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Date
21 April 1994
15th June 1994
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DECLARATION

I declare that this is indeed my own work and it has not been submitted before for any qualification and at any University.
DEDICATIONS

I dedicate this dissertation to my mother Tifwilemo (Chima) Zimba for being patient with my unending studies, I expect more of this behaviour from her. And I also dedicate this dissertation to my lovely wife Beatrice (Kapande) Zimba whose moral support and encouragement assisted in seeing me through the difficulties I experienced while I was doing this work.

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INTRODUCTION

Pioneering work in the determination of ordinary linear representations of the generalised symmetric group $B_n^m$ was first carried out independently by Osima [13] and Puttaswamaiah [14] by adhoc means. More recently Kerber [10], Read [16] and Hughes [8] determined the irreducible ordinary representations of $B_n^m$ by using Clifford’s theory applied to Wreath products.

The results of Osima, Puttaswamaiah, Kerber, Read and Hughes have recently been obtained in a much easier and more elegant way by M. Saeed-Ul-Islam [20], and without using Clifford’s theory.

Our object here is to give a detailed account of the inequivalent irreducible ordinary representations of $B_n^m$ by using Saeed-Ul-Islam’s approach. We also apply the results obtained to the construction of the irreducible ordinary representations of the hyperoctahedral group $B_n^2$ which is isomorphic to the Weyl group of type $B_n$. 
The following is a description of the arrangement of the chapters. In chapter I, we give the theory of ordinary representations of finite groups which will be used in the sequel. We describe the ordinary representation theory of the symmetric group $S_n$ and its Young subgroups in chapter II. The basic concepts of the generalised symmetric group $B^m_n$ and a description of its conjugacy classes are given in chapter III.

The main results of the ordinary representations of the generalised symmetric group $B^m_n$ are given in chapter IV. As a direct application of this work, we give an explicit construction of the ordinary representations of the hyperoctahedral group, $B^2_n$ in chapter V. We also give an illustration of the construction of the ordinary representations of the groups $B^2_3$ and $B^2_4$. The character tables of these groups are given in the appendix. In all that follows, results will be numbered $X.Y.Z$, where $X$ is the chapter number, $Y$ is the section number and $Z$ is the item number in section $Y$. The end of a proof will be marked by the symbol .
LIST OF SYMBOLS AND NOTATION

We shall use the following notation without further reference.

$G$ = a finite group with identity e.

$K$ = an algebraically closed field of characteristic zero.

$\mathbb{N}$ = the set of natural numbers.

$\mathbb{N}^* = \mathbb{N} \cup \{0\}$

$\mathbb{C}$ = the field of complex numbers

$\mathbb{C}^* = \mathbb{C}\setminus\{0\}$ = the multiplicative group of $\mathbb{C}$.

$V$ = a finite dimensional vector space over $\mathbb{C}$.

$\text{Hom}_\mathbb{C}(V,V)$ = the set of all endomorphisms of $V$ over $\mathbb{C}$.

$\text{GL}(V)$ = the general linear group, contained in $\text{Hom}_\mathbb{C}(V,V)$.

$\text{GL}_n(\mathbb{C})$ = the full group of invertible nxn matrices over $\mathbb{C}$.

$\text{GL}_n(\mathbb{C})$ is isomorphic to $\text{GL}(V)$ if the dimension $(V:\mathbb{C})$ of $V$ over $\mathbb{C}$ is $n$.

$\Omega$ = the set containing $n$ objects here represented by

$\{1,2,3,...,n\}$. 

CHAPTER 1

ORDINARY REPRESENTATION THEORY OF FINITE GROUPS

Our aim in this chapter is to establish much of the notation used in the ordinary representation theory of finite groups, and to give some basic results on the subject. For most of the material in this chapter we follow closely Burrow [1] and Curtis and Reiner [6]. All representations considered here are over the field \( \mathbb{C} \) of complex numbers, though the general theory used applies equally to any algebraically closed field \( K \) of characteristic zero.

1.1 Basic Concepts

Defn 1.1.1

Let \( \text{Hom}_\mathbb{C}(V,V) \) be the set of all endomorphisms \( T \) of a finite dimensional vector space \( V \) over the field \( \mathbb{C} \) of complex numbers. Then \( T \) is said to be reducible if \( V \) contains a non-trivial subspace \( U \) which is invariant under \( T \) (i.e \( Tu = u, \forall u \in U \)). \( T \) is said to be irreducible if no non-trivial subspace of \( V \) is invariant under \( T \). \( T \) is said to be completely reducible if whenever \( U \) is a non-trivial invariant subspace of \( V \), there exists
another non-trivial invariant subspace $W$ of $V$ such that $V$ is a direct sum of $U$ and $W$ (i.e. $V = U \oplus W$).

**Defn 1.1.2**

An ordinary (linear) representation of $G$ with representation space $V$ is a homomorphism $T$ of $G$ into $GL(V)$ (i.e. $T(gh) = T(g)T(h)$ and $T(e) = I_V \forall g, h, e \in G$ where $I_V$ is the identity element of $V$).

Two ordinary representations $T$ and $T'$ of $G$ with representation spaces $V$ and $V'$ respectively are said to be equivalent if there exists a $\mathbb{C}$ - isomorphism $S$ of $V$ into $V'$ such that $T'(g)S = ST(g)$ for all $g \in G$.

The dimension $(V: \mathbb{C})$ of $V$ over $\mathbb{C}$ is called the degree of $T$ denoted by $\deg T$.

An ordinary representation $T$ is said to be reducible or completely reducible if its representation space is so.

The following is an alternative definition to definition 1.1.2.
DEFN 1.1.3

A matrix representation of G of degree n is a homomorphism A of the group G into the full matrix group \( GL_n(C) \) (i.e. \( A(gh) = A(g)A(h) \) and \( A(e) = I_n \) \( \forall g, h, e \in G \) were \( I_n \) is the identity element of \( GL_n(C) \)).

Two matrix representations A and A' are equivalent if there exists a fixed invertible matrix P in \( GL_n(C) \) such that \( A'(g)P = PA(g) \) for all \( g \in G \).

Equivalent matrix representations have the same degree.

The concepts of reducibility, irreducibility and complete reducibility given above are immediately applicable to matrix representations of G.

REMARK 1.1.4

Let a matrix representation A of G be equivalent to

\[
A'(g) = \begin{pmatrix}
A_1(g) & B(g) \\
0 & A_2(g)
\end{pmatrix}
\]

for all \( g \in G \). Then

(i) A is reducible

(ii) A is completely reducible if \( B(g) = 0 \).
In all that follows, a representation of $G$ shall mean an ordinary representation (here abbreviated by o.r) of $G$. An irreducible ordinary representation of $G$ shall be abbreviated by i.o.r.

**DEFN 1.1.5**

Let $G$ be a group. Let $CG = \{ \sum_{g \in G} \alpha_g g | \alpha_g \in \mathbb{C} \}$ be a set of formal sums of elements of $G$. Define addition and scalar multiplication on $CG$ component-wise and multiplication by

$$\left( \sum_{g \in G} \alpha_g g \right) \left( \sum_{g' \in G} \alpha_{g'} g' \right) = \sum_{g', g \in G} \alpha_g \alpha_{g'} (gg').$$

Then $CG$ is an associative algebra called the **group algebra** of $G$. In what follows a representation of a group algebra $CG$ shall mean an algebra homomorphism of $CG$.

**LEMMA 1.1.6**

There is a bijection between the o.r's of a group $G$ and those of its group algebra $CG$. Furthermore there exists a bijection between the o.r's of $CG$ and finite dimensional left $CG$-modules $M$.

**PROOF**

Let $T'$ be an o.r. of $CG$ with representation space $M$. Define $T: G \to GL(M)$ by $T(g) = T'(\alpha_g g)$ $\forall g \in G$, i.e. a restriction of $T'$ to $G$. 

4
Then

\[ T(gh) = T'(a g h) = T'(a_g g) T'(a_h h) = T(g) T(h) \]

i.e. \( T \) is a homomorphism. Also \( T(e) = T'(a_e e) = I_M \). Thus \( T \) is an o.r. of \( G \) corresponding to \( T' \). Conversely let \( T \) be an o.r. of \( G \) with representation space \( M \). Define

\[ T': CG \rightarrow GL(M) \]

by

\[ T'(a) = \sum_{g \in G} a_g T(g), \]

where \( a = \sum_{g \in G} a_g g, \ a_g \in \mathbb{C} \). Then if \( a' = \sum_{h \in G} a_h h \),

we have

\[ T'(aa') = \sum_{gh \in G} a_g a_h T(gh), \text{ where } a_g = a_a a_h \]

\[ = \sum_{g, h \in G} a_g a_h T(g) T(h) \]

\[ = \left( \sum_{g \in G} a_g T(g) \right) \left( \sum_{h \in G} a_h T(h) \right) = T'(g) T'(h). \]

If now this \( T' \) is restricted to \( G \) we get back \( T \). Thus \( T' \) corresponds to \( T \).

Further, consider an o.r. \( T \) of \( G \) with representation space \( M \). Let \( r = \sum_{g \in G} a_g g \in CG \). Let \( m \in M \). Define \( rm \) by

\[ rm = \left( \sum_{g \in G} a_g T(g)m \right) \in M. \]

Then since \( M \) is an additive abelian group it is easily seen to be a \( CG \)-module corresponding to \( T \).

Conversely, let \( \{m_i / i = 1, \ldots, n\} \) be a \( \mathbb{C} \)-basis of a left \( CG \)-module \( M \) and \( r \in CG \). Let \( G = \{g_j / j = 1, \ldots, n\} \).
We have

\[ rm_1 = \left( \sum_{g_j \in G} \alpha_{g_j} g_j \right) \left( \sum_{i=1}^{n} a_i m_i \right) \]

\[ = \sum_{i=1}^{n} \left( \sum_{g_j \in G} (\alpha_{g_j} g_j) a_i m_i \right) \]

\[ = \sum_{i=1}^{n} \left( \sum_{g_j \in G} a_i \alpha_{g_j} g_j m_i \right) \]

\[ = \sum_{i=1}^{n} \left( \sum_{g_j \in G} (a_{1j}) g_j m_i \right) \]

Now take \( T(g_j) = (a_{1j}) \)

Then \( T \) is an o.r. of \( G \) with representation space \( M \).

Thus by Lemma 1.1.6, the problem of classifying o.r's of \( G \) is equivalent to the problem of classifying finite dimensional left \( CG \)-modules up to isomorphism.

We have the following theorem on the reducibility of representations of a group \( G \).

**Theorem 1.1.7 (Maschkes)**

Let \( G \) be a group, then every \( CG \)-module \( M \) is completely reducible.
PROOF

Let \( M_1 \) be a non-trivial \( CG \)-submodule of \( M \). We construct a non-trivial \( CG \)-submodule \( M_2 \) of \( M \) such that \( M = M_1 \oplus M_2 \), a direct sum. Considering \( M \) as a vector space there exists a \( CG \)-subspace \( U \) such that \( M = M_1 \oplus U \).

For if \( \phi: M \to M \) is a linear map given by \( \phi(m) = m_1 \), \( \forall m \in M, \ m_1 \in M_1 \). Then \( \phi(M) = M_1 \subset M \) and \( \phi^2 = \phi \) by definition of \( \phi \).

Now define \( T: M \to M \) by

\[
T(m) = \frac{1}{|G|} \sum_{g \in G} g\phi g^{-1}(m).
\]

Then \( \forall h \in G \), we have

\[
hTh^{-1} = \frac{1}{|G|} \sum_{g \in G} hg\phi g^{-1} h^{-1}
\]

\[
= \frac{1}{|G|} \sum_{hg \in G} hg\phi(hg)^{-1}
\]

\[
= T,
\]

as \( hg \) runs through \( G \) as \( g \) does. Now since \( T \) is a linear combination of \( \phi \), it follows that \( T^2(m) = T(T(m)) = T(m_1) = m_1 = T(m) \forall m \in M \) that is \( T^2 = T \) on \( M \).
We have
\[ T((1_m - T)(m)) = T(m - T(m)) = T(m) - T^2(m) = m_1 - m_1 = 0 \]
that is \( \ker T = (1_m - T)(M) \). We also have
\[ (1_m - T)(T(m)) = T(m) - T^2(m) = m_1 - m_1 = 0 \]
that is \( T(m) = \ker (1_m - T) \). Hence \( \ker(T) \cap T(M) = \{0\} \)
and \( M = T(M) \oplus \ker(T) \). Now let \( M_1 = T(M) \)
and \( M_2 = \ker(T) \) which gives the result \( M = M_1 \oplus M_2 \).

The above result has the following consequence in terms of o.r's of \( G \).

**Corollary 1.1.8**

Let \( T: G \to GL_n(\mathbb{C}) \) be an o.r of \( G \) with
representation space \( M \). Then there exists an
invertible matrix \( S \) in \( GL_n(\mathbb{C}) \) such that \( \forall g \in G \), we have

\[
S^{-1}T(g)S = \begin{pmatrix}
T_1(g) & 0 \\
0 & T_2(g) \\
& \ddots & \ddots \\
& & 0 & T_r(g)
\end{pmatrix}
\]

where \( T_i \) (\( i = 1, \ldots, r \)) are i.o.r's of \( G \). We write \( T \cong T_1 + \ldots + T_r \).
Let $M_{n_1}(\mathbb{C})$, where $i = 1, 2, \ldots, s$
be the full matrix algebra of $n_1 \times n_1$ matrices over $\mathbb{C}$.
Since $CG$ is completely reducible as a $CG$-module, it is
a direct sum: $CG \cong M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_s}(\mathbb{C})$
and this decomposition of $CG$ is unique (see [6]).
Since matrix multiplication is non-abelian, the
summands $M_{n_1}(\mathbb{C})$ of $CG$ are non-abelian so that $CG$ is
semisimple. Furthermore each $M_{n_1}(\mathbb{C})$ is completely
reducible by theorem 1.1.7. Thus there exists
inequivalent $CG$-modules $M_i(i = 1, \ldots, s)$ such that $M_{n_1}(\mathbb{C})$
is isomorphic to a direct sum of $m_i$ copies of $M_i$, where
$m_i = (M_i : \mathbb{C})$, that is $M_{n_1}(\mathbb{C}) = M_i \oplus \ldots \oplus M_i$ ($m_i$ copies).
we need the following definition.

**Defn 1.1.9**

An element $\varphi$ of $CG$ is called an **idempotent** if
$\varphi^2 = \varphi$. An idempotent $\varphi$ is called a **primitive idempotent**
of $CG$ if $\varphi \neq \varphi_1 + \varphi_2$, where $\varphi_1^2 = \varphi_1, \varphi_2^2 = \varphi_2$
and $\varphi_1 \varphi_2 = \varphi_2 \varphi_1 = 0$. It is clear that an idempotent
$\varphi$ of $CG$ is primitive if and only if the ideal
generated by $\varphi$ is minimal (see [1] or [6]).

The following combinatorial lemma is due to John Von Neumann (see Burrow [1]).
**Lemma 1.1.10**

Let $H_1$ and $H_2$ be two subgroups of $G$ with representations $T_1$ and $T_2$ respectively, both of degree one. Suppose that for all $g \in G$, $g \notin H_2H_1$ if and only if there exists $h_1 \in H_1, h_2 \in H_2$ such that $gh_1g^{-1} = h_2$ and $T_1(h_1) \neq T_2(h_2)$. Then $\varphi = PN$ is a scalar multiple of a primitive idempotent, where

$$P = \sum_{h_1 \in H_1} h_1 T_1(h_1)$$

and

$$N = \sum_{h_2 \in H_2} h_2 T_2(h_2).$$

**Proof**

Note that $P h_1 T_1(h_1) = \sum_{h' \in H_1} h_1 h' T_1(h'_1)$

$$= \sum_{h_1 h'_1 \in H_1} h_1 h'_1 T_1(h_1 h'_1)$$

$$= P,$$

for $h'_1 \in H_1$.

Similarly we can show that $N h_2 T_2(h'_2) = N$. Now consider $PNGPN$. If $g \in H_2 H_1$, $g = h_2 h_1$ say, then we have

$$PNGPN = P h_1 PN = T_1^{-1}(h_1) T_2^{-1}(h_2) P(h_2 T_2(h_2)) (h_1 T_1(h_1) P) N$$

$$= T_1^{-1}(h_1) T_2^{-1}(h_2) (PN)^2,$$

where $T_1^{-1}(h_1)$ is the reciprocal of the scalar $T_1(h_1)$, $T_i$ being of degree one, $i=1,2$. If $g \notin H_2 H_1$, then
\[ \text{PngPN} = T_1(h_1)\text{Pngh}_1 g^{-1} g^{\text{PN}} = T_1(h_1)\text{PN} h_2 g^{\text{PN}} \]
\[ = T_1(h_1)T_2^{-1}(h_2)\text{PngPN}. \]
Hence \( \text{PngPN}(1-T_1(h_1)T_2^{-1}(h_2)) = 0 \). This implies \( \text{PngPN} = 0 \)

since \( T_1(h_1) \neq T_2(h_2) \), and this with \( \varphi = \text{PN} \) yields
\[ \varphi \cdot \text{CG} \varphi = \mathbb{C} \varphi^2 \]

Now \( \varphi \neq 0 \) since the coefficient of the identity

I in \( \text{PN} \) is

\[ k = \sum_{h_2 \in H_2} T_1(h_1)T_2(h_2) \]
\[ h_1 \in H_1 \]

\( \forall h_1, h_2 \text{ such that } h_1, h_2 \in I, \text{ that is all } h_1, h_2 \in H_1 \cap H_2 \).
But \( I \cap I^{-1} = h_2^{-1} \) and since \( I \in H_2 \cap H_1 \), we have
\( T_1(h_1) = T_2(h_2^{-1}) \). So that \( k = |H_1 \cap H_2| \neq 0 \). Also \( \varphi^2 \neq 0 \)

otherwise \( \text{CG} \varphi \cdot \text{CG} \varphi = 0 \), implying \( \text{CG} \varphi \) is a non-zero
abelian ideal of \( \text{CG} \) contradicting the fact that \( \text{CG} \) is semi-simple.

Similarly on considering \( \text{Pg}^{-1}N \) we obtain
\( \text{PCGN} = \text{CPN} = \mathbb{C} \varphi \). Since \( \text{PCGN} \supset \text{PNCGPN} \), we get
\( \mathbb{C} \varphi \supset \varphi \cdot \text{CG} \varphi \). Now this with \( \varphi \cdot \text{CG} \varphi = \mathbb{C} \varphi^2 \) yields \( \varphi^2 = k \varphi \),
where \( k \in \mathbb{C} \). Now \( \varphi \) is primitive (otherwise \( \varphi = \varphi_1 + \varphi_2 \) with
\( \varphi_1 \varphi_2 = 0 \). But \( \varphi \neq 0 \) and by \( \varphi^2 = k \varphi \), we have \( \varphi_2 = 0 \), a
contradiction). Thus if \( G \) satisfies the conditions of
the lemma, then \( \varphi = \varphi^2 = k \varphi \) is a primitive idempotent of \( \text{CG} \).
We now wish to determine the number of inequivalent i.o.r's of \( G \) over the field \( \mathbb{C} \) of complex numbers. First we prove some preliminary lemmas.

**Lemma 1.1.11**

Let \( G \) be a group. Let \( \mathbb{C} \) be the complex field and \( Z(\mathbb{C}G) \) be the centre of \( \mathbb{C}G \). Then

\[
Z(\mathbb{C}G) = Z(M_{n_1}(\mathbb{C})) \oplus \cdots \oplus Z(M_{n_s}(\mathbb{C})) \quad \text{and} \quad (Z(\mathbb{C}G):\mathbb{C}) = s.
\]

**Proof**

From Corollary 1.1.8 \( \mathbb{C}G \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_s}(\mathbb{C}) \).

Hence \( Z(\mathbb{C}G) = Z(M_{n_1}(\mathbb{C})) \oplus \cdots \oplus Z(M_{n_s}(\mathbb{C})) \). Further the only matrices which commute with all matrices \( M_{n_1}(\mathbb{C}) \) are scalar multiples of \( I_{n_1} \). Thus \( (Z(M_{n_1}(\mathbb{C})):\mathbb{C}) = 1 \). Hence

\[
(Z(\mathbb{C}G):\mathbb{C}) = s
\]

**Lemma 1.1.12**

Let \( \{h_1, h_2, \ldots, h_t\} \) be the set of conjugacy classes of \( G \). Define \( C_i = \sum_{g \in h_i} g \) (\( i = 1, \ldots, t \)). Then \( \{C_1, \ldots, C_t\} \) is a \( \mathbb{C} \)-basis of \( Z(\mathbb{C}G) \).
PROOF

First we show that the elements of \( \{C_1, \ldots, C_t\} \) belong to \( Z(\mathbb{C}G) \). For all \( g \in G \), \( gC_1g^{-1} = \sum_{x \in h_1} gxg^{-1} = C_1 \).

Furthermore since the \( C_i \)'s are sums of group elements of disjoint sets \( h_i \), the set \( \{C_1, \ldots, C_t\} \) is linearly independent over \( \mathbb{C} \). Now let \( h = \sum_{g \in G} \alpha_g g \in Z(\mathbb{C}G) \). Then for each \( y \in G \), \( \sum_{g \in G} \alpha_g g = h = yhy^{-1} = \sum_{g \in G} \alpha_g ygy^{-1} \).

Therefore \( \alpha^{-1}_y gy = \alpha_g \) for all \( y \in G \). Hence \( \alpha_g = \alpha_y \), if and only if \( g \) and \( g' \) belong to the same conjugacy class \( h_1 \) of \( G \). Therefore \( h \) is a \( \mathbb{C} \)-linear combination of the set \( \{C_1, \ldots, C_t\} \).

We now prove the following theorem.

THEOREM 1.1.13

Keeping the notation above, the number of inequivalent irreducible \( \mathbb{C}G \) - modules \( M \) is equal to the number of conjugacy classes of \( G \).

PROOF

By lemma 1.1.11 \( (Z(\mathbb{C}G) : \mathbb{C}) = s \). And the number of elements of the \( \mathbb{C} \)-basis of \( Z(\mathbb{C}G) \) is \( t \), which is equal to the number of conjugacy classes of \( G \). Therefore \( t = s \).
Lemma 1.1.14 (Schur's Lemma).

Let $T$ and $T'$ be irreducible matrix representations of $G$ of degree $f$ and $f'$ respectively. Let $P$ be an $fxf'$ matrix such that $T(g)P = PT'(g)$, for all $g \in G$. Then either $P = 0$ and $T$ and $T'$ are not equivalent or $f' = f$ and $|P| \neq 0$ in which case $T$ and $T'$ are equivalent.

Proof

Suppose that $P \neq 0$. Then there exists non-singular matrices $A$ and $B$ such that $APB = N$ where

$$N = \begin{pmatrix} I_{rr} & O_{rt} \\ O_{sr} & O_{st} \end{pmatrix}_{fxf'}.$$

Now let $S(g) = AT(g)A^{-1}$ and $S'(g) = B^{-1}T'(g)B$ for all $g \in G$. Then since $T(g)P = PT'(g)$, we have

$$A^{-1}S(g)AP = PBS'(g)B^{-1}$$

if and only if $S(g)N = NS'(g)$, where $N = APB$, i.e. $S(g)$ is of the form

$$\begin{pmatrix} S_{rr}(g) & S_{rs}(g) \\ O_{sr}(g) & S_{ss}(g) \end{pmatrix}.$$
and $S'(g)$ is of the form

$$\begin{pmatrix}
S'_{rr}(g) & O_{rt}(g) \\
S'_{tr}(g) & S'_{tt}(g)
\end{pmatrix}.$$ 

Hence $S$ and $S'$ are reducible representations, which contradicts the irreducibility of $T$ and $T'$ unless $r = f = f'$. 

Therefore rank $P = f$, so that $|P| \neq 0$, and we can therefore write

$$T'(g) = P^{-1}T(g)P, \text{ for all } g \in G$$

i.e $T$ and $T'$ are equivalent.

**Corollary 1.1.15**

Let $T$ be an irreducible representation of $G$ of degree $f$ over $\mathbb{C}$ such that $P^{-1}T(g)P = T(g)$ for all $g \in G$, where $P$ is an invertible matrix. Then $P = \lambda I_f$, where $\lambda \in \mathbb{C}^\ast$.

**Proof**

Since we are working in the field of complex numbers, $P$ has an eigenvalue, $\lambda$ say. i.e $|P-\lambda I| = 0$. 

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Hence by hypothesis we have

\[ T(g)(P-\lambda I_f) = (P-\lambda I_f)T(g) \quad \forall \ g \in G. \]

The result now follows from theorem 1.1.14.

We need the following definition.

**DEFN 1.1.16**

Let \( A \) be a matrix representation of \( G \) of degree \( n \). Define \( \chi_A \) by \( \chi_A(g) = \text{trace}(A(g)) \) for all \( g \in G \). Then \( \chi_A \) is called a character of \( G \) afforded by \( A \). For brevity this shall be known simply as the character of \( A \).

We now prove the following result.

**THEOREM 1.1.17**

Let \( \chi_T \) denote the character of a representation \( T \) of \( G \). Then

(i) \( \chi_T \) is a class function on \( G \)

(ii) if \( S \) and \( T \) are equivalent matrix representations of \( G \) then \( \chi_S(g) = \chi_T(g) \) for all \( g \in G \).
PROOF

(i) Since similar matrices have the same characteristic polynomial, they have the same trace. Now since $T(x^{-1}gx)$ as a matrix is similar to $T(g)$, we have
\[ \chi_T(x^{-1}gx) = \text{Trace}(x^{-1}gx) = \text{Trace}(T(g)) = \chi_T(g) \]
i.e. $\chi_T$ is a class function.

(ii) If $S$ and $T$ are equivalent representations, then by definition $S(g)$ and $T(g)$ are similar matrices for all $g \in G$. The result is now immediate from (i) above.

THEOREM 1.1.18 (SCHUR'S RELATIONS)

Let $T_1$ and $T_2$ be irreducible matrix representations of $G$, where
\[ T_1(g) = (a_{ij}(g))_{n \times n} \quad \text{and} \quad T_2(g) = (b_{ij}(g))_{m \times m}. \]

Then if $T_1$ and $T_2$ are inequivalent,
\[ \sum_{g \in G} a_{is}(g^{-1})b_{tj}(g) = 0 \quad \forall i, s, t, j \]

Otherwise if $T_1$ and $T_2$ are equivalent,
\[ \sum_{g \in G} a_{is}(g^{-1})a_{tj}(g) = \frac{|G|}{n} \delta_{i,j} \delta_{s,t} \quad \forall i, j, s, t. \]
**Proof**

Let $S$ be any nxm matrix and define an nxm matrix $f(S)$ by $f(S) = \sum_{g \in G} T_1(g^{-1})ST_2(g)$

Then $T_1(h)f(S) = f(S)T_2(h)$, $\forall$ $h \in G$. Thus if in particular we set $S = E_{st}$, the nxm matrix with 1 in the $(s,t)$ - position and 0 elsewhere.

Then

$$f(E_{st}) = \sum_{g \in G} (\alpha_{1j}(g^{-1})(E_{st}))(\beta_{1j}(g)).$$

Therefore by Schurs lemma

(i) if $T_1$ and $T_2$ are not equivalent then $f(E_{st}) = 0$ so that

$$\sum_{g \in G} \alpha_{ls}(g^{-1})B_{lj}(g) = 0, \forall i,j,s,t.$$

(ii) If $T_1$ and $T_2$ are equivalent, then $f(E_{st})$ commutes with $T_1$, so that by corollary 1.1.15

$$f(E_{st}) = \lambda_{st} I_n, \text{ for some } \lambda_{st} \in \mathbb{C}^*$$

$$= \lambda \delta_{st} \delta_{ij}.$$ 

Now $(\alpha_{1j}(g^{-1})) (\alpha_{1j}(g)) = I_n$ if and only if

$$\sum_{s=1}^{n} \alpha_{ls}(g^{-1}) \alpha_{sj}(g) = \delta_{ij} \text{ and }$$

$$n \lambda = \sum_{s=1}^{n} \lambda = \sum_{s=1}^{n} (\sum_{g \in G} \alpha_{ls}(g^{-1})\alpha_{s1}(g)) = \sum_{g \in G} 1 = |G|$$

Therefore $\lambda = \frac{n}{|G|}$ so that $f(E_{st}) = \frac{n}{|G|} \delta_{st} \delta_{ij}$. giving the result.
**DEFN 1.1.19**

Let \( T_1 \) and \( T_2 \) be o.r's of \( G \) with characters \( \chi_{T_1} \) and \( \chi_{T_2} \) respectively. Then the inner product \( (\chi_{T_1}, \chi_{T_2}) \), of the characters of \( G \) is defined by

\[
(\chi_{T_1}, \chi_{T_2}) = \frac{1}{|G|} \sum_{g \in G} \chi_{T_1}(g)\chi_{T_2}(g^{-1}).
\]

we have the following lemma.

**LEMMA 1.1.20**

Let \( T_1 \) and \( T_2 \) be i.o.r's of \( G \) with characters \( \chi_{T_1} \) and \( \chi_{T_2} \) respectively. Then

(i) \( (\chi_{T_1}, \chi_{T_2})_G = 0 \), if \( T_1 \) is not equivalent to \( T_2 \).

(ii) \( (\chi_{T_1}, \chi_{T_2})_G = 1 \), if \( T_1 \) is equivalent to \( T_2 \).

**PROOF**

Let \( T_1(g) = (\alpha_{ij}(g))_{n \times n} \) and \( T_2(g) = (\beta_{ij}(g))_{m \times m} \).

Then \( \chi_{T_1}(g) = \sum_{i=1}^{n} \alpha_{i1}(g) \) and \( \chi_{T_2}(g) = \sum_{i=1}^{n} \beta_{i1}(g) \)

(i) if \( T_1 \) is not equivalent to \( T_2 \), then

\[
(\chi_{T_1}, \chi_{T_2})_G = \frac{1}{|G|} \sum_{g \in G} \chi_{T_1}(g)\chi_{T_2}(g^{-1}) = 0
\]

by theorem 1.1.18.

(ii) Now suppose \( T_1 \) is equivalent to \( T_2 \) then \( \chi_{T_1} = \chi_{T_2} \).
and we have

\[
(\chi_{T_1}, \chi_{T_2}) = \frac{1}{|G|} \sum_{g \in G} \chi_{T_1}(g) \chi_{T_2}(g^{-1})
\]

\[
= \frac{1}{|G|} \sum_{i=1}^{n} \left( \sum_{g \in G} \alpha_{i1}(g) \alpha_{i1}(g^{-1}) \right)
\]

\[
= \frac{1}{|G|} \sum_{i=1}^{n} \frac{|G|}{n} = \sum_{i=1}^{n} \frac{1}{n} = 1.
\]

We now prove the following theorem.

**Theorem 1.1.21**

Let T be an o.r of G with character \( \chi_T \). Then T is an irreducible representation of G if and only if

\[
(\chi_T, \chi_T)_G = 1.
\]

**Proof**

If T is irreducible then lemma 1.1.20 gives the result.

If \((\chi_T, \chi_T)_G = 1\), then by Maschke's theorem T is completely reducible, being a direct sum of irreducible representations

\[
i.e. \ T(g) \cong \begin{pmatrix} T_1(g) & 0 \\ 0 & T_2(g) \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & T_s(g) \end{pmatrix}
\]

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and we write $T(g) = T_1(g) + T_2(g) + \ldots + T_s(g)$. Let $\chi_{T_1}$ be a character of $T_1$ of $G$.

Then $(\chi_T, \chi_T)_G = \frac{1}{|G|} \sum_{g \in G} \chi_T(g)\chi_T(g)^{-1}$

$$= \frac{1}{|G|} \sum_{g \in G} \left( \sum_{i=1}^{s} a_i \chi_{T_1}(g) \right) \left( \sum_{i=1}^{s} a_i \chi_{T_1}(g)^{-1} \right)$$

$$= \sum_{i=1}^{s} a_i^2 \left( \frac{1}{|G|} \sum_{i=1}^{s} \chi_{T_1}(g)\chi_{T_1}(g^{-1}) \right)$$

$$= \sum_{i=1}^{s} a_i^2 , \text{ since the inner sum is } 1 \text{ by lemma 1.1.20.}$$

But $(\chi_T, \chi_T) = 1 = \sum_{i=1}^{s} a_i^2$ implies that $a_i = 1$ for only one $i$ such that $1 \leq i \leq s$ and $a_j = 0, \forall j \neq i$. By lemma 1.1.20, $T$ has one irreducible constituent, i.e. $T$ is irreducible.

**Lemma 1.1.22**

Every irreducible representation of $G$ is a component of the regular representation of $G$.

**Proof**

Let $r_G$ be the regular character of $G$. Since for $g' \neq e, g \in G, eg = g, gg' \neq g$, $r_G(e) = |G|$ and $r_G(g) = 0 \forall g(\neq(e)) \in G$.  

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Let $\chi_T$ be the character of any irreducible representation $T$ of $G$, then

$$
(r_g, \chi_T) = \frac{1}{|G|} \left( \sum_{g \in G} r_g(g) \chi_T(g^{-1}) \right)
$$

$$
= \frac{1}{|G|} \left( r_g(e)\chi_T(e) + \sum_{g \in G} r_g(g)\chi_T(g^{-1}) \right)
$$

$$
= \chi_T(e) + 0
$$

= Trace (T(e)) = Trace $I_f = f$

where $f$ is the degree of $T$. We have $f = (r_g, \chi_T)$ is the number of times $\chi_T$ appears as a component of $r_g$.

We now obtain an alternative to theorem 1.1.18.

**Theorem 1.1.23 (Second Orthogonality Relations)**

Let $\{T_1, T_2, \ldots, T_s\}$ be a complete set of i.o.r’s of $G$ with characters $\chi_{T_1}, \ldots, \chi_{T_s}$ of degrees $n_1, \ldots, n_s$ respectively. Let $Cl_i$ denote the $i$th conjugacy class of $G$, and $Cl_i^*$ be the class containing the inverse of $g \in Cl_i$. Let $h_i = |Cl_i|$ be the order of $Cl_i$. Let $\chi_i^j$ denote the value of $\chi_i$ on $Cl_j$. Then

(i) if $Cl_j$ is not inverse to $Cl_k$,

$$
\sum_{i=1}^s \chi_i^j \chi_i^k = 0
$$

(ii) if $Cl_j$ is inverse to $Cl_k$,

$$
\sum_{i=1}^s \chi_i^j \chi_i^k = \frac{|G|}{h_j}
$$
PROOF

Let \( C_j = \sum g \). Then \( C_j g = gC_j \) \( \forall g \in G \) by lemma 1.1.12. Therefore we have \( T_1(C_j g) = T_1(gC_j) \) if and only if \( T_1(C_j)T_1(g) = T_1(g)T_1(C_j) \), \( \forall g \in G \). Thereby by Schur's lemma, we have \( T_1(C_j) = w_j^i I_{n_1} \). And on computing the traces we have \( h_j^j \chi_1(g) = n_1 w_j^i \) (where \( n_1 = \text{trace } I_{n_1} \)). If now \( \chi_1 \) denotes the value of \( \chi_1 \) on an element \( g \) of \( C_1 \), then we have

\[
W_j^i = \frac{h_j^j \chi_1^i}{n_1}.
\]

Now since \( \{C_1, \ldots, C_s\} \) is a \( C \)-basis for \( Z(G) \), then

\[
C_j C_k = \sum_{1=1}^{s} \alpha_{jk1} C_1, \quad \text{where } \alpha_{jk1} \in C.
\]

Hence \( T_1(C_j)T_1(C_k) = \sum_{1=1}^{s} \alpha_{jk1} T_1(C_1) \) if and only if

\[
W_j^i W_k^l = \sum_{1=1}^{s} \alpha_{jk1} W_1^l \quad \text{if and only if}
\]

\[
\frac{h_j^j \chi_1^i}{n_1} \cdot \frac{h_k^k \chi_1^l}{n_1} = \sum_{1=1}^{s} \alpha_{jk1} \frac{h_1 \chi_1^l}{n_1}
\]

Now if \( r_G \) is the regular character of \( G \) then lemma 1.1.22 implies that \( r_G = \sum_{1=1}^{s} n_1 \chi_1^i \).

Hence \( h_j h_k \sum_{1=1}^{s} \chi_1^j \chi_1^k = \sum_{1=1}^{s} \alpha_{jk1} h_1 r_G^l \)

\[= \alpha_{jk1} |G|, \]

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since \( r^1 = \begin{cases} |G| & \text{if } Cl_1 = \{e\} \\ 0 & \text{otherwise.} \end{cases} \)

Now \( |Cl^*_j| = |Cl_j| = h_j \), we have

\[
\alpha_{jk1} = \begin{cases} h_j & \text{if } Cl_j = Cl^*_k \\ 0 & \text{otherwise.} \end{cases}
\]

Therefore \( \sum_{l=1}^{s} \chi^j_i \chi^k_i = 0 \) if \( Cl_j \) is not inverse to \( Cl_k \).

\[
\frac{h_j}{|G|} \sum_{l=1}^{s} \chi^j_i \chi^k_i = 1 \text{ if } Cl_k \text{ is inverse to } Cl_j.
\]

### 1.2 Construction of New Representations of Finite Groups

In this section we describe the methods of constructing new o.r’s of a finite group from old ones to be used here.

**Defn 1.2.1**

Let \( A = (a_{ij}) \) and \( B = (b_{ij}) \) be two matrices of degree \( s \) and \( r \) respectively. Then the **tensor (Kronecker) product** of matrices \( A \) and \( B \) is defined by

\[
A \otimes B = \begin{bmatrix}
    a_{11}B & a_{12}B & \cdots & a_{1s}B \\
    a_{21}B & a_{22}B & \cdots & a_{2s}B \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{s1}B & a_{s2}B & \cdots & a_{ss}B
\end{bmatrix}
\]
Now let $T_1$ and $T_2$ be representations of $G$ of degrees $s$ and $r$ respectively. Define $T$ by

$$T(g) = T_1(g) \otimes T_2(g) \in \text{GL}(sr, \mathbb{C}).$$

Then $T$ as defined is a representation of $G$ of degree $sr$ called the tensor (Kronecker) product representation of $G$. We shall write $T = T_1 \otimes T_2$.

The representation $T = T_1 \otimes T_2$ is in general reducible. Its reduction may yield new i.o.r's of $G$. The following construction gives rise to new i.o.r's of $G$. We have a definition.

**Defn 1.2.2**

Let $G$ be a group of form $H \times H'$, where $H$ and $H'$ are groups. Let $T$ and $T'$ be o.r's of $H$ and $H'$ respectively. Then the map $T \# T'$ defined by

$$T \# T' (h, h') = T(h) \otimes T'(h')$$

is a representation of $G$ called the outer tensor product representation.
**Lemma 1.2.3**

Let $G = H \times H'$ be a group with outer tensor product representation $T \# T'$. Then $T \# T'$ is irreducible if and only if $T$ and $T'$ are both irreducible. Further if $\{T_1, \ldots, T_r\}$ and $\{T'_1, \ldots, T'_s\}$ are complete sets of inequivalent i.o.r's of $H$ and $H'$ respectively then $\{T_i \# T'_j \mid i = 1, \ldots, r; j = 1, \ldots, s\}$ is a complete set of inequivalent i.o.r's of $G$.

**Proof**

It suffices to show the second part. Let $\chi^T_{i \# j}$ be the character of $T_i \# T'_j$. Then

$$\frac{1}{|H \times H'|} \sum_{(h, h') \in G} \chi^T_{i \# j}(h, h') \chi^{T'}_{k \# l}(h, h')^{-1}$$

$$= \frac{1}{|H| |H'|} \sum_{h \in H} \sum_{h' \in H'} \chi^T_{1}(h) \chi^{T'}_{j}(h') \chi^{T'}_{k}(h^{-1}) \chi^T_{1}(h')^{-1}$$

where $\chi^T_{1}$ is the character of $T_1$ and $\chi^{T'}_{j}$ is that of $T'_j$ $(i, k = 1, \ldots, r)$, $(j, l = 1, \ldots, s)$

$$= \left \{ \begin{array}{ll} 
\frac{1}{|H|} \sum_{h \in H} \chi^T_{1}(h) \chi^{T'}_{k}(h^{-1}) \chi^T_{j}(h') & \frac{1}{|H'|} \sum_{h' \in H'} \chi^{T'}_{j}(h') \chi^T_{1}(h')^{-1} \\
1 & \text{if } i = k, j = 1 \\
0 & \text{otherwise,}
\end{array} \right. , \text{ by theorem 1.1.18 and theorem 1.1.21.}$$

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We now consider the relationship between the o.r.'s of G and those of its subgroup H. Consider i.o.r.'s T and t of degree n and m of G and H respectively. We may limit consideration to those matrices of the i.o.r T of G associated with the elements of H. By such a restricting process we obtain an o.r. of H which we denote by $T_h$ or $T \downarrow H$. Similarly, we may obtain an o.r of G from that t of a subgroup H of G as follows.

**Defn 1.2.4**

Let H be a subgroup of G such that $G = \bigcup_{i=1}^{s} Hg_i$ is a decomposition of G into right cosets modulo H. Define

$$T(x) = \begin{pmatrix}
    t(g_1xg_1^{-1}) & t(g_1xg_2^{-1}) & \ldots & t(g_1xg_s^{-1}) \\
    t(g_2xg_1^{-1}) & t(g_2xg_2^{-1}) & \ldots & t(g_2xg_s^{-1}) \\
    \vdots & \vdots & \ddots & \vdots \\
    t(g_sxg_1^{-1}) & t(g_sxg_2^{-1}) & \ldots & t(g_sxg_s^{-1})
\end{pmatrix}$$

$$\implies t(g_ixg_j^{-1}) = \begin{cases} 
    t(g_ixg_j^{-1}) & \text{if } g_ixg_j^{-1} \in H \\
    0 & \text{if } g_ixg_j^{-1} \notin H.
\end{cases}$$
Then $T$ is easily seen to be a representation of $G$ of degree $m$ called a representation induced from $t$ or the induced representation of $G$. We denote the induced representation of $G$ by $t^{G}$ or $t^{\uparrow}G$.

Now let $\theta$ and $\chi$ be characters of $H$ and $G$ respectively. We shall write $\theta^{G}$ or $(\theta \uparrow G)$ for the character of $t^{G}$ (or $t^{\uparrow} G$) and $\chi_{\downarrow} H$ (or $\chi \downarrow H$) for the character of $G$ restricted to $H$. For the formula for the character of $G$ induced from that of $H$ we have

\[
\theta^{G}(g) = \text{trace}(t(x_j g x_j^{-1})) = \sum_{j=1}^{s} \theta(x_j g x_j^{-1})
\]

Where $\theta(g') = \theta(g')$ if $g' \in H$, or 0 if $g' \notin H$.

Now if $g \in G$, then $g$ has the form $H_g_{j}$ for some $j \in \{1, \ldots, s\}$, where $s = [G:H]$. Thus $g = hg_{j}$, for some $h \in H$, and we have

\[
\theta(x_j g x_j^{-1}) = \theta(g_j g g_j^{-1})
\]

Therefore $\theta^{G}(g) = \frac{1}{|G|} \sum_{x \in G} \theta(xgx^{-1})$

Now as $x$ ranges over $G$, $xgx^{-1}$ are in the conjugacy class $C_{1_{i}}$ of $G$ containing $g$. Thus as $x$ ranges over $G$, $xgx^{-1}$ ranges over $C_{1_{i}}$ and each element $y \in C_{1_{i}}$ appears $\frac{|G|}{h_{1}}$ times, where $h_{1} = |C_{1_{i}}|$.
Thus alternatively we have

$$\theta^G(g) = \frac{|G|}{|H|h_1} \sum_{y \in c_1 \cap H} \theta(y), g \in c_1$$

The representation $t^G$ is in general reducible.

The following results are associated with induced o.r's of $G$.

**Theorem 1.2.5 (Frobenius Reciprocity)**

Let $H < G$ with characters $\theta$ and $\chi$ respectively. Then $(\chi, \theta^G)_G = (\chi, \theta)_H$.

**Proof**

$$(\chi, \theta^G)_G = \frac{1}{|G|} \sum_{g \in G} \chi(g) \theta^G(g^{-1})$$

$$= \frac{1}{|G|} \sum_{t \in G} \left( \frac{1}{|H|} \sum_{g \in G} \chi(g) \theta(t^{-1} g^{-1} t) \right)$$

where $\theta(t^{-1} g^{-1} t) = 0$ if $t^{-1} g^{-1} t \notin H$. Since $\chi(t^{-1} g t) = \chi(g)$, $\forall$ $t \in G$, we have

$$(\chi, \theta^G)_G = \frac{1}{|G||H|} \sum_{t \in G} \left( \sum_{g \in G} \chi(t^{-1} g t) \theta(t^{-1} g^{-1} t) \right)$$

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Now for a fixed \( t \in G \), \( g \) varies through \( G \) as \( t^{-1}gt \) varies in a conjugacy class containing \( g \), so that

\[
(\chi, \theta^G)_g = \frac{1}{|G||H|} \sum_{t \in G} \sum_{g' \in G} \chi(g') \theta(g'^{-1})
\]

\[
= \frac{1}{|H|} \sum_{g' \in H} \chi(g') \theta(g'^{-1})
\]

\[
= \frac{1}{|H|} \sum_{g' \in H} \chi'(g') \theta(g'^{-1}) = (\chi', \theta)_H.
\]

Hence \( (\chi, \theta^G)_g = (\chi', \theta)_H \).

**Lemma 1.2.6** (Mackeys Subgroup Formula)

Let \( H \) and \( K \) be subgroups of \( G \) with characters \( \chi_1 \) and \( \chi_2 \) respectively. Let \( X \) be the set of representatives for the double cosets \( KxH \) in \( G \), (i.e. \( x \in X \)).

Then

\[
(\chi_1, (\chi_2^G)_H)_H = \sum_{x \in KxH} \left( \chi_1, ((\chi_2^x)_H x^{-1} \cap K)_H \right)_H
\]

where \( \chi_2^x(g) = \chi_2(x^{-1}gx) \).

**Proof** (see [11] or [6])  

The following result for the calculation of characters constructed in a special way, will be used in the sequel.
**Lemma 1.2.7 (Brauer)**

If a group $G$ has a pair of subgroups $H_1$ and $H_2$ with linear characters $\chi_1$ and $\chi_2$ respectively, such that the induced characters $\chi_1^G$ and $\chi_2^G$ have exactly one irreducible constituent in common in each case with multiplicity one, then this irreducible constituent character can be calculated using the formula

$$
\chi \ (= \chi_1^G \cap \chi_2^G, \text{ say}) \ (g) = \frac{n}{|\text{Ccl}(g)| |H_1 \cap H_2|} \sum_{h_1 \in H_1, h_2 \in H_2} \chi_1(h_1) \chi_2(h_2)
$$

where $|\text{ccl}(g)|$ is the order of the conjugacy class of $g$, $n$ is the degree of the irreducible constituent character and the summation is taken over all $h_1 \in H_1$, $h_2 \in H_2$ such that $h_1 h_2 \in \text{ccl}(g)$.

**Proof** (see [2])

We consider yet another application of the subgroup structure of $G$ to the construction of o.r.'s of $G$.

**Defn 1.2.8**

Let $H$ be a normal subgroup of index $s$ in $G$. Let $X = \{g_i, i = 1, \ldots, s\}$ be the set of representatives for the right cosets $Hg_i$ in $G$. Let $T': G/H \to \text{GL}_n(\mathbb{C})$ be an o.r. of the quotient group $G/H$ of $G$ modulo $H$. 

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Define
the composition \( T = T' \circ h \) of the homomorphism \( h: G \to G/H \)
and \( T' \) by \( T(g) = T'(Hg_1) \in \text{GL}_n(\mathbb{C}) \), where \( g_1 \) represents
\( g \) in \( X \). Then \( T = T' \circ h \) is an o.r. of \( G \) called a
representation of \( G \) lifted from \( T' \). We will denote
\( T = T' \circ h \) by \( (T')^* \). Since \( (T')^* \) takes the same values on \( G \)
as \( T' \) does on \( G/H \), the number of o.r.'s \( (T')^* \) of \( G \) is
equal to the number of the o.r.'s \( T' \) of \( G/H \).

**Lemma 1.2.9**

Let \( (T')^* \) be a representation of \( G \) lifted from \( T' \).
Then \( (T')^* \) is irreducible if and only if \( T' \) is irreducible.

**Proof**

Let \( \chi_{T'} \) and \( \chi_{(T')^*} \) be characters of \( T' \) and
\( (T')^* \) respectively. Then \( \chi_{(T')^*}(g) = \chi_{T'}(Hg) \) by
definition. We show that \( \langle \chi_{(T')^*}, \chi_{(T')^*} \rangle_G = 1 \) if and
only if \( \langle \chi_{T'}, \chi_{T'} \rangle_{G/H} = 1 \).

Now \( \langle \chi_{(T')^*}, \chi_{(T')^*} \rangle_G = \frac{1}{|G|} \sum_{g \in G} \chi_{(T')^*}(g) \chi_{(T')^*}(g^{-1}) \)

\( = \frac{1}{|G|} \sum_{g \in G} \chi_{T'}(Hg) \chi_{T'}(Hg^{-1}) \)

\( \langle \chi_{(T')^*}, \chi_{(T')^*} \rangle_G = \frac{1}{|G/H|} \sum_{hg \in G/H} \chi_{T'}(Hg) \chi_{T'}(Hg^{-1}) \)

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\[
= \left( \chi_{T'}, \chi_{T'} \right)_{G/H}.
\]

Thus if \( \left( \chi_{(T')}^*, \chi_{(T')}^* \right)_G = 1 \), then \( \left( \chi_{T'}, \chi_{T'} \right)_{G/H} = 1 \) and conversely.
CHAPTER II

REPRESENTATION THEORY OF THE SYMMETRIC GROUP, $S_n$

Here we give a brief outline of the ordinary representation theory of the symmetric group, $S_n$ which will be used in the sequel. The method of constructing the ordinary representations of $S_n$ discussed here is due to A. Young [21]. First we describe the symmetric group.

2.1 THE SYMMETRIC GROUP, $S_n$

**Defn 2.1.1**

A bijective mapping of a set $\Omega$ onto itself is called a permutation of $\Omega$. The set of all permutations of $\Omega$ together with the composition multiplication is a group called the symmetric group, $S_n$ of degree $n$.

A permutation $\theta \in S_n$ of the form

$$
\begin{pmatrix}
12\ldots k & k+1\ldots n \\
23\ldots 1 & k+1\ldots n
\end{pmatrix}
$$

(2.1)

is called a cycle of length $k$ (or a $k$-cycle, to emphasize the number of symbols of $\Omega$ moved by $\theta$) and is written $(123\ldots k)$. Each permutation $\theta \in S_n$ can be written as a product of uniquely determined disjoint
cycles up to their order of occurrence. Such an expression of \( \theta \) is called the cycle-notation of \( \theta \). The set of positive integers \( \alpha_i, i=1, \ldots, k \) representing the lengths of cycles in the cycle-notation of \( \theta \) is called the cycle-type or structure of \( \theta \).

Taking \( r_i = (i, i+1), (i=1, \ldots, n-1) \), then it is well known that the group \( S_n \) has a presentation given by

\[
S_n = \langle r_i \ (i=1, \ldots, n-1) | r_i^2 = 1 (i=1, \ldots, n-1), \\
(r_i r_{i+1})^3 = 1 \ (i=1, \ldots, n-2), \ (r_i r_j)^2 = 1, \\
j \neq i, i+1, (i=1, \ldots, n-1) \rangle \tag{2.2}
\]

\( S_n \) has order \( |S_n| = n! \)

**Defn 2.1.2**

Let \( (\alpha) = (t_1, \ldots, t_k) \) be a set of non-negative integers such that \( \sum_{i=1}^{k} t_i = n \) and \( t_1 \geq t_{i+1} \). Then \( (\alpha) \) is called a partition of \( n \). Here, we shall sometimes write \( (\alpha_n) \) to emphasize the number being partitioned. To each partition \( (\alpha) \) of \( n \) there corresponds a partition \( (\alpha)' = (t_1', t_2', \ldots, t_k') \) where \( t_1' = \sum_{j: j \geq 1} t_j \), called the associate or conjugate partition of \( n \).

The ordered lengths \( \alpha_1, \alpha_2, \ldots, \alpha_k \) (\( \alpha_i \geq \alpha_{i+1} \)) of the cyclic factors in the cycle-notation of \( \theta \) (including 1-cycles) form a partition \( (\alpha) = (\alpha_1, \ldots, \alpha_k) \) of \( n \). If
\( \theta \) and \( \lambda \) are conjugate elements in \( S_n \) then the cycle notation of \( \lambda \) is easily determined from that of \( \theta \).

**Lemma 2.1.3**

Let \( \theta, \mu \in S_n \). Then we obtain the conjugate \( \mu \theta \mu^{-1} \) of \( \theta \) by an application of \( \mu \) to the symbols of \( \Omega \) in \( \theta \).

**Proof**

This is clear since \( \mu \theta \mu^{-1} = \begin{bmatrix} i \\ \mu(i) \end{bmatrix} \begin{bmatrix} i \\ \theta(i) \end{bmatrix} \begin{bmatrix} \mu(i) \\ i \end{bmatrix} = \begin{bmatrix} \mu(i) \\ \mu \theta(i) \end{bmatrix} \).}

Thus conjugate elements in \( S_n \) have the same cycle-structure. Furthermore, it is well known that the conjugacy classes of \( S_n \) are in one-one correspondence with the partitions of \( n \)(see[10] or [19]).

In constructing the o.r.'s of \( S_n \) we use certain representations of a special class of subgroups of \( S_n \), called **young subgroups** in honour of A. Young for his pioneering work in this area.

**Defn 2.1.4**

Let \( (\alpha_1, \alpha_2, \ldots, \alpha_k) \) be a partition of \( n \). Define \( P_0 = 0, \ P_i = \sum_{j=1}^{i} \alpha_j \), (i=1,...,k). Denote by \( S_{\alpha_i} \) the symmetric group on the \( \alpha_i \) symbols \( \{P_{i-1}+1, \ldots, P_i\} \). Then the direct product \( S_{\alpha_1} \times S_{\alpha_2} \times \cdots \times S_{\alpha_k} \) is a subgroup of \( S_n \) called the **young subgroup**. Here we
shall denote the young subgroup of $S_n$ by $S(\alpha_1, \ldots, \alpha_k)$ or simply $S(\alpha)$.

We have the following analogous definition associated with the conjugate partition $(\alpha)'$ of $n$.

**Defn 2.1.5**

Let $(\alpha)'$ be a conjugate partition of $n$. Define $P'_o = 0$, $P'_1 = \sum_{j=1}^{l} \alpha'_j$ ($i=1, \ldots, k$). Denote by $S_{\alpha'_1}$ the symmetric group on the $\alpha'_1$ symbols $\{P'_{i-1} + 1, \ldots, P'_1\}$. Then the direct product $S_{\alpha'_1} \times S_{\alpha'_2} \times \ldots \times S_{\alpha'_k}$ is a subgroup of $S_n$ called the **conjugate young subgroup**. Here we shall denote the conjugate young subgroup of $S_n$ by $S(\alpha'_1, \alpha'_2, \ldots, \alpha'_k)$ or simply $S(\alpha)'$.

### 2.2. Representation Theory of Symmetric Groups

We require the following definition.

**Defn 2.2.1**

1. A **young diagram** associated with a partition $(\alpha)$ of $n$ (here written $(\alpha)$-diagram) is a diagram consisting of $n$ nodes in $k$-rows with $\alpha_i$ nodes in the $i$th row and all rows starting in the same column. An $(\alpha)$-diagram shall be denoted by $[\alpha]$. For example for $n = 3$ and $(\alpha) = (21)$, then $[21] = o$ is a $(21)$-diagram.
The **conjugate young diagram** \([\alpha]'\) (here written \((\alpha)'\)-diagram) is the diagram obtained by interchanging the rows and columns in the \((\alpha)\)-diagram \([\alpha]\).

Now A young has shown that for algebraically closed fields of characteristic zero, each \((\alpha)\)-diagram \([\alpha]\) corresponds to a uniquely determined i.o.r. of \(S_n\). This fact follows from definition 2.2.1 and our earlier remark that there is a bijection between the conjugacy classes of \(S_n\) and the partitions of \(n\). Thus here we may denote by \([\alpha]\), also the o.r. of \(S_n\) associated with the \((\alpha)\)-diagram \([\alpha]\). We proceed to show how this correspondence between the i.o.r's of \(S_n\) and the \((\alpha)\)-diagrams \([\alpha]\) actually arises. First the \((\alpha)\)-diagram leads to the following definition.

**Defn 2.2.2**

We keep the notation above. A **young-tableau** associated with a partition \((\alpha)\) (here written \((\alpha)\)-tableau) is one of the \(n!\) arrays of symbols in \(\Omega\) obtained by replacing each node in the \((\alpha)\)-diagram \([\alpha]\) by one of the symbols in \(\Omega\) with no repetitions. An \((\alpha)\)-tableau is said to be a **standard \((\alpha)\)-tableau** if the symbols in it increase from left to right along each row and from top to bottom down each column. For example for \([21] = \infty, \begin{array}{l|l} 1 & 2 \\ \hline 3 & 2 \end{array}\) and \(\begin{array}{l|l} 2 & 1 \\ \hline 3 & 3 \end{array}\) are the standard and non-standard \((21)\)-tableaux respectively.
If \( f(\alpha) \) is the number of the standard \((\alpha)\)-tableaux, then the standard \((\alpha)\)-tableaux will be denoted by
\[
T_1^{[\alpha]}, \ldots T_{f(\alpha)}^{[\alpha]}.
\]

As we shall see later, the number \( f(\alpha) \) of standard \((\alpha)\)-tableaux gives the degree of the i.o.r. \([
\alpha\]
) of \( S_n \) (see eg. [9] or [19]). Now consider any of the standard \((\alpha)\)-tableaux, say \( T_1^{[\alpha]} \). Then it is seen that the young subgroup \( S_{(\alpha)} \) permutes only the symbols within the rows of \( T_1^{[\alpha]} \) and the conjugate young subgroup \( S_{(\bar{\alpha})} \), permutes only the symbols within the columns of \( T_1^{[\alpha]} \) (see eg. [9]). Thus each standard \((\alpha)\)-tableau of an \((\alpha)\)-diagram yields a pair of young-subgroups \( S_{(\alpha)} \) and \( S_{(\bar{\alpha})} \) of \( S_n \). For example, the group \( S_3 \) has three \((\alpha)\)-diagrams with standard \((\alpha)\)-tableaux given by

\[
T_1^{[3]} =: \begin{array}{ccc}
1 & 2 & 3 \\
\hline
1 & 2 & 3 \\
\end{array},
T_1^{[21]} =: \begin{array}{cc}
1 & 2 \\
\hline
3 & 2 \\
\end{array},
T_1^{[13]} =: \begin{array}{cc}
1 & 2 \\
\hline
2 & 3 \\
\end{array}
\]

(2.3)

whose pairs of young subgroups are \( S_{(3)} = S_3, S_{(3)}' = S_1 \); \( S_{(21)} = \{e, (12)\}, S_{(21)}' = \{e, (13)\} \) and \( S_{(1^3)} = S_1, S_{(1^3)}' = S_3 \) respectively. Thus \( S_3 \) has three i.o.r.'s \([3]\) and \([1^3]\) each of degree 1 and \([21]\) of degree 2 (since \( T_2^{[21]} = \begin{array}{cc}
1 & 3 \\
\hline
2 & 2 \\
\end{array} \) is another standard \((21)\)-tableau).
Since our aim is to determine i.o.r's of $S_n$ associated with $(\alpha)$-diagrams, following from the remarks after lemma 1.1.6, we now determine the representation spaces or modules which afford the i.o.r's $[\alpha]$. To this end it is enough to determine a $\mathbb{C}$-basis of the representation space of the i.o.r $[\alpha]$. We have the following lemma.

**Lemma 2.2.3**

Let $\mathcal{C}S_n = A_1 \otimes \ldots \otimes A_s$ be a decomposition of the group algebra $\mathcal{C}S_n$ of $S_n$ into a unique direct sum of ideals $A_{(\alpha)} (\alpha = 1, \ldots, s)$. Then $A_{(\alpha)} = R_{(\alpha_1)} \otimes \ldots \otimes R_{(\alpha_f(\alpha))}$ where the $R_{(\alpha_i)}$'s are the representation spaces of the i.o.r $[\alpha]$ of $S_n$. Furthermore the $R_{(\alpha_i)}$'s are generated by $\varphi_{i1}^{(\alpha)} (i = 1, \ldots, f_{(\alpha)})$.

**Proof**

From corollary 1.1.8, since $\mathcal{C}S_n$ is completely reducible, it follows that $\mathcal{C}S_n = A_1 \otimes \ldots \otimes A_s$ and this decomposition is unique with each $A_{(\alpha)}$ being expressed as a direct sum of $f_{(\alpha)}$ isomorphic minimal left ideals of $\mathcal{C}S_n$ of degree $f_{(\alpha)}$ over $\mathbb{C}$. Thus there are $f_{(\alpha)}^2$ elements $\varphi_{ij}^{(\alpha)}$ of $A_{(\alpha)}$ ($i = 1, \ldots, f_{(\alpha)}$) such that $\varphi_{ij}^{(\alpha)} \varphi_{k1}^{(\alpha)} = \delta_{j}^{k} \varphi_{i1}^{(\alpha)}$ where $\delta_{jk}$ is the kronecker delta, and

for any $a \in A_{(\alpha)}$, $a = \sum_{i=1, j=1}^{f_{(\alpha)}} k_{ij} \varphi_{ij}^{(\alpha)}$ where $k_{ij} \in \mathbb{C}$. 
Also
\[ \varphi_{1j}^{(\alpha)} \varphi_{k1}^{(\beta)} = 0 \text{ for distinct ideals } A^{(\alpha)} \text{ and } A^{(\beta)} \text{ of } \mathbb{C} S_n. \]

We see that \( \sum_{\alpha=1}^{f^{(\alpha)}} \left( \sum_{i=1}^{s} \varphi_{1i}^{(\alpha)} \right) = 1 \in \mathbb{C} S_n \) where \( \sum_{i=1}^{f^{(\alpha)}} \varphi_{1i}^{(\alpha)} = 1 \in A^{(\alpha)}. \) Let \( R^{(\alpha)}_1 = \mathbb{C} S_n \varphi_{11}^{(\alpha)} (i=1, \ldots, f^{(\alpha)}). \) Then the \( R^{(\alpha)}_1 \)'s are left ideals of \( \mathbb{C} S_n. \) We have

\[
A^{(\alpha)} = \sum_{i=1}^{f^{(\alpha)}} A^{(\alpha)} \varphi_{1i}^{(\alpha)} = \sum_{i=1}^{f^{(\alpha)}} \left( \sum_{(\beta)=1}^{s} A^{(\beta)} \right) \varphi_{1i}^{(\alpha)}
\]

\[ = \sum_{i=1}^{f^{(\alpha)}} \mathbb{C} S_n \varphi_{1i}^{(\alpha)}
\]

[continue]

By the uniqueness of the decomposition of \( \mathbb{C} S_n, \) the \( R^{(\alpha)}_1 \)'s \( (i=1, \ldots, f^{(\alpha)}) \) are isomorphic minimal left ideals of \( \mathbb{C} S_n. \) Now by the remarks following lemma 1.1.9, the \( \varphi_{1j}^{(\alpha)} \) generating \( R^{(\alpha)}_1 \) are primitive elements of \( \mathbb{C} S_n. \) From lemma 1.1.12 we only need the central primitive idempotents of the two sided-ideals \( A^{(\alpha)} \) of \( \mathbb{C} S_n. \) The elements \( \varphi_{1i}^{(\alpha)} (i=1, \ldots, f^{(\alpha)})(i.e \varphi_{1j}^{(\alpha)}, \text{ such that } i=j) \) are the central primitive idempotents of \( A^{(\alpha)} \) and any one of these say \( \varphi_{11}^{(\alpha)} \) generates a minimal left ideal \( R^{(\alpha)}_1 = \mathbb{C} S_n \varphi_{11}^{(\alpha)} \) which is the representation space of the i.o.r \( [\alpha] \) of \( S_n. \) Furthermore \( \{ \varphi_{11}^{(\alpha)}, \ldots, \varphi_{1f^{(\alpha)}}^{(\alpha)} \} \) is a \( \mathbb{C} \)-basis of \( R^{(\alpha)}_1. \)
This confirms that the number \( f_\alpha \) is the degree of the i.o.r \([\alpha]\).

Now from above and the remarks following lemma 1.1.6, the problem of constructing i.o.r's of \( S_n \) thus reduces to that of finding primitive idempotents for each two sided ideal \( A_\alpha \) of \( CS_n \).

In the following result we show that the primitive idempotents of \( A_\alpha \) are obtained from the young subgroups \( S_\alpha \) and \( S_\alpha' \), of \( S_n \).

**Lemma 2.2.4**

Let \( IS_\alpha \) be the identity representation on \( S_\alpha \) and let \( AS_\alpha \), be the Alternating representation on \( S_\alpha' \), let \( f_\alpha \) be the number of standard \((\alpha)\)-tableaux. Then

\[
\varphi = \frac{f_\alpha}{n!} \sum_{\substack{\theta_1 \in S_\alpha \\theta_2 \in S_\alpha' \\theta_1, \theta_2}} IS_\alpha(\theta_1)AS_\alpha(\theta_2)
\]

is a primitive idempotent of the ideal \( A_\alpha \) of \( CS_n \).

**Proof**

It suffices to show that \( S_n \) with \( T_1 = IS_\alpha \) on \( S_\alpha \) and \( T_2 = AS_\alpha \), on \( S_\alpha' \), satisfies the conditions of lemma 1.1.10.
Now suppose $S_{(\alpha)} \cap \lambda S_{(\alpha)}', \lambda^{-1} \neq \{e\}$, then

$\lambda \in S_{(\alpha)} S_{(\alpha)}'$ (otherwise $\lambda \in S_{(\alpha)} S_{(\alpha)}'$ will imply that there exists some $\theta_1 \in S_{(\alpha)}$ and $\theta_2 \in S_{(\alpha)}$, such that for $\lambda = \theta^{11} \theta^{21} \in S_{(\alpha)} S_{(\alpha)}'$, $\theta^{11} \theta^{21} \theta^{-1} \theta_1^{-1} = \theta_1 \neq e$ and $\theta_1^{-1} \theta_2 \theta^{-1} = \theta_1^{-1} \theta_2 \theta^{-1}$, which is a contradiction since $\theta_2 \theta^{-1} \theta_1^{-1} \theta_1 \theta_2^{-1} \in S_{(\alpha)}$, while $\theta_1^{-1} \theta_2 \theta^{-1} \theta_1^{-1} \theta_2 \theta^{-1} \in S_{(\alpha)}$ and $S_{(\alpha)} \cap S_{(\alpha)}' = \{e\}$. Conversely we can similarly show that $S_{(\alpha)} \cap \lambda S_{(\alpha)}', \lambda^{-1} \neq \{e\}$ if $\lambda \notin S_{(\alpha)} S_{(\alpha)}'$. Now suppose that $S_{(\alpha)} \cap \lambda S_{(\alpha)}', \lambda^{-1} \neq \{e\}$. Let $\lambda \theta_2^{-1} = \theta_1 \neq e$ and let $(12\ldots k)$ be a disjoint cycle of $\theta_1$. Then there is a disjoint cycle $(1'2'\ldots k')$ of $\theta_2$ such that $\lambda \theta_2 = \theta_1$ $(i=1,\ldots,k)$. By lemma 2.1.3, for say $\theta_2 = (1'2')$, $\lambda (1'2') \lambda^{-1} = (12) = \theta_1$. In this case $AS_{(\alpha)}', (\theta_2) = -1$ and $IS_{(\alpha)}' (\theta_1) = 1$. Thus we have $AS_{(\alpha)}', (\theta_2) \neq IS_{(\alpha)}' (\theta_1)$ which satisfies the conditions of lemma 1.1.10. The multiple $k$ of the primitive idempotent $\varphi$ defined in lemma 1.1.10 is easily shown to be equal to $f(\alpha)/n!$ (see [1] or [10]).

As we have seen above, each $(\alpha)$-tableau of an $(\alpha)$-diagram $[\alpha]$ gives rise to a double coset $S_{(\alpha)} \lambda S_{(\alpha)}', containing \lambda$ such that $S_{(\alpha)} \cap \lambda S_{(\alpha)}', \lambda^{-1} = \{e\}$ if the conditions of lemma 1.1.10 are to be satisfied.
Now as in lemma 2.2.4, above if we denote by
\( IS(\alpha) \) the identity representation on \( S(\alpha) \) (and that
on \( S(n) \) by \([n]\)), similary if we denote the alternating
representation on \( S(\alpha) \), by \( AS(\alpha) \) (and that on \( S(n) \) by
\([1^n]\)), then denoting the Induced o.r's of \( S_n \) by
\( IS(\alpha) \uparrow S_n \) and \( AS(\alpha) \uparrow S_n \) respectively, we have the
following lemma.

**Lemma 2.2.5**

The o.r's \( IS(\alpha) \uparrow S_n \) and \( AS(\alpha) \uparrow S_n \) have one
irreducible constituent in common in each case with
multiplicity 1.

**Proof**

We show that
\[
\left\{ \chi_{IS(\alpha) \uparrow S_n}, \chi_{AS(\alpha) \uparrow S_n} \right\} = 1.
\]

By lemma 1.2.6, the above inner product of characters
of \( S_n \) is equal to
\[
\sum_{\lambda \in S(\alpha) \cap \lambda S(\alpha)} \chi_{IS(\alpha) \downarrow S_{\lambda}}, \chi_{AS(\alpha) \downarrow S_{\lambda}} \]
where \( S_\lambda = S(\alpha) \cap \lambda S(\alpha), \lambda^{-1} \) and \( S(\alpha) \lambda S(\alpha), \) is a double
coset of \( S_n \) and \((AS(\alpha),)^\lambda \) is an o.r of \( \lambda S(\alpha), \lambda^{-1} \)
conjugate to \( AS(\alpha) \), defined by
\[(AS(\alpha),)^\lambda(\lambda \theta \lambda^{-1}) = AS(\alpha), (\theta), \text{ for all } \theta \in S(\alpha). \]
The sum above is now equal to
\[
\sum_{\lambda \in S(\alpha) \cap \lambda S(\alpha)} \chi_{IS_{\lambda}}, \chi_{AS_{\lambda}} \cdot
\]
Since \( S(\alpha) \cap \lambda S(\alpha), \lambda^{-1} = \{e\} \) if \( \lambda \in S(\alpha) \lambda S(\alpha) \) (see Lemma 2.2.4), we have

\[
\chi^I_{S(\alpha) \cap \lambda S(\alpha), \lambda^{-1}} = \chi^A_{S(\alpha) \cap \lambda S(\alpha), \lambda^{-1}}
\]

and the above sum equals 1.

We remark that when the conditions of lemma 2.2.5 hold then the conditions of lemma 1.2.7 hold; and hence the characters \( \chi^{IS(\alpha)}_{S_n} \) and \( \chi^{AS(\alpha)}_{S_n} \) give rise to an irreducible character of \( S_n \). We shall denote the common constituent in lemma 2.2.5 by

\[
[\alpha] = (IS(\alpha) \uparrow S_n) \cap (AS(\alpha) \uparrow S_n).
\]  \hspace{1cm} (2.4)

Now since \( S_{(1^n)} \) is a direct product of \( n \) copies of \( S_1 \) which is isomorphic to \( S_1 \), we have

\( IS_{(1^n)} = I(e) = AS_{(1^n)} \). Thus the induced o.r.'s of \( S_n, IS_{(1^n)} \uparrow S_n \) and \( AS_{(1^n)} \uparrow S_n \) are the regular representation \( rS_n \) of \( S_n \). We have

\[
(IS_{(1^n)} \uparrow S_n) \cap (AS_{(1^n)} \uparrow S_n) = rS_n \cap AS_n = AS_n = [1^n] \\
(IS_{(n)} \uparrow S_n) \cap (AS_{(1^n)} \uparrow S_n) = IS_n \cap rS_n = IS_n = [n]
\]  \hspace{1cm} (2.5)

Thus certain o.r.'s of \( S_n \) may be obtained as above. Now denoting the identity representation on \( S(\alpha)' \), by \( IS(\alpha)' \), and the alternating representation on \( S(\alpha) \) by \( AS(\alpha) \), we can similarly obtain an o.r. of \( S_n \) which is given by \([\alpha]' = (IS(\alpha)' \uparrow S_n) \cap (AS(\alpha) \uparrow S_n)\).

Since \([n] = [1^n] \odot [1^n] \) we see that \([\alpha] \) and \([\alpha]' \) differ only on the odd permutations by a sign.
We have

\[ [\alpha]' = [\alpha] \otimes [1^n] \]  

(2.6)

The next result, which is our main result in this chapter, shows that for all partitions (\(\alpha\)) of \(n\) the pairs of o.r's \([\alpha], [\alpha]' = [\alpha] \otimes [1^n]\) are the only inequivalent i.o.r's of \(S_n\) (see [10] page 70).

**Theorem 2.2.6**

The representations, \(\{[\alpha] = (IS_{(\alpha)} \uparrow S_n) \cap (AS_{(\alpha)} \uparrow S_n), [\alpha]' = [\alpha] \otimes [1^n] / (\alpha)'\) is a partition of \(n\) conjugate to \((\alpha)\}) form a complete system of pair-wise inequivalent i.o.r's of the symmetric group, \(S_n\).

**Proof**

We show that for distinct partitions (\(\alpha\)) and (\(\beta\)) of \(n\), the tableaux \(T^{[\alpha]}\) and \(T^{[\beta]}\) provide two distinct primitive idempotents, which thus generate non-isomorphic representation spaces of the i.o.r's \([\alpha]\) and \([\beta]\) respectively. Let \(\varphi_1 = k_1 P_1 N_1\) and \(\varphi_2 = k_2 P_2 N_2\) be the associated primitive idempotents of the Tableaux \(T^{[\alpha]}\) and \(T^{[\beta]}\) respectively. Consider \(P_1 N_1 \lambda P_2 N_2, \lambda \in S_n\). If for all \(\theta_2' = (ij) \in S_{(\beta)}\), there is no \(\theta_1 \in S_{(\alpha)}\) such that \(\lambda \theta_2' \lambda^{-1} = \theta_1\), then no pair of symbols \(i,j\) from any column of \(T^{[\beta]}\) can be taken by \(\lambda\) into symbols within a row of \(T^{[\alpha]}\) (otherwise for \((ij) \in S_{(\beta)}\)'', we would have \(\lambda(ij) \lambda^{-1} = (\lambda i \lambda j) \in S_{(\alpha)}\) which implies that the rows
of $T^{[\beta]}$ have the same lengths as those of $T^{[\alpha]}$, that is the diagrams $[\alpha]$ and $[\beta]$ are the same, which is a contradiction. This implies that there exists some $\theta'_2 = (ij) \in S_2(\beta)$, such that 

$\lambda \theta'_2 \lambda^{-1} = \theta_1 \in S_2(\alpha)$.

Now

$P_1 N_1 \lambda P_2 N_2 = \chi_{IS}(\alpha)(\theta_1) P_1 N_1 \lambda \theta'_2 \lambda^{-1} \lambda P_2 N_2$

$= P_1 N_1 \lambda P_2 N_2$, since $\chi_{IS}(\alpha)(\theta_1) = 1$

$= \left(\chi_{IS}(\alpha)(\theta_1)\right)^{-1} P_1 N_1 \lambda P_2 N_2$

Hence $\left[1-\chi_{IS}(\alpha)(\theta_1)\right] P_1 N_1 \lambda P_2 N_2 = 0$. Since $\theta_1$ is a transposition $\left(\chi_{IS}(\alpha)(\theta_1)\right)^{-1} \neq 0$, so that $P_1 N_1 \lambda P_2 N_2 = 0$ for all $\lambda \in S_n$. Hence $\varphi_1 \in S_n \varphi_2 = 0$. But $\varphi_1$ and $\varphi_2$ are primitive idempotents so that $\varphi_1 \in S_n \varphi_2 = 0$ only if $\varphi_1$ and $\varphi_2$ belong to non-isomorphic summands $A_{(\alpha)}$ and $A_{(\beta)}$ of $CS_n$ respectively. Thus the tableaux $T^{[\alpha]}$ and $T^{[\beta]}$ yield primitive idempotents $\varphi_1$ and $\varphi_2$ from non-isomorphic summands of $CS_n$ and are associated with distinct i.o.r's $[\alpha]$ and $[\beta]$ respectively. Hence the $(\alpha)$-tableaux of distinct $(\alpha)$-diagrams provide a complete system of i.o.r's of $S_n$.

For an example of the application of the above result we give an explicit construction of the i.o.r's of the group $S_4$.
**Example 2.1**

Since the number \( P(4) \) of partitions of \( n=4 \) is five it follows that the group \( S_4 \) has five inequivalent i.o.r's. For the partitions

\[
(\alpha) \in \{(4), (31), (2^2), (21^2), (1^4)\}
\]  

We give the \((\alpha)\)-diagrams \([\alpha]\) below:

\[
\begin{align*}
(\alpha)\text{-diagrams } [\alpha] \\
[4] &= : o o o o \\
[31] &= : o o o \\
[2^2] &= : o o \\
[21^2] &= : o o \\
[1^4] &= o o o o
\end{align*}
\]  

\[
(\alpha)'\text{-diagrams, } [\alpha]'
\begin{align*}
[4]' &= [1^4] =: o o o o \\
[31]' &= [21^2] =: o o o o \\
[2^2]' &= [2^2] =: o o o o \\
[21^2]' &= [31] =: o o o o \\
[1^4]' &= [4] =: o o o o
\end{align*}
\]  

Hence the five i.o.r's of \( S_4 \) are \([4], [31], [2^2], [1^4] \). The numbers \( f_{(\alpha)} \) of standard \((\alpha)\)-tableaux \( T_{1}^{[\alpha]} \) associated with each \( (\alpha) \) are given below:
That the above is a complete set of i.o.r's of $S_4$ follows from considering the sum of the squares of the degrees $f_{(\alpha)}$.

The young subgroups of $S_4$ are easily obtained from the $\langle \alpha \rangle$-tableaux to give:

<table>
<thead>
<tr>
<th>YOUNG SUBGROUP, $S_{(\alpha)}$</th>
<th>CONJUGATE YOUNG SUBGROUP $S_{(\alpha)'},$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_{(4)} \cong S_4$</td>
<td>$S_{(1)} \cong S_1$</td>
</tr>
<tr>
<td>$S_{(3,1)} \cong S_3$</td>
<td>$S_{(2,1)} \cong S_2$</td>
</tr>
<tr>
<td>$S_{(2,2)} \cong S_2 \times S_2$</td>
<td>$S_{(2)} = S_2 \times S_2$</td>
</tr>
<tr>
<td>$S_{(2)} \cong S_3$</td>
<td>$S_{(3)} \cong S_2$</td>
</tr>
<tr>
<td>$S_{(4)} \cong S_1$</td>
<td>$S_{(4)} \cong S_4$</td>
</tr>
</tbody>
</table>

(2.10)

By lemma 2.2.4 the primitive idempotents of the two sided ideals of $CS_4$ can be calculated using the young subgroups of $S_4$ given above.
The identity representations on $S(\alpha)$ and the alternating representations on $S(\alpha)'$, are given by

$$IS_{(2^2)} =: [2] \otimes [2] \quad AS_{(2^2)} =: [1^2] \otimes [1^2]$$

(2.11)

And the induced representations of $S_4$ are given by

$$IS(4) \uparrow S_4 =: [4] \uparrow S_4, \quad AS_{(1^4)} \uparrow S_4 =: [1] \uparrow S_4 = rS_4$$
$$IS_{(31)} \uparrow S_4 =: [3] \uparrow S_4, \quad AS_{(21^2)} \uparrow S_4 =: [1^2] \uparrow S_4$$
$$IS_{(2^2)} \uparrow S_4 =: ([2] \otimes [2]) \uparrow S_4, \quad AS_{(2^2)} \uparrow S_4 =: ([1^2] \otimes [1^2]) \uparrow S_4$$
$$IS_{(21^2)} \uparrow S_4 =: [2] \uparrow S_4, \quad AS_{(31)} \uparrow S_4 =: [1^3] \uparrow S_4$$
$$IS_{(1^4)} \uparrow S_4 =: [1] \uparrow S_4 = rS_4, \quad AS_{(4)} \uparrow S_4 =: [1^4] \uparrow S_4$$

(2.12)

By lemma 2.2.5 the above pairs of representations $IS(\alpha) \uparrow S_4$ and $AS(\alpha) \uparrow S_4$ have one irreducible constituent in common in each case with multiplicity 1. Using the notation above we have a complete set of i.o.r's of $S_4$ given below:

$$[1^4] = (IS_{(1^4)} \uparrow S_4) \cap (AS_{(4)} \uparrow S_4) = rS_4 \cap AS_4 = AS_4$$
$$[4] = (IS_{(4)} \uparrow S_4) \cap (AS_{(1^4)} \uparrow S_4) = IS_4 \cap rS_4 = IS_4$$
$$[31] = (IS_{(31)} \uparrow S_4) \cap (AS_{(21^2)} \uparrow S_4)$$
$$[21^2] = (IS_{(21^2)} \uparrow S_4) \cap (AS_{(31)} \uparrow S_4)$$
$$[2^2] = (IS_{(2^2)} \uparrow S_4) \cap (AS_{(2^2)} \uparrow S_4)$$

(2.13)
**Proof**

This is a consequence of definition 1.2.2 and theorem 2.2.6.

From our earlier remarks and theorem 2.2.6 the number of inequivalent i.o.r's of each $S_{(\alpha_1)}$ is $P(\alpha_1)$. It follows that the number of inequivalent i.o.r's of $S_{(\alpha)}$ is given by $P(\alpha_1)P(\alpha_2)...P(\alpha_k)$. 
CHAPTER III

THE GENERALISED SYMMETRIC GROUP, $B_n^m$, AND ITS CONJUGACY CLASSES

In this chapter we examine the basic results on the generalised symmetric group $B_n^m$ (elsewhere written as $C_{m^n}$ or $G(m,1,n)$) and give the conjugacy classes of $B_n^m$. We first describe the group $B_n^m$ and state some of the relevant concepts to be used in the sequel.

3.1 THE GENERALISED SYMMETRIC GROUP, $B_n^m$

The group $B_n^m$ has a presentation given by

$$B_n^m = <r_i, w_j, (i=1,\ldots,n-1, j=1,\ldots,n)/r_i^2=1;$$

$$(r_i r_{i+1})^3=1=(r_i r_j)^2, \quad |i-j| \geq 2; \quad w_j^m = 1,$$

$$w_j w_k = w_k w_j, \quad r_i w_j = w_{i+1} r_i, \quad r_i w_j = w_j r_i, \quad j \neq i, \quad i+1.$$  

(3.1)

The group $B_n^m$ is called the generalised symmetric group because we can identify $r_i$ ($i=1,\ldots,n-1$) with the transposition $(i,i+1)$ and $w_j$ ($j=1,\ldots,n$) with the map $j \rightarrow \xi^j$, where $\xi$ is some primitive $m^{th}$ root of unity. The group generated by the $r_i$'s is the symmetric group $S_n$. Denoting by $C_m^{(j)}$ the group generated by $w_j$, then the group $C_m^n$ generated by the $w_j$'s is the direct product $C_m^n = C_m^{(1)} \times \ldots \times C_m^{(n)}$ of $n$ cyclic groups each of
order $m$. Therefore $B^m_n$ is a group permuting the symbols in $\Omega$ as well as multiplying arbitrarily subsets of $\Omega$ by powers of $\xi$. The group $B^m_n$ may also be considered as the semi-direct product of $C^m_n$ and $S_n$ (see [8],[10] or [14]). The order of $B^m_n$ is given by

$$|B^m_n| = m^n n!$$  \hspace{1cm} (3.2)

On ignoring the multiples by powers of $\xi$, we may consider each member of $B^m_n$ as a permutation $\theta$ of $\Omega = \{1,2,\ldots,n\}$, which is simply a member of $S_n$. Therefore by definition 2.1.1 we can express $\theta$ uniquely as a product of disjoint cycles. Such an expression of $\theta$ is called the cycle notation of $\theta$. Thus if $\theta \in B^m_n$, then $\theta = \theta_1 \theta_2 \ldots \theta_t$, where $\theta_i$ is a cycle given by

$$\theta_i = \begin{bmatrix} a_{i1} & a_{i2} & \ldots & a_{it_i} \\ \xi_{i1} a_{i2} & \xi_{i2} a_{i3} & \ldots & \xi_{it_i} a_{i1} \end{bmatrix}$$

where $a_{ij} \in \Omega = \{1,\ldots,n\}$, $\xi_{ij} \in \{1,\ldots,m\}$ and $t_i$ is the length of $\theta_i$. The multiplication of elements in $B^m_n$ is defined in the usual way as in $S_n$.

The following result is crucial in our method of constructing representations of $B^m_n$. 
**Lemma 3.1.1**

Let \( \theta \in B_n^a \) be of the form
\[
\begin{bmatrix}
1 & 2 & \ldots & n \\
\xi^1 a_1 & \xi^2 a_2 & \ldots & \xi^n a_n
\end{bmatrix}
\]

Define \( \Phi : B_n^a \rightarrow S_n \) by
\[
\Phi(\theta) = \begin{bmatrix}
1 & 2 & \ldots & n \\
a_1 a_2 & \ldots & a_n
\end{bmatrix}.
\]

Then \( \Phi \) a surjective homomorphism.

**Proof**

Let
\[
\lambda = \begin{bmatrix}
1 & 2 & \ldots & n \\
\xi^1 b_1 & \xi^2 b_2 & \ldots & \xi^n b_n
\end{bmatrix} \in B_n^a.
\]

We show that \( \Phi(\theta \lambda) = \Phi(\theta) \Phi(\lambda) \)
\[
\theta \lambda = \begin{bmatrix}
1 & 2 & \ldots & n \\
\xi^1 a_1 & \xi^2 a_2 & \ldots & \xi^n a_n
\end{bmatrix} \begin{bmatrix}
1 & 2 & \ldots & n \\
\xi^1 b_1 & \xi^2 b_2 & \ldots & \xi^n b_n
\end{bmatrix}
\]
\[
= \begin{bmatrix}
1 & 2 & \ldots & n \\
\xi^1 c_1 & \xi^2 c_2 & \ldots & \xi^n c_n
\end{bmatrix}
\]

where \( c_i = \text{some } b_i \) if \( \lambda \) maps \( a_i \) onto \( b_i \),
\( k_i' = k_i + k_i' \), otherwise leave as disjoint cycles.

we have
\[ \phi(\theta \lambda) = \begin{pmatrix} 1 & 2 & \ldots & k \\ \xi^{1}c_1 & \xi^{2}c_2 & \ldots & \xi^{n}c_n \end{pmatrix} \begin{pmatrix} 1 & 2 & \ldots & n \\ c_1 & c_2 & \ldots & c_n \end{pmatrix} = \begin{pmatrix} 1 & 2 & \ldots & n \\ a_1 & a_2 & \ldots & a_n \end{pmatrix} \begin{pmatrix} 1 & 2 & \ldots & n \\ b_1 & b_2 & \ldots & b_n \end{pmatrix} = \phi(\theta)\phi(\lambda) \]

Furthermore for \( \theta \in S_n \) considered as an element of \( B_n^m \), \( \phi(\theta) = \theta \), proving \( \phi \) is surjective.

From the first isomorphism theorem and lemma 3.1.1, the group \( B_n^m/\ker \phi \) is isomorphic to \( S_n \). We determine the \( \ker \phi \) in the following result.

**Lemma 3.1.2**

Let \( \phi \) be as defined above. Then \( \ker \phi = C_m^n \).

**Proof**

If \( \theta \in \ker \phi \), then \( \phi(\theta) = e \in S_n \). Thus \( \theta \) must be of the form

\[
\begin{pmatrix}
1 \\
\xi^{1}1 \\
\xi^{2}2 \\
\vdots \\
\xi^{n}n
\end{pmatrix}
\]

which lies in \( C_m^n \). Thus

\[
\theta = (w_1)^{k_1} (w_2)^{k_2} \ldots (w_n)^{k_n}
\]

Now if \( \theta \in C_m^n \), then \( \theta \) is generated by the \( w_j \)'s so that \( \theta = (w_1)^{k_1}(w_2)^{k_2}...(w_n)^{k_n} \). Hence \( \phi(\theta) = e \in S_n \) so that \( \theta \in \ker \phi \). Thus \( C_m^n \subseteq \ker \phi \).
Therefore ker $\phi = C_n$.

We end this section with a somewhat obvious result on conjugation in $B_n^m$.

**Lemma 3.1.3**

Let $\theta, \lambda \in B_n^m$. Then $\theta$ and $\lambda \theta \lambda^{-1}$ have equal cyclic lengths.

**Proof**

Regarding $\theta$ and $\lambda$ as permutations with powers of $\xi$ ignored, we have by lemma 2.1.3

$$\lambda \theta \lambda^{-1} = \begin{pmatrix} i & \ stackrel{\lambda(i)}{\theta(i)} \end{pmatrix} \begin{pmatrix} i & \lambda(i) \end{pmatrix} = \begin{pmatrix} \lambda(i) & \lambda \theta(i) \end{pmatrix},$$

that is we obtain $\lambda \theta \lambda^{-1}$ from $\theta$ by applying $\lambda$ to the symbols of $\theta$. This retains the cyclic lengths of $\theta$ since each symbol has its own image under $\lambda$. Hence $\theta$ and $\lambda \theta \lambda^{-1}$ have equal cyclic lengths.

3.2 **The Conjugacy Classes of $B_n^m$**

To describe the conjugacy classes of the generalised symmetric group $B_n^m$, we shall need some preliminary results.
**Defn 3.2.1**

Let \( \theta \in B_n^m \). We fix its expression in terms of disjoint cycles as \( \theta = \theta_1 \theta_2 \ldots \theta_t \). Define a map \( f: B_n^m \to \mathbb{N} \) by \( f(\theta_i) = \sum_{j=1}^{k_i} k_{i,j} \) where the \( k_{i,j} \)'s have the same meaning as in lemma 3.1.1 and set \( f(\theta) = \sum_{i=1}^{t} f(\theta_i) \). Let \( a_{pq}(\theta) \) be the number of cycles \( \theta_i \) of \( \theta \) of length \( q \) such that \( f(\theta_i) \equiv p \pmod{m} \), \( 1 \leq p \leq m \), \( 1 \leq q \leq n \). Then the \( mxn \) matrix \( (a_{pq}(\theta)) \) is called the **Type of \( \theta \).** Here we shall denote the type of \( \theta \) by \( \text{Typ}(\theta) \). We make this definition precise as follows.

**Lemma 3.2.2**

Let \( \theta = \lambda \mu \) be a product of elements \( \lambda, \mu \) in \( B_n^m \). Then \( \text{Typ}(\theta) = \text{Typ}(\lambda) + \text{Typ}(\mu) \) if and only if \( \lambda \) and \( \mu \) are disjoint.

**Proof**

Suppose \( \lambda \) and \( \mu \) are disjoint. Then \( f(\theta) = f(\lambda) + f(\mu) \) by definition of \( f \). And \( \theta \) has the same number of disjoint cycles of the same length as \( \lambda \mu \). Thus we have \( \text{Typ}(\theta) = \text{Typ}(\lambda) + \text{Typ}(\mu) \).

Conversely, suppose \( \lambda \) and \( \mu \) have some symbols in common. Let \( \theta = \theta_1 \theta_2 \ldots \theta_t \) be a decomposition of \( \theta \) into disjoint cycles. Then multiplication of the cycles of \( \lambda \) with cycles of \( \mu \) will in general disturb
the cyclic lengths of the cycles in $\theta$ which will be different from the lengths of the cycles of $\lambda$ and $\mu$, and $f(\theta) \neq f(\lambda) + f(\mu)$. Hence $\text{Typ}(\theta) \neq \text{Typ}(\lambda) + \text{Typ}(\mu)$.

**Corollary 3.2.3**

(a) Keeping the notation above, $f(\theta) = f(\lambda) + f(\mu)$.

(b) We obtain the type of a product of disjoint permutations (any number of them) by adding their respective types.

(c) Two permutations of $B_n^m$ will have the same type if they have the same numbers of disjoint cycles of the same lengths; hence if they have the same lengths.

The following result gives a necessary and sufficient condition for the conjugation of elements in $B_n^m$.

**Lemma 3.2.4**

Let $\theta_1, \theta_2 \in B_n^m$. Then $\theta_1$ is conjugate to $\theta_2$ in $B_n^m$ if and only if $\text{Typ}(\theta_1) = \text{Typ}(\theta_2)$.

**Proof**

Let $\theta_1$ be conjugate to $\theta_2$ in $B_n^m$. Then some $\lambda$ exists in $B_n^m$ such that $\lambda \theta_1 \lambda^{-1} = \theta_2$. Let $\theta_1 = \theta_{11} \theta_{12} \ldots \theta_{1t}$ be the decomposition of $\theta_1$ as a product of disjoint cycles.
Then
\[ \lambda \theta_1 \lambda^{-1} = \lambda \theta_{11} \theta_{12} \ldots \theta_{1t} \lambda^{-1} = \lambda \theta_{11} \lambda^{-1} \lambda \theta_{12} \lambda^{-1} \ldots \lambda \theta_{1t} \lambda^{-1}. \]

Now the \( \lambda \theta_i \lambda^{-1} \)'s \((i=1, \ldots, t)\) are disjoint cycles of \( \lambda \theta_i \lambda^{-1} \) since the application of \( \lambda \) to each disjoint cycle \( \theta_{ii} \) of \( \theta_i \) moves every symbol in \( \theta_{ii} \) into a unique symbol. By lemma 3.1.3, the length of \( \theta_{ii} \) is equal to that of \( \lambda \theta_{ii} \lambda^{-1} \).

Let \( \text{Typ}(\theta_i) = (a_{pq}(\theta_i)) = \sum_{i=1}^{t} \text{Typ}(\theta_{ii}) \)

\[ = \sum_{i=1}^{t} (a_{pq}(\theta_{ii})). \]

Now as a consequence of lemma 3.2.2, it is enough to show that \( \text{Typ}(\theta_{ii}) = \text{Typ}(\lambda \theta_{ii} \lambda^{-1}) \).

\( \text{Typ}(\theta_{ii}) = (a_{pq}(\theta_{ii})) \) means that \( \theta_{ii} \) is of length \( p \) and \( f(\theta_{ii}) \equiv p \pmod{m} \) i.e. \( f(\theta_{ii}) = km + p \), for some \( k \in \mathbb{N}^* \). By corollary 3.2.3,

\[ f(\lambda \theta_{ii} \lambda^{-1}) = f(\lambda) + f(\theta_{ii}) + f(\lambda^{-1}) \]
\[ = f(\lambda) + f(\lambda^{-1}) + km + p \]
\[ = f(e) + km + p \]
\[ = lm + km + p, \text{ for some } l \in \mathbb{N}^* \]
\[ = k'm + p, \text{ for some } k' = l + k \in \mathbb{N}^*. \]

so \( f(\lambda \theta_{ii} \lambda^{-1}) \equiv p \pmod{m} \). It follows that \( \text{Typ}(\theta_{ii}) = \text{Typ}(\lambda \theta_{ii} \lambda^{-1}) \) and therefore \( \text{Typ}(\theta_i) = \text{Typ}(\theta_2) \).
Conversely, suppose \( \text{Typ}(\theta_1) = \text{Typ}(\theta_2) \).
Then \( \langle a_{pq}(\theta_1) \rangle = \langle b_{pq}(\theta_2) \rangle \), so that the number of \( \theta_{1i} \)'s involved in \( \theta_1 \) of length \( q \) such that \( f(\theta_{1i}) = p(\text{mod.}m) \)
is equal to that of the \( \theta_{2i} \)'s of length \( q \) involved in
\( \theta_2 \) such that \( f(\theta_{2i}) = p(\text{mod.}m) \). Hence equal numbers of
disjoint cycles of equal length's \( q \) are involved in
the decompositions of \( \theta_1 \) and \( \theta_2 \) into disjoint cycles.
Thus appealing to the homomorphism \( \phi \) given in lemma
3.1.1 we have \( \phi(\theta_1) \) is conjugate to \( \phi(\theta_2) \) in \( S_n \).

Now let \( f(\theta_{1i}) = km + p \) and \( f(\theta_{2i}) = k'm + p, k, k' \in \mathbb{N}^* \).
Without loss of generality, we may assume \( k' > k \), say by
setting \( k' = k + l, l \in \mathbb{N}^* \). Then \( f(\theta_{2i}) = f(\theta_{1i}) + lm \); and
from lemma 3.2.2 we have \( \theta_{2i} = \theta_{1i} \mu \), for some \( \mu \in B_n^m \)
where \( \theta_{1i} \) and \( \mu \) are disjoint and \( f(\mu) = 0(\text{mod.}m) \).
Hence \( \mu \) represents a sign change in \( B_n^m \), so that we
consider \( \mu \) to be of the form \( \mu = \left( \xi^m_{j \choose j} \right) \), where \( \xi \) is
some primitive \( m \)th root of unity. Rewriting \( \mu \) as a
product of cycles \( \rho_j \rho_j^{-1} \) as
\[
\mu = \rho_1 \rho_1^{-1} \rho_2 \rho_2^{-1} \cdots \rho_j \rho_j^{-1} \cdots = \left( \xi^m_{j \choose j} \right)
\]
where \( \rho_j \rho_j^{-1} = \begin{pmatrix} 1 & 2 & \cdots & k \\
\xi^t_{1} & \xi^t_{2} & \cdots & \xi^t_{k} \\
1 & 2 & \cdots & k \\
\xi^u_{k} & \xi^u_{2} & \cdots & \xi^u_{1} \end{pmatrix}
\begin{pmatrix} 1 \\
1 \\
1 \\
1 \\
k \\
k \\
k \end{pmatrix}
\begin{pmatrix} j \\
t^u \\
t^u \\
t^u \\
k(j+1) \end{pmatrix}
\]
i.e \( \rho_j \rho_j^{-1} = \begin{pmatrix} j \\
t^u \\
t^u \\
t^u \\
k(j+1) \end{pmatrix} \)
and \( t_i + u_i = l m \) for \( i=j \), in only one cycle \( \rho_j \rho_j^{-1} \) of \( \mu \), and \( t_i + u_i = m \) in all the other cycles of \( \mu \). Now write \( \mu \) in the form \((\rho_1 \rho_2 \ldots)(\rho_1^{-1} \rho_2^{-1} \ldots) = \rho \rho^{-1}\) where the \( \rho_j^{-1} \) 's are disjoint cycles of \( \mu \). We have \( \theta_{21} = \theta_{11} \mu = \rho \theta_{11} \rho^{-1}, \rho \in B_n^m \). Generalising this expression for \( \theta_{21} \) to \( \theta_2 \) we get \( \theta_2 = \rho \theta_1 \rho^{-1} \), that is \( \theta_1 \) is conjugate to \( \theta_2 \).

From above we conclude that \( B_n^m \) contains as many conjugacy classes as there are distinct types \( (a_{pq}(\theta)) \). We need a definition before proving the main theorem of this section.

**Defn 3.2.5**

An \( m \)-set of partitions of \( n \), \( (\theta(t_1), \ldots, \theta(t_m)) \) is a set of partitions \( \theta(t_p)(p=1, \ldots, m) \) such that \( \theta(t_p) = (a_{p1}, a_{p2}, \ldots, a_{pm}) \) is a partition of \( t_p \), where \( t_p = \sum_{q=1}^m q a_{pq} \) and \( \sum_{p=1}^m t_p = n \). Here we shall denote by \( (\alpha)_m \) an \( m \)-set of partitions of \( n \); and sometimes write \( \theta(t_p) \) simply as \( t_p \). A 2-set of partitions of \( n \), \( (\alpha)_2 \) will also be called a double partition of \( n \). We now prove the following theorem.
THEOREM 3.2.6

Let \((\alpha)_m = (\theta(t_1), \ldots, \theta(t_m))\) be an \(m\)-set of partitions of \(n\). Let \(p(t_p)\) be the number of partitions of \(t_p, t_p \in \mathbb{N}\) and \(p(0) = 1\), and let \(p_m(n)\) be the number of the \(m\)-sets of partitions of \(n\). Then the number of conjugacy classes of \(B^m_n\) is given by

\[
\sum_{j=1}^{p_m(n)} \left\{ p(t_1) p(t_2) \ldots p(t_m) \right\}_j ,
\]

where the summation is taken over all the \(m\)-sets of partitions of \(n\).

PROOF

To a conjugacy class \(CCl(\theta)\) of \(\theta\) of type \((a_{pq}(\theta))\) associate an \(m\)-set of partitions of \(n\) defined by \((\alpha)_m = (\theta(t_1), \ldots, \theta(t_m))\), where

\[
\theta(t_p) = (a_{p1}, a_{p2}, \ldots, a_{pn}) \quad \text{and} \quad t_p = \sum_{q=1}^{n} q a_{pq}. \quad \text{This}
\]

\(m\)-set of partitions of \(n\) is unique since by definition an \(m\)-set of partitions of \(n\) different from this \((\alpha)_m\) shall correspond to a different conjugacy class not containing \(\theta\).
Conversely, let \( (\alpha)_m = (\theta(t_1), \ldots, \theta(t_n)) \) be an \( m \)-set of partitions of \( n \), so that \( \theta(t_p) = (a_{p1}, \ldots, a_{pn}) \). Then the set \( \Omega = \{1, 2, \ldots, n\} \) is decomposed into a disjoint union of subsets of the form

\[
(\alpha)_m = \left\{ \theta(t_1), \ldots, \theta(t_p), \ldots, \theta(t_m) \right\}
\]

\[
\left\{ \theta(t_1), \ldots, (a_{p1}^{i}, a_{p2}^{i}, \ldots, a_{pn}^{i}) = \theta_i(t_p), \ldots, \theta(t_m) \right\}
\]

\[
\left\{ \theta(t_1), \ldots, (a_{p1}^{2}, a_{p2}^{2}, \ldots, a_{pn}^{2}) = \theta_2(t_p), \ldots, \theta(t_m) \right\}
\]

\[
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots
\]

\[
\left\{ \theta(t_1), \ldots, (a_{p1}^{l}, a_{p2}^{l}, \ldots, a_{pn}^{l}) = \theta_l(t_p), \ldots, \theta(t_m) \right\}
\]

(3.3)

where \( \sum_{p=1}^{m} t_p = n \), \( t_p = \sum_{q=1}^{n} a_{pq} \), and \( l = p(t_p) \).

This is done to each \( \theta(t_p) \) with the other \( \theta(t_p) \)'s fixed. We see that there are \( \sum_{p, q} a_{pq} \) subsets \( \theta(t_p) \) such that exactly \( \sum_{p=1}^{m} a_{pq} \) of them have \( q \) elements as seen in the diagram (3.3) above. To see this clearly, for each \( \theta(t_p) \) and \( q \) fixed, \( q \) appears \( p(t_p) \) times down (3.3) above. Now on each such subset with \( q \) elements, we may define a cycle \( \theta_s \) of length \( q \) by
\[ \theta_s = \begin{pmatrix} 1 & 2 & \ldots & q \\ a_1 & a_2 & \ldots & \xi^k a_q \end{pmatrix} \]

where \( S = \{1, 2, \ldots, q\} \) and we may take \( k = p \) for the \( p \)-th \( a_{pq} \) subsets of length \( q \).

Let \( \theta \) be a product of these cycles \( \theta_s \). Then

\[ \text{Typ}(\theta) = (a_{pq}(\theta)). \]

Thus a given \( m \)-set of partitions of \( n \) \( (a)_m \) defines a conjugacy class \( \text{CCL}(\theta) \) of \( \theta \) in \( B_n^a \)

of Type \( (a_{pq}(\theta)) \).

Furthermore, on varying the \( q \) this \( m \)-set of partitions of \( n \) corresponds to other conjugacy classes different from \( \text{CCL}(\theta) \). Now since there are

\[ p(t_p) \]

possibilities for row \( t_p \) to be a row of a type

the number of conjugacy classes of \( B_n^a \) associated with each ordinary partition \( (t_1, t_2, \ldots, t_m) \) of \( n \) is

\[ p(t_1)p(t_2) \ldots p(t_m). \]

Summing these numbers over all the \( m \)-sets of partitions of \( n \) gives the result.
CHAPTER IV

REPRESENTATION THEORY OF THE GENERALISED SYMMETRIC GROUP, $B_n^m$.

All representations discussed here are considered over the field $\mathbb{C}$ of complex numbers. Therefore the number of inequivalent i.o.r's of $B_n^m$ is equal to the number of conjugacy classes of $B_n^m$. We aim to give a construction of this number of inequivalent i.o.r's of $B_n^m$.

Since there is a 1-1 correspondence between the number of conjugacy classes of $B_n^m$ and the $m$-sets of partitions $(\alpha)_m$ of $n$, we obtain a complete set of inequivalent i.o.r's of $B_n^m$ by associating each of the $(\alpha)_m$'s with an i.o.r of $B_n^m$ accordingly as in theorem 3.2.6. To this end we first consider the o.r's of special subgroups of $B_n^m$ which play a similar role to the representation theory of $B_n^m$ as do the young subgroups to the representation theory of the symmetric group, $S_n$. 
4.1 Generalised Young Subgroups and Basic Representations

We have the following definition.

**Defn 4.1.1**

Let \((\alpha) = (t_1, t_2, \ldots, t_k)\), be a \(k\)-tuple such that \(t_i \in \{0, 1, \ldots, n\}\) and \(\sum_{i=1}^{k} t_i = n, \ k \leq m\). We call \((\alpha)\) a permissible \(k\)-tuple of \(n\). Define \(P_0 = 0\) and

\[
P_i = \sum_{j=1}^{i} t_j \quad (i=1, \ldots, k).
\]

If \(t_i \neq 0\), let \(B_{t_i}^m\) be the corresponding generalised symmetric group on the \(t_i\) symbols \(P_i = \{P_{i-1} + 1, \ldots, P_i\}\), where \(1 \leq i \leq k\) and \(B_{0_i}^m\) is the trivial generalised symmetric group. That is \(B_{t_i}^m\) is a subgroup of \(B_n^m\). Then the direct product \(B_{t_1}^m \times B_{t_2}^m \times \ldots \times B_{t_k}^m\) is called the generalised young subgroup of \(B_n^m\) determined by the \(k\)-tuple \((\alpha) = (t_1, \ldots, t_k)\). Here we shall denote the generalised young subgroup of \(B_n^m\) by \(B_{(t_1, \ldots, t_k)}^m\) or simply \(B_{(\alpha)}^m\).

We shall denote by \(|(\alpha)|\) the number of permissible \(m\)-tuples of \(n\).
**Proposition 4.1.2**

Let \( \pi \) be a map: \( B^m_{t_1} \rightarrow \mathbb{C}^* \) defined by

\[
\pi(\theta_1) = \xi^{1f(\theta_1)} \text{ for all } \theta_1 \in B^m_{t_1},
\]

where \( f \) is as defined in definition 3.2.1 and \( \xi \) is a primitive \( m \)th root of unity. Then

(i) \( \pi \) is well defined.

(ii) \( \pi \) gives rise to an irreducible linear representation \( P_{t_1} \) of \( B^m_{t_1} \).

**Proof**

(i) \( \pi \) is well defined: let \( \theta_1, \theta'_1 \in B^m_{t_1} \).

\[
\pi(\theta_1) = \xi^{1f(\theta_1)} \quad \pi(\theta'_1) = \xi^{1f(\theta'_1)}
\]

Then \( \pi(\theta_1) \neq \pi(\theta'_1) = \xi^{1f(\theta'_1)} \)

if \( f(\theta_1) \neq f(\theta'_1) \) which implies \( \theta_1 \neq \theta'_1 \), by definition 3.2.1.

(ii) For any \( \theta_1 \in B^m_{t_1} \), define a map

\[
P_{t_1} : B^m_{t_1} \rightarrow \mathbb{C}^* \text{ by } P_{t_1}(\theta_1) = \pi(\theta_1).
\]

Then for any \( \theta_1, \theta'_1 \in B^m_{t_1} \), we have

\[
P_{t_1}(\theta_1 \theta'_1) = \pi(\theta_1 \theta'_1) = \xi^{1f(\theta_1 \theta'_1)} = \xi^{1(f(\theta_1) + f(\theta'_1))}
\]

\[
= \xi \xi^{1f(\theta'_1)} \quad \pi(\theta_1) \pi(\theta'_1) = \pi(\theta_1 \theta'_1)
\]

so that \( P_{t_1} \) is a homomorphism from \( B^m_{t_1} \) to \( \mathbb{C}^* \). \( P_{t_1} \)

is irreducible since it is linear.
Now extending the map \( \pi \) linearly to the whole group \( B^m(\alpha) \), we have for any \( \theta = \theta_1 \theta_2 \ldots \theta_k \in B^m(\alpha) \),

\[
\pi(\theta) = \xi_{\sum_{i=1}^{k} i(\theta_i)}.
\]

we state the following result.

**Lemma 4.1.3**

We keep the above notation. \( P_{t_1} \# P_{t_2} \# \ldots \# P_{t_k} \) is an irreducible linear representation of \( B^m(\alpha) \).

**Proof** This follows from definition 1.2.2 since \( P_{t_1} \# \ldots \# P_{t_k} \) is an outer tensor product representation of \( B^m(\alpha) \).

In what follows the representation \( P_{t_1} \# \ldots \# P_{t_k} \) of \( B^m(\alpha) \) shall be denoted by \( P(t_1, \ldots, t_k) \) or simply \( P(\alpha) \). The number of the representations \( P(\alpha) \) of \( B^m(\alpha) \) is equal to the number of \( k \)-sets of partitions of \( n \). \( (\alpha)_k \) since there is a unique map \( \pi \) defined on each \( B^m_{t_i} \), and changing the \( t_1 \) leads to definition of \( P_{t_1} \) on another generalised symmetric subgroup which is already taken care of by \( (\alpha) \) being a permissible \( k \)-tuple. Thus the number of the \( P(\alpha) \) depends only on the number of permissible \( k \)-tuples \( (\alpha) \) and is equal to the number of
k-sets of partitions of n. Also for \( \lambda \in B_n^m \) and some \( x \in \ker \phi \) of lemma 3.1.1 we shall write \( P^{(\lambda)}_{(\alpha)}(x) \) for \( P_{(\alpha)}(\lambda x \lambda^{-1}) \).

**Theorem 4.1.4**

We keep the above notation. Then

(i) If \( (\alpha) \) and \( (\alpha') \) are any two distinct permissible k-tuples, \( P^{(\lambda)}_{(\alpha)}(x) \neq P^{(\lambda)}_{(\alpha')} (x) \), for some \( x \in \ker \phi \) and for all \( \lambda \in B_n^m \).

(ii) If \( (\alpha) = (\alpha') \), then \( P^{(\lambda)}_{(\alpha)}(x) \neq P_{(\alpha)}(x) \) for some \( x \in \ker \phi \) and for all \( \lambda \in B_n^m \setminus B_{(\alpha)}^m \).

**Proof**

(i) Let \( (\alpha) = (t_1', \ldots, t_k') \) and \( (\alpha') = (t_1', t_2', \ldots, t_k') \) be distinct k-tuples. Then let \( i \) be the least index such that \( t_i' \neq t_i' \). Then we may assume without loss of generality that \( t_i < t_i' \). By definition, 4.1.1 \( P_i \leq P_i' \) and from lemma 3.1.1 if \( \lambda \in B_n^m \) either \( \phi(\lambda)P_i = P_i \) or \( \phi(\lambda)P_i \neq P_i \). If \( \phi(\lambda)P_i = P_i \), let \( j \in P_i' \setminus P_i \) so that \( \phi(\lambda)(j) \in P_i \), some \( l \neq i \). If \( \phi(\lambda)P_i \neq P_i \), there exists some \( j \in P_i \subset P_i' \) such that \( \phi(\lambda)(j) \in P_i' \), \( 1 \leq l \leq k \), for some \( l \neq i \). In either case choose \( x = w_j \in \ker \phi \) for the \( j \) chosen. We have

\[
P_{(\alpha')}^{(\lambda)}(x) = P_{(\alpha')}^{(\lambda)} \left( \xi_j^j \right) = P_{t_j'}^{(\lambda)} \left( \xi_j^j \right) = \xi_j^{\phi(\lambda)(w_j)}
\]

\[
= \xi_j^1 \text{ (j = i for } P_i' > P_i \).
\]
And \( P_{(\alpha)}^\lambda (x) = P_{(\alpha)} \left( \lambda \left( \xi^j \right) \lambda^{-1} \right) = P_{(\alpha)} \left( \Phi(\lambda) \left( \xi^j \right) \Phi(\lambda)^{-1} \right) \).

By lemma 3.1.3 it follows that

\[ P_{(\alpha)}^\lambda (x) = P_{(\alpha)} \left( \frac{\Phi(\lambda)(j)}{\xi \Phi(\lambda)(j)} \right), \]

that is \( P_{(\alpha)}^\lambda (x) = P_{(\alpha)} \left( \xi^1 \right) = \xi^{1_{\Phi(w)}} = \xi^1. \)

Now \( l \neq i \), so \( P_{(\alpha)}^\lambda (x) \neq P_{(\alpha')} (x) \), proving (i).

(ii) If \( (\alpha') = (\alpha) \) and \( \lambda \in B_n^m \setminus B_\alpha^m \) there exists at least one \( i \), \( 1 \leq i \leq k \) and a symbol \( j \in P_1 \) such that \( \Phi(\lambda)(i) \in P_1 \), \( l \neq i \). Let \( x = \omega_j \in \ker \Phi \) as above.

We have

\[ P_{(\alpha)}^\lambda (x) = P_{(\alpha)}^\lambda (\xi^j) = \xi^1 \] as above. But \( P_{(\alpha)}(x) = \xi^1 \)

\( (j = \text{some } i \in P_1) \) and \( \xi^1 \neq \xi^1 \) since \( l \neq i \), proving (ii).

**Defn 4.1.5**

Keeping the notation above, we shall call the representation \( P_{(\alpha)} \) the **basic ordinary representation** of \( B_\alpha^m \).

4.2 **Representations of the Generalised Young Subgroups**

We obtain the i.o.r's of the generalised young subgroup \( B_\alpha^m \) by appealing to definition 1.2.8 and the homomorphism \( \Phi \) of lemma 3.1.1 restricted to the young subgroup \( B_\alpha^m \). First we have the following result.
**Theorem 4.2.1**

We keep the above notation. For \( \theta \in B^m_{(\alpha)} \) define \( P^*:B^m_{(\alpha)} \rightarrow GL_n(\mathbb{C}) \) by \( P^*(\theta) = P((\ker \Phi) \lambda) = P(\Phi(\lambda)) \in GL_n(\mathbb{C}) \), where \( \lambda \) represents \( \theta \) in the transversal of \( \ker \Phi \) in \( B^m_{(\alpha)} \). Then \( P^* \) is an i.o.r. of \( B^m_{(\alpha)} \).

**Proof**

This follows from definition 1.2.8, lemma 3.1.1 and theorem 2.3.1; and \( P^* \) is a representation of \( B^m_{(\alpha)} \) lifted from the i.o.r.'s \( P \) of the young subgroup \( S_{(\alpha)} \) of \( S_n \).

In all that follows we shall denote by \( P \) the representation \( P^* = ([\alpha_1] \otimes \ldots \otimes [\alpha_m])^* \) of \( B^m_{(\alpha)} \).

We now use the representations \( P_{(\alpha)} \) and \( P \) of \( B^m_{(\alpha)} \) to construct further representations of \( B^m_{(\alpha)} \). We have the following result.

**Lemma 4.2.2**

We keep the above notation. The tensor product \( P \otimes P_{(\alpha)} = ([\alpha_1] \otimes \ldots \otimes [\alpha_m]) \otimes P_{(t_1, \ldots, t_m)} \) is an o.r. of \( B^m_{(\alpha)} \).
PROOF

This follows directly from definition 1.2.1.

4.3 REPRESENATIONS OF THE GENERALISED SYMMETRIC GROUP

Our aim in this section is to obtain a full set of inequivalent i.o.r's of $B_n^m$. We do this by inducing the o.r's $P\otimes P(\alpha)$ of the generalised young subgroups $B_n^m(\alpha)$ to give a representation

$$(P\otimes P(\alpha))^\dagger B_n^m = (([\alpha_1] \otimes \ldots \otimes [\alpha_m])^* \otimes P(t_1, \ldots, t_m))^\dagger B_n^m \quad (4.3)$$

of $B_n^m$. It is remarkable that the full set of inequivalent i.o.r's of $B_n^m$ is obtained in this way.

We now give our main result.

THEOREM 4.3.1

We keep the above notation. A full set of inequivalent i.o.r's of $B_n^m$ is given by

$$\left\{ (P\otimes P(\alpha))^\dagger B_n^m = (([\alpha_1] \otimes \ldots \otimes [\alpha_m])^* \otimes P(t_1, \ldots, t_m))^\dagger B_n^m \right\}$$

PROOF

We show that the set above is complete by showing that its cardinality equals the number of the conjugacy classes of $B_n^m$. Consider the tensor product $P\otimes P(\alpha)$. Let $(\alpha) = (t_1, t_2, \ldots t_m)$ be an m-tuple, then by the remarks following lemma 4.1.3 and Keeping $P$ fixed, it is seen that the number of the resulting
o.r’s $P \circ P_{(\alpha)}$ of $B_{(\alpha)}^m$ is equal to $P(t_1)P(t_2)\ldots P(t_m)$. Thus this construction gives rise to a total

$$\sum_{j=1}^{P_m(n)} \left\{ P(t_1)P(t_2)\ldots P(t_m) \right\}$$

of representations of $B_n^m$, where $P_m(n)$ is the number of $m$-sets of partitions of $n$. This equals the number of conjugacy classes of $B_n^m$.

Now we appeal to character theory to show that the representations above are inequivalent and irreducible. Let $(\alpha) = (t_1, \ldots, t_m)$ and $(\alpha') = (t'_1, \ldots, t'_m)$ be any two distinct permissible $m$-tuple.

Let $P_{(\alpha)}$ and $P_{(\alpha')}$ be the basic i.o.r’s of $B_{(\alpha)}^m$ and $B_{(\alpha')}^m$. Let $P$ and $P'$ be i.o.r of $B_{(\alpha)}^m$ and $B_{(\alpha')}^m$ lifted from i.o.r’s $P$ and $P'$ of $S_{(\alpha)}$ and $S_{(\alpha')}$ respectively.

Let $\chi_1, \chi'_1, \chi_2, \chi'_2, \chi_3$ and $\chi'_3$ denote the characters of $P_{(\alpha)}$, $P_{(\alpha')}$, $P$, $P'$, $P$ and $P'$ respectively. We consider the following inner product of characters of $B_n^m$.

$$\left( \chi_2 \chi_1 \uparrow B_n^m, \chi'_2 \chi'_1 \uparrow B_n^m \right) \quad \text{(I)}$$

Now $(I) = 0$ if $(\alpha) \neq (\alpha')$ by the orthogonality relations, that is the representations above are inequivalent. Using theorem 1.2.5 we obtain

$$(I) = \left( \chi_2 \chi_1, ((\chi'_2 \chi'_1) \uparrow B_n^m) \downarrow B_{(\alpha)}^m \right) B_{(\alpha)}^m$$

which on using lemma 1.2.6 gives
\[(I)= \sum_{x \in B_{(\alpha)}^m \times B_{(\alpha')}^m} \left( (x_2 \chi_1), ((x_2' \chi_1')^x \downarrow Bx \uparrow B_{(\alpha)}^m) \right) B_{(\alpha)}^m. \]

where \(Bx = B_{(\alpha)}^m \cap xB_{(\alpha')}^m x^{-1}\)

We use theorem 1.2.5 again and obtain

\[ (I)= \sum_{x \in B_{(\alpha)}^m \times B_{(\alpha')}^m} \left( (x_2 \chi_1) \downarrow Bx, (x_2' \chi_1')^x \downarrow Bx \right) Bx \]

and \(B_{(\alpha)}^m \times B_{(\alpha')}^m\) is a double coset of \(B_n^m\). Now (I) equals zero if the representations above are inequivalent as shown. Suppose (I) is not zero, that is for some \(x \in B_{(\alpha)}^m \times B_{(\alpha')}^m\)

\[ \left( (x_2 \chi_1) \downarrow B_{(\alpha)}^m \cap xB_{(\alpha')}^m x^{-1}, (x_2' \chi_1')^x \downarrow B_{(\alpha)}^m \cap xB_{(\alpha')}^m x^{-1} \right) \neq 0, \]

we have \(\left( (x_3 \chi_1) \downarrow \ker \phi, (x_3' \chi_1')^x \downarrow \ker \phi \right) \neq 0\)

since \(\ker \phi \subseteq Bx\) and where \(x_2\) and \(x_2'\) become \(x_3\) and \(x_3'\) on \(\ker \phi\) respectively. By theorem 1.1.21, this implies that the products \(\left( x_3 \downarrow \ker \phi \right) \left( x_1 \downarrow \ker \phi \right)\) and \(\left( x_3' \downarrow \ker \phi \right) \left( x_1' \downarrow \ker \phi \right)\) have at least one constituent in common. But \(x_1 \downarrow \ker \phi, x_3 \downarrow \ker \phi, x_1' \downarrow \ker \phi, x_3' \downarrow \ker \phi\) are all irreducible, so that we must have \(x_1 \downarrow \ker \phi = x_1' \downarrow \ker \phi\) and \(x_3 \downarrow \ker \phi = x_3' \downarrow \ker \phi\).

This implies that \(P_{(\alpha)} = P_{(\alpha')}^x\) on \(\ker \phi\). Now theorem 4.1.4 implies that \((\alpha) = (\alpha')\) and \(x \in B_{(\alpha)}^m\).

Thus it suffices to work in \(B_{(\alpha)}^m\), we therefore consider the inner product

\[ \left( x_2 \chi_1, x_2' \chi_1 \right)_{B_{(\alpha)}^m} = \frac{1}{|B_{(\alpha)}^m|} \sum_{\theta \in B_{(\alpha)}^m} x_2 \chi_1(\theta) x_2' \chi_1(\theta)^* \quad (II) \]
where $\chi_1' = \chi_1$ since $(\alpha) = (\alpha')$, $x \in B^n_{(\alpha)}$ and 
\[
\chi_2'\chi_1 = (\chi_2'\chi_1)^x. \text{ We now have}
\]
\[
(II) = \frac{1}{|B^n_{(\alpha)}|} \sum_{\theta \in B^n_{(\alpha)}} \chi_2(\theta) \chi_2'(\theta) \chi_1(\theta)\chi_1'(\theta)
\]
\[
= \frac{1}{|B^n_{(\alpha)}|} \sum_{\theta \in B^n_{(\alpha)}} \chi_2(\theta)\chi_2'(\theta), \chi_1(\theta)\chi_1'(\theta).
\]

And on applying theorem 1.2.5 we have
\[
(II) = \frac{1}{|S_{(\alpha)}|} \sum_{\theta \in S_{(\alpha)}} \chi_3(\theta)\chi_3'(\theta),
\]
where $\chi_2 = \chi_3$ and $\chi_2' = \chi_3'$ when restricted to $S_{(\alpha)}$. 
Thus $(II) = (\chi_3, \chi_3)$. Now if $(II) \neq 0$, it must equal 1, 
that is $\chi_3 = \chi_3'$ since $\chi_3$ and $\chi_3'$ are irreducible. Thus 
the representations above are irreducible. This 
completes the proof of the main theorem.

**Remark 4.3.2**

The i.o.r's of the generalised symmetric group $B^n_n$ 
have been investigated by Osima [13], Puttaswamaiah 
[14] using a method similar to that used in obtaining 
the i.o.r's of $S_n$ as given in theorem 2.2.6. That is 
by inducing linear representations of the generalised 
young subgroups. Kerber [10], Read [16] and Hughes 
[8] have obtained i.o.r's of $B^n_n$ by using Clifford's 
theory. It is seen that the i.o.r's of $B^n_n$ obtained in 
all these cases correspond to our results in theorem 
4.3.1.
CHAPTER V

APPLICATIONS TO THE REPRESENTATION THEORY OF THE HYPEROCTAHEDRAL GROUP, $B_n^2$

In this chapter, we apply the results obtained so far to the generalised symmetric group, $B_n^2$ which is isomorphic to the Weyl group of type $B_n$. The group $B_n^2$ is also called the hyperoctahedral group, and was first studied by A. Young in 1930. We shall use the notation above without further reference.

5.1 The Hyperoctahedral Group, $B_n^2$

The group $B_n^2$ is a semi-direct product $C_n^2 : S_n$ of the normal subgroup $C_n^2$ of order $2^n$ generated by $w_1,w_2,...,w_n$, with the symmetric group $S_n$ generated by $r_i$ ($i=1,...,n-1$). The action of $r_i$ on $w_j$ is given by $r_i w_j = w_j r_i$ if $j \neq i,i+1$ and $r_i w_i = w_{i+1} r_i$ (see (3.1)). Thus each element of $B_n^2$ has a canonical form $w r$ as a product, where $w \in C_n^2$ and $r \in S_n$. Thus $B_n^2$ permutes the symbols of $\Omega = (1,...,n)$ as well as changing the sign of subsets of $\Omega$. 
The group $B_n^2$ has a faithful permutation representation of degree $2n$ obtained by mapping $w_i$ to the transposition $(i,i+n)$ and $r_i$ to the permutation $(i,i+1)(i+n,i+1+n)$. Now taking $s_i = r_i$ $(i=1,\ldots,n-1)$ and $S_n = w_n$, we have (see [14]),

$$s_i^2 = 1 \quad (i=1,\ldots,n)$$
$$\left(s_is_{i+1}\right)^3 = 1 (i=1,\ldots,n-2)$$
$$\left(s_is_j\right)^2 = 1 \quad (i=1,\ldots,n), \quad j \neq i, i+1$$
$$\left(s_n^{-1}s_n\right)^4 = 1.$$  \hfill (5.1)

Hence $B_n^2$ has order $2^n n!$. Since each member $\theta$ of $B_n^2$ is a permutation of $\Omega = \{1,\ldots,n\}$, we can uniquely express $\theta$ as a product of disjoint cycles (see (3.3)).

**Defn 5.1.1**

A $k$-cycle $\theta_i = \left[1 \ 2 \ \ldots \ k \right]_{i\implies i+1}$ of $\theta$ is said to be positive if $\theta_i^k = 1$ and negative otherwise. The ordered lengths $(x_i \geq x_{i+1})$ of the cyclic factors in the cycle-notation of $\theta$ together with their signs give a set of positive and negative integers called the signed-cycle type of $\theta$ e.g. an element $\theta \in B_n^2$ which has $a_1$ positive 1-cycles, $a_2$ positive 2-cycles etc, and $b_1$ negative 1-cycles, $b_2$ negative 2-cycles etc, has singed-cycle type given by

$$\left[ a_1^+ \ a_2^+ \ \ldots ; \ b_1^- \ b_2^- \ \ldots \right]$$  \hfill (5.2)
It is clear from (5.2) and the length of \( \theta \) that the signed-cycle type of \( \theta \) satisfies \( |t_1| + |t_2| = n \), where \( t_1 = \sum a_i \) and \( t_2 = \sum b_i \). Thus \( (t_1, t_2) \) is a permissible 2-tuple. We have a lemma.

**Lemma 5.1.2**

The elements of \( B^2_n \) of the same signed cycle-type are conjugate to each other. Furthermore there is a 1-1 correspondence between the conjugacy classes of \( B^2_n \) and double partitions of \( n \).

**Proof**

The signed-cycle type of \( \theta \) is simply a special form of describing the Type, \( (a_{pq}(\theta)) \), of \( \theta \) (see 3.2.1). The result then follows from lemma 3.2.4. Further since the signed-cycle types define double partitions of \( n \) as seen above, the other result follows (see also [3]).

We prove the following result which gives the number of conjugacy classes of \( B^2_n \).

**Lemma 5.1.3**

Let \( n \in \mathbb{N}^* \), then \( |(\alpha)| = n+1 \). Furthermore the number of conjugacy classes of \( B^2_n \) is given by

\[
\sum_{j=1}^{n+1} P(t_j)P(n-t_j)
\]
where \( t_j \leq n, \ t_j = j-1, \) and \( j \) ranges over all permissible 2-tuples of \( n. \)

**Proof**

Let \( (\alpha) = (t_1, t_2) \) be any permissible 2-tuple of \( n, \) that is \( t_1 + t_2 = n. \) Thus \( t_2 = n - t_1, \) so that \( (t_1, t_2) = (t_1, n-t_1). \) In general we can write \( (t_j, t_k) = (t_j, n-t_j) \) for all \( t_j, t_k \leq n, \ t_j + t_k = n. \)

Thus \( \{(0,n), (1,n-1), \ldots, (n-1,1), (n,0)\} \) is the complete set of permissible 2-tuples of \( n, \) giving a total of \( n+1 \) of them. Hence \( |(\alpha)| = n+1. \)

By theorem 3.2.6, the number of conjugacy classes of \( B_n^2 \) will be given by

\[
\sum_{j=1}^{p_2(n)} (p(t_1)p(t_2))j
\]

which is now equal to the following

\[
\sum_{j=1}^{n+1} p(t_j)p(n-t_j)
\]

where clearly from the set of permissible 2-tuples of \( n, \) \( t_j = j-1, \ j = 1, 2, \ldots, n+1. \)
5.2 REPRESENTATIONS OF THE GROUP $B_n^2$

The irreducible ordinary representations of $B_n^2$ have been described by A. Young (see [21]). Here we obtain these representations by applying the methods developed in chapter IV. First we will establish some notation.

Let $\theta = \theta_1 \theta_2$ be a decomposition of

\[ \theta \in B_{t_j}^2 \times B_{n-t_j}^2 \]

into disjoint cycles $\theta_1$ and $\theta_2$ where $\theta_1 \in B_{t_j}^2$, $\theta_2 \in B_{n-t_j}^2$, are given by $\theta_1 = \theta_{11} \theta_{12} \ldots \theta_{1t}$ and $\theta_2 = \theta_{21} \theta_{22} \ldots \theta_{2s}$. Let $\sigma(\theta)$ denote the number of symbols of $\Omega$ involved in $\theta_1$ which are mapped onto negative symbols by $\theta$. Define a map

\[ P(t_j, n-t_j) \text{ from } B_{(t_j, n-t_j)}^2 \text{ onto } \mathbb{Z}_2 \text{ by} \]

\[ P(t_j, n-t_j) (\theta) = \begin{cases} 1 & \text{if } \sigma(\theta) \text{ is even} \\ -1 & \text{if } \sigma(\theta) \text{ is odd.} \end{cases} \] (5.3)

We now have the following result.

**LEMMA 5.2.1**

The map $P(t_j, n-t_j)$ is an i.o.r of

$B_{(t_j, n-t_j)}^2$, $t_j = j-1$, $j=1, \ldots, n+1$.

**PROOF**

By lemma 5.1.3 $(t_j, n-t_j)$ is a permissible 2-tuple of $n$, so that $B_{(t_j, n-t_j)}^2$ is a generalised young subgroup of $B_n^2$. Now taking $\xi = -1$ and $\sigma(\theta) = f(\theta_1)$ where $f$ is as defined in definition 3.2.1, then lemma 4.1.3 gives the result.
If now \( \phi \) is the homomorphism in lemma 3.1.1, then \( \ker \phi = \mathbb{C}_2^n \), so that

\[
(B^2_{(t_j, n-t_j)}/\ker \phi) \cong S_{(t_j, n-t_j)}.
\]

Thus the i.o.r.'s of \( S_{(t_j, n-t_j)} \) may be lifted to give i.o.r.'s of \( B^2_{(t_j, n-t_j)} \). In the table below for each generalised young subgroup \( B^2_{(t_j, n-t_j)} \), we give the corresponding young subgroup \( S_{(t_j, n-t_j)} \) from which the representation \( P \) of \( B^2_{(t_j, n-t_j)} \) may be lifted. We have

\[
P = \left( \left[ \alpha_{t_j} \right] \otimes \left[ \alpha_{n-t_j} \right] \right)^* \tag{5.4}
\]

where \( \left[ \alpha_{t_j} \right] \) denotes the representation of \( S_{t_j} \) corresponding to the partition \( (\alpha_{t_j}) \) of \( t_j \), \( \left[ \alpha_{n-t_j} \right] \) is that of \( S_{n-t_j} \) corresponding to the partition \( (\alpha_{n-t_j}) \) of \( n-t_j \); and * denotes the lifting of representations.
GENERALISED YOUNG SUBGROUP

\[ B^{2}_{(t_j, n-t_j)} \]
\[ B^{2}_{(0, n)} \cong B^{2}_n \]
\[ B^{2}_{(1, n-1)} \]
\[ \vdots \]
\[ B^{2}_{(n-1, 1)} \]
\[ B^{2}_{(n, o)} \cong B^{2}_n \]

CORRESPONDING YOUNG SUBGROUP, \( S^{(t_j, n-t_j)} \)

\[ S^{(0, n)} \cong S^n \]
\[ S^{(1, n-1)} \]
\[ \vdots \]
\[ S^{(n-1, 1)} \]
\[ S^{(n, o)} \cong S^n \]

(5.5)

And we see that corresponding to each permissible 2-tuple \((t_j, n-t_j)\) of \(n\), there are \(P(t_j)P(n-t_j)\) o.r.'s \(P^\ast\) of \(B^{2}_{(t_j, n-t_j)}\) obtained in this way. Thus the total number of such o.r.'s \(P\) equals \(\sum_{j=1}^{n+1} P(t_j)P(n-t_j)\) where \(j\) ranges over all permissible 2-tuples of \(n\). Hence the number of the representations \(P\) of \(B^{2}_{(t_j, n-t_j)}\) is equal to the number of i.o.r.'s of \(S^{(t_j, n-t_j)}\) counted over all 2-tuples \((t_j, n-t_j)\), of \(n\).

Now if \(P_{(t_j, n-t_j)}\) is the o.r. of \(B^{2}_{(t_j, n-t_j)}\) obtained in lemma 5.2.1, for each o.r. \(P\) of \(B^{2}_{(t_j, n-t_j)}\) given by (5.4), we form the tensor products

\[ P \otimes P_{(t_j, n-t_j)} = \left( \left[ a_{t_j} \right] \otimes \left[ a_{n-t_j} \right] \right)^\ast \]

(5.6)

If \(P \otimes P_{(t_j, n-t_j)} \uparrow B^2_n\) denotes an induced representation of \(B^2_n\), we have our main result.
**Theorem 5.2.2**

A full set of inequivalent i.o.r's of $B_n^2$ is given by
\[ \left\{ \left( P \circ P_{(t_j, n-t_j)} \right) \uparrow B_n^2 \right\} \]
where $(\alpha) = (t_j, n-t_j)$ ranges over all permissible 2-tuples of $n$.

**Proof**

From the above construction the total number of these o.r's of $B_n^2$ is
\[ \sum_{j=1}^{n+1} P(t_j)P(n-t_j), \]
where $j$ ranges over all permissible 2-tuples of $n$.

This gives a full set of i.o.r's of $B_n^2$ by lemma 5.1.3 and theorem 4.3.1.

**Remark 5.2.3**

The i.o.r's of the Weyl group of type $B_n$ which is isomorphic to the hyperoctahedral group $B_n^2$ have been investigated by J.J. Mayer [12], Hughes [8] and Gessinger and Kinch [22]. It is seen that the results obtained there correspond to those given in theorem 5.2.2.

5.3 **Ordinary Representations of $B_3^2$ and $B_4^2$**

In this section we give an explicit construction of the i.o.r's of the hyperoctahedral groups $B_3^2$ and $B_4^2$ as a direct application of theorem 5.2.2.
5.3.1 **Representations of $B_3^2$**

The group $B_3^2$ has order 48. In the table below we give the conjugacy class representatives of $B_3^2$ together with their types and orders.

<table>
<thead>
<tr>
<th>Type</th>
<th>Class Representative(s)</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ccl_1 = \begin{pmatrix} 0 &amp; 0 &amp; 0 \ 1 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$e$</td>
<td>1</td>
</tr>
<tr>
<td>$ccl_2 = \begin{pmatrix} 0 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 12 \ 21 \end{pmatrix}$ or $\begin{pmatrix} 12 \ -2-1 \end{pmatrix}$</td>
<td>6</td>
</tr>
<tr>
<td>$ccl_3 = \begin{pmatrix} 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 123 \ 231 \end{pmatrix}$ or $\begin{pmatrix} 1 &amp; 2 &amp; 3 \ -2 &amp; 3 &amp; 1 \end{pmatrix}$</td>
<td>8</td>
</tr>
<tr>
<td>$ccl_4 = \begin{pmatrix} 1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 1 \ -1 \end{pmatrix}$</td>
<td>3</td>
</tr>
<tr>
<td>$ccl_5 = \begin{pmatrix} 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 1 &amp; 2 \ -2 &amp; 1 \end{pmatrix}$ or $\begin{pmatrix} 1 &amp; 2 \ 2 &amp; -1 \end{pmatrix}$</td>
<td>6</td>
</tr>
<tr>
<td>$ccl_6 = \begin{pmatrix} 0 &amp; 0 &amp; 1 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 1 &amp; 2 &amp; 3 \ -2 &amp; 3 &amp; 1 \end{pmatrix}$ or $\begin{pmatrix} 1 &amp; 2 &amp; 3 \ -2 &amp; -3 &amp; 1 \end{pmatrix}$</td>
<td>8</td>
</tr>
<tr>
<td>$ccl_7 = \begin{pmatrix} 2 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 1 \ -1 &amp; -2 \end{pmatrix}$</td>
<td>3</td>
</tr>
<tr>
<td>$ccl_8 = \begin{pmatrix} 3 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 1 &amp; 2 &amp; 3 \ -1 &amp; -2 &amp; -3 \end{pmatrix}$</td>
<td>1</td>
</tr>
<tr>
<td>$ccl_9 = \begin{pmatrix} 1 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 1 &amp; 2 \ -2 &amp; 1 \end{pmatrix}$ or $\begin{pmatrix} 1 &amp; 2 \ 2 &amp; -1 \end{pmatrix}$</td>
<td>6</td>
</tr>
<tr>
<td>$ccl_{10} = \begin{pmatrix} 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 1 &amp; 2 \ 2 &amp; 1 \end{pmatrix}$ or $\begin{pmatrix} 1 &amp; 2 \ -2 &amp; -1 \end{pmatrix}$</td>
<td>6</td>
</tr>
</tbody>
</table>

The permissible 2-tuples of $n=3$ are $(t_j, 3-t_j) \in \{(0,3), (1,2), (2,1), (3,0)\}$ so that $|\alpha|=4$. By lemma 5.1.3 the number of conjugacy classes of $B_3^2$ is $\sum_{j=1}^{4} P(t_j)P(3-t_j) = 10$. Hence $B_3^2$ has 10 inequivalent i.o.r's, we construct these step by step.
Representations of the Young Subgroups $S_{(t_j, 3-t_j)}$

The partition $(t_j, 3-t_j) = (2,1)$

The young subgroup $S_{(2,1)}$ has two i.o.r's given by $[2] \otimes [1]$ and $[1^2] \otimes [1]$.

The partition $(t_j, 3-t_j) = (3)$ or $(3,0)$


Thus the full set of relevant i.o.r's $P$ of the young subgroups of $S_3$ is

$$\{[2] \otimes [1], [1^2] \otimes [1], [3], [21], [1^3]\}$$

Representations of the Generalised Young Subgroups $B^2_{(t_j, 3-t_j)}$

Keeping the above notation, the i.or's $P$ of the young subgroups of $S_3$ are lifted to give the i.o.r's $P=P^*$ of the generalised young subgroups. The full sets of the i.o.r's $P$ of $B^2_{(t_j, 3-t_j)}$ are given below for each subgroup:
For \( B^2_{(2,1)} \)
\[ \{ ([2] \otimes [1])^*, ([1^2] \otimes [1])^* \} \]

For \( B^2_{(1,2)} \)
\[ \{ ([1] \otimes [2])^*, ([1] \otimes [1^2])^* \} \]

For \( B^2_{(0,3)} \cong B^2_{(3,0)} \cong B^2_{(3)} \)

**Basic Representations of the** \( B^2_{(t_j, 3-t_j)} \)

For the permissible 2-tuples of \( n=3 \) \( (t_j, 3-t_j) \) in \{ (0,3), (1,2), (2,1), (3,0) \}, we have the basic o.r.'s \( P_{(t_j, 3-t_j)} \) as in lemma 5.2.1 given by

\[
\begin{align*}
P_{(0,3)} &= P_0 \otimes P_3 \\
P_{(1,2)} &= P_1 \otimes P_2 \\
P_{(2,1)} &= P_2 \otimes P_1 \\
P_{(3,0)} &= P_3 \otimes P_0. \\
\end{align*}
\] (5.8)

As in (5.6) we tensor the o.r.'s \( P=P^* \) obtained earlier with the representations \( P_{(t_j, 3-t_j)} \) above to give the following o.r.'s

\[
\begin{align*}
P \otimes P_{(t_j, 3-t_j)} \text{ of } B^2_{(t_j, 3-t_j), t_j = j-1, (j=1,2,\ldots,4)} \\
([1] \otimes [2]) \otimes P_{(1,2)} &= [3]^* \otimes P_{(0,3)} \\
([1] \otimes [1^2]) \otimes P_{(1,2)} &= [21]^* \otimes P_{(0,3)} \\
([2] \otimes [1]) \otimes P_{(2,1)} &= [1^3]^* \otimes P_{(0,3)} \\
&\quad [3]^* \otimes P_{(3,0)} \\
([1^2] \otimes [1]) \otimes P_{(2,1)} &= [21]^* \otimes P_{(3,0)} \\
&\quad [1^3]^* \otimes P_{(3,0)} \end{align*}
\] (5.9)
The o.r's \( P \circ P_{(t_j, 3-t_j)} \) of \( B^2_{(t_j, 3-t_j)} \) are now induced to \( B^2_3 \) as in theorem 5.2.2, to give the full set of i.or's of \( B^2_3 \).

\[
T_1 = ([1] \circ [2]) \circ P_{(1, 2)} \uparrow B^2_3; \quad T_6 = ([3] \circ P_{(0, 3)} \uparrow B^2_3
\]

\[
T_2 = ([1] \circ [1^2]) \circ P_{(1, 2)} \uparrow B^2_3; \quad T_7 = ([21] \circ P_{(0, 3)} \uparrow B^2_3
\]

\[
T_3 = ([2] \circ [1]) \circ P_{(2, 1)} \uparrow B^2_3; \quad T_8 = ([1^3] \circ P_{(0, 3)} \uparrow B^2_3
\]

\[
T_4 = ([1^2] \circ [1]) \circ P_{(2, 1)} \uparrow B^2_3; \quad T_9 = ([3] \circ P_{(3, 0)} \uparrow B^2_3
\]

\[
T_5 = ([21] \circ P_{(3, 0)} \uparrow B^2_3; \quad T_{10} = ([1^3] \circ P_{(3, 0)} \uparrow B^2_3
\]

(5.10)

The characters of \( B^2_3 \) are given in Table III of the appendix.

### 5.3.2 Representations of \( B^2_4 \)

The group \( B^2_4 \) has order 384. The table below shows the representatives of the classes of \( B^2_4 \) together with their types and orders.
<table>
<thead>
<tr>
<th>Type</th>
<th>Class representatives</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>ccl₁₅</td>
<td>( \begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 1 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix} )</td>
<td>12</td>
</tr>
<tr>
<td>ccl₂₆</td>
<td>( \begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 1 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix} ) ( \begin{pmatrix} 1 &amp; 2 \ -2 &amp; -1 \end{pmatrix} ) or ( \begin{pmatrix} 1 &amp; 2 \ 2 &amp; 1 \end{pmatrix} )</td>
<td>12</td>
</tr>
<tr>
<td>ccl₃₇</td>
<td>( \begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \end{pmatrix} ) ( \begin{pmatrix} 1 &amp; 2 &amp; 3 \ -2 &amp; -3 &amp; 1 \end{pmatrix} ) or ( \begin{pmatrix} 1 &amp; 2 &amp; 3 \ 2 &amp; 3 &amp; 1 \end{pmatrix} )</td>
<td>32</td>
</tr>
<tr>
<td>ccl₄₈</td>
<td>( \begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 1 \end{pmatrix} ) ( \begin{pmatrix} 1 &amp; 2 &amp; 3 &amp; 4 \ -2 &amp; -3 &amp; 4 &amp; -1 \end{pmatrix} ) or ( \begin{pmatrix} 1 &amp; 2 &amp; 3 &amp; 4 \ 2 &amp; 3 &amp; 4 &amp; 1 \end{pmatrix} )</td>
<td>48</td>
</tr>
<tr>
<td>ccl₅₉</td>
<td>( \begin{pmatrix} 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix} ) ( \begin{pmatrix} 1 \ -1 \end{pmatrix} )</td>
<td>4</td>
</tr>
<tr>
<td>ccl₆₀</td>
<td>( \begin{pmatrix} 0 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix} ) ( \begin{pmatrix} 1 &amp; 2 \ 2 &amp; -1 \end{pmatrix} ) or ( \begin{pmatrix} 1 &amp; 2 \ -2 &amp; 1 \end{pmatrix} )</td>
<td>12</td>
</tr>
<tr>
<td>ccl₇₁</td>
<td>( \begin{pmatrix} 0 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix} ) ( \begin{pmatrix} 1 &amp; 2 &amp; 3 \ -2 &amp; 3 &amp; 1 \end{pmatrix} ) or ( \begin{pmatrix} 1 &amp; 2 &amp; 3 \ -2 &amp; -3 &amp; -1 \end{pmatrix} )</td>
<td>32</td>
</tr>
<tr>
<td>ccl₈₂</td>
<td>( \begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 1 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix} ) ( \begin{pmatrix} 1 &amp; 2 &amp; 3 &amp; 4 \ -2 &amp; 3 &amp; 4 &amp; 1 \end{pmatrix} ) or ( \begin{pmatrix} 1 &amp; 2 &amp; 3 &amp; 4 \ -2 &amp; -3 &amp; -4 &amp; 1 \end{pmatrix} )</td>
<td>48</td>
</tr>
<tr>
<td>ccl₉₃</td>
<td>( \begin{pmatrix} 1 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix} ) ( \begin{pmatrix} 1 &amp; 2 \ 2 &amp; -1 \end{pmatrix} ) ( \begin{pmatrix} 3 \ -3 \end{pmatrix} ) or ( \begin{pmatrix} 1 &amp; 2 \ -2 &amp; 1 \end{pmatrix} ) ( \begin{pmatrix} 3 \ -3 \end{pmatrix} )</td>
<td>24</td>
</tr>
<tr>
<td>ccl₁₀₄</td>
<td>( \begin{pmatrix} 1 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix} ) ( \begin{pmatrix} 1 &amp; 2 &amp; 3 \ -2 &amp; 3 &amp; 1 \end{pmatrix} ) ( \begin{pmatrix} 4 \ -4 \end{pmatrix} ) or ( \begin{pmatrix} 1 &amp; 2 &amp; 3 \ -2 &amp; -3 &amp; -1 \end{pmatrix} ) ( \begin{pmatrix} 4 \ -4 \end{pmatrix} )</td>
<td>32</td>
</tr>
<tr>
<td>ccl₁₁₅</td>
<td>( \begin{pmatrix} 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; 0 \end{pmatrix} ) ( \begin{pmatrix} 1 &amp; 2 \ 2 &amp; 1 \end{pmatrix} ) ( \begin{pmatrix} 3 \ -4 \end{pmatrix} ) or ( \begin{pmatrix} 1 &amp; 2 \ 2 &amp; 1 \end{pmatrix} ) ( \begin{pmatrix} 4 \end{pmatrix} )</td>
<td>24</td>
</tr>
<tr>
<td>ccl₁₂₆</td>
<td>( \begin{pmatrix} 1 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 \end{pmatrix} ) ( \begin{pmatrix} 1 &amp; 2 &amp; 3 \ 2 &amp; 3 &amp; 1 \end{pmatrix} ) ( \begin{pmatrix} 4 \ -4 \end{pmatrix} ) or ( \begin{pmatrix} 1 &amp; 2 &amp; 3 \ -2 &amp; -3 &amp; -1 \end{pmatrix} ) ( \begin{pmatrix} 4 \ -4 \end{pmatrix} )</td>
<td>32</td>
</tr>
<tr>
<td>ccl₁₃₇</td>
<td>( \begin{pmatrix} 0 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; 0 \end{pmatrix} ) ( \begin{pmatrix} 1 &amp; 2 \ -2 &amp; 1 \end{pmatrix} ) ( \begin{pmatrix} 3 &amp; 4 \ 4 &amp; 3 \end{pmatrix} ) or ( \begin{pmatrix} 1 &amp; 2 \ -2 &amp; -1 \end{pmatrix} ) ( \begin{pmatrix} 3 &amp; 4 \ -4 &amp; 3 \end{pmatrix} )</td>
<td>24</td>
</tr>
<tr>
<td>ccl₁₄₈</td>
<td>( \begin{pmatrix} 2 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix} ) ( \begin{pmatrix} 1 \ -2 \end{pmatrix} ) ( \begin{pmatrix} 2 \ -2 \end{pmatrix} )</td>
<td>6</td>
</tr>
<tr>
<td>ccl₁₅₉</td>
<td>( \begin{pmatrix} 0 &amp; 2 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix} ) ( \begin{pmatrix} 1 &amp; 2 \ -2 &amp; 1 \end{pmatrix} ) ( \begin{pmatrix} 3 &amp; 4 \ -4 &amp; 3 \end{pmatrix} )</td>
<td>12</td>
</tr>
<tr>
<td>ccl₁₆₀</td>
<td>( \begin{pmatrix} 0 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 2 &amp; 0 &amp; 0 \end{pmatrix} ) ( \begin{pmatrix} 1 &amp; 2 \ 2 &amp; 1 \end{pmatrix} ) ( \begin{pmatrix} 3 &amp; 4 \ 4 &amp; 3 \end{pmatrix} ) or ( \begin{pmatrix} 1 &amp; 2 \ -2 &amp; -1 \end{pmatrix} ) ( \begin{pmatrix} 3 &amp; 4 \ -4 &amp; -3 \end{pmatrix} )</td>
<td>12</td>
</tr>
<tr>
<td>ccl₁₇₁</td>
<td>( \begin{pmatrix} 2 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; 0 \end{pmatrix} ) ( \begin{pmatrix} 1 &amp; 2 \ -2 &amp; 1 \end{pmatrix} ) ( \begin{pmatrix} 3 \ -4 \end{pmatrix} ) or ( \begin{pmatrix} 1 &amp; 2 \ 2 &amp; -1 \end{pmatrix} ) ( \begin{pmatrix} 3 \ -4 \end{pmatrix} )</td>
<td>12</td>
</tr>
<tr>
<td>ccl₁₈₂</td>
<td>( \begin{pmatrix} 2 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; 0 \end{pmatrix} ) ( \begin{pmatrix} 12 \ 21 \end{pmatrix} ) ( \begin{pmatrix} 3 \ -4 \end{pmatrix} ) or ( \begin{pmatrix} 1 &amp; 2 \ -2 &amp; -1 \end{pmatrix} ) ( \begin{pmatrix} 3 \ -4 \end{pmatrix} )</td>
<td>12</td>
</tr>
</tbody>
</table>
\[ \text{cct}_{19} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{cct}_{20} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

\[ \begin{pmatrix} 1 & -2 & -3 \\ -1 & 2 & 3 \\ -1 & 2 & -3 & -4 \end{pmatrix} \]

(5.11)

The permissible 2-tuples of \( n=4 \) are:

\((t_j, 4-t_j) \in \{(0,4), (1,3), (2,2), (3,1), (4,0)\} \) so that \(|(\alpha)|=5\). By lemma 5.1.3 the number of conjugacy classes of \( B_4^2 \) is \( \sum_{j=1}^{5} P(t_j)P(4-t_j) \) which equals 20. Hence \( B_4^2 \) has 20 inequivalent i.o.r's.

We construct these as in (5.3.1).

**Representations of the Young Subgroups \( S_{(t_j, 4-t_j)} \)**

The partition \((t_j, 4-t_j) = (2,2)\)

The young subgroup \( S_{(2,2)} = S_2 \times S_2 \) has 4 i.o.r's given by \([2] \otimes [2], [2] \otimes [1^2], [1^2] \otimes [1^2], [1^2] \otimes [2]\).

The partition \((t_j, 4-t_j) = (3,1)\)

The Young subgroup \( S_{(3,1)} = S_3 \times S_1 \) is isomorphic to \( S_3 \) and has three i.o.r's given by \([3] \otimes [1], [21] \otimes [1], [1^3] \otimes [1]\).

The partition \((t_j, 4-t_j) = (4) = (4,0) = (0,4)\)

The young subgroup \( S_{(4)} \) is isomorphic to \( S_4 \) and has five i.o.r's given by \([4], [31], [21^2], [2^2] \) and \([1^4] \). (See example 2.1).
Thus the full set of relevant i.o.r's $P$ of the Young subgroups $S_{(t_j, 4-t_j)}$ of $S_4$ is

$$\begin{align*}
[21] \otimes [1], [1^3] \otimes [1], [4], [31], [2^2], [21^2], [1^4] \}
\end{align*}$$

(5.12)

**Representations of the Generalised Young Subgroups $B^2_{(t_j, 4-t_j)}$**

We keep the above notation. The i.o.r's $P$ of the young subgroups $S_{(t_j, 4-t_j)}$ of $S_4$ are lifted to give i.o.r's $P^*$ of the generalised young subgroups $B^2_{(t_j, 4-t_j)}$ of $B_4^2$. The full sets of the i.o.r's $P$ of $B^2_{(t_j, 4-t_j)}$ are given below for each subgroup.

For $B^2_{(2, 2)}$

$$\begin{align*}
\{ ([2] \otimes [2])^*, ([2] \otimes [1^2])^*, ([1^2] \otimes [2])^*, ([1^2] \otimes [1^2])^* \}
\end{align*}$$

For $B^2_{(3, 1)}$

$$\begin{align*}
\{ ([3] \otimes [1])^*, ([21] \otimes [1])^*, ([1^3] \otimes [1])^* \}
\end{align*}$$

For $B^2_{(1, 3)}$

$$\begin{align*}
\{ ([1] \otimes [3])^*, ([1] \otimes [21])^*, ([1] \otimes [1^3])^* \}
\end{align*}$$

For $B^2_{(4)}$ or $B^2_{(4, 0)}$ or $B^2_{(0, 4)}$

$$\begin{align*}
\end{align*}$$
For the permissible 2-tuples of n=4, \((t_j, 4-t_j)\) in \{(0,4), (1,3), (2,2), (3,1), (4,0)\} we have the basic o.r's \(P_{(t_j, 4-t_j)}\) as in lemma 5.2.1 given by

\[
\begin{align*}
P_{(0,4)} &= P_0 \otimes P_4; & P_{(3,1)} &= P_3 \otimes P_1 \\
P_{(1,3)} &= P_1 \otimes P_3; & P_{(4,0)} &= P_4 \otimes P_0. \\
P_{(2,2)} &= P_2 \otimes P_2
\end{align*}
\]

(5.13)

As in (5.6) we tensor the o.r's \(P = P^*\) obtained earlier with the \(P_{(t_j, 4-t_j)}\) above to give the following o.r's

\[
\begin{align*}
P \otimes P_{(t_j, 4-t_j)}, & \quad t_j = j-1, j = 1, \ldots, 5. \\
([2] \otimes [2])^* \otimes P_{(2,2)}; & \quad ([1^2] \otimes [1^2])^* \otimes P_{(2,2)} \\
([2] \otimes [1^2])^* \otimes P_{(2,2)}; & \quad ([3] \otimes [1])^* \otimes P_{(3,1)} \\
([1^2] \otimes [2])^* \otimes P_{(2,2)}; & \quad ([21] \otimes [1])^* \otimes P_{(3,1)} \\
([1^3] \otimes [1])^* \otimes P_{(3,1)}; & \quad ([1] \otimes [21])^* \otimes P_{(2,2)} \\
([1] \otimes [3])^* \otimes P_{(1,3)}; & \quad ([1] \otimes [1^3])^* \otimes P_{(1,3)} \\
[4]^* \otimes P_{(0,4)}; & \quad ([4]^* \otimes P_{(4,0)} \\
[31]^* \otimes P_{(0,4)}; & \quad [31]^* \otimes P_{(4,0)} \\
[21^2]^* \otimes P_{(0,4)}; & \quad [21^2]^* \otimes P_{(4,0)} \\
[2^2]^* \otimes P_{(0,4)}; & \quad [2^2]^* \otimes P_{(4,0)} \\
[1^4]^* \otimes P_{(0,4)}; & \quad [1^4]^* \otimes P_{(4,0)} \\
\end{align*}
\]

(5.14)
The o.r's \( P \circ P_{t_j^{4-t_j}} \) of \( B_{t_j^{4-t_j}}^2 \) are now induced to \( B_4^2 \) as in theorem 5.2.2 to give the full set of i.o.r's of \( B_4^2 \):

\[
T_1 = (([2] \circ [2])^* \circ P_{(2,2)} \uparrow B_4^2; 
T_5 = ([3] \circ [1])^* \circ P_{(3,1)} \uparrow B_4^2
T_2 = (([2] \circ [1^2])^* \circ P_{(2,2)} \uparrow B_4^2; 
T_6 = ([21] \circ [1])^* \circ P_{(3,1)} \uparrow B_4^2
T_3 = ([1^2] \circ [2])^* \circ P_{(2,2)} \uparrow B_4^2; 
T_7 = ([1^3] \circ [1])^* \circ P_{(3,1)} \uparrow B_4^2
T_4 = ([1^2] \circ [1^2])^* \circ P_{(2,2)} \uparrow B_4^2; 
T_8 = ([1] \circ [3])^* \circ P_{(1,3)} \uparrow B_4^2
T_9 = ([1] \circ [21])^* \circ P_{(1,3)} \uparrow B_4^2
T_{11} = ([4] \circ P_{(0,4)} \uparrow B_4^2; 
T_{10} = ([1] \circ [1^3])^* \circ P_{(1,3)} \uparrow B_4^2
T_{12} = ([31] \circ P_{(0,4)} \uparrow B_4^2; 
T_{14} = ([2^2] \circ P_{(0,4)} \uparrow B_4^2
T_{13} = ([21^2] \circ P_{(0,4)} \uparrow B_4^2; 
T_{15} = ([1^4] \circ P_{(0,4)} \uparrow B_4^2
T_{16} = ([4] \circ P_{(4,0)} \uparrow B_4^2; 
T_{17} = ([31] \circ P_{(4,0)} \uparrow B_4^2
T_{18} = ([21^2] \circ P_{(4,0)} \uparrow B_4^2; 
T_{19} = ([2^2] \circ P_{(4,0)} \uparrow B_4^2
T_{20} = ([1^4] \circ P_{(4,0)} \uparrow B_4^2;
\]

(5.15)

The characters of \( B_4^2 \) are given in table IV of the appendix.
### APPENDIX

<table>
<thead>
<tr>
<th>CLASS REPRESENTATIVES</th>
<th>e</th>
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<th>(123)</th>
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\[
\chi_{[3]} \quad \begin{array}{ccc}
1 & 1 & 1 \\
\end{array}
\]

\[
\chi_{[2]} \quad \begin{array}{ccc}
2 & 0 & -1 \\
\end{array}
\]

\[
\chi_{[1^3]} \quad \begin{array}{ccc}
1 & -1 & 1 \\
\end{array}
\]

**TABLE I:** CHARACTERS OF $S_3$

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\[
\chi_{[4]} \quad \begin{array}{ccc}
1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\chi_{[31]} \quad \begin{array}{ccc}
3 & 1 & 0 & -1 & -1 \\
\end{array}
\]

\[
\chi_{[2^2]} \quad \begin{array}{ccc}
2 & 0 & -1 & 2 & 0 \\
\end{array}
\]

\[
\chi_{[21^2]} \quad \begin{array}{ccc}
3 & -1 & 0 & -1 & 1 \\
\end{array}
\]

\[
\chi_{[1^4]} \quad \begin{array}{ccc}
1 & -1 & 1 & 1 & -1 \\
\end{array}
\]

**TABLE II:** CHARACTERS OF $S_4$
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**Classes of \( \xi^2 \)**

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TABLE IV: CHARACTERISTICS OF \( t \)
REFERENCES


[5] A.J. Coleman, Induced representations with applications to $S_n \text{ and } \text{GL}_n(K)$, Queen’s papers in pure and applied maths, Queen’s Univ, Kingston, Ontario (1966).


