_1 + 2\delta|t-t_o| + \int_{t_o}^{t} K\|\xi(s)\|ds.\end{aligned}$$

By Lemma 1.5.2

$$\|\xi(t)\| \leq [\delta_1 + 2\delta|t-t_o|]\exp\{K|t-t_o|\}.$$

Now the proof of theorem 2 follows.

Proof of Theorem 1.5.1

Since f is Lipschitz in $I \times D$, there is an open set $\Delta \subset I \times D$ such that $(t,\phi(t,t_0,\xi)) \in \Delta$ for $t \in [c,d]$ and f is Lipschitz in Δ.

The solution $\phi(t,t_0,\eta)$ exists. Using Lemma 4, so long as $(t,\phi(t,t_0,\eta)$ remains in Δ, and noting that $\delta = 0$ for solutions,

$$\|\phi(t,t_0,\xi) - \phi(t,t_0,\eta)\| \leq \|\xi-\eta\|\exp\{K|t-t_0|\}.$$

This implies that for sufficiently small $\|\xi-\eta\|$, $\phi(t,t_0,\eta)$ remains in Δ and is defined for $t \in [c,d]$. The above inequality also implies that as $\xi \to \eta$, $\phi(t,t_0,\eta) \to \phi(t,t_0,\xi)$ uniformly.

Theorem 1.5.1 states that for fixed t and t_0, $\phi(t,t_0,\xi)$ is continuous in ξ. It is indeed also true that $\phi(t,t_0,\xi)$ is continuous in both t_0 and ξ. For suppose that the hypotheses of Theorem 1.5.1 hold and $\|\xi-\eta\|$ and $|t_0-t_1|$ are sufficiently small. $\phi(t,t_1,\eta)$ is defined and if $z = \phi(t_0,t_1,\eta)$ when $t = t_0$, then $\phi(t,t_1,\eta) = \phi(t,t_0,z)$. Now

$$\begin{aligned}\|\eta-z\| &= \left\|\int_{t_1}^{t} f(s,\phi(t,t_1,\eta))ds\right\| \\ &\leq M|t-t_1|\end{aligned}$$

since f is continuous. Applying Lemma 1.5.3 we have

$$\begin{aligned}\|\phi(t,t_1,\eta) - \phi(t,t_0,\xi)\| &\leq \|\phi(t,t_1,\eta) - \phi(t,t_0,\eta)\| \\ &\quad + \|\phi(t,t_0,\eta) - \phi(t,t_0,\xi)\| \\ &= \|\phi(t,t_0,z) - \phi(t,t_0,\eta)\| \\ &\quad + \|\phi(t,t_0,\eta) - \phi(t,t_0,\xi)\| \\ &\leq [\|z-\eta\| + \|\eta-\xi\|]\exp\{K|t-t_0|\}\end{aligned}$$

so long as $(t,\phi(t,t_1,\eta))$ and $(t,\phi(t,t_0,\xi))$ remain in Δ.

Hence

$$\|\phi(t,t_1,\eta) - \phi(t,t_0,\xi)\| \leq [M|t-t_1| + \|\eta-\xi\|]\exp\{K|t-t_0|\},$$

from which the result follows.

It can also be shown (e.g. [3, page 25]) that $\boldsymbol{\phi}$ can be differentiat with respect to t_0 and ξ_i, $i=1,\ldots,n$ where $\xi = (\xi_1,\ldots,\xi_n)$ when $\frac{\partial f}{\partial x}$ exists and is continuous.

The ideas mentioned above apply with minor modifications to systems of the type

$$\dot{x} = f(t,x,\mu), \tag{1.5.2}$$

where μ is a vector parameter with real components μ_i, $i=1,\ldots,m$. For instance the initial point would be (t_0,ξ,μ°) and continuity or differentiability would apply under appropriate conditions like f be continuous in μ and for fixed μ, $f(t,x,\mu)$ satisfies the hypotheses of the uniqueness theorem.

1.6 Linear Systems

Linear systems play a special role in the theory of differential equations. The theory for linear equations, which is elegant and complete, is the basis for much of the study of nonlinear equations.

The equation

$$\dot{x} = A(t)x, \qquad (x \in \mathbb{R}^n) \tag{1.6.1}$$

is called a homogeneous equation. Its solutions, when found, are used to construct solutions of equation (1.2.6).

Suppose that $A(t)$ is a matrix of continuous functions defined on I. Then the set of all solutions of (1.6.1) is described by the following theorem.

Theorem 1.6.1

The set of all solutions of (1.6.1) form a vector space of dimension n over the complex field $\mathbb{C}$.

Proof

Let ϕ_1 and ϕ_2 be solutions of (1.6.1). Put

$$\psi(t) = \lambda_1\phi_1(t) + \lambda_2\phi_2(t).$$

Then

$$\frac{d}{dt}\psi(t) = \lambda_1\dot{\phi}(t) + \lambda_2\dot{\phi}(t)$$

$$= \lambda_1 A(t)\phi_1(t) + \lambda_2 A(t)\phi_2(t)$$

$$= A(t)[\lambda\phi_1(t) + \lambda_2\phi_2(t)].$$

Hence the set of all solutions is a vector space.

Let ξ_i, i=1,...,n be linearly independent points in $\mathbb{R}^n$. That is ξ_i = (0,0,...,1,0,...,0) where 1 is at ith position. By the existence theorem 1.3.1 if $t_0 \in I$ there exist solutions $\phi_i(t)$ such that

$$\phi_i(t_0) = \xi_i, \quad i=1,\ldots,n.$$

If now

$$\lambda_1\phi_1(t) + \lambda_2\phi_2(t)+\ldots+\lambda_n\phi_n(t) = 0 \text{ for all } t \in I,$$

then

$$\lambda_1\phi_1(t_0) + \lambda_2\phi_2(t_0)+\ldots+\lambda_n\phi_n(t_0) = 0$$

or

$$\lambda_1\xi_1 + \lambda_2\xi_2+\ldots+\lambda_n\xi_n = 0.$$

Hence

$$\lambda_1 = \lambda_2 = \dots = \lambda_n = 0,$$

since ξ_i, i=1,...,n are linearly independent.

If ϕ is any solution of (1.6.1) such that $\phi(t_o) = \xi$, then there exist constants λ_i, i=1,...,n such that $\xi = \lambda_1\xi_1+\dots+\lambda_n\xi_n$. Hence

$$\phi(t_o) = \lambda_1\xi_1+\dots+\lambda_n\xi_n.$$

By uniqueness,

$$\phi(t) = \lambda_1\phi_1(t) + \lambda_2\phi_2(t)+\dots+\lambda_n\phi_n(t).$$

Therefore every solution of (1.6.1) is a linear combination of $\phi_i(t)$, i=1,...,n, as required.

When n linearly independent solutions are found, a matrix whose columns are these n solutions can be formed. This matrix, known as a ***fundamental matrix***, can now be used to construct any solution of equation (1.6.1). Denoting such a matrix by Φ, any solution of (1.6.1) is of the form

$$\phi(t) = \Phi(t)C,$$

where C is a constant vector.

Associated with equation (1.6.1) is a matrix differential equation

$$\dot{Z} = A(t)Z \tag{1.6.2}$$

whose solutions are n×n matrices whose columns are solutions of equation (1.6.1). A fundamental matrix Φ satisfies (1.6.2). In fact, a necessary and sufficient condition that a solution matrix Φ of (1.6.2) be a fundamental matrix is that the determinant of Φ, det Φ, does not vanish for any t.

If Φ is a fundamental matrix, then its column vectors are linearly independent. It follows that det $\Phi(t) \neq 0$ for each t.

Suppose that $\Phi(t)$ is a solution matrix of (1.6.2) and that det $\Phi(t_o) \neq 0$ for some t_o. Then the column vectors of $\Phi(t_o)$ are linearly independent. Now, if at some other point t_1, det $\Phi(t_1) = 0$, let

$$\psi(t) = a_1\phi_1(t) + \ldots + a_n\phi_n(t)$$

where $\phi_i(t)$, i=1,...,n are the column vectors of $\Phi(t)$. Then at $t = t_1$, $\psi(t_1) = 0$. By uniqueness condition, $\psi(t) \equiv 0$ since $x \equiv 0$ is a solution of (1.6.1). This contradicts the fact that $\phi_1(t_o),\ldots,\phi_n(t_o)$ are linearly independent. Hence det $\Phi(t) \neq 0$ for each t. It follows that Φ has linearly independent solutions of (1.6.1).

When a fundamental matrix for (1.6.1) has been constructed, a solution of (1.2.6) can readily be found by using the variation-of-constants formula. If $\psi(t)$ is a solution of (1.6.1), then

$$\psi(t) = \Phi(t)C, \tag{1.6.3}$$

where C is a constant n-vector. We seek a solution of (1.2.6) of the form

$$\Phi(t)C(t). \tag{1.6.4}$$

Then considering (1.6.4) as a solution of (1.2.6) we have

$$\dot{\Phi}(t)C(t) + \Phi(t)\dot{C}(t) = A(t)\Phi(t)C(t) + b(t)$$

or

$$A(t)\Phi(t)C(t) + \Phi(t)\dot{C}(t) = A(t)\Phi(t)C + b(t)$$

which reduces to

$$\Phi(t)\dot{C}(t) = b(t).$$

Since $\det \Phi(t) \neq 0$, we have

$$\dot{C}(t) = \Phi^{-1}(t)b(t)$$

or

$$C(t) = \int_{t_0}^{t} \Phi^{-1}(s)b(s)ds + C(t_0).$$

Hence the solution $\Phi(t)$ of (1.2.6) is of the form

$$\phi(t) = \phi(t_0) + \int_{t_0}^{t} \Phi(t)\Phi^{-1}(s)b(s)ds.$$

The case where $A(t)$ is a constant matrix is of particular interest because a fundamental matrix associated with it can be readily constructed. Write e^{tA} in the form

$$e^{tA} = E + tA + \frac{t^2A^2}{2!} + \ldots + \frac{t^KA^K}{K!} + \ldots, \qquad (1.6.5)$$

where E is a unit matrix. Differentiating (1.6.5) with respect to t, it is easily seen that e^{tA} satisfies equation (1.6.2). Put $t = 0$ in (1.6.5) and evaluate its determinant. The determinant is 1. Hence

$$\Phi(t) = e^{tA}$$

is a fundamental matrix for (1.6.1) for a constant matrix A.

The case where $A(t)$ is periodic calls for a special mention as well. Suppose that w is a period of $A(t)$. That is

$$A(t+w) = A(t) \text{ for all } t.$$

The *characteristic roots* of C in (1.6.8) are called *characteristic multipliers* while those of R are called *characteristic exponents.* These roots play a significant role in determining the existence of periodic solutions and stability of solutions of periodic systems.

1.7 Autonomous Systems

Autonomous systems, like linear systems, occupy a special place in the theory of differential equations. Frequent occurrence of these systems in applications has led to extensive mathematical investigations. As a result the theory of dynamical systems of which autonomous systems are part and parcel developed.

Solutions of autonomous systems are often studied in the x-space called the phase space rather than the (t,x)-space as is often the case with non-autonomous systems. This is because when a solution of an autonomous system is unique, its curve or orbit will never intersect any orbit of another solution.

For suppose that ϕ and ψ are two solutions of (1.2.5) such that $\phi(t_1) = \psi(t_2)$, where $t_1 \neq t_2$. Define

$$\chi(t) = \psi(t+t_2-t_1).$$

Then

$$\dot{\chi}(t) = \dot{\psi}(t+t_2-t_1) = f(\psi(t+t_2-t_1))$$
$$= f(\chi(t)),$$

implying that $\chi(t)$ is also a solution of (1.2.5). But $\chi(t_1) = \psi(t_2) = \phi(t_1)$. Hence by uniqueness $\chi(t) = \phi(t)$ for all t in the maximal interval of existence of ϕ. Since the orbits of $\psi(t)$ and $\chi(t)$ are identically the same, it follows that the orbits of ϕ and ψ coincide.

Three types of orbits occur in the phase space, namely a *single poi* which corresponds to a constant or trivial solution a *simple closed curve* corresponding to a nontrivial periodic solution, and a *simple arc* which represents a nontrivial, nonperiodic solution.

CHAPTER 2

CRITICAL POINTS OF 2-DIMENSIONAL AUTONOMOUS SYSTEMS

2.1 Introduction

Critical points, sometimes called equilibrium points, stationary or singular points, are quite significant in the study of systems of differential equations. Cunningham [6, page 85] points out the importance of these points. He states: "The singular points of a differential equation are fundamental in determining properties of its solutions. Considerable insight into the qualitative aspects of the solutions, and some quantitative information as well, can be had through the study of singularities."

2.2 Nondegenerate Critical Point

Critical points are those points $x(\varepsilon\ \mathbb{R}^n)$ such that $f(x) = 0$ in (1.2.5).

Consider the Jacobian matrix

$$J = \left(\frac{\partial f_i}{\partial x_j}\right) \qquad i,j=1,\dots,n.$$

Let P be a critical point of (1.2.5). P is called a *nondegenerate* critical point if

$$\det J(P) \neq 0. \tag{2.2.1}$$

Otherwise it is called a *degenerate* critical point. A nondegenerate critical point is sometimes called an elementary or simple critical point.

The consequence of (2.2.1) is that a nondegenerate critical point is isolated, since if (2.2.1) holds and P is not isolated, then there exists a sequence of critical points P_n such that $P_n \to P$ as $n \to \infty$.

Let $h_n = P_n - P$ or $P_n = P + h_n$. Then we have

$$f(P_n) = f(P) + h_n \left(\frac{\partial f_i}{\partial x_j}(P)\right) + o(h_n), \text{ as } h_n \to 0,$$

where

$$g(r) = o(r) \text{ as } r \to 0$$

means

$$\frac{\|g(r)\|}{\|r\|} \to 0 \text{ as } r \to 0.$$

But $f(P_n) = f(P) = 0$.

Hence

$$h_n \left(\frac{\partial f_i}{\partial x_j}(P)\right) = o(h_n).$$

Since (2.2.1) holds $\left(\frac{\partial f_i}{\partial x_j}(P)\right)^{-1}$ exists so that

$$h_n = o(h_n).$$

This is a contradiction.

It should be pointed out that though a nondegenerate critical point is isolated, the converse is not always true. This is easily verified by the following example:

Let

$$\dot{x}_1 = x_1$$
$$\dot{x}_2 = x_2^2.$$

Then (0,0) is the only critical point. Hence it is isolated. But

$$\det\begin{pmatrix}1 & 0\\ 0 & 2x_2\end{pmatrix} = \begin{vmatrix}1 & 0\\ 0 & 2x_2\end{vmatrix} = 2x_2,$$

and it vanishes at $x = (0,0)$. Hence it is not simple.

2.3 Type of 2-dimensional Systems Considered

Consider now the system

$$\begin{aligned}\dot{x}_1 &= P(x_1,x_2),\\ \dot{x}_2 &= Q(x_1,x_2),\end{aligned} \tag{2.3.1}$$

where P and Q are continuous and are locally Lipschitz. Without loss of generality the critical point of system (2.3.1) may be taken to be the origin. This is because if (x_1^0,x_2^0) is any critical point, a linear transformation of coordinates (x_1,x_2) by

$$\begin{aligned}U_1 &= x_1-x_1^0\\ U_2 &= x_2-x_2^0\end{aligned}$$

translates the critical point into the origin in (U_1,U_2)-space.

Suppose further that system (2.3.1) is of the form

$$\begin{aligned}\dot{x}_1 &= ax_1+bx_2+f_1(x_1,x_2),\\ \dot{x}_2 &= cx_1+dx_2+f_2(x_1,x_2),\end{aligned} \tag{2.3.2}$$

where $f_i = o(r)$ as $r \to 0$, $r^2 = x_1^2+x_2^2$, i=1,2, . That is f_i tends to zero faster than the linear terms as $r \to 0$. When $f_i = 0$, i=1,2, system (2.3.2) takes the linear form

$$\begin{aligned}\dot{x}_1 &= ax_1+bx_2\\ \dot{x}_2 &= cx_1+dx_2.\end{aligned} \tag{2.3.3}$$

For system (2.3.3) the origin is a simple critical point if $ad-bc \neq 0$.

The linear system (2.3.3) is quite important in determining the nature of solutions of the system (2.3.2). When the origin is a simple critical point, it is generally true to expect the behaviour of the orbits of solutions of (2.3.2) near the origin to be very similar to the behaviour of the orbits of (2.3.3) provided f_1 and f_2 satisfy certain minimum assumptions.

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be the coefficient matrix in system (2.3.3). Suppose that λ_1 and λ_2 are the characteristic roots of A. Let P be a nonsingular matrix. The transformation

$$x = Py$$

reduces system (2.3.3) to the form

$$y = P^{-1}APy. \qquad (2.3.4)$$

P can be chosen such that

$$B = P^{-1}AP$$

has one of the following forms:-

1. $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ ($\lambda_1 \neq \lambda_2$, both real),

2. $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ ($\lambda = \lambda_1 = \lambda_2$, real),

3. $\begin{pmatrix} \lambda & \gamma \\ 0 & \lambda \end{pmatrix}$ ($\lambda = \lambda_1 = \lambda_2$ real, and $\gamma > 0$),

4. $\begin{pmatrix} \alpha+i\beta & 0 \\ 0 & \alpha-i\beta \end{pmatrix}$ (λ_1, λ_2 complex conjugates;

Since stability or instability of the origin plays an important part in classifying critical points, it will therefore be mentioned here.

The zero solution $\psi(t) \equiv 0$ of (2.3.3) is stable if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that all solutions $\phi(t)$ for which

$$\|\phi(t_0)\| < \delta, \tag{2.3.5}$$

imply that

$$\|\phi(t)\| < \varepsilon \tag{2.3.6}$$

holds for all $t \geq t_0$.

This means that the origin is stable if any solution whose initial value lies in the spherical region of radius δ will remain in the spherical region of radius ε.

If in addition

$$\lim_{t \to \infty} \|\phi(t)\| = 0, \tag{2.3.7}$$

then the origin is said to be *asymptotically stable.* However, should (2.3.6) fail to hold, then the origin is unstable.

It is also convenient to consider the polar functions

$$r(t) = (\phi_1^2(t) + \phi_2^2(t))^{\frac{1}{2}} \tag{2.3.8}$$

and

$$\theta(t) = \tan^{-1} \frac{\phi_2(t)}{\phi_1(t)}, \tag{2.3.9}$$

where $\phi(t) = (\phi_1(t), \phi_2(t))$, when discussing critical points of 2-dimension systems. They help to trace the position and direction of the orbits.

One other technique employed in the analysis of orbits of solutions of 2-dimensional system (2.3.2) is to rewrite (2.3.2) in the form

$$\frac{dx_2}{dx_1} \quad (\text{or } \frac{dx_1}{dx_2}) \tag{2.3.10}$$

where $\dot{x}_1 \neq 0$ (or $\dot{x}_2 = 0$).

2.4 Classification of Critical Points of the Linear System (2.3.3)

Case 1: λ_1 and λ_2 are real and distinct.

(a) $\lambda_1 < 0, \lambda_2 < 0.$

Equation (2.3.4) becomes

$$\dot{y}_1 = \lambda_1 y_1,$$
$$\dot{y}_2 = \lambda_2 y_2.$$

The solution through (y_1^o, y_2^o) [$\neq (o,o)$] at $t = 0$ is

$$y_1(t) = y_1^o e^{\lambda_1 t},$$
$$y_2(t) = y_2^o e^{\lambda_2 t}.$$

Since $\lambda_i < 0$, $i=1,2$,

$$\lim_{t \to \infty} y_i(t) = 0, \quad i=1,2.$$

If $y_1^o = 0$ and $y_2^o \neq 0$, then $(0, y_2(t))$ is the solution. Similarly $(y_1(t), 0)$ is the solution through $(y_1^o, 0)$. Using (2.3.10) we have

$$y_2 = y_1^k, \qquad k = \frac{\lambda_2}{\lambda_1}.$$

The orbits near the origin are as shown in Figure 1(a).

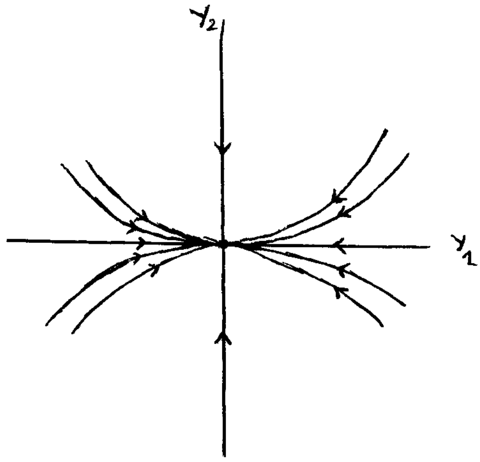


Figure 1(a)

When returning to the original coordinates (x_1, x_2), the figures are skewed but retain the qualitative features. For example, a straight line in (y_1, y_2) coordinates will be a straight line in (x_1, x_2) coordinates. For if $y_2 = my_1 + c$ and

$$A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

then the equivalent equation in (x_1, x_2) coordinates is

$$x_2 = \frac{\gamma + \delta m}{\alpha + \beta m} x_1 + \frac{\alpha\delta - \beta\gamma}{\alpha + \beta m} c$$

which is an equation of a straight line. Also a closed curve in a (y_1, y_2)-plane will be a closed curve in a (x_1, x_2)-plane.

Thus, Figure 1(a) above is equivalent to Figure 1(a).1 below.

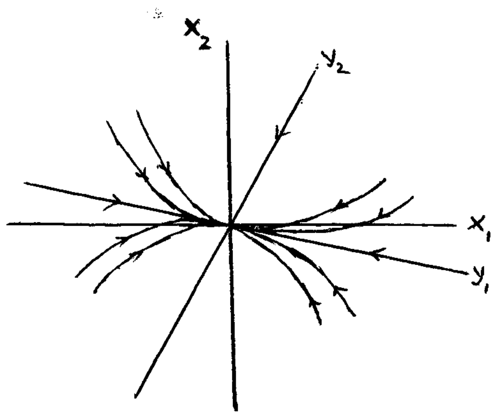


Figure 1(a).1

Every orbit in Figure 1(a) (and hence Figure 1(a).1) has the same limiting direction at the origin except the orbit along the y_2 axis. The origin is called an *ordinary stable node.*

(b) $\underline{\lambda_1 > 0,\ \lambda_2 > 0}$

Here

$$\lim_{t \to \infty} y_i(t) = \infty \quad i=1,2,...$$

The orbits are as shown in Figures 1(b) and 1(b).1.

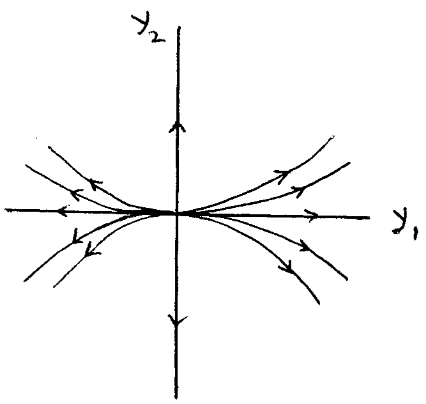


Figure 1(b)

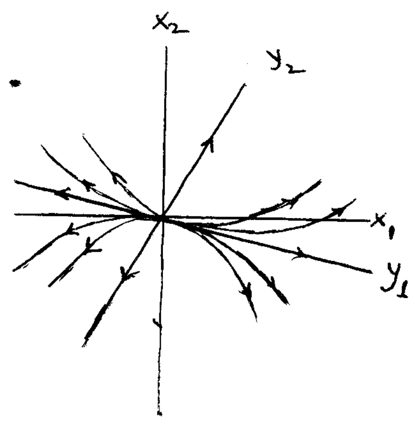


Figure 1(b).1

The origin is called an *unstable ordinary node.*

(c) $\underline{\lambda_2 < 0 < \lambda_1}$

$$y_1(t) = y_1^o e^{\lambda_1 t}$$

$$y_2(t) = y_2^o e^{\lambda_2 t} .$$

$\lim_{t \to \infty} y_1(t) = \pm \infty$ according to whether $y_1^o > 0$ or $y_1^o < 0$. However, $\lim_{t\to \infty} y_2(t) = 0$. In general, the orbits resemble hyperbolas as shown in Figures 1(c) and 1(c).1.

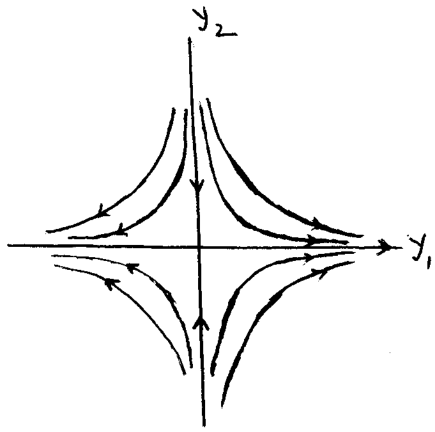


Figure 1(c)

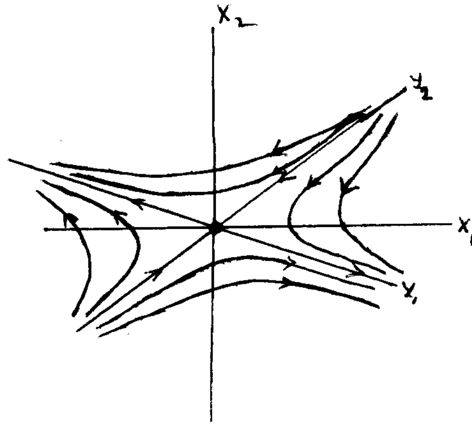


Figure 1(c).1

The origin is called a *saddle point* or a *col*. It is unstable.

Case 2: $\lambda = \lambda_1 = \lambda_2, \gamma = 0$

(a) $\lambda < 0$

System (2.3.4) becomes

$$\dot{y}_1 = \lambda y_1$$
$$\dot{y}_2 = \lambda y_2,$$

giving

$$y_1(t) = y_1^o e^{\lambda t}, \; y_2(t) = y_2^o e^{\lambda t}.$$

$\lim_{t \to \infty} \gamma(t) = 0$ and using (2.3.10)

$$y_2 = y_1.$$

Hence the orbits are straight lines

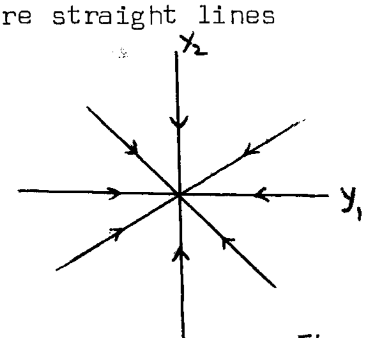


Figure 2(a)

terminating at the origin. The origin is called a *singular node*. It is stable.

(b) $\lambda > 0$

As in 2(a), the orbits are straight lines but this time originatin from the origin. The critical point is known as an unstable singular node.

Case 3: $\lambda = \lambda_1 = \lambda_2,\ \gamma > 0$

System (2.3.4) is of the form

$$\dot{y}_1 = \lambda y_1 + \gamma y_2$$

$$\dot{y}_2 = \lambda y_2.$$

So

$$y_2(t) = y_2^o e^{\lambda t},$$

and hence $$\dot{y}_1 - \lambda y_1 = \gamma y_2^o e^{\lambda t},$$

giving $$y_1(t) = (y_1^o + \gamma y_2^o t)e^{\lambda t}.$$

If $\lambda < 0$, then both $y_1(t)$ and $y_2(t)$ tend to zero as $t \to \infty$. Also

$$\frac{dy_1}{dy_2} = \frac{(y_1^o+\gamma y_2^o t)e^{\lambda t}+\gamma y_2^o e^{\lambda t}}{\lambda y_2^o e^{\lambda t}}$$

$$= \frac{y_1^o}{y_2^o} + \gamma t + \frac{\gamma}{\lambda}$$

$$= \frac{y_1^o}{y_2^o} + \gamma(t + \frac{1}{\lambda}).$$

Hence $\lim\limits_{t \to \pm\infty} \frac{dy_1}{dy_2} = \pm\,\infty.$

If $y_2^0 > 0$, both $y_1(t)$ and $y_2(t)$ are positive for t positive and large enough.

If $y_2^0 = 0$, $y_2(t) = 0$, $y_1(t)$ is either positive if $y_1^0 > 0$ or negative if $y_1^0 < 0$. In the case of $\lambda > 0$, $y_1(t)$ and $y_2(t)$ tend to infinity as $t \to \infty$. Hence the orbits are as shown in Figures 3.1 and 3.2.

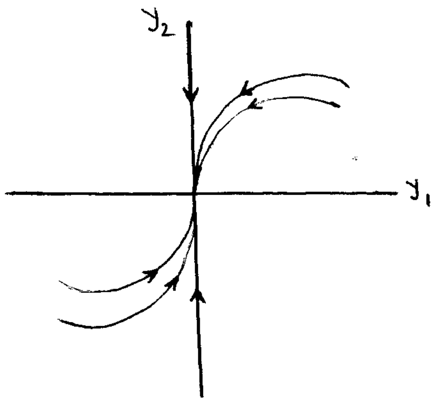


Figure 3.1

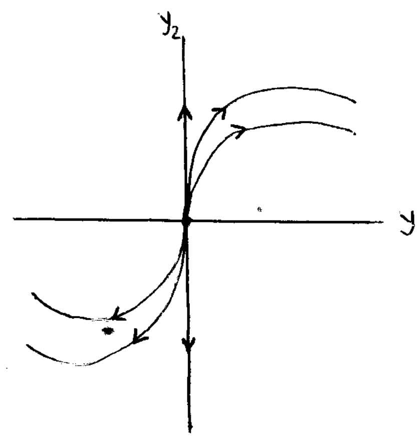


Figure 3.2

The origin is called a *stable degenerate node* when $\lambda < 0$ and an *unstable degenerate node* when $\lambda > 0$.

Case 4: $\lambda_1 = \alpha+i\beta$, $\lambda_2 = \alpha-i\beta$, $\beta \neq 0$

(a) $\alpha < 0$

The real canonical form of

$$\begin{pmatrix} \alpha+i\beta & 0 \\ 0 & \alpha-i\beta \end{pmatrix}$$

is

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

Hence

$$\begin{aligned} \dot{y}_1 &= \alpha y_1 + \beta y_2 \\ \dot{y}_2 &= -\beta y_1 + \alpha y_2. \end{aligned} \qquad (2.3.11)$$

Using polar coordinates

$$y_1 = r \cos \theta, \; y_2 = r \sin \theta,$$

then (2.3.11) becomes

$$\dot{r} = \alpha r$$

$$\dot{\theta} = -\beta.$$

Hence

$$r(t) = r_o e^{\alpha t}$$

$$\theta(t) = w(t) = -\beta t.$$

$$\lim_{t \to \infty} r(t) = 0$$

$$\lim_{t \to \infty} w(t) = \pm \infty \quad \text{if } \beta < 0 \text{ or } \beta > 0.$$

The orbits are as shown in Figures 4.1 and 4.2.

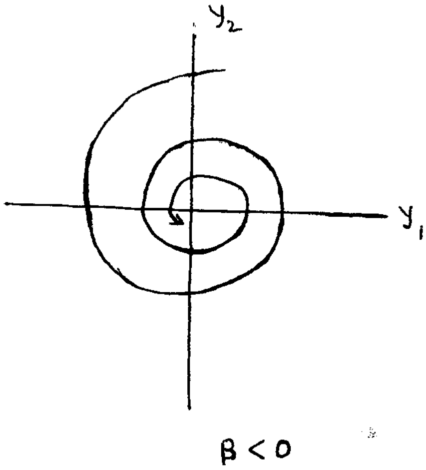


Figure 4.1

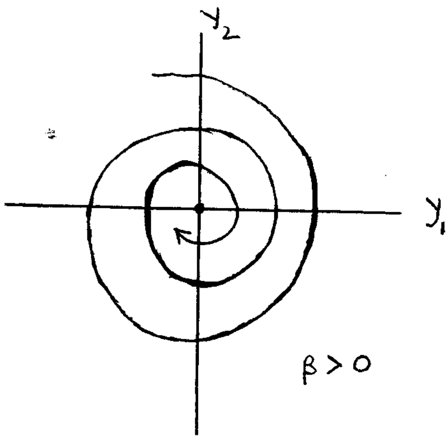


Figure 4.2

The origin is called a *focus*. It is stable.

(b) $\underline{\alpha > 0}$

The arguments are as those in (a). However, since r(t) tends to infinity as $t \to \infty$, it is called an *unstable focus* and it spirals outwardly.

(c) $\underline{\alpha = 0}$

In this case the system is

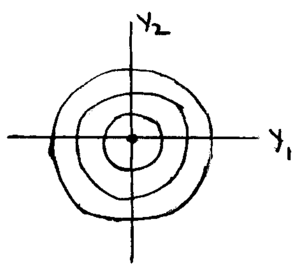
$$\dot{y}_1 = \beta y_1$$

$$\dot{y}_2 = -\beta y_2.$$

Using (2.3.10), then

$$y_1^2 + y_2^2 = c.$$

Hence the orbits are circles



and the origin is known as a *centre*. It is stable but not asymptotical stable.

The above discussion shows that if the real parts of the character istic roots are all negative or zero then the origin is stable. Otherwise it is unstable.

2.5 Critical Points in Nonlinear System (2.3.2)

The problem which arises when considering system (2.3.2) is to be able to predict the extent to which the behaviour of the orbits near the critical point of (2.3.2) can be determined by the linearized system (2.3.3). There are a few examples, given below, which show that the behaviour of orbits near a critical point of a linearized system can be different from that of orbits of the corresponding nonlinear system.

Despite some difficulties which may be encountered when relating system (2.3.2) to (2.3.3), one property is always carried over from the linearized system to the nonlinear (2.3.2). This is that if the origin is an asymptotically stable critical point for the linearized system (2.3.3) then it is one for system (2.3.2). This arises in cases where both characteristic roots have negative real parts.

Consider first the case where all characteristic roots are real and distinct. The canonical form of (2.3.2) is then

$$\begin{aligned} \dot{y}_1 &= \lambda_1 y_1 + g_1(y_1, y_2) \\ y_2 &= \lambda_2 y_2 + g_2(y_1, y_2) \end{aligned} \qquad (2.5.1)$$

where $g_i = o(r)$ as $r\ (=\sqrt{y_1^2+y_2^2}) \to 0$.

Changing to polar coordinates (2.5.1) becomes

$$\begin{aligned} r\dot{r} &= y_1\dot{y}_1 + y_2\dot{y}_2 \\ &= \lambda_1 y_1^2 + \lambda_2 y_2^2 + r\cos\theta\, g_1(r\cos\theta, r\sin\theta) \\ &\qquad + r\sin\theta\, g_2(r\cos\theta, r\sin\theta) \\ &\le \mu r^2 + rR(r,\theta), \end{aligned}$$

where $\mu = -\min(|\lambda_1|, |\lambda_2|)$.

Hence

$$\dot{r} \leq \mu r + o(r), \quad (\mu < 0). \qquad (2.5.2)$$

The case where $\lambda_1 = \lambda_2$ $(=\mu)$ and $\gamma = 0$ also gives (2.5.2). When $\gamma \neq 0$, let $\gamma = \mu$, then system (2.3.2) becomes

$$\dot{y}_1 = \mu y_1 + \mu y_2 + g_1(y_1, y_2)$$
$$\dot{y}_2 = \mu y_2 + g_2(y_1, y_2),$$

so that in the polar coordinates it becomes

$$\begin{aligned} r\dot{r} &= y_1\dot{y}_1 + y_2\dot{y}_2 \\ &= \mu y_1^2 + \mu y_1 y_2 + y_1 g_1(y_1, y_2) \\ &\quad + \mu y_2^2 + y_2 g_2(y_1, y_2) \\ &= \mu r^2 + \mu r^2 \cos\theta \sin\theta + rR(r,\theta) \end{aligned}$$

or

$$\dot{r} \leq \frac{\mu}{2} r + o(r),$$

which is of the form (2.5.2). Finally, when $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$, $\beta \neq 0$ and $\alpha < 0$, the system in polar coordinates is also of the form (2.5.2).

Hence if r is small enough, say $0 < r < r_1$ then all the cases considered give

$$r(t) \leq r(0)e^{\mu t}. \qquad (2.5.3)$$

So if $r(0) < r_1$, $r(t) < r_1$, for all $t \geq 0$. From (2.5.3) it follows that $r(t)$ tends to the origin as t tends to infinity. In other words all orbits approach the origin as t increases. Similarly, if the real parts are all positive it can be shown that the origin is unstable for system (2.3.2).

Before we can carry on further analysis, the critical points of system (2.3.2) will be precisely defined.

The origin is called an ***attractor*** if there exists a $\delta > 0$ such that for any orbit of a solution of (2.3.2) which has at least one point in $0 < r < \delta$, the solution exists over a t half line and the orbit tends to the origin as $t \to \infty$ or as $t \to -\infty$.

Thus if $f_1 = f_2 = 0$ in (2.3.2) then nodes and spirals are attractor whereas saddles and centres are not.

The origin is called a ***node*** for system (2.3.2) if it is an attracto and orbits arrive at the origin in a definite direction. This definition embraces *ordinary* and *degenerate* nodes. If orbits arrive in a definite direction at the origin and every straight line through the origin is tangent to some orbit, then the origin is called a *singular node*.

The origin is called a *focus* if it is an attractor such that $|\theta(t)| \to +\infty$ as $t \to \infty$ (or $-\infty$), where $\theta(t)$ is as defined in (2.3.9). I there exists a neighbourhood U of the origin such that all orbits in U are closed, then the origin is called a *centre*.

If there exists exactly one solution tending to the origin as $t \to \infty$ and this solution lies on a curve $y_2 = \psi(y_1)$ where $\psi(y_1)$ has a continuous first derivative and $\dot{\psi}(o) = 0$, then the origin is called a *col* or a *saddle point*. The roles of y_1 and y_2 are interchanged for $t \to -\infty$.

Theorem 2.5.1

If the origin is a focus for the linearized system (2.3.3), then it is a focus for the nonlinear system (2.3.2).

Proof

It has already been shown that the origin is an attractor for (2.3.2). The equations (taken to be in canonical form) for (2.3.2) are

$$\begin{aligned} \dot{y}_1 &= \alpha y_1 + \beta y_2 + g_1(y_1,y_2) \\ \dot{y}_2 &= -\beta y_1 + \alpha y_2 + g_2(y_1,y_2). \end{aligned} \qquad (2.5.4)$$

The equation for θ in polar coordinates is given by

$$\begin{aligned} r^2\dot{\theta} &= y_1\dot{y}_2 - \dot{y}_1 y_2 \\ &= y_1[-\beta y_1+\alpha y_2+g_2(y_1,y_2)] \\ &\quad - y_2[\alpha y_1+\beta y_2+g_1(y_1,y_2)] \\ &= -\beta(y_1^2+y_2^2) + y_1 g_2(y_1,y_2) - y_2 g_1(y_1,y_2) \\ &= -\beta r^2 + o(r^2), \qquad \text{as } r \to 0, \end{aligned}$$

or

$$\dot{\theta} = -\beta + o(1), \qquad \text{as } r \to 0.$$

But $r \to 0$ as $t \to \infty$ (for $\alpha < 0$). Hence as $t \to \infty$

$$\dot{\theta} = -\beta + o(1).$$

Therefore for any solution $(y_1(t), y_2(t))$ sufficiently near the origin

$$\theta(t) = -\beta t + o(t).$$

It follows that

$$|\theta(t)| \to \infty \quad \text{as } t \to \pm\infty .$$

This completes the proof.

Though attractors of (2.3.3) go into attractors of (2.3.2) it is not generally true that a node or a centre for (2.3.3) goes into a node or a centre respectively of (2.3.2). The following two examples illustrate the situation.

Example 1

Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 - \frac{x_2}{\log_e (x_1^2+x_2^2)^{\frac{1}{2}}} \\ \dot{x}_2 &= -x_2 + \frac{x_1}{\log_e (x_1^2+x_2^2)^{\frac{1}{2}}} . \end{aligned} \tag{2.5.5}$$

The linear system is

$$\begin{aligned} \dot{x}_1 &= -x_1 \\ \dot{x}_2 &= -x_2 , \end{aligned} \tag{2.5.6}$$

and the origin is a singular node for (2.5.6). The polar equations corresponding to (2.5.5) are

$$\begin{aligned} \dot{r} &= -r \\ \dot{\theta} &= \frac{1}{\log_e r} , \end{aligned}$$

giving

$$r(t) = ce^{-t} \text{ and hence}$$

$$\theta(t) = -\log_e(t-\log_e c) + k$$

where c,k are constants.

So $$\theta(t) \to -\infty \text{ as } t \to \infty .$$

Hence the origin is a focus.

Example 2

$$\begin{aligned} \dot{x}_1 &= -x_2 - x_1\sqrt{x_1^2+x_2^2} \\ \dot{x}_2 &= x_1 - x_2\sqrt{x_1^2+x_2^2} . \end{aligned} \tag{2.5.7}$$

The linear system is

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 . \end{aligned} \tag{2.5.8}$$

The origin is a centre for (2.5.8).

Changing (2.5.7) to polar coordinates we get

$$\begin{aligned} r\dot{r} &= x_1\dot{x}_1 + x_2\dot{x}_2 \\ &= -x_1x_2 - x_1^2(x_1^2+x_2^2)^{\frac{1}{2}} + x_1x_2 - x_2^2(x_1^2+x_2^2)^{\frac{1}{2}} \\ &= -(x_1^2+x_2^2)^{3/2} \\ &= -r^3, \end{aligned}$$

or

$$\dot{r} = -r^2$$

and

$$r^2\dot{\theta} = x_1\dot{x}_2 - x_2\dot{x}_1$$

$$= x_1^2 - x_1x_2(x_1^2+x_2^2)^{\frac{1}{2}} + x_2^2 + x_1x_2(x_1^2+x_2^2)^{\frac{1}{2}}$$

$$= x_1^2 + x_2^2$$

$$= r^2.$$

Hence $\dot{\theta} = 1.$

The solution through (r_o,θ_o) at $t = 0$, $(r_o \neq 0)$, is given by

$$r(t) = (t + \frac{1}{r_o})^{-1}$$

$$\theta(t) = t + \theta_o.$$

Therefore as $t \to \infty$

$$r(t) \to 0$$

and

$$\theta(t) \to \infty.$$

Hence the origin is a focus for (3.5.7).

Theorem 2.5.2

If the origin is a centre for (2.3.3), then it is either a centre or a focus for (2.3.2).

Proof

The system in canonical form is

$$\begin{aligned} \dot{y}_1 &= \beta y_2 + g_1(y_1,y_2) \\ \dot{y}_2 &= -\beta y_1 + g_2(y_1,y_2). \end{aligned} \qquad (2.5.8)$$

In polar coordinates (2.5.8) reduces to

$$\dot{r} = \cos\theta\, g_1(r\cos\theta, r\sin\theta) + \sin\theta\, g_2(r\cos\theta, r\sin\theta) \tag{2.5.9}$$

$$\dot{\theta} = -\beta + \frac{\cos\theta}{r} g_2(r\cos\theta, r\sin\theta) - \frac{\sin\theta}{r} g_1(r\cos\theta, r\sin\theta)$$

or

$$\dot{r} = o(r) \qquad (2\cdot5\cdot10)$$

$$\dot{\theta} = -\beta + o(1)$$

as $r \to 0$. It follows therefore that for sufficiently small r, $\theta(t)$ is either positive for $\beta < 0$ or negative for $\beta > 0$, so that $\theta(t) \to \pm\infty$ as $t \to \infty$.

Theorem 2.5.3

If f_1 and f_2 are such that

$$|f_i| = o(r^{1+\varepsilon}) \quad (i=1,2) \text{ as } r \to 0 \tag{2.5.11}$$

where $\varepsilon > 0$, then if the origin is a singular node for (2.3.3), it is a singular node for (2.3.2).

Proof

The system in canonical form is

$$\begin{aligned} \dot{y}_1 &= \lambda y_1 + g_1(y_1, y_2) \\ \dot{y}_2 &= \lambda y_2 + g_2(y_1, y_2). \end{aligned} \tag{2.5.12}$$

In polar coordinates (2.5.12) is

$$\begin{aligned} \dot{r} &= \lambda r + \cos\theta\, g_1(r\cos\theta, r\sin\theta) \\ &\qquad + \sin\theta\, g_2(r\cos\theta, r\sin\theta) \\ \dot{\theta} &= \frac{\cos\theta}{r} g_2(r\cos\theta, r\sin\theta) - \frac{\sin\theta}{r} g_1(r\cos\theta, r\sin\theta) \end{aligned} \tag{2.5.13}$$

from which

$$\frac{d\theta}{dr} = \frac{\frac{1}{r}\,[\cos\theta\, g_2(r\cos\theta, r\sin\theta) - \sin\theta\, g_1(r\cos\theta, r\sin\theta)]}{\lambda r + \cos\theta\, g_1(r\cos\theta, r\sin\theta) + \sin\theta\, g_2(r\cos\theta, r\sin\theta)} \cdot \tag{2.5.14}$$

Let the right hand side of (2.5.14) be $F(r,\theta)$.

By (2.5.11)

$$|F(r,\theta)| \leq 2C\, r^{1+\varepsilon}, \quad C > 0 \text{ as } r \to 0.$$

Thus

$$\left|\int_0^{r_0} F(r,\theta(r))dr\right| \leq 2C \int_0^{r_0} r^{\varepsilon+1} dr < \infty, \tag{2.5.16}$$

for r small enough. It follows therefore that $\dot{\theta} = \theta(r) = \theta(r(t))$ tends to a finite angle as $r \to 0$, as required.

Theorem 2.5.4

If the origin is an ordinary node for (2.3.3) then every orbit of (2.3.2) near the origin has a limiting direction which makes an angle of 0, $\pi/2$, π or $3\pi/2$.

Proof

The system in canonical form is given by

$$\begin{aligned} \dot{y}_1 &= \lambda_1 y_1 + g_1(y_1, y_2) \\ \dot{y}_2 &= \lambda_2 y_2 + g_2(y_1, y_2). \end{aligned} \tag{2.5.17}$$

In polar coordinates (2.5.17) reduces to

$$\begin{aligned} \dot{r} &= r(\lambda_1\cos^2\theta + \lambda_2\sin^2\theta) + o(r) \\ \dot{\theta} &= (\lambda_2 - \lambda_1)\cos\theta\sin\theta + o(1). \end{aligned} \tag{2.5.18}$$

Consider the following regions for $\varepsilon > 0$ $(0 < \varepsilon < \pi/4)$:

$$R_1 : |\theta| \leq \varepsilon$$

$$R_2 : |\theta - \pi/2| \leq \varepsilon$$

$$R_3 : |\theta - \pi| \leq \varepsilon$$

$$R_4 : |\theta - 3\pi/2| \leq \varepsilon$$

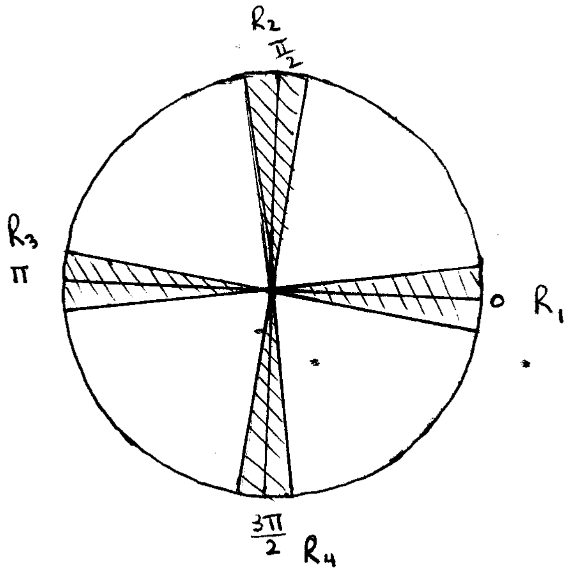


Figure 2.5.1

In the canonical coordinates (y_1, y_2), the regions are as shown in Figure 2.5.2.

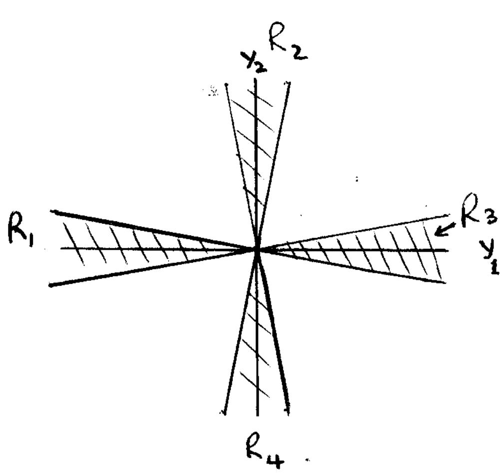


Figure 2.5.2

Now, from (2.5.17)

$$\dot{\theta} = \tfrac{1}{2}(\lambda_2-\lambda_1)\sin 2\,\theta + o(1). \qquad (2.5.18)$$

Let $\lambda_2 < \lambda_1 < 0$. The other cases can be treated similarly. On the line $\theta = \varepsilon$, $\sin 2\varepsilon > 0$. By (2.5.18) $\dot{\theta} < 0$ for sufficiently small r. Also, on the line $\theta = -\varepsilon$, $\dot{\theta} > 0$. Hence if r is sufficiently small then any orbit getting in R_1 stays in R_1. A similar argument holds for R_3. On the other hand, on line $\theta = \varepsilon + \pi/2$,

$$\sin 2(\pi/2 + \varepsilon) = -\sin 2\varepsilon,$$

Similarly

$$\sin 2(\pi/2 - \varepsilon) = + \sin 2\varepsilon.$$

It follows that any orbit which is outside R_2 cannot get into R_2 since the direction of any orbit on the boundary of R_2 is towards the exterior of R_2. This is also true for region R_4. Now, select $\delta > 0$ small enough such that orbits starting in $0 < r \leq \delta$ behave as outlined above.

We show that if an orbit C starts inside $0 < r < \delta$ it approaches the origin at an angle of 0, $\pi/2$, π or $3\pi/2$. Suppose that this is not true. Then C does not lie in any of the four regions R_1,R_2,R_3 or R_4 for some ε_o. But if C is in the region $\varepsilon_o < \theta < \pi/2 - \varepsilon_o$, it eventually enters R_1, since for $\theta = \varepsilon_o$

$$\tfrac{1}{2}(\lambda_2-\lambda_1)\sin 2\varepsilon_o < 0$$

implying that $\dot{\theta} < 0$. Hence C enters R_1 for every ε. It follows therefore that C approaches the origin at an angle of π. A similar procedure holds if C is in any region other than R_1,R_2,R_3 or R_4, thus completing the proof.

It can be shown (see for example [3; page 384]) that if $\frac{\partial f_1}{\partial x_1}$ and $\frac{\partial f_2}{\partial x_1}$ exist and are continuous in $0 \leq r \leq \delta$, then there exists exactly one orbit tending to the origin in the directions $\pi/2$ and $3\pi/2$.

In the case of a saddle point for (2.3.3) the following theorem describes the geometry of the orbits of (2.3.2) near the origin.

Theorem 2.5.5

There exists at least one orbit tending to the origin at each of the angles 0 and π. If in addition $\frac{\partial f_1}{\partial x_2}$ and $\frac{\partial f_2}{\partial x_2}$ exist and are continuous in $0 \leq r \leq \delta$, then there exists exactly one orbit tending to the origin at each of the angles 0 and π. Any orbit starting sufficiently near either of these orbits in the neighbourhood of the origin tends away from them as $t \to \infty$.

The proof can be found, for example, in [3, page 387].

CHAPTER 3

STABILITY

3.1 Definitions

In Chapter 2 the concept of stability was mentioned. Though stability was introduced when discussing critical points, nontrivial solutions can equally be said to be stable or unstable. As in the case of critical points, a change of coordinates may be applied to the system such that the discussion is based on the zero solution.

Consider for instance a system of differential equations

$$\dot{y} = g(t,y), \tag{3.1.1}$$

such that g is continuous and satisfies a Lipschitz condition. Suppose that $\phi(t)$ is a nonzero solution of (3.1.1) under discussion. By setting

$$x = y - \phi(t)$$

or

$$y = x + \phi(t),$$

system (3.1.1) is transformed into

$$\dot{x} = g(t,x+\phi(t)) - g(t,\phi(t)). \tag{3.1.2}$$

Denote by f(t,x) the right-hand side of (3.1.2). Then

$$\dot{x} = f(t,x), \; f(t,0) = 0, \tag{3.1.3}$$

and the zero solution x(t) = 0 of (3.1.3) corresponds to the nonzero solution $\phi(t)$. In this regard, when discussing a zero solution, we shall have in mind system (3.1.3).

While the above descriptions of stability may apply to solutions of nonautonomous and some solutions of autonomous systems such as critical points and nonperiodic solutions, complications do arise when periodic solutions of autonomous systems are discussed. For example, if $\phi(t)$ is a nontrivial periodic solution of (1.2.5) it cannot be asymptotically stable. For suppose $\phi(t)$ is asymptotically stable. Since $\phi(t+\mu)$ is also a solution of (1.2.5), given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < r < \|\phi(t_0) - \phi(t_0+\mu)\| < \delta \qquad (3.1.4)$$

and

$$\lim_{t \to \infty} \|\phi(t)-\phi(t+\mu)\| = 0. \qquad (3.1.5)$$

Let T be a period of $\phi(t)$. Put $t = t_0 + nT$. Then $t \to \infty$ as $n \to \infty$, but

$$0 < r < \|\phi(t_0+nT)-\phi(t_0+nT+\mu)\|.$$

It is in this respect that certain concepts of stability have been formulated specifically to describe periodic solutions. Two such concepts are orbital and phase asymptotic stability.

In the definition of orbital stability, the distance from an orbit is significant. Let C be a closed orbit. The distance between a point x and C, denoted by d(x,C), is

$$d(x,C) = \text{infimum}\{d(x,y), y \in C\}. \qquad (3.1.4)$$

C is said to be ***orbitally stable*** if any solution whose orbit at some point comes close to C stays within the region near C. That is, given $\varepsilon > 0$, there exists a $\delta > 0$ such that for any solution $\psi(t)$ which at some point t_0

$$d(\psi(t_0),C) < \delta$$

implies that

$$d(\psi(t),C) < \varepsilon$$

for all $t \geq t_0$. If in addition

$$\lim_{t \to \infty} d(\psi(t),C) = 0,$$

then C is said to be ***asymptotically orbitally stable***.

Orbital stability is not very close to the definitions of stability introduced earlier on. The following definition is much closer.

Let $\phi(t)$ be a solution of an autonomous system (1.2.5). $\phi(t)$ is said to be ***uniformly stable*** if, given $\varepsilon > 0$, there exists δ depending only on ε, such that if $\psi(t)$ is a solution of (1.2.5) which at some t_1 and t_2 satisfies

$$\|\phi(t_1)-\psi(t_2)\| < \delta,$$

then

$$\|\phi(t+t_1)-\psi(t+t_2)\| < \varepsilon$$

for all $t \geq 0$. If in addition there exists a constant T such that

$$\lim_{t \to \infty} \|\phi(t+T)-\psi(t)\| = 0,$$

then ϕ is said to be *phase asymptotically stable*. T is called an *asymptotic phase*.

3.2 Liapunov Functions

The stability of the zero solution can be established through the use of Liapunov functions. The systems to be discussed are (3.1.3) and

$$\dot{x} = f(x), \quad f(0) = 0. \tag{3.2.1}$$

where f is defined and satisfies a Lipschitz condition locally.

Let V(x) be a real-valued function from $\mathbb{R}^n$ into $\mathbb{R}$ and suppose that V(x) satisfies the following two conditions:

(i) V(x) has continuous first order partial derivatives with respect to x_i, i=1,...,n in an open region Ω about the origin;

(ii) V(0) = 0.

The function V is said to be *positive semi-definite* on Ω if for all $x \in \Omega$

$$V(x) \geq 0. \tag{3.2.2}$$

If

$$V(x) > 0, \tag{3.2.3}$$

then V is said to be *positive definite* on Ω. *Negative semi-definite* and *negative definite* are when (3.2.2) and (3.2.3) are reversed.

V is called a *Liapunov function* if it satisfies a definiteness condition and the function

$$\dot{V}(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \dot{x}_i \tag{3.2.4}$$

satisfies a definiteness condition of the opposite sign, where $x = \phi(t)$ is a solution of (3.2.1).

In the case of a nonautonomous system (3.1.3) the definition of a Liapunov function has to be slightly modified. Let $W(x)$ satisfy conditions (i) and (ii) and a definiteness condition. Let $V(t,x)$ satisfy the following conditions:

(iii) $V(t,x)$ is defined in Ω for all $t \geq 0$;

(iv) $V(t,x)$ has continuous first order partial derivatives

$$\frac{\partial V}{\partial t}, \frac{\partial V}{\partial x_i}, \quad i=1,\ldots,n;$$

(v) $V(t,0) = 0$ for all $t \geq 0$;

(vi) $W(x) \leq V(t,x)$ [or $W(x) \leq -V(t,x)$ if W satisfies a negative definiteness condition].

Then $V(t,x)$ is a Liapunov function if

$$\dot{V}(t,x) = \frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} \dot{x}_i \tag{3.2.5}$$

satisfies a definiteness condition of the opposite sign to $V(t,x)$.

Theorem 3.2.1

If $V(x)$ is positive definite and $\dot{V}(x) \leq 0$ in some neighbourhood Ω of the origin, then the zero solution is stable. If in addition $-\dot{V}(x) > 0$ for all $x \in \Omega$, $x \neq 0$, then the zero solution is asymptotically stable.

The theorem holds if $V(x)$ is replaced by $V(t,x)$ for a non-autonomous system.

Proof

Given $\varepsilon > 0$, define

$$H_\varepsilon(0) = \{x : \|x\| = \varepsilon\} \subset \Omega$$

and

$$S_\varepsilon(0) = \{x : \|x\| < \varepsilon\} \subset \Omega.$$

$H_\varepsilon(0)$ is a compact set. Since $V(x)$ is continuous and positive definite on $H_\varepsilon(0)$ there exists a positive number K such that $V(x) \geq K$ on $H_\varepsilon(0)$. Since $V(x)$ is continuous at $x = 0$ and $V(0) = 0$, there exists $\delta > 0$ such that $0 < \delta < \varepsilon$ and $V(x) < K$ for $x \in S_\delta(0)$.

Let $\phi(t)$ be a solution which at $t = t_o$, $\phi(t_o) = x_o$ and $x_o \in S_\delta(0)$. Since $\dot{V}(\phi(t)) \leq 0$ for $t \geq t_o$, we must have $V(\phi(t))$ nonincreasing for $t \geq t_o$. That is $V(\phi(t)) \leq V(\phi(t_o)) < K$. This implies that $\phi(t) \in S_\delta(0)$ for all $t \geq t_o$.

Now if $-\dot{V}(x) > 0$, then $V(x)$ is decreasing along the orbit of $x = \phi(t)$ and tends to the origin. For if not, then $-\dot{V}(x)$ has a zero not identical with the origin, to which $-\dot{V}(x)$ tends. But this cannot happen since $-\dot{V}(x)$ is positive definite in Ω and for $x \neq 0$. Hence $\phi(t)$ tends to the origin as $t \to \infty$.

Theorem 3.2.2

Let $V(x)$, with $V(0) = 0$, have first order partial derivatives in Ω. Suppose that $\dot{V}(x)$ is positive definite and $V(x)$ assumes positive values arbitrarily near the origin. Then the origin is unstable.

Proof

Take $\varepsilon > 0$. Choose $\delta \leq \varepsilon$ and define $S_\delta(0)$ and $H_\varepsilon(0)$ as in the proof of Theorem 3.2.1. Let $x_0 \in S_\delta(0)$ and consider a solution $\phi(t)$ which at $t = t_0$ has the value x_0. The point x_0 can be chosen such that $V(x_0) > 0$. Since $\dot{V}(x)$ is positive definite $V(x)$ increases along the orbit of $\phi(t)$. Hence such an orbit cannot tend to the origin. Also $\dot{V} \geq K > 0$ in Ω and therefore $V(x)$ cannot approach a fixed value in $S_\varepsilon(0)$. The orbit of $\phi(t)$ therefore leaves $S_\varepsilon(0)$.

Theorem 3.2.3

Under the same assumptions for $V(x)$ as in Theorem 3.2.2 with

$$\dot{V}(x) = \lambda V(x) + V^*(x)$$

where $V^*(x)$ is non-negative in Ω and $\lambda > 0$, the zero solution is unstable.

Proof

As in the proof of Theorem 3.2.2, let $x_0 \in S_\delta(0)$ be an initial value of $\phi(t)$ at $t = 0$ such that $V(x_0) > 0$. Now

$$\frac{d}{dt} V(\phi(t)) = \lambda V(\phi(t)) + V^*(\phi(t))$$

or

$$\frac{d}{dt} [e^{-\lambda t} V(\phi(t))] = e^{-\lambda t} V^*(\phi(t)).$$

So

$$e^{-\lambda t} V(\phi(t)) = V(x_0) + \int_0^t e^{-\lambda \tau} V^*(\phi(\tau)) d\tau.$$

Since $V^*(\phi(t)) \geq 0$ in Ω,

$$e^{-\lambda t} V(\phi(t)) \geq V(x_0) > 0$$

or

$$V(\phi(t)) \geq V(x_0) e^{\lambda t} > 0.$$

Hence for all t, V is increasing on the orbit of ϕ. Therefore $\phi(t)$ cannot stay in any neighbourhood $S_\varepsilon(0)$ contained in Ω.

3.3 Stability of a System

It is of interest to examine whether a system of differential equations which is known to have stable systems will retain stability under perturbations.

Consider the system

$$\dot{x} = f(t,x). \tag{1.2.4}$$

The perturbed system is of the form

$$\dot{x} = f(t,x) + g(t,x). \tag{3.3.1}$$

One approach used is to place conditions on f and find out what perturbations g(t,x) will preserve stability. For example, suppose that (1.2.4) is the homogeneous equation

$$\dot{x} = Ax, \quad (x \in \mathbb{R}^n) \tag{1.6.1}$$

where A is a real n×n constant matrix with all its characteristic roots having negative real parts. The following theorem holds.

Theorem 3.3.1

If A satisfies the conditions above and if g(t,x) is continuous and

$$\frac{\|g(t,x)\|}{\|x\|} \to 0 \quad \text{as} \quad \|x\| \to 0,$$

then the zero solution of

$$\dot{x} = Ax + g(t,x) \tag{3.3.2}$$

is asymptotically stable.

Proof

Recall that e^{tA} is a fundamental matrix for (1.6.1). Let $\phi(t)$ be a solution of (3.3.2) which exists for all t with $\phi(0)$ sufficiently small. Now

$$\phi(t) = e^{tA}\phi(0) + \int_0^t e^{(t-s)A}g(s,\phi(s))ds.$$

Since A has all its characteristic roots with negative real parts, there exist positive numbers K and σ such that

$$\|e^{tA}\| \leq Ke^{-\sigma t}.$$

Then

$$\|\phi(t)\| \leq Ke^{-\sigma t}\|\phi(0)\| + \int_0^t Ke^{-(t-s)\sigma}\|g(s,\phi(s))\|ds.$$

Since

$$\frac{\|g(t,x)\|}{\|x\|} \to 0 \quad \text{as} \quad \|x\| \to 0,$$

for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|g(t,x)\| < \frac{\varepsilon}{K}\|x\|$$

whenever

$$\|x\| \leq \delta.$$

Therefore, so long as $\phi(t)$ remains small enough so that

$$\|\phi(t)\| \leq \delta\ ,$$

then

$$\|\phi(t)\| \leq Ke^{-\sigma t}\|\phi(0)\| + \varepsilon \int_0^t e^{-(t-s)\sigma}\|\phi(s)\|ds$$

or

$$e^{\sigma t}\|\phi(t)\| \leq K\|\phi(0)\| + \varepsilon \int_0^t e^{\sigma s}\|\phi(s)\|ds.$$

By Lemma 1.5.2

$$e^{\sigma t}\|\phi(t)\| \leq K\|\phi(0)\|e^{\varepsilon t}$$

or

$$\|\phi(t)\| \leq K\|\phi(0)\|e^{(\varepsilon-\sigma)t}.$$

Since ε can be chosen such that $\varepsilon < \sigma$, then provided

$$\|\phi(0)\| < \frac{\delta}{K}$$

we have

$$\|\phi(t)\| < \delta \quad \text{for all } t \geq 0,$$

and

$$\lim_{t \to \infty} \|\phi(t)\| = 0.$$

The proof is now complete.

On the other hand, if not all the real parts of the characteristic roots are negative, the zero solution is not stable (cf. [3; page 317]). However, if some characteristic roots have negative real parts, some solutions of (3.3.2) will tend to zero as $t \to \infty$. In fact, if A has K characteristic roots, with negative real parts, there exists in the x-space, a real K-dimensional manifold S containing the origin such that any solution $\phi(t)$ of (3.3.2) with $\phi(t_0)$ on S for some large t_0 will tend to the origin as $t \to \infty$. On the other hand, if $\phi(t_0)$ is not on S though $\phi(t_0)$ may be near the origin $\phi(t)$ will not tend to the origin as $t \to \infty$. Details may be seen for example in [3; page 330].

The results above apply well to the case where the constant matrix A is replaced by a real periodic matrix A(t) of period w. The characteristic exponents replace the characteristic roots of A. In particular the following theorem holds.

Theorem 3.3.2

If A in (1.6.1) is replaced by a real periodic matrix A(t) of period w and (1.6.1) has all its characteristic exponents with negative real parts then the zero solution of (3.3.2) is asymptoticall stable.

Proof

A fundamental matrix of (1.6.1), Φ, is now given by

$$\Phi(t) = P(t)e^{tR}$$

where P(t) is a periodic nonsingular matrix of period w and R is a constant matrix with all its characteristic roots having negative real parts. If we let

$$x = P(t)y,$$

then

$$\begin{aligned} \dot{x} &= \dot{P}(t)y + P(t)\dot{y} \\ &= (\dot{\Phi}e^{-tR} - \Phi e^{-tR}R)y + P(t)\dot{y} \\ &= (A(t)\Phi e^{-tR} - \Phi e^{-tR}R)y + P(t)\dot{y} \\ &= A(t)P(t) - P(t)R)y + P(t)\dot{y} \end{aligned}$$

so that (3.3.2) becomes

$$\dot{y} = Ry + P^{-1}(t)g(t,P(t)y), \qquad (3.3.3)$$

and Theorem 3.3.1 applies now to (3.3.3).

The other approach used when studying the perturbed systems is to identify systems which will preserve stability under specific perturbations. Chow and Yorke [2] have shown that under interval bounded perturbations, system (3.1.3) will preserve stability.

CHAPTER 4

PERIODIC SOLUTIONS

4.1 Introduction

A physical system may be described by a system of differential equations. The problem which may arise therefore is to find out whether there exists a solution of the differential equations which will show some oscillatory behaviour of the physical system represented by the system of differential equations. This leads to the search of periodic solutions. Naturally, asymptotically stable solutions would be the most suitable to seek because they show that when a physical system works under slightly varied conditions it will eventually regain its 'normal' behaviour.

This chapter looks at some of the literature on the existence of periodic solutions.

4.2 2-dimensional Autonomous Systems

Consider the system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1,x_2) \\ \dot{x}_2 &= f_2(x_1,x_2), \end{aligned} \qquad (4.2.1)$$

where f_1 and f_2 are continuous and satisfy a Lipschitz condition in some open subset D of $\mathbb{R}^2$. Recall that $x^o = (x_1^o, x_2^o)$ is called a critical point of $f = (f_1, f_2)$ if $f(x^o) = 0$. A point at which $f(x) \neq 0$ is called a regular point.

Suppose $\phi(t)$ is a solution of an n-dimensional autonomous system. Denote by $\gamma(\phi)$ its orbit. Taking $\mathbb{R}$ as its interval of existence, $\gamma^{+}(\phi)$ will represent a semiorbit for $0 \leq t < \infty$ and $\gamma^{-}(\phi)$ another semiorbit for $-\infty < t \leq 0$. Let $\Omega(\phi)$ be the set of limit points of $\phi(t)$ as $t \to \infty$. The set of limit points of $\phi(t)$ as $t \to -\infty$ can be represented by $\Lambda(\phi)$. Some of the properties of $\Omega(\phi)$ (or $\Lambda(\phi)$) are summarized by the following two theorems. Let $D \subset \mathbb{R}^{n}$.

<u>Theorem 4.2.1</u>

<u>If</u> $\gamma^{+}(\phi)$ <u>is contained in a closed and bounded subset</u> K <u>of</u> D, <u>then</u> $\Omega(\phi)$ <u>is nonempty, closed and connected.</u>

<u>Proof</u>

Since $\gamma^{+}(\phi)$ is contained in a closed and bounded subset K, the sequence of points defined by

$$P_n = \phi(t+n), \qquad n=1,2,...$$

is bounded and has a convergent subsequence in K. The limit of such a convergent subsequence is in K since K is closed. Hence $\Omega(\phi)$ is nonempty.

Let α be a limit point of $\Omega(\phi)$. We show that α is in $\Omega(\phi)$. There exists a sequence of points α_n in $\Omega(\phi)$ such that $\alpha_n \to \alpha$. That is, $d(\alpha_n,\alpha) \to 0$ as $n \to \infty$. Since α_n are in $\Omega(\phi)$ there exists a sequence of points t_{n_m} such that $\phi(t_{n_m}) \to \alpha_n$ as $m \to \infty$. That is $d(\phi(t_{n_m}),\alpha_n) \to 0$ as $m \to \infty$. Hence for each n, as $m \to \infty$,

$$d(\phi(t_{n_m}),\alpha_n) < \frac{1}{n}.$$

Therefore

$$d(\phi(t_{n_m}),\alpha) \leq d(\phi(t_{n_m}),\alpha_n) + d(\alpha_n,\alpha)$$

$$< \frac{1}{n} + d(\alpha_n,\alpha).$$

So, as $n \to \infty$, $\phi(t_{n_m}) \to \alpha$. Hence $\alpha \in \Omega(\phi)$.

Finally, suppose that $\Omega(\phi)$ is not connected. Then there exist two closed nonempty sets A and B in $\mathbb{R}^n$ such that

$$\Omega(\phi) = A \cup B$$

and

$$A \cap B = \emptyset.$$

Since $\Omega(\phi)$ is bounded, A and B are bounded, and consequently A and B are a positive distance apart. Let

$$d(A,B) = \inf\{d(x,y) : x \in A,\ y \in B\} = \delta,$$

where $\delta > 0$. Since the points in $\Omega(\phi)$ are limit points, there exists a sequence of points $t_n \to \infty$ as $n \to \infty$ such that

$$d(\phi(t_n),A) < \frac{\delta}{4} \quad \text{for } n \text{ odd}$$

and

$$d(\phi(t_n),B) < \frac{\delta}{4} \quad \text{for } n \text{ even.}$$

Now $\phi(t)$ is continuous in t; so if we define

$$g(t) = d(\phi(t),A)$$

and

$$h(t) = d(\phi(t),B)$$

we have that $g(t)$ and $h(t)$ are continuous functions. For large t

$$g(t_{2m-1})-h(t_{2m-1}) < 0$$

since

$$\begin{aligned} g(t_{2m-1})-h(t_{2m-1}) &= d(\phi(t_{2m-1}),A) \\ &\quad - d(\phi(t_{2m-1}),B) \\ &\leq \frac{\delta}{4} - d(\phi(t_{2m-1}),B), \end{aligned}$$

and

$$d(\phi(t_{2m-1}),B) > \frac{\delta}{4}.$$

A similar argument shows that

$$g(t_{2m})-h(t_{2m}) > 0.$$

Hence there exists a sequence of points t_m' such that $t_m' \in (t_{2m-1}, t_{2m})$ and

$$g(t_m') - h(t_m') = 0. \tag{4.2.2}$$

Since $\phi(t_m')$ is bounded in K, $(\phi(t_m'))$ has a convergent subsequence, say $\phi(t_m') \to \xi$. By (4.2.2)

$$d(\xi,A) = d(\xi,B). \tag{4.2.3}$$

Since $\xi \in \Omega(\phi)$, $\xi \in A \cup B$. Now, $A \cap B = \emptyset$, so $\xi \notin A \cap B$. If $\xi \in A$, then

$$d(\xi, A) = 0$$

and

$$d(\xi, B) = \delta,$$

contradicting (4.2.3). Similarly, if $\xi \in B$, (4.2.3) is contradicted. Hence $\Omega(\phi)$ is connected. This completes the proof.

Before we start the other theorem, we define the concept of invariance. A subset E of D is said to be *invariant* if for each point $x^0 \in E$, $\phi(t, t_0, x^0) \in E$ for all t.

Theorem 4.2.2

If $\gamma^+(\phi)$ is contained in a closed and bounded subset K of D, then $\Omega(\phi)$ is invariant.

Proof

Let $x^0 \in \Omega(\phi)$. If x^0 is a critical point then the theorem holds trivially. Suppose x^0 is not a critical point. Let $\psi(t)$ be a solution through x^0 at $t = t_0$. Since $x^0 \in \Omega(\phi)$, there exists a sequence of points t_n with $t_n \to \infty$ as $n \to \infty$ such that $\phi(t_n) \to x^0 = \psi(t_0)$. Let t_1 be in the domain of ψ. We show that there exists a sequence of points on $\gamma^+(\phi)$ which converge to $\psi(t_1)$. Let $\tau = t_1 - t_0$. By Theorem 1.5.1, provided x^0 and $\psi(t_1)$ are sufficiently close

$$\lim_{n \to \infty} \phi(t_n + \tau) = \psi(t_0 + \tau)$$
$$= \psi(t_1),$$

as required.

We now return to system (4.2.1). The aim in this section is to prove the Poincaré-Bendixson theorem which is one of the main theorems in 2-dimensional systems.

A concept which plays a prominent role is that of a transversal. A *transversal* with respect to f is a finite closed straight line segment L with the following defining properties:-

(a) every point of L is a regular point of f;

(b) the vector field f is never tangent to L.

The following are other properties which can be proved (see for example [3; pages 392-3]):-

(1) Every regular point of f can be made an interior point of some transversal.

(2) All orbits which meet a transversal cross it in the same direction as t increases.

(3) Given any interior point P of a transversal L and any $\varepsilon > 0$, there exists a circle C_ε centred at P such that any orbit which passes through C_ε at $t = 0$ crosses L for some t, where $|t| < \varepsilon$.

(4) A finite closed arc B of an orbit meeting a transversal L does so in a finite number of points whose order on B is the same as the order on L.

(5) A transversal meets a periodic orbit once.

Lemma 4.2.3

If all the points in $\Omega(\phi)$ are regular and the semi-orbits $\gamma^+(\phi)$ and $\Omega(\phi)$ have a point in common, then $\gamma(\phi)$ is a periodic orbit.

Proof

Let $\xi \in \gamma^+(\phi) \cap \Omega(\phi)$. Let $\xi = \phi(t_1)$. ξ is regular and therefore can be made an interior point of a transversal L. Since

$\xi \varepsilon \Omega(\phi)$, there exists a circle C_ε centred at ξ such that for some t_n, $\eta = \phi(t_n) \varepsilon L$ and $|t_1 - t_n| < \varepsilon$. Suppose that $\eta \neq \xi$. The arc $\widehat{\xi\eta}$ meets L in a finite number of points. Furthermore successful intersections of the orbit of ϕ and L form a monotonic sequence tending away from η. Hence $\eta \notin \Omega(\phi)$. This implies that the orbit does not meet L in any other point except at ξ. Therefore the orbit is periodic.

Remark: The proof of Lemma 4.2.3 also shows that a transversal meets the limit set once only.

Lemma 4.2.4

If a limit set $\Omega(\phi)$ contains a periodic orbit $\gamma(\psi)$, then it is identical with it.

Proof

Suppose that $\gamma(\psi)$ is properly contained in $\Omega(\phi)$. Let $\Omega(\phi)-\gamma(\psi)$ be the complement of $\gamma(\psi)$. Since $\Omega(\phi)$ is connected, $\gamma(\psi)$ must conta a limit point of $\Omega(\phi)-\gamma(\psi)$, say P. P is a regular point and hence let L be a transversal through P. There exist a circle C_ε around P and a regular point Q in $\Omega(\phi)-\gamma(\psi)$ such that $Q \varepsilon C_\varepsilon$ and $Q \varepsilon L$. An orbit through Q exists and it is unique. Furthermore this orbit is entirely contained in $\Omega(\phi)$. P and Q are distinct and $\Omega(\phi)$ meets L at these two points. This contradicts the remark following Lemma 4.2.3.

Hence $\Omega(\phi) = \gamma(\psi)$.

Theorem 4.2.5 (Poincaré-Bendixson)

Let $\gamma^+(\phi)$ be a semi orbit contained in a closed and bounded subset K of D. If $\Omega(\phi)$ contains regular points only, then either $\gamma(\phi)$ is a periodic orbit in which case it is identical with $\Omega(\phi)$, or $\Omega(\phi)$ is itself a periodic orbit. (In the case where $\Omega(\phi)$ is the periodic orbit, it is called a limit cycle.)

Proof

If the orbit of ϕ is periodic then there is nothing to prove. Suppose therefore that $\gamma^+(\phi)$ is not periodic. Since $\Omega(\phi)$ has regular points only, there is an orbit $\gamma(\psi)$ contained in $\Omega(\phi)$. Let P be a limit point of $\gamma(\psi)$. Then P is in $\Omega(\phi)$. Let L be a transversal through P. Then L meets $\Omega(\phi)$ at P only. Since P is a limit point of $\gamma(\psi)$, L meets $\gamma(\psi)$ at P otherwise it would meet $\Omega(\phi)$ twice. Hence P is a common point of $\gamma(\psi)$ and $\Omega(\phi)$. So $\gamma(\psi)$ is periodic. But $\gamma(\psi) \subset \Omega(\phi)$. Hence $\gamma(\psi) = \Omega(\phi)$, as required.

When $\Omega(\phi)$ contains a finite number of critical points, a third possibility may arise. That is $\Omega(\phi)$ will contain orbits each of these tending to one critical point.

There are a variety of ways of showing that a given region has no closed orbits or limit cycles. One of them is known as the *Bendixson Criterion*. It states that there are no closed orbits in a simply connected region on which

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \quad (\neq 0) \qquad (4.2.4)$$

is of one sign. This is due to the fact that if C is a closed orbit in a simply connected region R, by Green's theorem

$$\iint_R \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\right) dx_1 dx_2 = \oint_C f_1 dx_1 - f_2 dx_2$$
$$= 0$$

which is not compatible with the fact that the integrand on the left hand side is nonzero and of one sign.

When f_1 and f_2 are everywhere differentiable and (4.2.1) can be transformed into a polar coordinates system of equations

$$\begin{aligned} \dot{r} &= R(r,\theta) \\ \dot{\theta} &= \theta(r,\theta) \end{aligned} \tag{4.2.5}$$

where

(a) $R(0,\theta) = 0$ for all θ

(b) $\theta(r,\theta) > 0$ for all θ and $r \neq 0$,

(c) $\frac{\partial}{\partial r}\left(\frac{R}{\theta}\right) \neq 0$ $(r \neq 0)$

then Lloyd [11] has shown that (4.2.5) has no limit cycles.

Another method used to establish nonexistence of a limit cycle is based on the concept of an *index number* defined below.

If C is a simply closed curve, the vector $f(x_1,x_2)$ can be observed changing in angle, say θ, as $x = (x_1,x_2)$ traverses the curve once.

Let the total change be $\Delta\theta$. Certainly

$$\Delta\theta = 2\pi I,$$

where I is called the ***index of C*** with respect to f.

The index of a simply closed curve is associated with the critical points contained within it. If a curve C does not enclose any critical point, its index is always zero. If two closed curves enclose one and the same critical point, they have the same index. The index of a critical point is the index of a closed curve enclosing only that critical point. Thus, it has been possible to work out the indices of the critical points classified in Chapter 2. A node, stable or unstable, has index 1, a centre 1, a spiral 1 and a saddle -1.

When C is an orbit of a periodic solution of systym (4.2.1), its index has been found to be +1. This implies that a periodic orbit always encircles at least one critical point. In a region where the only critical point is a saddle (4.2.1) cannot have a periodic orbit.

4.3 n-dimensional Systems

While the Poincaré-Bendixson theorem, the Bendixson criterion, etc., apply well to 2-dimensional systems, difficulties arise when higher dimensional systems are considered. This is mainly due to the fact that some concepts applied to 2-dimensional systems, like the Jordan curve principle applied in the Poincaré-Bendixson theorem, fail to extend to higher dimensions. Sell [15] successfully *obtains results which form a kind of extension of the Poincaré-* Bendixson theorem to higher dimensional systems. Sell's theorem also applies to 2-dimensional systems.

Let

$$\dot{x} = f(x) \tag{1.2.5}$$

be an n-dimensional system where f is continuous and satisfies the Lipschitz condition in $I \times D$. Let E be a subset of D. We shall define a few concepts required in the proof of Sell's theorem.

E is called a ***minimal set*** relative to system (1.2.5) if it is nonempty, closed and invariant and E has no proper subset with these three properties. If in addition E is bounded, then it is called a ***compact minimal set.***

A solution $\phi(t)$ of (1.2.5) is ***recurrent*** if, given any $\varepsilon > 0$, there exists a number $T > 0$ such that for any two numbers t_1 and t_2 there is a number w such that $t_1 < w < t_1 + T$ and

$$\|\phi(t_2) - \phi(w)\| < \varepsilon.$$

That is, a solution ϕ is recurrent if an arc of the orbit of ϕ of time length T approximates the entire orbit of ϕ.

Let g be a real-valued function mapping $\mathbb{R}$ into $\mathbb{R}^n$. g is ***almost periodic*** if for every $\varepsilon > 0$ the set

$$\{T \in \mathbb{R} : \|g(t+T)-g(t)\| < \varepsilon \text{ for all } t \in \mathbb{R}\}$$

is dense in $\mathbb{R}$. This means that given $\varepsilon > 0$ there exists a number L such that any interval (t_0, t_0+L) contains a number T for which

$$\|g(t+T)-g(t)\| < \varepsilon$$

for all t in $\mathbb{R}$.

Theorem 4.3.1 (Sell)

If $\phi(t)$ is a bounded phase asymptotically stable solution of (1.2.5), then there is a phase asymptotically stable periodic solution $\psi(t)$ of (1.2.5) such that $\gamma(\psi) = \Omega(\phi)$.

The proof of theorem 4.3.1 requires the following:

Theorem 4.3.2 (Sell-Deysach)

If $\phi(t)$ is a bounded uniformly stable solution of (1.2.5), then there exists an almost periodic uniformly stable solution $\psi(t)$ of (1.2.5) such that

$$\gamma(\psi) \subset \Omega(\phi).$$

The proof of Theorem 4.3.2 follows through the following lemmas and Birkhoff's theorem.

Theorem 4.3.3 (Birkhoff)

Every orbit in a compact minimal set is recurrent.

Proof of Theorem 4.3.3

Let $\gamma(\phi)$ be an orbit of a solution $\phi(t)$ where $\gamma(\phi)$ is contained in a compact minimal set M but $\phi(t)$ is not recurrent. Then given any $\varepsilon > 0$ there exist sequences $(\bar{t}_n)$ and $((t_n, T_n))$ such that

$$\|\phi(\bar{t}_n) - \phi(t)\| \geq \varepsilon \qquad \text{for all } t \in (t_n, t_n + T_n).$$

The sequence $(\phi(\bar{t}_n))$ has a convergent subsequence, say the sequence itself converges. That is

$$\phi(\bar{t}_n) \to x.$$

Also $\phi(t_n)$ has a convergent subsequence, say

$$\phi(t_n) \to y.$$

Let $\psi(t)$ be a solution of (1.2.5) such that $\psi(0) = y$. By Theorem 1.5.1, if $z(t)$ is a solution such that

$$\|y-z(0)\| < \delta$$

for $\delta > 0$, then

$$\|\psi(t)-z(t)\| < \frac{\varepsilon}{3}$$

for all $t \in (0,T)$ where $T > 0$.

Select n such that $T_n > T$,

$$\|y-\phi(t_n)\| < \delta$$

and

$$\|x-\phi(\bar{t}_n)\| < \frac{\varepsilon}{3}.$$

Then

$$\|\psi(t)-\phi(t_n+t)\| < \frac{\varepsilon}{3}$$

for a fixed $t \in (0,T)$. But for $t \in (0,T)$

$$\|\phi(\bar{t}_n)-\phi(t_n+t)\| \geq \varepsilon.$$

So

$$\begin{aligned}\|\psi(t)-x\| &= \|\psi(t)-\phi(t_n+t)+\phi(t_n+t)-\phi(\bar{t}_n)+\phi(\bar{t}_n)-x\| \\ &\geq \|\phi(\bar{t}_n)-\phi(t_n+t)\| \\ &- \|\psi(t)-\phi(t_n+t)\| \\ &- \|\phi(\bar{t}_n)-x\| \\ &\geq \frac{\varepsilon}{3}.\end{aligned}$$

It follows that

$$\|\psi(t)-x\| \geq \frac{\varepsilon}{3}$$

for all $t \geq 0$. This result means that $\Omega(\psi) \subset M-\Omega(\phi) \subset M$. Since $\Omega(\psi)$ is nonempty, closed and invariant, it is a contradiction. This completes the proof.

The proof of Theorem 4.3.2 follows now through the following lemmas whose proofs can be found for instance in [4; pages 235-237].

Lemma 4.3.4

If $\phi(t)$ is a bounded solution of (1.2.5), then there exists a recurrent solution $\psi(t)$ of (1.2.5) such that

$$\gamma(\psi) \subset \Omega(\phi).$$

Proof

$\Omega(\phi)$ is nonempty, closed and invariant and hence contains a minimal set [13; page 374]. By Birkhoff's theorem every orbit in $\Omega(\phi)$ is recurrent.

Lemma 4.3.5

If $\phi(t)$ *is a bounded uniformly stable solution of (1.2.5) and* $\psi(t)$ *is a solution of (1.2.5) such that* $\gamma(\psi) \subset \Omega(\phi)$, *then* $\psi(t)$ *is uniformly stable. Furthermore, if* $\phi(t)$ *is phase asymptotically stable then so is* $\psi(t)$.

Proof

See [4; pages 235 and 238].

Lemma 4.3.6

If $\phi(t)$ *is a recurrent uniformly stable solution of (1.2.5) then* $\phi(t)$ *is almost periodic.*

Proof

Since $\phi(t)$ is recurrent, given $\delta > 0$ there exists a relatively dense set $\{w\}$ such that

$$\|\phi(0)-\phi(w)\| < \delta \ . \tag{4.3.1}$$

But $\phi(t)$ is uniformly stable. So taking the δ given in the recurrence as our δ in uniform stability, then (4.3.1) implies that

$$\|\phi(t+w)-\phi(t)\| < \varepsilon \tag{4.3.2}$$

for all $t \geq 0$. Since $\{w\}$ is dense in $\mathbb{R}$, the result follows.

Thus Lemma 4.3.6 completes the proof of the Sell-Deysach theorem.

Lemma 4.3.7

If $\phi(t)$ is a recurrent solution of (1.2.5) such that $\phi(t)$ is not periodic, then

$$\gamma(\phi) \not\subset \Omega(\phi).$$

Proof

See for instance [4; page 240].

Proof of Sell's theorem

By Sell-Deysach theorem and Lemma 4.3.5 there is a solution $\psi(t)$ which is phase asymptotically stable and almost periodic and such that $\gamma(\psi) \subset \Omega(\phi)$. By almost periodicity, we get $\gamma(\psi) \subset \Omega(\psi)$. So by Lemma 4.3.7, ψ is periodic. Since ψ is periodic $\Omega(\psi) \subset \gamma(\psi)$. Then $\Omega(\psi) \subset \Omega(\phi)$, which contradicts the minimality of $\Omega(\phi)$. Hence $\Omega(\phi) = \gamma(\psi)$.

4.4 1- and 2-dimensional Nonautonomous Systems

Generally, the literature on nonautonomous systems is not as prolific as that of autonomous systems. Within the nonautonomous systems, prominence has been given to periodic systems, hence this review is on periodic systems.

For a system

$$\dot{x} = f(t,x), \quad f(t+w,x) = f(t,x), \tag{1.2.7}$$

what is required is to find out when it will have a periodic solution of the same period w. This has the added advantage that a period is fixed already, unlike in autonomous systems where a period is unknown in the first place.

In the case where (1.2.7) is 1-dimensional, the following theorem due to Massera [12] holds. Suppose that f satisfies the Lipschitz condition so that uniqueness holds.

Theorem 4.4.1

If the 1-dimensional system (1.2.7) has a solution which remains bounded as $t \to \infty$, then the (1.2.7) has a periodic solution with period w.

Proof

Let $\phi(t)$ be a bounded solution of (1.2.7) for $t \geq 0$. Define

$$\phi_n(t) = \phi(t+nw), \quad n=1,2,\ldots$$

with $\phi_n(0) = \phi(nw)$ as the initial value at $t = 0$. $\phi_n(t)$ so defined is a solution of (1.2.7) since

$$\begin{aligned} \dot{\phi}_n &= \dot{\phi}(t+nw) \\ &= f(t+nw, \phi(t+nw)) \\ &= f(t, \phi(t+nw)) \\ &= f(t, \phi_n). \end{aligned}$$

By uniqueness of solutions, if $\phi_n(0) = \phi_{n+1}(0)$ then $\phi_n(t) = \phi_{n+1}(t)$ for all t. Hence if $\phi_n(0) < \phi_{n+1}(0)$ then $\phi_n(t) < \phi_{n+1}(t)$ for all t. Similarly if $\phi_{n+1}(0) < \phi_n(0)$, $\phi_{n+1}(t) < \phi_n(t)$ for all t. Since (ϕ_n) is a bounded monotonic sequence it has a uniformly convergent subsequence. Assuming that (ϕ_n) is itself convergent, say to $\psi(t)$, then $\psi(t)$ is a solution of (1.2.7) since the convergence is uniform and f in (1.2.7) is continuous.

Now $\psi(0) = \lim\limits_{n \to \infty} \phi_n(0) = \lim\limits_{n \to \infty} \phi(nw)$,

also $\psi(0) = \lim\limits_{n \to \infty} \phi_{n+1}(0) = \lim\limits_{n \to \infty} \phi(nw+w)$.

Hence $\psi(t)$ is periodic with period w, as required.

For 2-dimensional systems, Massera [12] gave the following example to show that a system may have a bounded solution yet fail to have a periodic solution of the same period.

Example

$$\begin{aligned} \dot{x} &= f(u,v)\cos^2 \pi t - g(u,v)\sin \pi t \cos \pi t - \pi y \\ \dot{y} &= g(u,v)\cos^2 \pi t + f(u,v)\sin \pi t \cos \pi t + \pi x \end{aligned} \tag{4.4.1}$$

where

$x, y \in \mathbb{R}$

$u = x \cos \pi t + y \sin \pi t$

$v = y \cos \pi t - x \sin \pi t$

and f and g satisfy the following conditions:-

(1) f, g have continuous first partial derivatives;

(2) $f(-u,-v) = f(u,v)$, $g(-u,-v) = g(u,v)$;

(3) $f(1,0) = g(1,0) = 0$,

$f(0,v) = 0$, $g(0,v) > 0$ for all v.

(4) $\displaystyle\int_{-\infty}^{\infty} \left(\frac{1}{g(0,v)}\right) dv < \frac{2}{\pi}$.

In (u,v) coordinates (4.4.1) is

$$\dot{u} = f(u,v)\cos \pi t,$$
$$\dot{v} = g(u,v)\cos \pi t. \tag{4.4.2}$$

The orbits of (4.4.2) are the integral curves of solutions of

$$\frac{dv}{du} = \frac{g(u,v)}{f(u,v)} . \tag{4.4.3}$$

Now, system (4.4.1) is periodic with period 1. Since $u = \pm 1$, $v = 0$ are solutions of (4.4.2) then $x = \pm\cos \pi t$, $y = \pm \sin \pi t$ are solutions of (4.4.1) and they are definitely bounded. Of course, they are periodic, but with period 2. The question is: can there be a solution with period 1?

$u = 0$ is an orbit of (4.4.3) and because of uniqueness if at some point $t = t_o$, $u(t_o) > 0$, then $u(t) > 0$ for all t. Such a solution when reverted to the original coordinates cannot give rise to a periodic solution of period π since u would change sign. Indeed

$$\begin{aligned} u &= x \cos \pi(t+1) + y \sin \pi(t+1) \\ &= -x \cos \pi t - y \sin \pi t \\ &= -(x \cos \pi t + y \sin \pi t). \end{aligned}$$

Similarly if $u(t_o) < 0$, the same argument holds. Finally, if $u = 0$ and $v(t)$ is a solution of (4.4.2) then $v(t)$ is given by

$$\int_{v(t_0)}^{v(t)} \frac{dv}{g(0,v)} = \int_{t_0}^{t} \cos \pi\tau \; d\tau$$

$$= \frac{1}{\pi} [\sin \pi t - \sin \pi t_0] \; .$$

Letting t vary between $-\frac{1}{2}$ and $+\frac{1}{2}$ then

$$\int_{v(-\frac{1}{2})}^{v(+\frac{1}{2})} \frac{dv}{g(0,v)} = \frac{1}{\pi} [\sin \frac{\pi}{2} - \sin(- \frac{\pi}{2})]$$

$$= \frac{2}{\pi}$$

which contradicts (4) above. Hence $v(t)$ cannot be defined for all t. Therefore it cannot give rise to a periodic solution in (x,y) coordinates.

Massera [12] however has the following theorem:

Theorem 4.4.2

If all solutions of a 2-dimensional system (1.2.7) exist in the future and one of them is bounded in the future, then (1.2.7) has a periodic solution of period w.

The proof uses a topological lemma due to Brouwer recast as a fixed point theorem. The lemma states that given a simply connected plane-open domain G and a homeomorphism T of G into itself, T being orientation-preserving and there exists a point x_0 in G and a sub-sequence of its successive images $x_1 = Tx_0$, $x_2 = Tx_1$, $x_n = Tx_{n-1}$,... which converges to a point in G, then T has a fixed point in G.

Proof of Theorem 4.4.2

Let ϕ be a solution of (1.2.7). Define T as

$$T : x^0 \to \phi(w,0,x^0).$$

T so defined is orientation-preserving and maps $\mathbb{R}^2$ into itself. If ϕ is a bounded solution, then its successive images $\phi(w)$, $\phi(2,w)$,... form a bounded sequence and therefore has a convergent subsequence. Applying Brouwer's lemma, T has a fixed point. That is, there is a point $x \in \mathbb{R}^2$ such that

$$Tx = x$$

or

$$\begin{aligned} \phi(w,0,x) &= x \\ &= \phi(0,0,x). \end{aligned}$$

Hence a periodic solution exists.

4.5 n-dimensional Nonautonomous Systems

We first define one boundedness concept to be used in a theorem to follow.

Solutions of (1.2.4) are said to be *ultimately bounded* if there exist a positive number B which is independent of a particular solution, and T > 0 which may depend on a particular solution, such that every solution $\phi(t,t_0,\xi)$ satisfies

$$\|\phi(t,t_0,\xi)\| < B \quad \text{for all } t \geq t_0+T.$$

The following theorem (see for instance Yoshizawa [16, page 172]) holds for the case where n is any positive integer.

Theorem 4.5.1

If the solutions of (1.2.7) are ultimately bounded for bound B, then there exists a periodic solution $\phi(t)$ of period w such that $\|\phi(t)\| \leq B$ for all t.

The proof uses Browder's fixed point theorem (see [16, page 163]). A mapping

$$Tx^0 = x(w,0,x^0)$$

is shown to have a fixed point in suitable convex sets which are chosen using the property of boundedness enjoyed by the system (1.2.7).

4.6 Perturbed Nonautonomous Systems

Consider the system

$$\dot{x} = g(t,x) + \mu h(t,x,\mu). \qquad (4.6.1)$$

Suppose that for $\mu = 0$ (4.6.1) has a periodic solution $\psi(t)$ of period Conditions are sought under which (4.6.1) with $\|\mu\|$ sufficiently small has a periodic solution, say $\phi(t,\mu)$ which as $\mu \to 0$ approaches $\psi(t)$.

A frequently used device employs an equation commonly called the *variational equation* derived as follows:-

Suppose

$$\dot{x} = f(t,x) \qquad (1.2.4)$$

is a system of differential equations where f is continuous and has first partial derivatives with respect to x_i, $i=1,\ldots,n$ in a region Δ containing the solution $u(t)$. Let $\phi(t)$ be another solution of (1.2.4) and let

$$y = \phi(t) - u(t).$$

Then

$$\begin{aligned} \dot{y} &= \dot{\phi}(t) - \dot{u}(t) \\ &= f(t,y+u(t)) - f(t,u(t)) \\ &= f_x(t,u(t))y + o(\|y\|). \end{aligned} \tag{4.6.2}$$

The equation

$$\dot{y} = f_x(t,u(t))y, \tag{4.6.3}$$

is called the variational equation of (1.2.4) with respect to $u(t)$. f_x is a matrix

$$\left(\frac{\partial f_i}{\partial x_j}\right), \quad j,i=1,2,\ldots,n.$$

For convenience, consider (4.6.1) in the form

$$\dot{x} = f(t,x,\mu), \tag{4.6.4}$$

and suppose that f is continuous and has first partial derivatives. We suppose then that when $\mu=0$ (4.6.4) has a periodic solution $\psi(t)$. The following theorem is due to Poincaré.

Theorem 4.6.1

If the variational equation of (4.6.4) with respect to a periodic solution $\psi(t)$ has no solution of period w for $\mu=0$, then for sufficiently small $\|\mu\|$ (4.6.4) has a periodic solution $\phi(t,\mu)$ which is such that

$$\lim_{\|\mu\|\to 0} \phi(t,\mu) = \psi(t).$$

Proof

Let $\phi(t)$ be a solution of (4.6.4) which has the value $\psi(0) + \alpha$, $\|\alpha\|$ sufficiently small, at $t = 0$. Thus $\phi(t)$ can be written as $\phi(t,\alpha,\mu)$. If $\phi(t,\alpha,\mu)$ is periodic with period w, then

$$\phi(w,\alpha,\mu) = \phi(0,\alpha,\mu). \tag{4.6.5}$$

But $\phi(0,\alpha,\mu) = \psi(0) + \alpha$. Hence

$$\phi(w,\alpha,\mu) - \psi(0) + \alpha = 0. \tag{4.6.6}$$

Equation (4.6.6) has a trivial solution when $\mu=0$ which is $\alpha=0$. By the Implicit Function theorem (see [14, page 224]), α is a function of μ if at $(\mu,\alpha) = (0,0)$ the Jacobian

$$\frac{\partial(F_1,\ldots,F_n)}{\partial(\alpha_1,\ldots,\alpha_n)} \neq 0, \tag{4.6.7}$$

where $F_i = \phi_i(w,\alpha,\mu)-\alpha_i-\psi_i(0)$, $i=1,\ldots,n$. But at $(0,0)$, (4.6.7) is the determinant

$$\det[\phi_\alpha(w,0,0)-E], \tag{4.6.8}$$

where E is a unit matrix.

Thus if (4.6.8) does not vanish, then there exists a unique function $\alpha = \alpha(\mu)$ such that (4.6.6) holds. This precisely means that $\phi(t,\alpha(\mu),\mu)$ is periodic.

How does the variational equation relate to the existence of periodic solutions in the above case? Consider the solution $\phi(t,\alpha,\mu)$ above. Since it is a solution of (4.6.4) we have

$$\dot{\phi}(t,\alpha,\mu) = f(t,\phi(t,\alpha,\mu),\mu). \tag{4.6.9}$$

Differentiating (4.6.9) with respect to α_i, $i=1,\dots,n$, gives

$$\dot{\phi}_\alpha(t,\alpha,\mu) = f_x(t,\phi(t,\alpha,\mu),\mu)\phi_\alpha(t,\alpha,\mu). \quad (4.6.$$

Hence $\phi_\alpha(t,\alpha,\mu)$ satisfies the variational equation with respect to $\phi(t,\alpha,\mu)$. At $\mu = 0$, $\alpha = 0$ (4.6.10) gives

$$\dot{\phi}_\alpha(t,0,0) = f_x(t,\phi(t,0,0),0)\phi_\alpha(t,0,0).$$

Realising that

$$\phi(t,0,0) = \psi(t),$$

then

$$\dot{\phi}_\alpha(t,0,0) = f_x(t,\psi(t),0)\phi_\alpha(t,0,0). \quad (4.6.11)$$

Hence the matrix

$$\Phi(t) = \phi_\alpha(t,0,0)$$

satisfies the variational equation with respect to the periodic solution $\psi(t)$. Furthermore Φ is a fundamental matrix since $\Phi(0) = E$. Consider now

$$\det[\Phi(w)-\lambda E] = 0. \quad (4.6.12)$$

The roots of (4.6.12) are the *characteristic multipliers* associated with the variational equation with respect to $\psi(t)$. (4.6.12) is (4.6 when $\lambda = 1$. Thus (4.6.8) vanishes if and only if $\lambda = 1$ is a root of (4.6.12). Since $\lambda = 1$ is a root of (4.6.12) if

$$\dot{y} = f_x(t,\psi(t),0)y \quad (4.6.13)$$

has a periodic solution of period w, then the theorem is proved.

Theorem 4.6.2

If the characteristic multipliers associated with (4.6.13) are all less than one in magnitude, then (4.6.4) has a periodic solution $\phi(t,\mu)$ which is asymptotically stable provided $\|\mu\|$ is sufficiently small.

Proof

The existence of $\phi(t,\mu)$ as a periodic solution is evident from the previous theorem.

Let

$$\dot{y} = f_x(t,\phi(t,\mu),\mu)y \tag{4.6.14}$$

be the variational equation with respect to $\phi(t,\mu)$. If $\Psi(t,\mu)$ is a fundamental matrix of (4.6.14) with $\Psi(0,\mu) = E$, then the characteristic multipliers of (4.6.14) are the characteristic roots of $\Psi(w,\mu)$. Since the characteristic roots of $\Psi(w,0)$ are less than one in magnitude and $\Psi(t,\mu)$ is continuous in μ for sufficiently small $\|\mu\|$, the characteristic roots of $\Psi(t,\mu)$ are also less than one in magnitude for sufficiently small $\|\mu\|$. Using Theorem 3.3.1 and recalling that (4.6.14) is a linear equation of (4.6.2), the theorem is proved after realising that the real parts of the characteristic exponents are negative if and only if the characteristic multipliers are less than one in magnitude.

4.7 Perturbed Autonomous Systems

In the case of

$$\dot{x} = f(x,\mu), \tag{4.7.1}$$

a slight modification is called for. This is because if $\psi(t)$ is a periodic solution of period w when $\mu = 0$, its derivative $\dot{\psi}(t)$ is a solution of the variational equation with respect to $\psi(t)$. This can easily be seen from the fact that if $\psi(t)$ is a solution of (4.7.1) when $\mu = 0$ then

$$\dot{\psi} = f(\psi, 0). \tag{4.7.2}$$

Now, differentiating (4.7.2) with respect to t gives

$$\frac{d}{dt}\dot{\psi} = f_x(\psi, 0)\dot{\psi}. \tag{4.7.3}$$

But $\dot{\psi}$ is periodic with the same period w. Hence Theorem 4.6.1 cannot apply. This situation is resolved by the following theorem whose proof follows similar steps as that in the proof of Theorem 4.6.1.

Theorem 4.7.1

If the variational equation

$$\dot{y} = f_x(\psi, 0)y \tag{4.7.4}$$

with respect to the periodic solution ψ has 1 as a simple characteristic multiplier, then for sufficiently small $\|\mu\|$, there exists a solution $\phi(t,\mu)$ of (4.7.1) with period w such that $\phi(t,0) = \psi(t)$. If n-1 of the characteristic multipliers associated with the variational equation (4.7.4) of (4.7.1) are less than one in magnitude, then $\phi(t,\mu)$ is phase asymptotically stable for sufficiently small $\|\mu\|$.

One other theorem called Hopf-Bifurcation Theorem is used to show that periodic solutions exist for the case where μ is real. Hsü [7] recasts the theorem in the following way:-

Theorem 4.7.2

Let f be analytic in $D \times [-c,c]$, $c > 0$ and suppose that there exists a vector function g defined on $[-c,c]$ such that $f(g(\mu),\mu) = 0$.

$f(x,\mu)$ can be expanded as

$$f(x,\mu) = f(g(\mu),\mu) + [x-g(\mu)]f_x(g(\mu),\mu) + O([x-g(\mu)]^2)$$

$$= [x-g(\mu)]f_x(g(\mu),\mu) + O([x-g(\mu)]^2).$$

If there exist exactly two complex conjugate characteristic roots $\alpha(\mu)$ and $\overline{\alpha(\mu)}$ of $f_x(g(\mu),\mu)$ with the properties

$$Re[\alpha(0)] = 0 \text{ and } Re[\alpha'(0)] \neq 0, \quad (' = \frac{d}{d\mu})$$

then there exists a periodic solution $\phi(t,\varepsilon)$ of period $w(\varepsilon) \neq g(\mu(\varepsilon))$ for all sufficiently small $\varepsilon \neq 0$.

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