SOME CLASSES OF LINEAR AND NONLINEAR OPERATORS

BY

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I declare that this dissertation represents my own work and it has not been previously submitted for a degree at this or another University.

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TO MY "MOTHER" MAUREEN NSHINKA

ABSTRACT

The work presented here is a survey of some known results in functional analysis, particularly in the field of Operator theory.

The study begins with definitions of linear spaces and some topological results which form the necessary background.

Chapter two is the heart of the work and deals with operators defined on a Hilbert space with mention of operators which are generalization of linear operators on finite dimensional spaces.

Chapter three looks at two classes of nonlinear operators known as Lipschitz and α-Lipschitz which frequently occur in applications. The last chapter is a brief look at the spectral theory of operators with no emphasis on a particular type of operator.

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CONTENTS

1. LINEAR SPACES AND TRANSFORMATIONS

1.1 INTRODUCTION

Linear transformations make it easier to study abstract spaces. Linear transformations basically preserve the operations of addition and multiplication by a scalar from one space to another. Thus it is possible to study an abstract space in terms of matrices if we can find a transformation between two spaces.

Sometimes it is necessary to study transformations between two abstract spaces. In such cases, the importance of transformations will be in their applications to the study of certain equations rather than the study of linear spaces.

LINEAR SPACES:

Let X be a non empty set. Assume elements of X can be added and multiplied to yield an element of the same set. Then X will be called a **linear space** over a field K if

(i) $x + y = y + x \qquad x, y \in X$

(ii) $x + (y + z) = (x + y) + z \qquad x, y, z \in X$

(iii) we can find a unique element $0 \in X$ such that

$x + 0 = x \qquad x \in X$

(iv) for each $x \in X$, $\exists$ a unique $x' \in X$ such that

$x + x' = 0$

(v) $\alpha(x + y) = \alpha x + \alpha y \qquad \alpha \in K, \quad x, y \in X$

(vi) $(\alpha + \beta)x = \alpha x + \beta x \qquad \alpha, \beta \in K, \quad x \in X$

(vii) $(\alpha\beta)x = \alpha(\beta x)$

(viii) $1.x = x \qquad 1 \in K, \quad x \in X$

The field of scalars K can be real or complex. A linear space is also called a **vector space**.

NORMED LINEAR SPACE:

A normed linear space is a linear space on which a norm is defined i.e. a function which assigns to x in a linear space, a real number $\|x\|$ such that

(i) $\|x\| = 0$ iff $x = 0$

(ii) $\|x\| \geq 0 \quad \forall x$

(iii) $\|\alpha x\| = |\alpha| . \|x\|$

(iv) $\|x + y\| \leq \|x + y\|$

BANACH SPACES:

A normed linear space which is complete as a metric space is called a **Banach space**.

1.2 LINEAR TRANSFORMATIONS

Let X and Y be linear spaces over a field K. A mapping $f: X \longrightarrow Y$ is called a **linear transformation** if

$$f(x + y) = f(x) + f(y) \qquad x, y \in X$$

$$f(\alpha x) = \alpha f(x) \qquad \alpha \in K, \quad x \in X$$

In case of X and Y being normed linear spaces, linear transformations can be identified by shaper results because of the algebraic and metric structures on these spaces. The following are well known equivalent results for linear transformations $f: X \to Y$

(i) f is continuous

(ii) f is continuous at the origin

(iii) $\|f(x)\| \leq M \cdot \|x\|$ for some scalar M, $x \in X$

A transformation satisfying (iii) above is called a **bounded** linear transformation or simply a **linear operator**.

NORM OF A TRANSFORMATION:

For a continuous linear transformation f, the norm $\|f\|$ is given by

$$\|f\| = \sup\{ \|f(x)\| : \|x\| \leq 1\}$$

Alternatively, the norm is given by

$$\|f\| = \sup\{ \|f(x)\| : \|x\| = 1\}$$

provided the domain of f is non empty and does not contain the origin only.

1.3 SPACES OF LINEAR OPERATORS

Let X and Y be linear spaces over the same field K. The set of all linear operators $T:X \longrightarrow Y$ form a linear space if we define addition and multiplication as

$$(T + T')x = T(x) + T'(x)$$

$$T(\alpha x) = \alpha T(x)$$

We denote the space of all continuous linear operators from X into Y by B(X,Y). Thus $T \in B(X,Y)$ iff $\|T\| < \infty$. If X and Y are normed linear spaces so is B(X,Y).

UNIFORM TOPOLOGY

Suppose (T_n) is a sequence of operators in B(X,Y) and $T \in$ B(X,Y). Then

$$\|T_n - T\| \longrightarrow 0$$

is equivalent to

$$\|T_n(x) - T(x)\| \longrightarrow 0 \qquad \forall x \in X \text{ such that } \|x\| \leq 1$$

Thus the topology defined by

$$\|T\| = \sup\{ \|T(x)\| : \|x\| \leq 1\}$$

is called the **uniform topology** for B(X,Y).

If we can find a function P on a linear space X such that

$$P(x + y) \leq P(x) + P(y) \qquad \forall x, y \in X$$

$$P(\alpha x) = |\alpha| \cdot P(x) \qquad \alpha \in K, \quad x \in X$$

then P will be called a **seminorm.**

STRONG TOPOLOGY

The strong operator topology is the locally convex topology defined by the family of all seminorms of the form

$$P_x(T) = \|T(x)\| \qquad x \in X.$$

It is interesting to note that B(X,Y) becomes a Banach space when the norm is defined on it. For this reason, we shall henceforth use the uniform topolgy.

PRINCIPLE OF UNIFORM BOUNDEDNESS:

Let X be a Banach space and B(X,Y) be a family of bounded linear operators from X to the normed space Y. Suppose that for each $x \in X$ we can find a constant C such that $\|T(x)\| \leq C$, $T \in B(X,Y)$. Then the operators in B(X,Y) are uniformly bounded i.e we can find a constant M such that $\|T\| \leq M$, $T \in B(X,Y)$. This is a well known result and for an easy proof we refer to Royden [11].

REMARK: 1

(i) To avoid many braces, we will be writing Tx to indicate the action of a function on an element instead of T(x).

(ii) If a transformation is acting on a space, say X, then we will be writing B(X) instead of B(X,X).

We define the product of two operators TT' by

$$(TT')x = T(T'x) \qquad x \in X, \quad T,T' \in B(X).$$

Clearly, $TT' \in B(X)$ and this turns $B(X)$ into an algebra.

An operator $T \in B(X)$ is said to be **invertible** if it is both one to one (injective) and onto (surjective).

1.4 CLOSED LINEAR OPERATORS

It is sometimes useful to consider linear operators which are not continuous. Many operators which are discontinuous have the property defined below which make up for this deficiency.

Definition 1.4.1

Let X and Y be topological spaces. A function $f:X \longrightarrow Y$ is said to be **closed** if its graph $G(T) = \{ (x,f(x)) : x \in X \}$ is closed in the product topology (X,Y).

REMARK:2

Suppose X is a topological space and Y is a Hausdorff space. If $f:X \longrightarrow Y$ is continuous with a closed domain, then f is itself closed.

Theorem 1.4.2

Let X and Y be normed linear spaces. Let Y be complete and D be a subspace of X. If $T:D \longrightarrow Y$ is a closed and continuous linear operator, then D is closed.

Proof

Suppose x belongs to the closure of D. Then we can find a sequence (x_n) in D such that $x_n \to x$. (Tx_n) is Cauchy for

$$\|Tx_n - Tx_m\| \le \|T\| \cdot \|x_n - x_m\|$$

Thus (Tx_n) has a limit $y \in Y$. Since T is closed, $x \in D$ and $Tx = y$. Q.E.D.

Let C(X,Y) be the class of all continuous functions $f:X \longrightarrow Y$.

Theorem 1.4.3

Let X and Y be normed linear spaces. $f \in C(X,Y)$ iff $f^{-1}(B)$ is an open subset of X whenever $B \subset f(X)$ is open.

<u>Proof</u>

Let $f \in C(X,Y)$. Let B be an open subset of f(X) and $x \in f^{-1}(B)$. Since $f(x) \in B$ and B is open, $\exists\ \varepsilon > 0$ such that the open ball $S_\sigma(f(x)),\varepsilon) \subset B$ where σ is the metric induced by the norm. Also $\exists\ \delta = \delta(x,\varepsilon) > 0$ such that $\|x - x'\| < \delta$ implies $\|f(x) - f(x')\| < \varepsilon$ i.e

$$S_\rho(x,\ \delta) \subset f^{-1}(S_\sigma(\ f(x),\varepsilon\ )$$

Hence $S_\rho(x,\delta) \in f^{-1}(B)$ and so every point in $f^{-1}(B)$ is an interior point. Thus $f^{-1}(B)$ is open.

Conversely, suppose $f^{-1}(B)$ is open in X and B open in X. For $x \in X$ and $\varepsilon > 0$ let $N(\varepsilon) = S_\sigma(f(x),\varepsilon)$. Then $N(\varepsilon)$ is open and therefore $f^{-1}(N(\varepsilon))$ is open. But $f(x) \in N(\varepsilon) \Rightarrow x \in f^{-1}(N(\varepsilon))$. So we can find $\delta > 0$ such that $S_\sigma(x,\delta) \subset f^{-1}(N(\varepsilon))$ i.e

$$\|x - x'\| < \delta \qquad \text{whenever} \qquad \|f(x) - f(x')\| < \varepsilon$$

<u>COMMENT</u>

$f:X \longrightarrow Y$ is an **open mapping** if f(B) is open in Y whenever B is is open in X. If X and Y are Banach spaces and f is continuous then f is an open mapping. This is the open mapping theorem (see [15] page 236).

Theorem 1.4.4 (CLOSED GRAPH THEOREM)

Let X and Y be Banach spaces and $T: X \longrightarrow Y$ be a closed operator. Then T is continuous.

<u>Proof</u>

Suppose X and Y are complete metric spaces. The product $X \times Y$ is a Banach space if the norm is given by

$$\| (x,y) \|_1 = \|x\| + \|y\| \qquad (x,y) \in X \times Y$$

The graph, $G(T) = \{ (x,f(x)) : x \in X \}$ of T is a closed linear subspace of $X \times Y$ and therefore can be regarded as a Banach space. Define a function $A: G(T) \longrightarrow X$ as follows:

$$A(x,Tx) = x$$

Clearly A is linear. Since $\|A(x,Tx)\|_1 = \|x\| \le \|(x,Tx)\|_1$ A is continuous and thus closed (by remark 2). The inverse A^{-1} defined by

$$A^{-1}x = (x,Tx)$$

exists and is clearly continuous. If B is defined by

$$B(x,Tx) = Tx$$

then B is continuous for $\|B(x,Tx)\|_1 = \|Tx\| \le \|(x,Tx)\|_1$. So $T = BA^{-1}$ is continuous from X into Y.

1.5 CONJUGATE OPERATORS

Let X be an arbitrary normed linear space over a field K. Denote by X^* the set of all continuous linear transformations $T: X \longrightarrow K$. Then X^* is called the **conjugate space** of X and the elements of X^* are called functionals. X^* is seen to be a Banach space if we define addition and multiplication pointwise and the norm of $T' \in X^*$ by

$$\|T'\| = \sup\{ \|T'x\| : \|x\| = 1 \}$$

Most of the theory of conjugate operators depend on the Hahn-Banach theorem. This theorem says that any functional on a linear subspace can be extended to the whole space without altering its norm. Before proving the Hahn-Banach theorem we state the analytic form which we will use.

Theorem 1.5.1

Let X be a linear space and h be a seminorm on X. Let M be a subspace of X and h′ be a linear functional on M such that $|h'(x)| \le h(x)$ if $x \in M$. Then we can find a linear functional f defined on X such that

$$|f(x)| \le h(x) \quad \text{if } x \in X \quad \text{and}$$

$$f(x) = h'(x) \quad \text{if } x \in M$$ [see [16] page 131]

Theorem 1.5.2 (HAHN-BANACH)

Let M be a linear subspace of a normed linear space X and let f be a functional defined on M. Then f can be extended to a functional f′ defined on X such that $\|f'\| = \|f\|$, and $f'(x) = f(x)$ $\forall\ x \in M$

Proof

Define $P(x) = \|f\| \cdot \|x\|$ $x \in X$. Then P is a seminorm and $|f(x)| \le P(x)$ if $x \in M$. Thus we can find a linear functional f′ on X that is an extension of f such that

$$|f'(x)| \le \|f\| \cdot \|x\| \qquad x \in X$$

by theorem (1.5.1). This implies that f′ is an extension of f. So $\|f\| \le \|f'\|$. Hence the result.

Theorem 1.5.3

If X is a normed linear space and x is a non zero vector in X, we can find a functional $h \in X^*$ such that $h(x) = \|x\|$ and $\|h\| = 1$

Proof

Let $M = \{ \alpha x \}$ be a linear subspace of X spanned by x. Define f on M by

$$f(\alpha x) = \alpha\|x\|$$

Then f is a functional on M such that $f(x) = \|x\|$ and $\|f\| = 1$. By the Hahn-Banach theorem, f can be extended to $h \in X^*$ with the required properties.

If X is a normed linear space, then it is possible to form a conjugate space of X^* since X^* is itself a normed linear space. We denote the conjugate space of X^* by X^{**} and call it the **second conjugate** space. The importance of X^{**} lies in the fact that each vector $x \in X$ gives rise to a functional in X^{**}.

Let $x \in X$. Define a function T_x on X^* by

$$T_x(f) = f(x)$$

Then
$$\begin{aligned} T_x(\alpha f + \beta g) &= (\alpha f + \beta g)x \\ &= \alpha f(x) + \beta g(x) \\ &= \alpha T_x(f) + \beta T_x(g) \end{aligned}$$

and
$$\begin{aligned} \|T_x(f)\| &\leq \sup\{ T_x(f) : \|f\| \leq 1 \} \\ &= \sup\{ |f(x)| : \|f\| \leq 1 \} \\ &\leq \sup\{ \|f\|\cdot\|x\| : \|f\| \leq 1 \} \\ &\leq \|x\| \end{aligned}$$

Hence $\|T_x\| \leq \|x\|$. Using (1.5.3) it is easy to show that

$$\|T_x\| = \|x\|$$

Thus $x \longrightarrow T_x$ is a norm preserving linear operator. It is called the **canonical** mapping of X into X^{**}. Functionals of type T_x are called **induced** functionals.

Also

$$T_{x+y}(f) = (T_x + T_y)f$$

$$T_{\alpha x}(f) = \alpha T_x(f)$$

Thus the mapping $x \longrightarrow T_x$ is an isometric isomorphism of X into X^{**}. This isometric isomorphism is called the *natural imbedding* of X into X^{**}. The space X can therefore be identified as a subspace of X^{**}.

Definition 1.5.4

A normed linear space X is said to be **reflexive** if $X = X^{**}$.

Definition 1.5.5

Let X and Y be normed linear spaces with conjugate spaces X^* and Y^* respectively. For a given bounded linear transformation $T: X \longrightarrow Y$ we define the **conjugate (adjoint)** $T^*: Y^* \longrightarrow X^*$ by

$$T^*(f)x = f(Tx) \qquad f \in Y^*, \quad x \in X.$$

The study of linear algebra involves the study of transformations, between abstract spaces, which preserve linear structures. In Banach spaces, use is made of the metric structure to study transformations. Still Banach spaces are rather too general to yield a really rich theory of operators. Spaces which possess useful additional structure are Hilbert spaces.

Definition 1.5.6

A **Hilbert space** is a Banach space with an inner product defined on it.

REMARK: 4

We denote the inner product of two vectors x and y in a Hilbert space by $<x,y>$. In a Hilbert space it is possible to tell whether two vectors are orthogonal or not. There is also a natural correspondence between a Hilbert space and its conjugate making it easy to understand the importance of operators which are related to their adjoints in simple ways.

Let T be an operator on a Hilbert space H. We can find a unique mapping T^* of H into itself which satisfies

$$<Tx,y> = <x,T^*y> \qquad x,y \in H. \qquad (i)$$

Now,
$$\begin{aligned} <x,T^*(\alpha y + \beta z)> &= <Tx, \alpha y + \beta z> \\ &= <Tx,\alpha y> + <Tx,\beta z> \\ &= \bar{\alpha}<x,T^*y> + \bar{\beta}<x,T^*z> \\ &= <x,\alpha T^*y> + <x,\beta T^*z> \\ &= <x,\alpha T^*y + \beta T^*z> \end{aligned}$$

which implies that $T^*(\alpha y + \beta z) = \alpha T^*y + \beta T^*z$. Also

$$\|T^*x\|^2 = <T^*x, T^*x> = <TT^*x, x> \le \|TT^*x\|\cdot\|x\| \le \|T\|\cdot\|T^*x\|\cdot\|x\|$$

implies that $\|T^*x\| \le \|T\|\cdot\|x\| \quad \Rightarrow \|T^*\| \le \|T\|$.

Thus T^* as given in (i) is a linear operator called the **adjoint** operator.

Theorem 1.5.7

The adjoint operator has the following properties:

(i) $(T_1 + T_2)^* = T^*_1 + T^*_2$

(ii) $(\alpha T)^* = \bar{\alpha} T^*$

(iii) $(T_1 T_2)^* = T_2^* T_1^*$

(iv) $T^{**} = T$

(v) $\|T\| = \|T^*\|$

(vi) $\|T^* T\| = \|T\|^2$

(vii) if T^{-1} or $(T^*)^{-1}$ exists, so does the other and

$$(T^*)^{-1} = (T^{-1})^*$$

<u>Proof</u>

We supply proofs for (iii) and (v) for other proofs use essentially similar arguments.

(iii) $\langle x, (T_1 T_2)^* y\rangle = \langle (T_1 T_2)x, y\rangle$

$= \langle T_2 x, T_1^* y\rangle$

$= \langle x, T_2^* T_1^* y\rangle$

$\Rightarrow \quad (T_1 T_2)^* = T_2^* T_1^*$

(v) $\|T^* x\|^2 = \langle T^* x, T^* x\rangle = \langle TT^* x, x\rangle \le \|TTx^{**}\| \cdot \|x\|$

$\le \|T\| \cdot \|T^* x\| \cdot \|x\|$

$\Rightarrow \quad \|T^* x\| \le \|T\| \cdot \|x\|$

$\therefore \quad \|T^*\| \le \|T\|$

Using (iv) we have $\|T\| = \|T^{**}\| \le \|T^*\|$

The result then follows.

2. SOME OPERATORS ON HILBERT SPACE

There is an interesting analogy between the set B(H) of all operators on a Hilbert space H and the set of all complex numbers. Each is a complex algebra together with the mapping of the algebra into itself. The only significant difference is that multiplication in B(H) is in general not commutative. We now look at some subsets of B(H).

Definition 2.1.1

Let T be an operator on a Hilbert space H. T is said to **Self-adjoint** *or* **Hermitian** if $T = T^*$

REMARK: 5

If T_1 and T_2 are self-adjoint and α and β are real numbers

$$(\alpha T_1 + \beta T_2)^* = \bar{\alpha}T_1^* + \bar{\beta}T_2^* = \alpha T_1 + \beta T_2$$

Thus $\alpha T_1 + \beta T_2$ is also self-adjoint. If (T_n) is a sequence of self-adjoint operators converging to T, then

$$\begin{aligned} \|T - T^*\| \le \|T-T_n\| + \|T_n-T_n^*\| + \|T_n^*-T^*\| &= \|T-T_n\| + \|(T_n-T)^*\| \\ &= \|T-T_n\| + \|T_n-T\| \\ &= 2\|T_n-T\| \end{aligned}$$

As $n \to \infty$, $\|T_n-T\| \to 0$.

Therefore, $T-T^* = 0$ or $T = T^*$

From this we conclude that the set of self-adjoint operators $B^*(H)$ is a closed real linear subspace of B(H).

Theorem 2.1.2

Let operators T_1 and T_2 be self-adjoint on a Hilbert space H. Then the product T_1T_2 is self-adjoint iff T_1 and T_2 commute.

Proof

Suppose T_1 , T_2 and T_1T_2 are self-adjoint. Then

$$(T_1T_2)^* = T_2^* T_1^* \qquad \text{(using theorem 1.5.7)}$$
$$= T_2T_1$$

Since T_1T_2 is self-adjoint, $(T_1T_2)^* = T_1T_2$. Therefore $T_1T_2 = T_2T_1$

Assume $T_1T_2 = T_2T_1$. Then

$$(T_1T_2)^* = T_2^* T_1^* = T_2T_1 = T_1T_2$$
$$\Rightarrow \quad (T_1T_2)^* = (T_1T_2)$$

COMMENT:

For an operator T on a Hilbert space H, $\langle Tx, x\rangle = 0 \quad \forall x \in H$ implies that $T = 0$. (See for instance [15, page 267])

The next theorem shows a connection between self-adjoint operators and real numbers.

Theorem 2.1.3

An operator T on H is self-adjoint iff $\langle Tx, x\rangle$ is real for all $x \in H$.

Proof

Suppose T is self-adjoint and let $x \in H$. Then

$$\langle Tx, x\rangle = \langle x, T^*x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle}$$

$\Rightarrow$ $\langle Tx, x\rangle$ is real since it equals its conjugate.

Conversely, suppose $\langle Tx,x\rangle = \overline{\langle Tx,x\rangle}$, $\forall$ x ε H. Then

$$\langle Tx,x\rangle = \overline{\langle x,T^{*}x\rangle} = \langle T^{*}x,x\rangle$$

$$\Rightarrow \langle Tx,x\rangle = \langle T^{*}x,x\rangle \text{ and } \langle (T-T^{*})x,x\rangle = 0 \quad \forall x \quad \Rightarrow \quad T = T^{*}$$

So just as on real numbers, we can impose an order relation on self-adjoint operators. Thus $T_1 \le T_2$ will mean

$$\langle T_1x,x\rangle \le \langle T_2x,x\rangle \qquad \forall x \in H.$$

Self-adjoint operators obey the following linear and order structure:

(i) $T_1 \le T_2 \Rightarrow T_1 + T \le T_2 + T$, $\forall$T self-adjoint.

(ii) $T_1 \le T_2$ and $\alpha \ge 0 \Rightarrow \alpha T_1 \le \alpha T_2$

Now, suppose we have $T_1 \le T_2$ and $T_2 \le T_1$ where both operators are distinct. This will imply that

$$\langle (T_1-T_2)x,x\rangle = 0 \qquad \forall x \in H$$

or $\quad T_1-T_2 = 0 \qquad \Rightarrow T_1 = T_2$

We can conclude then, that the real Banach space of all self-adjoint operators is a partially ordered set.

Definition 2.1.4

An operator P on H is said to be **positive** if

$$\langle Px,x\rangle \ge 0 \qquad \forall x \ \varepsilon \ H$$

Clearly, a positive operator P on a Hilbert space is self-adjoint and so is the product $P^{*}P$ since

$$\langle P^{*}Px,x\rangle = \langle Px,Px\rangle = \|Px\|^2 \ge 0.$$

Theorem 2.1.4

If T is a positive operator on H, then I+T is non singular, and range of I+T is H.

Proof

Suppose that $(I+T)x = 0$. Since

$$<(I+T)x, x> = <x, x> + <Tx, x>$$

then $<Tx, x> = -\|x\|^2 \geq 0 \quad \Rightarrow \quad \|x\| = 0$ i.e $x = 0$.

The only vector mapped into the origin is zero implies that I+T is one to one.

Now, suppose that the range M of I+T is not the whole of H. Let $y_n \in M$ such that $y_n \to y$.

$y_n \in M$ implies that $y_n = (I+T)x_n$. Now

$$\|(I+T)x_n\|^2 = <x_n + Tx_n, x_n + Tx_n>$$
$$= \|x_n\|^2 + 2<Tx_n, x_n> + \|Tx_n\|^2 \geq \|x_n\|^2$$

Therefore,

$$\|x_n\| \leq \|(I+T)x_n\|$$

So $x_n \to x$ and $y = (I+T)x$. In this form, then $y \in M$ and as such M is closed. Now $M \neq H$ implies that we can find $x_o \neq 0$ such that x_o is orthogonal to M (i.e $<x_o, y> = 0 \quad \forall y \in M$) (see e.g [15, page 249]).

Now $\quad <(I+T)x_o, x_o> = <x_o, x_o> + <Tx_o, x_o> = 0$

$\Rightarrow \quad <Tx_o, x_o> = -\|x_o\|^2 \geq 0 \quad \Rightarrow \quad x_o = 0$

This is a contradiction. Thus M = H.

2.2 NORMAL AND UNITARY OPERATORS

Normal operators are generalizations of self-adjoint operators. The concept of normality is very important in connection with the spectral theory of operators.

Definition 2.2.1

An operator T on a Hilbert space is said to be **normal** if and only if $TT^* = T^*T$.

REMARK: 6

If T is a normal operator and α is a real number, then clearly αT is normal. If we take a sequence of normal operators (T_n) converging to T then $T_n^* \longrightarrow T^*$ also. Now,

$$\begin{aligned} \|TT^* - T^*T\| &\leq \|TT^* - T_nT_n^*\| + \|T_nT_n^* - T_n^*T_n\| + \|T_nT_n^* - T^*T\| \\ &= \|TT^* - T_nT_n^*\| + \|T_n^*T_n - T^*T\| \longrightarrow 0 \end{aligned}$$

$\Rightarrow\ TT^* = T^*T.$

The following theorem is a consequence of the preceeding remark.

Theorem 2.2.2

The set of all normal operators on a Hilbert space H is a closed subset of B(H) and contains the set of all self-adjoint operators.

Under certain conditions, we can say something about the sum and product of normal operators for it is not always automatic that the sum and product are normal too. The following theorem illustrates this.

Theorem 2.2.3

Let T_1 and T_2 be normal operators such that each commutes with the adjoint of the other. Then $T_1 + T_2$ and T_1T_2 are normal.

Proof

By hypothesis $T_1T_2^* = T_2^*T_1$ and $T_2T_1^* = T_1^*T_2$. Now,

$$(T_1 + T_2)(T_1 + T_2)^* = (T_1 + T_2)(T_1^* + T_2^*)$$
$$= T_1T_1^* + T_1T_2^* + T_2T_1^* + T_2T_2^* \quad \ldots\ldots\ldots (i)$$

and

$$(T_1 + T_2)^*(T_1 + T_2) = (T_1^* + T_2^*)(T_1 + T_2)$$
$$= T_1^*T_1 + T_1^*T_2 + T_2^*T_1 + T_2^*T_2 \quad \ldots\ \ldots (ii)$$

From (i) and (ii) we infer that $T_1 + T_2$ is normal. We can prove that T_1T_2 is normal in a similar way.

We may wish to characterize a normal operator T by the norms of Tx and T^*x. In this line, we have:

Theorem 2.2.4

An operator T on a Hilbert space is normal iff

$$\|T^*x\| = \|Tx\| \qquad \forall x \in H$$

Proof

$\|T^*x\| = \|Tx\|$ iff $\|T^*x\|^2 = \|Tx\|^2$ iff $<T^*x, T^*x> = <Tx, Tx>$ iff $<TT^*x, x> = <T^*Tx, x>$ iff $<(TT^* - T^*T)x, x> = 0 \Leftrightarrow TT^* - T^*T = 0$

The analogy between self-adjoint operators and real numbers suggest that for any operator $T \in B(H)$, we form

$$T_1 = \frac{1}{2}(T+T^*)$$
$$T_2 = \frac{1}{2i}(T-T^*)$$

The operators T_1 and T_2 are self-adjoint and, $T = T_1 + iT_2$. We call the self-adjoint operators T_1 and T_2 *real* and *imaginary* parts of T respectively. We now discuss T in terms of its real and imaginary parts.

Theorem 2.2.5

If T is an operator on H, then T is normal iff its real and imaginary parts commute.

Proof

Let $T = T_1 + iT_2$. Then $T^* = T_1 - iT_2$. Now,

$$TT^* = (T_1 + iT_2)\cdot(T_1 - iT_2) = T_1^2 + T_2^2 + i(T_2T_1 - T_1T_2)$$

Also,

$$T^*T = T_1^2 + i(T_1T_2 - T_2T_1) + T_2^2$$

If $T_1T_2 = T_2T_1$ then $TT^* = T^*T$.

Conversely, if $TT^* = T^*T$ then

$$T_1T_2 - T_2T_1 = T_2T_1 - T_1T_2$$

i.e $$T_1T_2 = T_2T_1$$

Definition 2.2.6

An operator T on a Hilbert space is said to be **unitary** if

$$T^*T = TT^* = I$$

Clearly, unitary operators are normal and we can infact compare them to complex numbers with unit absolute value. In other words, we can say, unitary operators are those non-singular operators whose inverses equal their adjoints.

Theorem 2.2.7

The following statements are equivalent for an operator T on a Hilbert space H

(i) $T^{*}T = I$

(ii) $\langle Tx, Ty\rangle = \langle x, y\rangle \qquad \forall x, y \in H$

(iii) $\|Tx\| = \|x\| \qquad x \in H$

Proof

(i) $\Rightarrow$ (ii) Assume $T^{*}T = I$. Then

$$\langle x, y\rangle = \langle T^{*}Tx, y\rangle = \langle Tx, Ty\rangle$$

(ii) $\Rightarrow$ (iii) If $\langle Tx, Ty\rangle = \langle x, y\rangle$, and $x = y$, we have

$$\langle Tx, Tx\rangle = \langle x, x\rangle \quad \text{or} \quad \|Tx\|^2 = \|x\|^2$$

Therefore $\|Tx\| = \|x\|$

(iii) $\Rightarrow$ (i) $\|Tx\|^2 = \|x\|^2 \Rightarrow \langle Tx, Tx\rangle = \langle x, x\rangle$

$$\Rightarrow \langle T^{*}Tx, x\rangle = \langle x, x\rangle \quad \text{i.e} \quad \langle (T^{*}T-I)x, x\rangle = 0 \qquad \forall x \in H$$

$$\Rightarrow T^{*}T = I$$

REAMRK: 7

An operator with property (iii) is simply an **isometric isomorphism of** of H into itself.

2.3 PROJECTIONS

We briefly look at operators which have a very simple yet beautiful theory.

Let X be a linear space. Given two subspaces M and N of X we define

$$M + N = \{ m+n : m \in M, \quad n \in N \}$$

The set $M+N$ is called the sum of M and N and it is the smallest subspace of X containing both M and N. When $M \cap N = \{ 0 \}$ we write $M \oplus N$ in place of $M+N$ and call it the **direct sum** of M and N

Theorem 2.3.1

Let M and N be subspaces of a linear space X. Then $X = M \oplus N$ iff each $x \in X$ can be written uniquely in the form $x = m + n \qquad m \in M, \quad n \in N.$

<u>Proof</u>

Suppose $X = M \oplus N$. Then $X = M + N$ and $x \in X$ can be written as $x = m + n$. If $x = m_1 + n_1 = m_2 + n_2$ where $m_1, m_2 \in M$, $n_1 n_2 \in N$ then $m_1 - m_2 = n_2 - n_1$. This implies that $m_1 - m_2$ belong both to M and N. But $M \cap N = \{ 0 \}$ implies $m_1 - m_2 = 0$ or $m_1 = m_2$. Similarly, $n_1 = n_2$.

Conversely, suppose $x \in X$ has a unique representation $m + n$. Then $X = M + N$. If $y \in M \cap N$, then we can write $y = y + 0$ and $y = 0 + y$. By hypothesis these representations are the same. Hence $y = 0$. Therefore $M \cap N = \{ 0 \}$ so that $X = M \oplus N$.

It follows that if $X = M \oplus N$, then $\dim X = \dim M + \dim N$ where $\dim X$ denotes the dimension of X. When $X = M \oplus N$, then M and N are called **complementary** subspaces.

Definition 2.3.2

Let $T: X \longrightarrow Y$ be a general mapping. Define

$$N(T) = \{ x \in X : T(x) = 0 \}$$

$$R(T) = \{ y \in Y : y = T(x),\ x \in X \}$$

We call N(T) the **nullity** of T and R(T) the **range** of T. Clearly N(T) and R(T) are subspaces of X and Y respectively. The relationship on dimensions now becomes

$$\dim X = \dim N(T) + \dim R(T)$$

Definition 2.3.3

A **projection** on a Hilbert space H is an opeartor $P \in B(H)$ such that $P^2 = P$.

<u>NOTE</u>: The term **idempotent** is also used for the relationship $E^2 = E$ for an operator E.

Definition 2.3.4

Two vectors in an Hilbert space are said to be **orthogonal** if their inner product equals zero.

If M is a subset of H, we will denote by $M^{\perp}$ the set of all vectors in H orthogonal to M. If M is a closed subspace of a Hilbert space H, then $M^{\perp}$ is also a closed subspace and is disjoint from M in the sense that

$$M \cap M^{\perp} = \{0\},$$

Moreover, $H = M \oplus M^{\perp}$. Define a mapping $P' : H \longrightarrow H$ by $P'x = y$ where $x \in H$, $x = y + z$ and $y \in M$, $z \in M^{\perp}$.

Theorem 2.3.5

If M is a closed subspace of H, then P' is a projection having range M. Also if P is a projection on H, we can find a closed subspace M such that $P = P'$.

Proof

Suppose $x_1, x_2 \in H$ and α, β are complex numbers. Then $x_1 = y_1 + z_1$ and $x_2 = y_2 + z_2$ where $y_1, y_2 \in M$ and $z_1, z_2 \in M^{\perp}$.

$$\alpha x_1 + \beta x_2 = (\alpha y_1 + \beta y_2) + (\alpha z_1 + \beta z_2)$$

By uniqueness of such a decomposition

$$\begin{aligned} P'(\alpha x_1 + \beta x_2) &= \alpha y_1 + \beta y_2 \\ &= \alpha P' x_1 + \beta P' x_2 \end{aligned}$$

and

$$\|P'x_1\|^2 = \|y_1\|^2 \le \|y_1\|^2 + \|z_1\|^2 = \|x_1\|^2$$

$$\Rightarrow \qquad \|P'x_1\| \le \|x_1\|$$

Hence P' is a linear operator. Moreover

$$\begin{aligned} \langle P'x_1, x_2 \rangle = \langle y_1, y_2 + z_2 \rangle &= \langle y_1, y_2 \rangle + \langle y_1, z_2 \rangle \\ &= \langle y_1 + y_2 \rangle \\ &= \langle y_1 + z_1, y_2 \rangle \\ &= \langle x_1, P'x_2 \rangle \end{aligned}$$

and so P' is self-adjoint. If $x \in M$, $x = x + 0$ is the required decomposition of x and hence

$$P'x = x$$

Since $R(P') = M$, then $(P')^2 = P'$ and so P' is idempotent or P'

is a projection with range M.

Next, suppose P is a projection on H. Set R(P) = M. If (Px_n) is a Cauchy sequence in H such that $(Px_n) \longrightarrow y$ then

$$y = \lim_{n\to\infty} Px_n = \lim_{n\to\infty} P^2 x_n = P(\lim_{n\to\infty} Px_n) = Py$$

So $y \in M$ implying that M is closed. For $y \in M^{\perp}$,

$$\|Py\|^2 = \langle Py, Py\rangle = \langle y, P^2 y\rangle = 0$$

$$\text{i.e} \qquad Py = 0.$$

If $x \in H$, then $x = Px + z$ where $z \in M^{\perp}$ and hence

$$= P'x = P^2 x + Pz = Px$$

$$\therefore \qquad P' = P.$$

The above result shows that a projection P on a linear space X determines a decomposition

$$X = M \oplus N \tag{i}$$

where $M = \{ P(x) : x \in X \}$ and $N = \{ x : P(x) = 0 \}$. On the other hand, a pair of linear subspaces M and N such that (i) holds determines a projection whose range and null spaces are M and N respectively.

Theorem 2.3.6

Let P be a projection on H with range M and null space N. Then $M \perp N$ (read M is perpendicular to N) if and only if P is self-adjoint. In this case $N = M^{\perp}$.

<u>Proof</u>

Suppose $M \perp N$. Let $z = x + y$, where $x \in M$, $y \in N$. Then

$$\langle P^* z, z\rangle = \langle z, P^{**} z\rangle = \langle z, Pz\rangle$$

$$\Rightarrow \quad \langle P^* z, z\rangle = \langle x + y, x\rangle = \langle x, x\rangle + \langle y, x\rangle$$

$$= \langle x, x\rangle \qquad \text{a real number.}$$

Therefore,

$$\overline{\langle P^* z, z\rangle} = \langle Pz, z\rangle$$

$$\Rightarrow \quad \langle P^* z, z\rangle = \langle Pz, z\rangle$$

$$\Rightarrow \quad \langle (P^* - P)z, z\rangle = 0$$

$$\Rightarrow \quad P^* - P = 0 \quad \text{or } P^* = P.$$

Conversely, suppose $P^* = P$. For $x \in M$, $y \in N$,

$$\langle x, y\rangle = \langle Px, y\rangle = \langle x, Py\rangle = \langle x, 0\rangle$$

$$\Rightarrow \quad M \perp N$$

Clearly $N \subseteq M^{\perp}$. Suppose the inclusion is proper. Then we can find $x \in M^{\perp}$, $x \neq 0$ such that $x \perp N$. But by definition $x \in M^{\perp}$ implies that $x \perp M$. Thus $x \perp N$ and $x \perp M$. Since $H = M \oplus N$ then $x \perp H$. This is impossible for a non zero x. Thus $N = M^{\perp}$.

<u>OBSERVATION</u>

If P is a projection on M, then $I-P$ is a projection on $M^{\perp}$. Further more

$$\|x\|^2 = \|Px + (I-P)x\|^2 = \|Px\|^2 + \|(I-P)x\|^2$$

shows that

$$\|Px\|^2 \leq \|x\|^2$$

or that $\qquad \|Px\| \leq \|x\|$.

So $\qquad \|P\| < 1$.

Also for any $x \in H$,

$$\langle Px, x\rangle = \langle PPx, x\rangle = \langle Px, P^* x\rangle = \|Px\|^2 \geq 0.$$

Thus a projection is a positive operator. Since I-P is a projection, we must have

$$I-P > 0 \qquad \text{i.e } \ 0 \le P < 1.$$

There is a relationship between the concept of invariance of a subspace and projections which we now look at.

Definition 2.3.7

A closed linear subspace M is said to be **invariant** under an operator T if $T(M) \subseteq M$.

Theorem 2.3.8

A closed linear subspace $M \subset H$ is invariant under an operator T iff $M^{\perp}$ is invariant under T^{*}.

Proof

Since $T^{**} = T$ and $M = M^{\perp\perp}$ it is sufficient to prove the necessary condition only.

If M is invariant under T, $x \in M$ and $y \in M^{\perp}$, then

$$\langle x, T^{*}y\rangle = \langle Tx, y\rangle = 0$$

$\Rightarrow T^{*}y \in M^{\perp}$. Hence $M^{\perp}$ is invariant under T^{*}.

If both M and $M^{\perp}$ are invariant under T we say that M reduces T. From the above theorem it follows that M reduces T if and only if M is invariant under both T and T^{*}. The next theorem demonstrates the relationship between an operator and a projection.

Theorem 2.3.9

Let P be a projection on a closed linear subspace M and T be any operator. M is invariant under T iff TP = PTP.

Proof

Suppose M is invariant under T. Let $z \in H$. Then

$$TPz \in M$$

and $$TPz = PTPz \quad \Rightarrow \quad TP = PTP$$

Conversely, let TP = PTP and $x \in M$. Then

$$Tx = TPx = PTPx$$
$$= P(Tx)$$

Thus M is invariant.

Theorem 2.3.10

If M and N are closed subspaces of H with projections P and Q respectively, then the following statements are equivalent.

(i) $P \le Q$

(ii) $\|Px\| \le \|Qx\| \qquad \forall x \in H$

(iii) $M \subseteq N$

(iv) $QP = P$

(v) $PQ = P$

Proof

If $P \le Q$ then $\|Px\|^2 = \langle Px, x\rangle \le \langle Qx, x\rangle = \|Qx\|^2 \quad \forall x \in H,$

If $\|Px\| \le \|Qx\| \quad \forall x$, consider $x \in M$. Then

$$\|x\| = \|Px\| \le \|Qx\| \le \|x\| \qquad (\text{since } \|Q\| \le 1)$$

i.e $\|Qx\| = \|x\|$ and $\therefore$ $Qx = x$ or $x \in N$.

If $M \subseteq N$, then $Px \in N$ and $QPx = Px$ for all x.

If $QP = P$ then forming adjoints on both sides,

$$PQ = P.$$

Lastly, if $PQ = P$ then

$$\langle Px, x\rangle = \|Px\|^2 = \|PQx\|^2 \le \|Qx\|^2 = \langle Qx, x\rangle \quad \forall x.$$

We can express the orthogonality of subspaces in terms of projections.

Theorem 2.3.11

Let P and Q be projections on a closed linear subspaces M and N respectively. Then $M \perp N$ iff $PQ = 0$.

Proof

Suppose $M \perp N$. Then $N \subseteq M^{\perp}$ so that for all x, $Qx \in M^{\perp}$ and hence $(PQ)x = P(Qx) = 0$, i.e $PQ = 0$.

Conversely, suppose $PQ = 0$. Let $x \in N$. Then

$$0 = PQx = P(Qx) = Px \quad \Rightarrow \quad x \in M^{\perp}.$$

Therefore $N \subseteq M^{\perp}$. Hence $N \perp M$.

Theorem 2.3.12

Let P_1 and P_2 be projections with ranges M_1 and M_2 respectively. Then $P = P_1 - P_2$ is a projection with range M iff $P_2 \le P_1$ and $M = M_1 - M_2$.

Proof

Assume P is a projection. Then

$$\langle P_1x, x\rangle - \langle P_2x, x\rangle = \langle Px, x\rangle = \|Px\|^2 \ge 0 \quad \forall x$$

implies

$$\langle P_1x,x\rangle \geq \langle P_2x,x\rangle \quad \text{or} \quad P_1 \geq P_2$$

On the other hand, if $P_2 \leq P_1$ then $P_1P_2 = P_2P_1 = P_2$ by theorem 2.3.10. So

$$(P_1 - P_2)^2 = P_1 - P_1P_2 - P_2P_1 + P_2 = P_1 - P_2$$

Now $P_1 \leq P_2 \Rightarrow P_1$ commutes with $I-P_2$ since $P_1-P_2 = P_1(I-P_2)$. The range of $I-P$ is M_2 and therefore

$$M = M_1 \cap M_2^{\perp} = M_1 - M_2$$

A projection defined on a closed subspace behaves exactly like an identity operator on that subspace. The next theorem shows that under certain conditions, a sum of projections defines an identity operator on H.

Theorem 2.3.13

If H is a Hilbert space, M_i $(i=1,2,3,\dots,n)$ are closed subspaces of H and P_i $(i=1,2,3,\dots,n)$ are projections onto the M_i's then $P_1 + P_2 + \dots + P_n = I$ iff the subspaces $[M_i]$ are pairwise orthogonal and span H.

<u>Proof</u>

If $P_1 + P_2 + \dots + P_n = I$ then each $x \in H$ has a unique representation $x = P_1x + P_2x + \dots + P_nx$. Hence M_i span H.

Conversely, suppose M_i's span H and $P_1 + P_2 + \dots + P_n$ is a projection. To show that $P_1 + P_2 + \dots + P_n$ is an identity operator, it is sufficient to show that $P_1 + P_2 + \dots + P_n$ is projection iff $[M_i]$ are pairwise orthogonal.

Suppose P_1, P_2 and $P_1 + P_2$ are projections. For $x \in M_1$ we have

$$\begin{aligned}
<(P_1+P_2)x,x> &= <(P_1+P_2)^2x,x> \\
&= <P_1x,P_1x> + <P_1x,P_2x> + <P_2x,P_1x> + <P_2x,P_2x> \\
&= <P_1x,x> + <x,P_2x> + <P_2x,x> + <P_2x,x> \\
&= <(P_1+P_2)x,x> + 2<x,P_2x>
\end{aligned}$$

which implies that $<x,P_2x> = 0$. But

$$\|P_2x\|^2 = <P_2x,P_2x> = <P_2x,x>.$$

Thus for $x \neq 0$, we conclude that $M_1 \perp M_2$. If P_1 and P_2 are such that $R(P_1) \perp R(P_2)$, then

$$\begin{aligned}
(P_1 + P_2)^2x &= (P_1 + P_2)P_1x + (P_1 + P_2)P_2x \\
&= P_1^2x + P_2^2x \\
&= (P_1 + P_2)x \quad (\text{since } P_2P_1x = P_1P_2x = 0),
\end{aligned}$$

By induction we conclude that $P_1 + P_2 + \ldots + P_n$ is a projection.

REMARK: 8

The previous theorem constitutes what is called the **spectral** theorem. We have seen that for any closed linear subspace M of a Hilbert space H we can always find a projection on H whose range will be this closed subspace and $H = M \oplus M^{\perp}$. This is not usually the case for any general Banach space.

2.4 COMPACT OPERATORS

Linear operators on finite dimensional spaces are easy to study because they can be represented by matrices whose theory is well understood. There is a class of bounded operators called compact operators which are in many respects analogous to operators on finite dimensional spaces. Even closer to finite dimensional linear operators is a subclass of compact operators known as degenerate operators.

Definition 2.4.1

A set is said to be **relatively compact** if its closure is compact.

Since compact sets are always closed, it follows that compact sets are relatively compact.

Definition 2.4.2

Let X and Y be Banach spaces and $T: X \longrightarrow Y$ be a continuous operator. T is said to be **compact** if $T(S)$ is a relatively compact subset of Y whenever S is a bounded subset of X.

REMARK: 9

Since Y is a complete metric space, we say T is compact if for any bounded sequence (x_n) in X the sequence (Tx_n) contains a Cauchy subsequence in Y.

Many operators that arise in the study of integral equations are compact and this accounts for their importance from the application point of view.

The spaces considered in the rest of this section are generally Banach Spaces.

Theorem 2.4.3

The set $K(X,Y)$ of all compact operators from X to Y is a closed linear subspace of the Banach space $B(X,Y)$.

Proof

If T is a compact operator, and α is a scalar then obviously αT is compact. Let T and S be compact operators. Let (x_n) be a sequence in X and (x_n^1) be a subsequence of (x_n) such that (Tx_n^1) is Cauchy in Y. Take a subsequence (x_n^2) of (x_n^1) such that (Sx_n^2) is Cauchy. Then $\{ (T+S)x_n^2 \}$ is a Cauchy sequence. Thus $T+S$ is compact and $K(X,Y)$ is a linear subspace.

Now, let (T_k) be a sequence of operators such that $\|T_k - T\| \longrightarrow 0$ as $k \longrightarrow \infty$. We show that T is compact. Take a subsequence (x_n^1) of (x_n) such that $(T_1 x_n^1)$ is Cauchy. Take a subsequence (x_n^2) of (x_n^1) such that $(T_2 x_n^2)$ is Cauchy. Continuing in this way, we get a diagonal sequence $(x_n^n) = \omega_n$ such that $(T_n \omega_n)$ is Cauchy. Since for $k \leq n$ (ω_n) is a subsequence of every sequence (x_n^k), each $(T_k \omega_n)$ is Cauchy for fixed k. For $\varepsilon > 0$ take k so large that $\|T_k - T\| < \varepsilon$ and then take N so large that $\|T_k \omega_n - T_k \omega_{n+p}\| < \varepsilon$ $\quad n > N, \quad p > 0$. Then

$$\|T\omega_n - T\omega_{n+p}\| \leq \|(T - T_k)(\omega_n - \omega_{n+p})\| + \|T_k(\omega_n - \omega_{n+p})\|$$
$$\leq 2(M + 1)\varepsilon$$

where $M = \sup \|\omega_n\| < \infty$. The above is true whenever $n \in N$ and so $(T\omega_n)$ is Cauchy. Thus T is compact.

REMARK: 10

The product of a compact operator with a bounded operator is compact as continuous operators take bounded and relatively compact sets into bounded and relatively compact sets.

Definition 2.4.4

A subset F of B(X) is **equicontinuous** if for every $\varepsilon > 0$ and $x \in X$, we can find a neighbourhood $N = N(x)$ of x such that

$$\sup_{f \in F} \sup_{t \in N} \|f(x) - f(t)\| < \varepsilon .$$

If F contains only one element , equicontinuity is the same as continuity. The continuity of an element of F is a consequence of the equicontinuity of F.

We recall that a subset F of a metric space is said to be **totally bounded** if F is contained in the union of a finite number of open balls of radius $\varepsilon > 0$. Note that a relatively compact subset of a complete metric space is totally bounded and vice-versa.

Theorem 2.4.5 (Arzela-Ascoli)

If X is compact and $F \subset B(X)$, then F is relatively compact iff F is bounded and equicontinuous.

Proof

Let $F \subset B(X)$ be equicontinuous and bounded. For $\varepsilon > 0$, we can find a finite number of neighbourhoods $N_1, N_2, \ldots, N_m$ which covers X (since X is compact). Thus

Since

$$\sup_{f\in F}\ \sup_{x\in N_i} \|f(x_i) - f(x)\| < \varepsilon$$

then F is relatively compact.

Conversely, let F be relatively compact. Then F is totally bounded and hence bounded. If $\varepsilon > 0$, $\exists$ $f_1, f_2, \ldots, f_n$ in F such that $f \in F$ has distance $< \frac{\varepsilon}{3}$ from one of $f_1, f_2, \ldots, f_n$. For $x \in X$ choose $N(x)$ such that

$$\|f_i(x) - f_i(t)\| < \frac{\varepsilon}{3} \qquad t \in N, \quad i = 1, 2, \ldots, n$$

Then

$$\|f(x) - f(t)\| \le \|f(x) - f_i(x)\| + \|f_i(x) - f_i(t)\| + \|f_i(t) - f(t)\|$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

for each $f \in F$, $t \in N$ and $i \le n$. Hence F is equicontinuous.

Theorem 2.4.6 (Schauder)

An operator in $B(X,Y)$ is compact iff its adjoint is compact.

<u>Proof</u>

Suppose T is compact. Let (y_n^*) be a sequence in Y^* such that $\|Y^*\| \le 1$, and N be a unit ball in X. Define $f_n : Y \longrightarrow K$ by

$$f_n(y) = \langle y, y_n^* \rangle \quad y \in Y \text{ (to mean image of } y \text{ under } y_n^*)$$

Since $\|f_n(y) - f_n(y')\| \le \|y - y'\|$, (f_n) is equicontinuous. Also, since $T(N)$ has a compact closure in Y, Arzela-Ascoli theorem implies that (f_n) has a subsequence (f_{n_i}) that converges uniformly on $T(N)$. Now

$$\|T^* y^*_{n_i} - T^* y^*_{n_j}\| = \sup\|\langle Tx, y^*_{n_i} - y^*_{n_j}\rangle\| = \sup\|f_{n_i}(Tx) - f_{n_j}(Tx)\|$$

the sup taken over $x \in N$. Completeness of X^* implies that $(T^* y^*_{n_i})$ converges. Hence T^* is compact. The reverse can be proved similarly.

Let us call the dimension of $R(T)$ the rank of T.

Definition 2.4.7

An operator $T \in B(X,Y)$ is said to be **degenerate** if rank T is finite.

Since a finite dimensional space is locally compact, a degenerate operator is compact. The set of all degenerate operators is a subspace of $B(X,Y)$ though not generally closed. Also, the conjugate of a degenerate operator is degenerate.

2.5 FREDHOLM OPERATORS

We may want to classify an operator in terms of the dimension of its null space and range. Fredholm operators are identified in this way

Let $T \in B(X,Y)$. A complex number λ is called an *eigenvalue* of T if $\exists$ $x \in X$ such that

$$Tx = \lambda x \qquad\qquad x \neq 0$$

Here x is called an *eigenvector* belonging to λ. The zero vector together with all eigenvectors of T is called the *eigenspace* of T. The dimension of the eigenspace is called the (geometric) *multiplicity* of λ.

Definition 2.5.1

The **nullity**, $n(T)$ of an operator $T : X \longrightarrow Y$ is defined as the dimension of the null space of T.

Since $N(T)$ is the geometric eigenspace of T for the eigenvalue zero, $n(T)$ is the geometric multiplicity of this eigenspace.

Definition 2.5.2

The **defect** $d(T)$ of T is the codimension in Y of $R(T)$.

Note that each of $n(T)$ or $d(T)$ can take on values $1, 2, \ldots$ or ∞ .

Definition 2.5.3

If at least one of $n(T)$ or $d(T)$ is finite, we define the **index** $i(T)$ of T to be

$$i(T) = n(T) - d(T)$$

Definition 2.5.4

An operator $T \in B(X, Y)$ is said to be **Fredholm** if $n(T) < \infty$ and $d(T) < \infty$.

For a Fredholm operator T, the solution of the equation

$$Tx = y$$

is usually equivalent to determining the orthogonality of y to the finite subspace of the kernel of the conjugate operator. It is easier to study boundary value problems which are formulated in this way.

Definition 2.5.5

An operator T is said to be **semifredholm** if the range of T is closed and at least one of n(T) or d(T) is finite.

It is worth noting that every bijective operator in B(X,Y) is Fredholm.

3. NON LINEAR OPERATORS

3.1 INTRODUCTION

The need to study nonlinear functions and nonlinear equations in particular stems from the fact that most equations in real life are nonlinear. Some examples can be cited in fields like elasticity, acoustics, fluid dynamics and oscillations. Though most of the equations have been solved by linearization, there are cases when this process is unsatisfactory. Therefore other methods of solving nonlinear problems ought to be employed, for instance the use of fixed point theorems. The results thus obtained reveal properties which are closer to real situations. So we now look at the class of important nonlinear operators, namely: Lipschitz.

3.2 LIPSCHITZ OPERATORS

Let X and Y be linear spaces over a field of real or complex numbers K. We denote by Op(X,Y) the class of all functions $T:X \longrightarrow Y$ and call members of Op(X,Y) operators. In the case where X = Y, we will write Op(X). The rules of addition, multiplication, scalar multiplication, inverse etc. follow as in the case of linear operators.

Let D be a subspace of X and denote by F(D,Y) the class of all operators $T \in Op(X,Y)$ such that the domain of T (D(T) = D) is a linear space over K. We will denote by F(D) the set of all $T \in F(X,X)$ such that $Tx \in D$, $x \in D$. We note that if D is not a linear subspace of X, then F(D) is no longer a linear subspace.

Definition 3.2.1

Let X and Y be normed linear spaces. Denote by Lp(X,Y) the class of all members $T \in F(D,Y)$ such that

$$\|T\| = \sup\{ \|Tx-Ty\| \,/\, \|x-y\| : x,y \in D,\ x \neq y \} < \infty .$$

We call members of Lp(D,Y) **Lipschitz** operators and $\|T\|$ the Lipschitz constant of T. Thus $T \in Lp(D,Y)$ if there exists a constant $L \geq 0$ such that

$$\|Tx-Ty\| \leq L\|x-y\| \qquad \forall x,y \in D$$

Since linear operators also satisfy the above condition, the class of Lipschitz operators include linear operators as well. The following relations are immediate consequences of the definition of $\|T\|$

(i) $\|T\| = 0$ iff T is constant on D

(ii) $\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|$

(iii) $\|\alpha T\| = |\alpha|\,\|T\| \qquad \alpha \in K.$

In particular, we can say that $\|\cdot\|$ is a seminorm on Lp(D,Y). Suppose we have an infinite sequence (T_n) in Lp(D,Y) such that

$$\lim_{n,m\to\infty} \|T_n-T_m\| = 0 \text{ and } \lim_{n,m\to\infty} \|T_nx'-T_mx'\| = 0 , \quad \text{where } x' \in D.$$

Then

$$\lim_{n,m\to\infty} \|T_nx-T_mx\| = 0, \qquad \text{uniformly over D where D is bounded.}$$

If Y is a Banach space then $\lim_{n\to\infty} T_nx = Tx$ each $x \in D$, uniformly. Thus $T \in Lp(D,Y)$ and $\lim_{n\to\infty} \|T_n-T\| = 0$. For $x \in D$ we define υ_x by

$$\upsilon_x(T) = \|Tx\| + \|T\| \qquad T \in Lp(D,Y).$$

$\upsilon_X(\cdot)$ is a norm on Lp(D, Y). If Y is a Banach space, then Lp(D, Y) is complete. If T comes from the class B(X, Y) of linear members of Lp(D, Y), then

$$\|T\| = \sup\{ \|Tx\| / \|x\| : x \neq 0 \}$$

Since Lp(X, Y) is a vector space and B(X, Y) has been extracted from Lp(X, Y), then B(X, Y) is a vector subspace. B(X, Y) is also closed in Lp(X, Y). The following remark compares B(X, Y) with $L_p(X, Y)$;

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Uniform boundedness does not hold in Lp(X, Y) as the example below (see Martin [7]) shows.

Let X = Y = ℝ. Define

$$Tx = \sqrt{x} \quad \text{if} \quad 0 \le x \le 1$$
$$Tx = 0 \qquad x \le 0$$
$$Tx = 1 \qquad x \ge 1$$

Let (T_n) be a sequence of polynomials that converge uniformly to $\sqrt{x}$ on [0, 1]. Extend T_n to ℝ by defining

$$T_n x = T_n \cdot 1 \qquad \text{if } x \ge 1$$
$$T_n x = T_n \cdot 0 \qquad \text{if } x \le 0$$

Clearly, T_n is bounded on the real line and $\lim_{n\to\infty} T_n x = Tx \quad x \in \mathbb{R}$.

However the sequence $(\|T_n\|)$ is not bounded and even T does not belong to Lp(ℝ, ℝ).

We recall that for any two operators S and T, multiplication is defined as

$$(S\cdot T)x = S(Tx)$$

Now if $S, T \in Lp(X,Y)$ then

$$\|(S\cdot T)x - (S\cdot T)y\| = \|S(Tx) - S(Ty)\| \le \|S\|\cdot\|Tx - Ty\|$$
$$\le \|S\|\cdot\|T\|\cdot\|x-y\|.$$

Thus $S\cdot T \in Lp(X,Y)$ and $\|S\cdot T\| \le \|S\|\cdot\|T\|$

We deviate a little and discuss $B(X,Y)$. Suppose X is a Banach space and (T_n) is a sequence in $B(X,Y)$ such that $\lim\limits_{n\to\infty} T_n x = Tx$ exists for all $x \in X$. Then $T \in B(X,Y)$ and $\|T\| \le \liminf\limits_{n\to\infty}\|T_n\|$. This follows since $\{ T_n x : n \ge 1 \}$ is bounded in Y for each $x \in X$. Thus we can find a constant C such that $\|T_n\| \le C$, $n \ge 1$. Hence $\liminf\limits_{n\to\infty} \|T_n\| = L$ for some limit L.

Now if $x \in X$,

$$\|Tx\| = \lim_{n\to\infty} \|T_n x\| \le \liminf_{n\to\infty}\|T_n\|\cdot\|x\| = L\cdot\|x\|$$

Therefore, $T \in B(X,Y)$ and $\|T\| \le \liminf\limits_{n\to\infty}\|T_n\|$.

Theorem 3.2.2

Let X be a Banach space, $T \in Lp(X)$ and $\|T\| < 1$. Then $1-T$ is invertible in $Lp(X)$ and

$$\|(1-T)^{-1}\| \le (1-\|T\|)^{-1}$$

Proof

It follows from

$$\|(1-T)x - (1-T)y\| \ge \|x - y\| - \|Tx - Ty\| \ge (1-\|T\|)\cdot\|x - y\|$$

$x, y \in X$ that $1-T$ is injective. Now, define $B_o = 1$ and $B_n = 1 + TB_{n-1}$ $n = 1,2,3,...$ Since $\|T\| < 1$ and X is complete $\lim\limits_{n\to\infty} B_n x = Lx$ exists . and

$$Lx = \lim_{n\to\infty} B_n x = \lim_{n\to\infty}(1-TB_{n-1})x = x + TLx \quad .$$

Then $L = 1 + TL$ or $(1-T)L = 1$. So L is the inverse of $1-T$ and hence $1-T$ is surjective. We get the estimate from

$$\|(1-T)^{-1}r - (1-T)^{-1}s\| \le (1-\|T\|)^{-1} \|r - s\|, \qquad r,s \in R(1-T).$$

Note that this result holds even in the case of linear operators.

3.3 α-LIPSCHITZ OPERATORS

Let X and Y be normed linear spaces over a field K. Suppose U is a bounded subset of X. Define a *diameter* $\delta[\cdot]$ on X as

$$\delta[U] = \sup \{ \|x - y\| : x,y \in U \}.$$

Define $\alpha[\cdot]$ also by

$$\alpha[U] = \inf \{ \varepsilon > 0 : U \text{ can be covered by a finite number of sets with maximum diameter less than } \varepsilon \}$$

We call $\alpha[\cdot]$ the measure of non compactness. Note that $\delta[U] = 0$ if and only if U consists of exactly one point and $0 \le \delta[U] < \infty$.

Definition 3.3.1

A mapping $T:D \longrightarrow Y$ is said to be α-Lipschitz if

(i) T is continuous

(ii) T is bounded

(iii) We can find $K \ge 0$ such that $\alpha[T(U)] \le K\alpha[U]$ for all bounded $U \subset D$

We denote by α-$Lp(D,Y)$ the class of α-Lipschitz mappings. If $T \in \alpha$-$Lp(D,Y)$ the α-Lipschitz constant, $\|T\|_\alpha$ is the smallest number K such that condition (iii) above holds.

The class α-Lp(D,Y) can be shown to be a linear subspace of F(D,Y). Suppose that $\alpha[\beta T(U)] = |\beta|\alpha[T(U)]$. Then the following properties of α-Lipschitz operators follow

(i) $\|\beta T\|\alpha = |\beta| \cdot \|T\|\alpha$

(ii) $\|T + T'\|\alpha \le \|T\|\alpha + \|T'\|\alpha$

If X and Y are infinite dimensional, α-Lp(D,Y) is precisely the class of bounded continuous operators T such that $\|T\|\alpha = 0$.

Let $T: D \longrightarrow Y$ and $K \ge 0$ be such that

$$\delta[T(U)] \le K\delta[U] \quad \text{for all bounded } U \subset D \qquad (a)$$

If we take $U = \{x,y\}$ then $\delta[U] = \|x-y\|$ and $\delta[T(U)] = \|Tx - Ty\|$. Thus if (a) holds then $T \in$ Lp(D,Y) with $\|T\| \le K$. On the other hand, let $T \in$ Lp(D,Y). Choose two sequences (x_k) and (y_k) in U such that

$$\delta[T(U)] = \lim_{k \to \infty} \|Tx_k - Ty_k\|$$

Then $\delta[T(U)] \le \lim \|T\| \cdot \|x_k - y_k\| \le \|T\| \cdot \delta[U]$. Thus $T \in$ Lp(D,Y) implies that (a) holds for $K \ge \|T\|$. It follows that if $T \in$ Lp(D,Y) then, $T \in \alpha$-Lp(D,Y) with $\|T\|\alpha \le \|T\|$. Apart from Lp(D,Y), there is another class of operators contained in α-Lp(D,Y).

Definition 3.3.2

A mapping is said to be **completely continuous** if it is both continuous and compact.

The class of completely continuous operators is a subspace of α-Lp(D,Y). We actually identify completely continuous operators with operators $T \in \alpha$-Lp(D,Y) such that $\|T\|\alpha = 0$.

Finally,

Definition 3.3.3

A function $\psi : X \longrightarrow Y$ is **Frechet** differentiable at $x \in X$ if we can find a linear continuous function $d\psi(x,\cdot):X \longrightarrow Y$ called the F-differential of ψ at x, such that

$$\lim_{\|y\|\to 0} \frac{1}{\|y\|} \|\psi(x+y) - \psi(x) - d\psi(x,y)\| = 0 \qquad y \in X$$

Usually $d\psi(x,\cdot)$ is identified with $d\psi(x)$ or simply $\psi'(x)$. The class of all operators $\psi: D \longrightarrow Y$ such that F-derivatives exists for each $x \in D$ with $\|d\psi(x)\| < \infty$ is a vector space over K. In this case the F-derivative mappings are identified with Lp(D,Y).

4. SPECTRAL THEORY

4.1 INTRODUCTION

Given an operator T, the inverse of T and $\lambda I-T$ (I is the identity operator) if they exist exhibit very interesting properties. For instance, if the inverse of $\lambda I-T$ exists, then the equation

$$\lambda x - Tx = y$$

has a solution of the form $x = (\lambda-T)^{-1}y$. Spectral theory as we shall see is the study of such functions together with certain sets. We have seen that if λ is a scalar and x is a nonzero vector such that

$$Tx = \lambda x$$

then x has been called an *eigenvector* and λ the *eigenvalue* of T corresponding to x. We briefly examine some of the results in spectral theory.

4.2 SPECTRUM AND RESOLVENT

Given λ, either $\lambda I-T$ is invertible or not. Hence scalars of such type can be grouped into disjoint sets.

Definition 4.2.1

For any operator T, the **resolvent** set of T is given by

$$\rho(T) = \{ \lambda \in K : (\lambda I-T)^{-1} \text{ exists in } Lp(X) \}$$

The complement of $\rho(T)$ is called the **spectrum** of T and is denoted by $\sigma(T)$. To simplify our notation, instead of $\lambda I-T$, we will be writing $\lambda-T$.

Definition 4.2.2

Let T be an operator and $\rho(T)$ be non empty. The **resolvent** of T is a function $R(.,T)$: $\rho(T) \longrightarrow Lp(X)$ defined by

$$R(\lambda,T) = (\lambda-T)^{-1} \qquad \text{for all } \lambda \in \rho(T).$$

It follows that λ belongs to the resolvent set in the case the resolvent belongs to Lp(X) and $\|R(\lambda,T)\| = \|(\lambda-T)^{-1}\|$. If the resolvent set is non empty, then T is necessarily closed.

We recall that if $T \in Lp(X)$ and $\|T\| < 1$, then the inverse of $(I-T)$ exists in Lp(X) and

$$\|(I-T)^{-1}\| \leq (I-\|T\|)^{-1} .$$

Under the same conditions for T, it can be shown that

$$(I-T)^{-1} = \sum_{n=o}^{\infty} T^n \qquad \text{(a)}$$

Now suppose $T \in Lp(X)$ and λ, $\mu \in \rho(T)$. Then

$$\mu-T = (\lambda-T) + (\mu - \lambda)$$

$$= [I + (\mu - \lambda) R(\lambda,T)] (\lambda-T) \qquad \text{(b)}$$

Multiply (b) on the left side by $R(\lambda,T)$ and then by $R(\mu,T)$ on the right, on both sides in each case. What we get is an equation known as the **nonlinear resolvent**:

$$R(\lambda,T) = R(\mu,T) [I - (\mu-\lambda)R(\lambda,T)] \qquad \text{(c)}$$

With the given background, we are now in a position to prove our main theorem.

Theorem 4.2.3

Let T be a member of Lp(X). Then $\rho(T)$ is an open set and, $\sigma(T)$ being its complement, is therefore closed.

Proof

Let $\lambda \in \rho(T)$ and $\mu \in K$ be such that

$$|\lambda - \mu| < \|R(\lambda,T)\|^{-1}$$

Then $|\lambda - \mu|\cdot\|R(\lambda,T)\| < 1$ and therefore $1-(\lambda-\mu)R(\lambda,T)$ is invertible. Hence

$$\| [1-(\lambda-\mu)R(\lambda,T)]^{-1}\| \le (1-|\lambda-\mu|\,\|R(\lambda,T)\|)^{-1}$$

It now follows from (b) ,if we invert it, that

$$R(\mu,T) = R(\lambda,T)[\ 1-(\lambda-\mu)R(\lambda,T)]^{-1} \tag{d}$$

which implies that $\rho(T)$ contains the neighbourhood $\{\ \mu \in K: \ |\lambda- \mu| < \|R(\lambda,T)\|^{-1}\ \}$. Hence $\rho(T)$ is open and its complement $\sigma(T)$ is closed.

Let (μ,y) and $(\lambda,x) \in \rho(T) \times X$. Let $|\lambda-\mu| \le \|R(\mu,T)\|^{-1}$. Then

$$\|R(\lambda,T)x - R(\mu,T)y\| = \|R(\lambda,T)x - R(\mu,T)x + R(\mu,T)x - R(\mu,T)y\|$$

$$\le \|R(\lambda,T)x - R(\mu,T)x\| + \| [R(\mu,T)x - R(\mu,T)y\| \tag{*}$$

Now, using (d), we have

$$\|R(\lambda,T)x - R(\mu,T)x\| = \|R(\mu,T)[\ 1-(\lambda-\mu)R(\mu,T)]^{-1}x - R(\mu,T)x\|\cdot$$

$$\le \|R(\mu,T)\|\cdot\| \ [1-(\lambda-\mu)R(\mu,T)]^{-1}x - x\|$$

Substituting this in (*) we get

$$\|R(\lambda,T)x-R(\mu,T)y\| \le \|R(\mu,T)\|\cdot\|[1-(\lambda-\mu)R(\mu,T)]^{-1}x-x\|+\|R(\mu,T)\|\cdot\|x-y\|$$

$$\le |\lambda-\mu|\cdot\|R(\mu,T)x\|\cdot\|R(\mu,T)\| \ [1-|\lambda-\mu|\,\|R(\mu,T)\|]^{-1} + \|R(\mu,T)\|\cdot\|x-y\|$$

Thus $\lim\limits_{(\lambda,x)\to(\mu,y)} R(\lambda,T)x = R(\mu,T)y$.

So the function $(\lambda,x)\longrightarrow R(\lambda,T)x$ is continuous.

The next result shows that the resolvent set is a subset of the complex plane which contains the set $\{ z : |z| > \|T\| \}$.

Theorem 4.2.4

Let $T \in Lp(X)$. Then $\rho(T) \supset \{ \lambda \in K: |\lambda|>\|T\| \}$ and $\|R(\lambda,T)\| < (|\lambda| - \|T\|)^{-1}$ for all $\lambda \in K$ such that $|\lambda| > \|T\|$.

Proof

Now, $|\lambda| > \|T\|$ implies $1 > \|\lambda^{-1}T\|$. So $(1-\lambda^{-1}T)^{-1}$ is a member of $Lp(X)$ and

$$\|(1-\lambda^{-1}T)^{-1}\| \le (1-|\lambda^{-1}|\cdot\|T\|)^{-1}$$

Thus $\lambda \in \rho(T)$. Also $\lambda-T = \lambda(1-\lambda^{-1}T)$ and therefore

$$\begin{aligned}\|(\lambda-T)^{-1}\| &= \|(1-\lambda^{-1}T)^{-1}(\lambda^{-1})\| \\ &\le |\lambda^{-1}|\ (1-|\lambda^{-1}|\cdot\|T\|)^{-1} \\ &= (|\lambda| - \|T\|)^{-1}\end{aligned}$$

4.3 LINEAR RESOLVENT

The results we have looked at for general operators hold for linear operators too. In particular we have the result that the resolvent set is open and the spectrum is closed and bounded.

We now assume that the space X is complete and the operators dealt with are closed. This condition makes the resolvent to be linear as well. We can therefore rewrite (c) as

$$R(\lambda,T) - R(\mu,T) = (\mu-\lambda)R(\mu,T)R(\lambda,T)$$

Given an operator T, on a space X, there are various reasons why $\lambda-T$ may fail to be invertible. May be

(i) $\lambda-T$ is not one-one which means that we can find a non zero vector x such that $(\lambda-T)x = 0$.

(ii) $\lambda-T$ is one-one and though $(\lambda-T)^{-1}$ is defined on the dense subspace of X, it fails to be continuous.

(iii) $(\lambda-T)^{-1}$ exists but its domain is not dense in X.

The three cases above lead us to the following definitions:

Definition 4.3.1

The **point spectrum** $\sigma_p(T)$ of an operator $T \in B(X)$ is the set of all eigenvalues of T.

Definition 4.3.2

The **continuous spectrum** $\sigma_c(T)$ consists of all $\lambda \in K$ such that $\lambda-T$ is a one-one mapping of X onto a denser proper subset of X, where $(\lambda-T)^{-1}$ is discontinuous.

Definition 4.3.3

The set of all λ such that the domain of $(\lambda-T)^{-1}$ is not dense in X is called the **residual spectrum** and is denoted by $\sigma_r(T)$

<u>REMARK: 12</u>

The spectra σ_p, σ_c and σ_r divide the spectrum σ into three parts. In a finite dimensional space however, $\sigma(T) = \sigma_p(T)$.

Definition 4.3.4

Let $T \in B(X)$. The **spectral radius** of T denoted by $r(T)$ is given by

$$r(T) = \sup\{ |\lambda| : \lambda \in \sigma(T) \}.$$

The above number exists since we well know that $\sigma(T)$ is non empty. There is a strong relationship between the spectrum and a sequence of powers of an operator.

Theorem 4.3.5

For any operator T, $\sigma(T^n) = [\sigma(T)]^n$.

Proof

Suppose $\lambda \in \sigma(T)$. Then

$$\lambda^n - T^n = (\lambda - T)(\lambda^{n-1} + \lambda^{n-2}T + \ldots + T^{n-1})$$

The factors in the above equation commute and so $\lambda^n - T^n$ has no inverse. Thus $\lambda^n \in \sigma(T^n)$.

If $\mu \in \sigma(T^n)$ and λ is an n^{th} root of μ, factoring $\lambda^n - T^n$ shows that $\lambda \in \sigma(T)$ for at least one λ.

The previous theorem implies that $r(T^n) = [r(T)]^n$. Now $r(T^n) \le \|T^n\|$. Hence $r(T) \le \|T^n\|^{1/n}$ which implies that

$$r(T) \le \lim_{n\to\infty} \inf\|T^n\|^{1/n} .$$

Let $r(T) < 1$ and $\|T^n\| \to 0$. If we take n so large, $\|T^n\| < 1$ and $\|T^n\|^{1/n} < 1$. Thus

$$\lim_{n\to\infty} \sup\|T^n\|^{1/n} \le 1$$

Define $S = (r(T) + \varepsilon)^{-1}T$ where T is any operator. Then $r(S) < 1$ and $\lim_{n\to\infty} \sup \|S^n\|^{1/n} \leq 1$

$$\Rightarrow \quad \lim_{n\to\infty} \sup \|T^n\|^{1/n} \leq r(T)$$

Combining all the steps above, we have

$$r(T) \leq \lim \inf \|T^n\|^{1/n} \leq \lim \sup \|T^n\|^{1/n} \leq r(T)$$

or that $\quad r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$

What we have above is just another way of defining the spectral radius $r(T)$ of any operator T. We have actually shown that the series $\{ \|T^n\|^{1/n} \}$ is convergent.

The following example (Martin [7]) shows that it is not for any operator that the mapping : $\lambda \longrightarrow R(\lambda, T)$ is continuous.

<u>Example</u>

Take T to be an operator over $\mathbb{R}$ and $D(T) = \mathbb{R}$ itself. Define

$$Tx = x \,, \qquad x \leq 2$$

$$Tx = 2 \,, \qquad x \geq 2$$

Then T is bounded, $\|T\| = 1$ and $\rho(T) = \{ \lambda \in \mathbb{R} : \lambda \notin [0,1] \}$.

Case of $\lambda < 0$.

$$R(\lambda,T)x = (1-\lambda)^{-1}x \qquad \text{if } x \leq 2(1-\lambda)$$

$$R(\lambda,T)x = \lambda^{-1}(2-x) \qquad \text{if } x > 2(1-\lambda)$$

Case of $\lambda > 1$.

$$R(\lambda,T) = \lambda^{-1}(2-x) \qquad \text{for } x \leq 2(1-\lambda)$$

$$R(\lambda,T) = (1-\lambda)^{-1}x \qquad \text{for } x \geq 2(1-\lambda)$$

Thus the map $\lambda \longrightarrow R(\lambda, T)$ is not continuous since

$$\lim_{n\to\infty} \sup \|R(\lambda_n, T) - R(\lambda, T)\| \geq \frac{1}{2} .$$

We have previously seen that $r(T) = \lim \|T^n\|^{1/n}$ exists and that $r(T) \leq \|T\|$. We can now give the series form of the resolvent. Since $R(\lambda,T)$ is analytic when $|\lambda| > r(T)$, it must have a Laurent expansion convergent outside the circle of radius $r(T)$. Therefore

$$R(\lambda,T) = \sum \lambda^{-k}\, T^{k-1} \qquad\qquad |\lambda| > r(T)$$

It is interesting to note that for self-adjoint operators, the eigenvectors corresponding to different eigenvalues are orthogonal. For suppose that $\lambda_1 \neq \lambda_2$ and $Tx_1 = \lambda_1 x_1$, $Tx_2 = \lambda_2 x_2$ Then

$$\lambda_1 \langle x_1, x_2 \rangle = \langle Tx_1, x_2 \rangle = \langle x_1, Tx_2 \rangle = \lambda_2 \langle x_1, x_2 \rangle$$

which can only happen if $\langle x_1, x_2 \rangle = 0$ implying that the two vectors are orthogonal.

The calculation of $r(T)$ for self-adjoint operator T is also simple. We know that $\|T^2\| = \|T\|^2$ and

$$\|T\| = \|T^2\|^{1/2} = \|T^4\|^{1/4} = \ldots$$

Therefore $r(T) = \|T\|$.

The above result is true for a normal operator T over the space of complex numbers.

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