THE UNIVERSITY OF ZAMBIA
SCHOOL OF NATURAL SCIENCES
DEPARTMENT OF MATHEMATICS & STATISTICS

2018/2019 Academic Year Final Examinations NOVEMBER, 2019
MAT 3100 ADVANCED CALCULUS

Total time allowed: Three (3) hours
Instructions:

- Answer any five (5) questions.
- Show detailed working to earn full marks.
- Each question carries 20 marks.
- The total marks for this paper is 100.

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1. (a) (i) State the Divergence theorem without proof.

[4 Marks]

(ii) Hence or otherwise calculate the outward flux of $\boldsymbol{F}$, $\displaystyle\int\int_S \boldsymbol{F}\cdot d\boldsymbol{S}$, across the boundary of the region S: $\boldsymbol{F}(x,y,z) = x^2\boldsymbol{i} + y^2\boldsymbol{j} + z^2\boldsymbol{k}$ and S is the region cut from the solid cylinder $x^2+y^2 \leq a^2$ by the planes $z=0$ and $z=1$.

[6 Marks]

(b) Use Lagrange Multipliers to find the maximum and minimum values of the function subject to the given constraints:

$$f(x,y,z) = x+2y,\ x+y+z=1,\ y^2+z^2=4.$$

[6 Marks]

(c) Let $\boldsymbol{F}(x,y,z) = e^x \sin y\boldsymbol{i} + e^x \cos y\boldsymbol{j} + z\boldsymbol{k}$. Find $\nabla^2\boldsymbol{F}$.

[4 Marks]

2. (a) (i) Explain why the function $f(x,y) = ye^{xy}$ is differentiable at $(0,1)$.

[4 Marks]

(*ii*) Find an approximation of $f(0.2, 1.2)$.

[2 Marks]

(*b*) Obtain the complex Fourier series for the function $f(t) = t$, $-\pi \leq t \leq \pi$, given that $f(t + 2\pi) = f(t)$.

[5 Marks]

(*c*) (*i*) State the Inverse Function theorem without proof.

[4 Marks]

(*ii*) Suppose that $\hat{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$\hat{f} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y^4 e^x + x - 4y \\ 2xy + 3y \end{pmatrix},$$

show that $\hat{f}$ has a differentiable inverse near $(0, 1)$

[3 Marks]

(*iii*) Find the Jacobian matrix of the inverse mapping $\hat{f}^{-1}$.

[2 Marks]

3. (*a*) Find the general solution of the differential equation

$$t^2 y'' - 2y = 3t^2 - 1$$

using the method of variation of parameters.

[10 Marks]

(*b*) Evaluate

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2 + y^2) dz dy dx.$$

[5 Marks]

(*c*) Given that

$$\boldsymbol{F}(x, y, z) = y\boldsymbol{i} + x\boldsymbol{j} + z\boldsymbol{k},$$

find

(*i*) $\text{curl}(\boldsymbol{F}) = \nabla \times \boldsymbol{F}$

[3 Marks]

(*ii*) $\text{div}(\boldsymbol{F}) = \nabla \bullet \boldsymbol{F}$

[2 Marks]

4. (a) (i) State Stokes' theorem without proof.

[4 Marks]

(ii) Hence or otherwise find $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ given that $\boldsymbol{F}(x, y, z) = yz\boldsymbol{i} + 2xz\boldsymbol{j} + e^{xy}\boldsymbol{k}$ and C is the circle $x^2 + y^2 = 16, z = 5$.

[6 Marks]

(b) Evaluate the integral

$$\int\int_R 2e^{\frac{x+y}{x-y}} dA,$$

where R is the trapezoidal region with vertices $(1, 0)$, $(2, 0)$, $(0, -2)$ and $(0, -1)$.

[6 Marks]

(c) Given that $z = f(x, y)$ has continuous second order partial derivatives and $x = r^2 + s^2$, $y = 2rs$, find $\frac{\partial^2 z}{\partial s^2}$.

[4 Marks]

5. (a) (i) Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

[5 Marks]

(ii) Hence or otherwise prove that

$$\int_0^\infty \frac{\cos w \sin w}{w} dw = \frac{\pi}{4}.$$

[4 Marks]

(b) Obtain the Fourier sine transform of the function $f(x) = e^{-ax}$, $a > 0$.

[7 Marks]

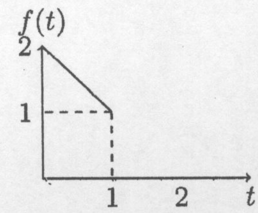
(c) Find the directional derivative of $f(x, y) = 3x^3 - 4xy$ along the parabola $y = 2x^2 - x + 2$ at the point $((1, 3)$.

[4 Marks]

6. (*a*) Given that $\mathcal{L}(f) = \dfrac{2(e^{-s} - e^{-3s})}{s^2 - 4}$. Find the inverse Laplace transform.

[7 Marks]

(*b*) (i) Obtain the even periodic extension of the function below



[5 Marks]

(ii) Sketch the corresponding periodic extension of f for $0 \leq t \leq 5$

[2 Marks]

(*c*) Given that $\boldsymbol{F}(x, y, z) = z\boldsymbol{i} + y\boldsymbol{j} + x\boldsymbol{k}$,

(*i*) Show that $\boldsymbol{F}(x, y, z)$ is conservative.

[2 Mark]

(*ii*) Find the work done by $\boldsymbol{F}$ in moving a particle along the curve C given by

$$\boldsymbol{r}(t) = 3t\boldsymbol{i} - 5t\boldsymbol{j} + t^2\boldsymbol{k}, \quad -1 \leq t \leq 1.$$

[4 Marks]

7. (*a*) Solve the shifted data Initial Value Problem by the Laplace transform:

$$y'' + 3y' - 4y = 6e^{2t-3}, \quad y(1.5) = 4, \quad y'(1.5) = 5.$$

[5 Marks]

(*b*) Find the general solution of the non- homogeneous system

$$\begin{cases} y_1' = y_1 - 2y_2 + \cos t \\ y_2' = -2y_1 + y_2 - \sin t. \end{cases}$$

[10 Marks]

(*c*) Evaluate the line integral

$$\int_C x^2y^2dx + 4xy^3dy$$

where C consists of line segments from $(0,0)$ to $(0,3)$, $(0,3)$ to $(1,3)$ and $(1,3)$ to $(0,0)$

[5 Marks]

-END OF EXAM-

The University of Zambia
School of Natural Sciences
Department of Mathematics and Statistics

2018/19 Academic Year Examinations

MAT 3110 Engineering Mathematics II

November 18, 2019.

Duration: THREE Hours

Instructions:

- Do not open this booklet until you are told to do so.
- This paper consists of seven questions, each carry 20 marks. Attempt any FIVE questions. Total marks is 100.
- Show all the essential working to earn full marks.
- Non programmable calculators are allowed.

The University of Zambia

School of Natural Sciences

Department of Mathematics and Statistics

2018/19 Academic Year Final Examinations

MAT3622 - Linear Regression

Time allowed : 3 Hours

Instructions:

- There are **five (5)** questions in this examination paper. Attempt **any four (4)** questions. **All** questions carry **equal** marks.
- Indicate your **Computer Number** on all your answer booklets.
- You are required to show all **necessary** steps in your solutions.

This paper consists of 4 pages of questions.

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1. (a) Consider the multiple linear regression model $Y = X\beta + \varepsilon,\ \varepsilon \sim N(0, \sigma^2 I)$.

   (i) Find the maximum likelihood estimates of β.

   (ii) Show that the residuals from a linear regression model can be expressed as $e = (I - H)\varepsilon$, where H is the "hat" matrix.

   (iii) Show that H and $I - H$ are symmetric and idempotent.

   (iv) Find Var[e].

   (v) Find COV($\hat{\beta}, Y - X\hat{\beta}$).

   (vi) Find $Var(\hat{y})$

   (vii) Find $E(e'e)$

   (b) Suppose we fit the model $\mathbf{y} = \mathbf{X_1}\beta_1 + \varepsilon$ when the true model is actually given by $\mathbf{y} = \mathbf{X_1}\beta_1 + \mathbf{X_2}\beta_2 + \varepsilon$. For both models, assume $E(\varepsilon) = 0$ and $Var(\varepsilon) = \sigma^2 I$. Find the expected value of the ordinary least squares estimate, $\hat{\beta}_1$. Under what conditions is this estimate unbiased?

2. (a) Suppose that a one-way analysis of variance involves four treatments but that a different number of observations (e.g., n_i) has been taken under each treatment. Asuming that $n_1 = 3, n_2 = 2, n_3 = 4$ and $n_4 = 3$.

   (i) Write down the y vector and X matrix for analysing these data as a multiple regression model.

(ii) Obtain the least squares estimates of the model parameters.

(b) The data below were used to relate annual regional advertising expenses to annual regional concentrate sales for a soft drink company. The table below presents the twenty years of these data. It is assumed that a straight-line relationship was appropriate.

Year (t)	**Sales** y_t	**Expenditures** x_t(10^3 Kwachas)	**Residuals**
1	3083	75	-32.3298
2	3149	78	-26.6027
3	3218	80	2.2154
4	3239	82	-16.9665
5	3295	84	-1.1484
6	3374	88	-2.5123
7	3475	93	-1.9671
8	3569	97	11.6691
9	3597	99	-0.5128
10	3725	104	27.0324
11	3794	109	-4.4224
12	3959	115	40.0318
13	4043	120	23.5770
14	4194	127	33.9403
15	4318	135	-2.7874
16	4493	144	-8.6060
17	4683	153	0.5753
18	4850	161	6.8476
19	5005	170	-18.9710
20	5236	182	-29.0625

(i) Fit a simple linear regression model to these data.

(ii) Plot residuals versus time and explain whether you find any evidence of positive autocorrelation.

(iii) Conduct a formal test for positive autocorrelation using a significance level of 0.05. Clearly stating the alternatives, decision rule, and conclusion.

(iv) Is the residual analysis in part (ii) in accord with the test result in (iii)?

3. (a) Suppose the experimenter postulates a model $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i \quad (i = 1, 2, \ldots, n)$ and $\varepsilon_i \sim N(0, \sigma^2)$.

(i) Find the least squares estimates of β_0 and β_1.

(ii) Find the variances of the estimators of β_1 and β_0.

(iii) Prove that the estimates in (i) are uncorrelated if and only if $\bar{x} = 0$.

(iv) Show that $E\left[\sum_{i=1}^{n}(\varepsilon_i - \bar{\varepsilon})^2\right] = (n-1)\sigma^2$.

(b) Suppose that we want to fit the no-intercept model $y_i = \beta x_i + \varepsilon_i$ using weighted least squares. Assume that the observations are uncorrelated but have unequal variances.

(i) Find a general formula for the weighted least-squares estimator of β.

(ii) Find the variance of the weighted least-squares estimator in (i)?

(iii) Suppose that $Var(y_i) = cx_i$ that is, the variance of y_i is proportional to the corresponding x_i. Using the results of parts (i) and (ii), find the weighted least-squares estimator of β and the variance of this estimator.

(iv) Suppose that $Var(y_i) = cx_i^2$ that is, the variance of y_i is proportional to the square of the corresponding x_i. Using the results of parts (i) and (ii), find the weighted least-squares estimator of β and the variance of this estimator.

4. (a) Consider a simple linear regression model with first-order autoregressive errors $y_t = \beta_0 + \beta_1 x_t + \varepsilon_t$ where $\varepsilon_t = \rho\varepsilon_{t-1} + a_t$.

(i) Define autocorrelation.

(ii) Show that $\varepsilon_t = \sum_{j=0}^{\infty} \rho^j a_{t-j}$.

(iii) Find $E(\varepsilon_t^2)$.

(iv) Find $Cov(\varepsilon_t, \varepsilon_{t-2})$.

(b) In a small-scale regression study, the following data were obtained:

Y_i:	42	33	75	28	91	55
X_{i1}:	7	4	16	3	21	8
X_{i2}:	33	41	7	49	5	31

(i) Define $\mathbf{Y}, \mathbf{X}, \beta$, and ε for a model involving both independent variables and an intercept.

(ii) Compute $\mathbf{X}'\mathbf{X}$ and $\mathbf{X}'\mathbf{Y}$.

(iii) $(\mathbf{X}'\mathbf{X})^{-1}$ for this problem is

$$(\mathbf{X}'\mathbf{X})^{-1} = \begin{bmatrix} 34.57855744 & -1.65089268 & -0.65704022 \\ -1.65089268 & 0.08030796 & 0.03112763 \\ -0.65704022 & 0.03112763 & 0.01268501 \end{bmatrix}.$$

Verify that this is the inverse of $\mathbf{X}'\mathbf{X}$.

(iv) Compute $\hat{\beta}$ and write the regression equation.

(v) Test the significance of the regression at $\alpha = 0.05$.

(vi) Construct a 95% confidence interval for each of the parameters in the model.

5. (a) The number of pounds of steam used per month at a plant is thought to be related to the average monthly ambient temperature. The past year's usages and temperatures follow:

Month	Temperature	Usage/1000	Month	Temperature	Usage/1000
Jan.	21	185.79	Jul.	68	621.55
Feb.	24	214.47	Aug.	74	675.06
Mar.	32	288.03	Sep.	62	562.03
Apr.	47	424.84	Oct.	50	452.93
May	50	454.68	Nov.	41	369.95
Jun.	59	539.03	Dec.	30	273.98

(i) Compute $\hat{\beta}_0$ and $\hat{\beta}_1$ for the linear regression of ~~Hardness~~ Usage on ~~time~~ temp and write the regression equation.

(ii) Interpret the meaning of the values of $\hat{\beta}_0$ and $\hat{\beta}_1$ calculated in part (i).

(iii) Obtain the estimate of σ^2 and find the estimated standard errors of $\hat{\beta}_0$ and $\hat{\beta}_1$.

(iv) Construct 95% confidence intervals on β_1 and β_0.

(v) Test for significance of regression at 5% level of significance.

(vi) Plant management believes that an increase in average ambient temperature of 1 degree will increase average monthly steam consumption by 10,000 lbs. Do the data support this statement?

(vii) Construct a 99% prediction interval on steam usage in a month with average ambient temperature of 58°.

(b) Consider the model

$$Y_i = \beta_0 + \varepsilon_i, \qquad i = 1, \ldots, n,$$

where $\varepsilon_i = \varepsilon_1^* + \varepsilon_2^* + \cdots + \varepsilon_i^*$ and ε_i^*'s are uncorrelated $(0, \sigma^2)$ random variables.

(i) Find the ordinary least squares estimator $\hat{\beta}_0$.

(ii) Compute $Var(\hat{\beta}_0)$.

(iii) Find the appropriate transformation Y^* that can stabilise the variance.

END OF EXAM

(b) Use Stoke's theorem to evaluate the integral (6)

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

where $\mathbf{F}(x, y, z) = (zx^3 - 2z)\ \mathbf{i} + xz\ \mathbf{j} + xy\ \mathbf{k}$ and C is the triangle with vertices $(0,0,4)$, $(0,2,0)$ and $(2,0,0)$. Direction is $(2,0,0)$ to $(0,0,4)$, then $(0,0,4)$ to $(0,2,0)$ and then $(0,2,0)$ to $(2,0,0)$.

(c) Evaluate the surface integral (6)

$$\int\int_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

where $\mathbf{F}(x, y, z) = -xy\ \mathbf{i} + (z-1)\ \mathbf{j} + z^3\ \mathbf{k}$ and S the surface of the solid bounded by $y = 4x^2 + 4z^2 - 1$ and the plane $y = 7$. Note that both of the surfaces of this solid are included in S.

7. During an epidemic of a certain disease a doctor is consulted by 110 people suffering from symptoms commonly associated with the disease. Of the 110 people, 45 are female of whom 20 actually have the disease and 25 do not. Fifteen males have the disease and the rest do not.

(a) A person is selected at random, The event that this person is female is denoted by A and the event that this person is suffering from the disease is denoted by B. Evaluate

i. $P(A)$ (1)

ii. $P(A \cup B)$, (2)

iii. $P(B \mid A)$, (2)

iv. $P(A \mid B)$. (2)

(b) If three different people are selected at random without replacement, what is the probability of

i. all three having the disease, (2)

ii. exactly one of the three having the disease, (2)

iii. one of the three being a female with the disease, one a male with the disease and one a female without the disease? (3)

(c) Of the people with the disease 96% react positively to a test for diagnosing the disease as do 8% of people without the disease. 'What is the probability of a person selected at random

i. reacting positively, (3)

ii. having the disease given that he or she reacted positively. (3)

——— *End of Examination* ———

THE UNIVERSITY OF ZAMBIA
SCHOOL OF NATURAL SCIENCES
DEPARTMENT OF MATHEMATICS & STATISTICS

2018/19 Academic Year
End of Year Final Examinations

15 November 2019 (am)

MAT3300 – Real Analysis

Time allowed : **Three (3) hours** Full marks : **100**

Instructions:
- There are **six (6)** questions in this paper. Attempt **any five (5)** questions.
- Mark allocations are shown in brackets.
- **Full credit** will only be given when **necessary work** is shown.
- Indicate your **computer number** on all answer booklets used.

This paper consists of 4 pages of questions.

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1. (a) Let A be a subset of $\mathbb{R}$. When is a point $x \in \mathbb{R}$ called

   (i) an interior point of A, (1)

   (ii) a limit point of A, (1)

   (iii) a boundary point of A. (1)

   (b) Let $A = \bigcap_{n=1}^{\infty} \left[a, b + \frac{1}{n}\right)$, where a and b are real numbers such that $a < b$.

   (i) Discuss whether A is open or closed in $\mathbb{R}$. Justify each statement you make. (5)

   (ii) Determine the interior, closure, and boundary of A. (1)

(c) Let $K_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ be obtained from $[0, 1]$ by removing the middle third $(\frac{1}{3}, \frac{2}{3})$. Repeat the process to obtain $K_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. In general, K_{n+1} is obtained from K_n by removing the middle third of each interval in K_n.

Let $K = \cap_{n=1}^{\infty} K_n$, also known as the Cantor set. Prove that

(i) K is compact, **(2)**

(ii) the interior of K is empty, **(2)**

(iii) K has infinitely many points, **(2)**

(iv) the total length of the intervals removed is equal to 1. **(2)**

(d) (i) State what is meant by a set $E \subset \mathbb{R}$ is connected. **(1)**

(ii) Let K be the Cantor set as defined in (c). Prove that K is totally disconnected. **(2)**

2. (a) (i) State the $\varepsilon - \delta$ definition that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $c \in \mathbb{R}$. **(2)**

(ii) Prove that if f is continuous at $c \in \mathbb{R}$ and $f(c) \neq 0$, then there exists a $\delta > 0$ such that for all $x \in \mathbb{R}$,

$$|x - c| < \delta \Longrightarrow |f(x)| > \frac{|f(c)|}{2}.$$

(3)

(b) Every rational x can be written in the form $x = m/n$, where $n > 0$, and m and n are integers without any common divisors . When $x = 0$, we take $n = 1$.

Consider the function f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ \frac{1}{n} & \text{if } x = \frac{m}{n}. \end{cases}$$

Prove that

(i) f is continuous at every irrational point.

(ii) f has a simple discontinuity at every rational point.

(7)

(c) Use the Intermediate Value Theorem for continuous functions to prove that

$$\frac{2}{1 + x^2} + 3\cos(\pi x) = 0$$

has at least two solutions in $[-1, 1]$. **(4)**

(d) Use Rolle's Theorem to prove that

$$\frac{2}{1+x^2} + 3\cos(\pi x) = 0$$

has exactly one solution in $[-1, 1]$. **(4)**

3. (a) (i) Suppose $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ is uniformly continuous. Prove that if $\{x_n\}$ is a Cauchy sequence in A, then $\{f(x_n)\}$ is Cauchy too. **(3)**

(ii) Find a continuous function $f : A \to \mathbb{R}$ and a Cauchy sequence $\{x_n\}$ such that $\{f(x_n)\}$ is not Cauchy. **(2)**

(b) Let A be a compact subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$ a continuous function on A. Prove that $f(A)$ is compact. **(4)**

(c) Suppose that $f : [a, b] \to \mathbb{R}$ is a Lipschitz function with a Lipschitz constant $c > 0$. Prove that f is absolutely continuous on $[a, b]$. **(4)**

(d) In the closed interval $[0, 1]$, a function f is defined as follows: $f(0) = 0, f(1) = 1$ and $f(m/n) = 1/n^3$, where m and n are positive integers such that $(m, n) = 1$. Furthermore, for irrational x, we define $f(x) = 0$.

Show that f is of bounded variation. (*Hint:* $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$.) **(7)**

4. (a) (i) State the Interior Minimum Theorem. **(1)**

(ii) Suppose a real-valued function f is differentiable on $[a, b]$ and $f'(a) < \lambda < f'(b)$. Prove that there exists a point $c \in (a, b)$ such that $f'(c) = \lambda$. **(4)**

(b) (i) State the Cauchy Mean Value Theorem. **(2)**

(ii) Suppose that a real function f is continuous on $[a, b]$ and differentiable in (a, b). Show that if $f > 0$ and $f' \neq 0$ on $[a, b]$, then there exists $c \in (a, b)$ such that

$$f(b) = f(a) + \ln\left(\frac{f(b)}{f(a)}\right) f(c).$$

Remark: No derivative appears on the right-hand side! **(4)**

(c) Suppose that the derivative f' is continuous on the closed interval $[a, b]$ and $\varepsilon > 0$. Prove that there exists $\delta > 0$ such that

$$\left| \frac{f(y) - f(x)}{y - x} - f'(x) \right| < \varepsilon$$

whenever $0 < |y - x| < \delta$, $a \leq x \leq b$ and $a \leq y \leq b$. **(5)**

(d) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a C^2 function that satifies $f \geq 0$ and $f'' \leq 0$ everywhere. Use Taylor's theorem to show that f is a constant function. **(4)**

(d) Use Rolle's Theorem to prove that

$$\frac{2}{1+x^2} + 3\cos(\pi x) = 0$$

has exactly one solution in $[-1, 1]$. **(4)**

3. (a) (i) Suppose $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ is uniformly continuous. Prove that if $\{x_n\}$ is a Cauchy sequence in A, then $\{f(x_n)\}$ is Cauchy too. **(3)**

   (ii) Find a continuous function $f : A \to \mathbb{R}$ and a Cauchy sequence $\{x_n\}$ such that $\{f(x_n)\}$ is not Cauchy. **(2)**

   (b) Let A be a compact subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$ a continuous function on A. Prove that $f(A)$ is compact. **(4)**

   (c) Suppose that $f : [a, b] \to \mathbb{R}$ is a Lipschitz function with a Lipschitz constant $c > 0$. Prove that f is absolutely continuous on $[a, b]$. **(4)**

   (d) In the closed interval $[0, 1]$, a function f is defined as follows: $f(0) = 0, f(1) = 1$ and $f(m/n) = 1/n^3$, where m and n are positive integers such that $(m, n) = 1$. Furthermore, for irrational x, we define $f(x) = 0$.

   Show that f is of bounded variation. (*Hint:* $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$.) **(7)**

4. (a) (i) State the Interior Minimum Theorem. **(1)**

   (ii) Suppose a real-valued function f is differentiable on $[a, b]$ and $f'(a) < \lambda < f'(b)$. Prove that there exists a point $c \in (a, b)$ such that $f'(c) = \lambda$. **(4)**

   (b) (i) State the Cauchy Mean Value Theorem. **(2)**

   (ii) Suppose that a real function f is continuous on $[a, b]$ and differentiable in (a, b). Show that if $f > 0$ and $f' \neq 0$ on $[a, b]$, then there exists $c \in (a, b)$ such that

$$f(b) = f(a) + \ln\left(\frac{f(b)}{f(a)}\right) f(c).$$

   Remark: No derivative appears on the right-hand side! **(4)**

   (c) Suppose that the derivative f' is continuous on the closed interval $[a, b]$ and $\varepsilon > 0$. Prove that there exists $\delta > 0$ such that

$$\left| \frac{f(y) - f(x)}{y - x} - f'(x) \right| < \varepsilon$$

   whenever $0 < |y - x| < \delta$, $a \leq x \leq b$ and $a \leq y \leq b$. **(5)**

   (d) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a C^2 function that satifies $f \geq 0$ and $f'' \leq 0$ everywhere. Use Taylor's theorem to show that f is a constant function. **(4)**

5. (a) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Define what is meant by

(i) a partition P of $[a, b]$, (1)

(ii) the upper and lower sums, $U(P, f)$ and $L(P, f)$ respectively, (2)

(iii) the upper and lower integrals, $\overline{\int_a^b} f$ and $\underline{\int_a^b} f$ respectively, (2)

(iv) f is Riemann integrable over $[a, b]$. (1)

(b) (i) State the Riemann–Stieltjes integrability criterion theorem. (1)

(ii) Consider the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & 0 \leq x < 1/2, \\ 1, & 1/2 \leq x \leq 1. \end{cases}$$

If $\alpha = x^2$, prove that for the partition $P = \{0, 2/5, 3/5, 1\}$ of $[0, 1]$, (6)

$$U(P, f, \alpha) - L(P, f, \alpha) < \frac{1}{3}.$$

(c) (i) Suppose f is continuous and nonnegative on $[0, 1]$, $0 \leq x_0 \leq 1$, and $f(x_0) > 0$. Prove that f is integrable on $[0, 1]$, and that $\int_0^1 f \, \mathrm{d}x > 0$. (5)

(ii) Construct a nonnegative function f on $[0, 1]$ with $f(\frac{1}{3}) > 0$ but $\int_0^1 f \, \mathrm{d}x = 0$. (2)

6. (a) (i) State the fundamental theorem of calculus. (2)

(ii) Assume f is a real continuously differentiable function on $[a, b]$, $f(a) = f(b) = 0$, and $\int_a^b f^2(x) \, \mathrm{d}x = 1$. Prove that (6)

$$\int_a^b x f(x) f'(x) \, \mathrm{d}x < 0 \quad \text{and that} \quad \int_a^b \left[f'(x)\right]^2 \mathrm{d}x \cdot \int_a^b x^2 f^2(x) \, \mathrm{d}x > 0.$$

(b) Suppose f is a real continuously differentiable function on $[a, b]$ and $f(a) = f(b) = 0$. Prove that if f has integrable first and second derivatives then $\int_a^b f f'' \, \mathrm{d}x \leq 0$.
Hint: Use the theorem for integration by parts. (4)

(c) Consider the improper integral $\int_0^\infty e^{-x} \sin x \, \mathrm{d}x$.

(i) Prove that the integral converges. (5)

(ii) Does the integral converge absolutely? (*Hint:* $\left|e^{-x} \sin x\right| \leq e^{-x}$) (3)

END OF PAPER

The University of Zambia

School of Natural Sciences

Department of Mathematics and Statistics

2018-19 Academic Year

End of Year University Examinations

MAT4022: Pension Design and Valuation

Time: Three hours. [Maximum Marks : 100]

1. Attempt All questions. All questions carry equal marks.
2. Write clearly your computer number on each answer booklet.
3. Show all essential working

Some Useful Formulas:

Probabilities

Joint life

$$
\begin{aligned}
{}_tp_{x:y} &= {}_tp_x\,{}_tp_y \\
{}_tq_{x:y} &= {}_tq_x + {}_tq_y - {}_tq_{\overline{x:y}}
\end{aligned}
$$

last survivor

$$
\begin{aligned}
{}_tq_{\overline{x:y}} &= {}_tq_x\,{}_tq_y \\
{}_tp_{\overline{x:y}} &= {}_tp_x + {}_tp_y - {}_tp_{x:y}
\end{aligned}
$$

Joint Assurances and annuities

$$
\begin{aligned}
A_{\overline{x:y}} &= A_x + A_y - A_{x:y} \\
\ddot{a}_{\overline{x:y}} &= \ddot{a}_x + \ddot{a}_y - \ddot{a}_{x:y}
\end{aligned}
$$

Contingent Assurances

$$A^1_{x:y} + A_{x:\overset{1}{y}} = A_{x:y}$$

Reversionary annuities

$$\ddot{a}_{x|y} = \ddot{a}_y - \ddot{a}_{x:y} \cong a_y - a_{x:y} = a_{x|y}$$

Independent and Dependent joint life functions

$$
\begin{aligned}
(aq)^k_x &= \frac{(ad)^k_x}{(al)_x} \\
(al)_{x+1} &= (al)_x - (ad)^k_x - (ad)^{-k}_x \\
(aq)^k_x &= \frac{\mu^k_x}{\sum_j \mu^j_x}\left(1 - e^{\sum_j \mu^j_x}\right) \\
q^k_x &= \frac{(aq)^k_x}{1 - \frac{1}{2}(aq)^{-k}_x} \qquad \text{where } -k \text{ indicates the state not } k \\
(aq)^k_x &= q^k_x \left[1 - \frac{1}{2}\sum_{i \neq k} q^i_x + \frac{1}{3}\sum_{i \neq k}\sum_{i \neq j \neq k} q^i_x q^j_x + \cdots\cdots\right]
\end{aligned}
$$

QUESTION ONE

(a) List any two roles and two responsibilities of pension fund trustees. [2]

(b) Outline any two possible challenges that arise in trying to achieve the above for trustees for pension funds in Zambia. [2]

(c) The government of a developing SADC country is keen to ensure the adequacy of its future pension provision. The country has had only private pension provision through occupational schemes and private individual pension purchased through life assurance companies. Two proposals have been made.

Proposal A: To introduce a national social security scheme in line with existing global arrangements.

Proposal B: To introduce some form of compulsory private pensions provision

(i) Outline the key issues associated with proposal A. [5]

(ii) List the approaches that a state may take to maximise the security of benefits that are provided through proposal B. [5]

(d) (i) Define the term net replacement ratio. [1]

(ii) Discuss how the scheme design affects net replacement ratio for an individual member. [5]

[Total 20 marks]

QUESTION TWO

(a) Distinguish between the *joint life* and *last survivor* statuses for two independent lives. Write down an expression for the future life time of each status in terms of the future life time of the two individual lives. Define all notation used. [5]

(b) Assuming that the AM92 life table is appropriate for both lives and that the survival of each is independent. Calculate

(i) ${}_3p_{62:65}$, [2]

(ii) ${}_2p_{\overline{45:47}}$ [3]

(c) A pension is payable at the rate of K24,000 p.a. (in arrears) to a married couple. On the first death, the pension continues to be paid at two thirds of its original level. Find the value of this pension if it is payable to a couple where the husband is 65 and the wife is aged 60. Use PFA92C20 and PMA92C20 at 4% interest [4]

(d) Explain the following terms and give an example of each:

(i) class selection

(ii) spurious selection

(iii) time selection

[6]

[Total 20 marks]

QUESTION THREE

(a) (i) Describe three distinct methods of averaging salary that might be defined in the scheme rules of a pension fund [3]

(ii) Define s_x and z_x, in the context of a pension fund. [2]

(iii) In order to value the benefits in a final salary pension scheme as at 1 January 2008, a salary scale, s_x, has been defined so that s_{x+t}/s_x is the ratio of a member's total earnings between ages $x+t$ and $x+t+1$ to the member's total earnings between ages x and $x+1$.

Salary increases take place on 1 July every year. One member, whose date of birth is l April 1961, has an annual salary rate of K75 000 on the valuation date. Write down an expression for the member's expected earnings during 2008. [5]

(b) An amount of K300 000 is to be divided between the survivors of the three males aged 30, 35 and 40 in 20 years time. Find the expected values of the amount received by (40). Use AM92 Ultimate and 4% interest. [10]

[Total 20 marks]

QUESTION FOUR

(a) A population is subject to three modes of decrement, a, b and c, all uniformly distributed over each year of age in a single decrement table.

(i) Starting from the definition

$$(aq)_x^a = \int_0^1 {}_t(ap)_x.(a\mu)_{x+t}^a dt$$

and using the fact that ${}_t(ap)_x = {}_tp_x^a.{}_tp_x^b.{}_tp_x^c$ and $(a\mu)_{x+t}^a = \mu_{x+t}^a$ together with the assumption of uniform distribution of decrements, demonstrate that the probability of a life aged x exiting from an active life is

$$(aq)_x^a = q_x^a \left[1 - \frac{1}{2}\left(q_x^b + q_x^c\right) + \frac{1}{3} q_x^b q_x^c\right]$$

for integer ages x, where q_x^i and $(aq)_x^i$ are the independent and dependent rates of mortality for decrement i, respectively. [4]

(ii) In the particular case that

$$q_{40+t}^a = 0.010 + 0.002t, \qquad q_{40+t}^b = 0.050 + 0.005t, \qquad q^c = 0.002$$

for $t = 0, 1, \cdots, 5$, calculate ${}_{2|}(aq)_{40}^a$ defining the meaning of this quantity.

[6]

(b) The table extract below shows the death statistics in a population due to cancer and other causes.

Age (x)	Population $(al)_x$	Cancer $(ad)_x^c$	Others $(ad)_x^o$
61	10,000	55	248
62	9,697	175	533
63	8,989		

Recent changes have resulted in an estimate that the annual independent cancer death rate is reduced by 20% than that previously used.
Calculate a revised table assuming no changes to the other independent death rates.

(i) Calculate dependent rates $\left[(aq)_x^k\right]$ from the information given. [1]

(ii) Calculate approximate independent rates $\left[q_x^k\right]$ [3]

(iii) Make the necessary adjustment given and calculate approximate dependent rates $\left[(aq)_x^k\right]$ from the adjusted indepndent rates. [4]

(iv) Reconstruct the Revised Table. [2]

[Total 20 marks]

QUESTION FIVE

(a) Explain in general terms, how a life office may become susceptible to adverse selection. [4]

(b) A life insurance company issues a three-year unit-linked endowment assurance contract to a male life aged 45 exact under which level annual premiums of K6, 500 are payable at the start of each year. 90% of the premium is allocated to units. The units are subject to a bid-offer spread of 3% and an annual management charge of 1% of the bid value of the units is deducted at the end of each year before any benefits are paid.

If the policyholder dies during the term of the policy, the death benefit of K20, 000 or the bid value of the units after the deduction of the management charge, whichever is higher, is payable at the end of the year of death. On survival to the end of the term, the bid value of the units plus a bonus of 10% is awarded.

Basis:

- Mortality: AM92 Select
- Initial expenses: K250 incurred at outset
- Renewal expenses: K70 incurred at the start of each of years 2 and 3
- Non-unit fund interest rate: 4% per annum
- Unit fund growth rate: 7% per annum Calculate the profit vector, before allowing for non-unit reserves. [16]

[Total 20 marks]

======END OF MAT4022 EXAMINATION ======

THE UNIVERSITY OF ZAMBIA

SCHOOL OF NATURAL SCIENCES

DEPARTMENT OF MATHEMATICS & STATISTICS

2018/19 Academic Year
End of Year Final Examinations

7 November 2019 (am)

MAT4032 – Financial Engineering

Time allowed : **Three (3) hours** Full marks : **100**

Instructions:
- There are **six (6)** questions in this paper.
- Attempt **any five (5)** questions. **All** questions carry **equal** marks.
- Mark allocations are shown in brackets.
- **Full credit** will only be given when **necessary work** is shown.
- Indicate your **computer number** on all answer booklets used.

Formulae and Tables are provided.

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1. (a) (i) Give four defining characteristics of a Brownian Motion W_t, such that $W_0 = 0$. **(2)**

(ii) Derive the value of a which makes

$$\exp(\sigma W_t - at)$$

a Martingale when W_t is a standard Brownian Motion. **(3)**

(b) Consider a stochastic differential equation

$$\mathrm{d}X_t = Y_t \, \mathrm{d}W_t + A_t \, \mathrm{d}t \,,$$

where A_t is a deterministic process and Y_t is a process adapted to the natural filtration of W_t.

(i) Write down Ito's lemma for $f(t, X_t)$, where f is a suitable function. **(2)**

(ii) Determine $\mathrm{d}f(t, X_t)$ where $f(t, X_t) = e^{4t^2 X_t}$. **(3)**

(c) Let $(X_t; t \geq 0)$ be a stochastic process satisfying

$$X_t = X_0 + \int_0^t \mu_s \, \mathrm{d}s + \int_0^t \sigma_s \, \mathrm{d}W_t \,,$$

where W_t is a standard Brownian motion.

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function, twice partially differentiable with respect to x, once with respect to t.

(i) State the stochastic differential equation for $f(t, X_t)$. **(2)**

(ii) Let $\mathrm{d}X_t = -\gamma(X_t - \mu) \, \mathrm{d}t + \sigma \, \mathrm{d}W_t$.

Prove that the solution of this stochastic differential equation is given by

$$X_t = X_0 e^{-\gamma t} + \mu\big(1 - e^{-\gamma t}\big) + \sigma \int_0^t e^{-\gamma(t-s)} \, \mathrm{d}W_s \,.$$

(8)

2. (a) State how the price at time t of a zero-coupon bond paying K1 at T (denoted by $P(t, T)$) is related to

(i) spot rate curve,

(ii) instantaneous forward rate curve,

(iii) instantaneous risk free rate.

Define all notation used. **(6)**

(b) Let $B(t, T)$ be the price at time t of a zero-coupon bond paying K1 at time T, r_t the short-rate of interest, P the real world probability measure and Q the risk neutral probability measure.

(i) Write down two equations for the price of a zero-coupon bond, one of which uses the risk-neutral approach to pricing and the other of which uses the state-price-deflator approach to pricing. **(2)**

(ii) State the Stochastic Differential Equation (SDE) of the short rate r_t under Q for the Vasicek model and the general type of process this SDE represents. **(3)**

(iii) Solve the SDE for the short rate r_t from part (ii). **(6)**

(iv) Deduce the form of the distribution of the zero-coupon bond price under this model. **(3)**

3. (a) Explain what is meant by an option. **(3)**

(b) Consider a three-period binomial model for a stock with the following parameters:

$$u = 1.2, \quad d = 0.9 \quad \text{and} \quad S_0 = 60.$$

Assume the discretely compounded risk-free rate of interest is $r = 11\%$ per period.

(i) α) Verify that there is no arbitrage in the market. **(2)**

β) Construct the binomial tree. **(2)**

(ii) Calculate the price of a standard European call option with maturity date in three periods and strike price $K = 60$. **(8)**

(c) Consider the problem in part (b). A new "knock-in" option is introduced which has the following characteristics: If the value of the stock crosses the level 80 during the whole life of the option, the contract holder has the right to obtain the difference between the value of the stock at maturity (in three periods) and 60. Calculate the price of this new option **(5)**

4. (a) Describe what is meant by an arbitrage opportunity. **(3)**

(b) (i) List five factors that affect the price of a European put option on a non-dividend paying share. **(3)**

(ii) State how the premium for a European put option would change if each of these factors increased. **(3)**

(c) Discuss whether one-factor models are good models for the short-rate of interest (instantaneous risk free rate). Include discussion of extensions that may be considered to improve the model. Illustrate your discussion by defining and referring to particular models. **(7)**

(d) Consider an asset with price S_t at time, paying a dividend at a constant dividend yield D. Dividends are paid at the end of each year and are immediately reinvested in the asset. The continuously compounded risk-free rate of interest is r pa.

Derive the forward price of a contract issued at time 0 with maturity at time T, to trade one unit of the asset, where T is an integer number of years. State any assumptions you make. **(4)**

5. (a) Explain what is meant by a "risk-neutral probability measure" and state mathematically what it implies about the pricing of derivatives relative to the price of the underlying asset. **(2)**

(b) A non-dividend paying stock has a current price of $S_0 = 140$ and trades in a market which is arbitrage free and has a constant effective risk-free rate of interest r. After one year the price of the stock could increase to 270, or decrease to 110. Over the following year the price could increase from 270 either to 410 or to 312. If the stock price had decreased to 110, then over the following year it could increase to 158 or decrease to 102.

(i) Determine the range of values that the annual risk-free interest rate could take. **(3)**

(ii) Assume that r takes the value 30% per annum. Calculate the price at time 0 of a non-standard derivative which pays off

$$(S_2 - 100)^2$$

at the end of two years. **(7)**

(c) Consider a binomial tree model for the non-dividend paying stock with price S_t.

Assume this price either rises by 20% or falls by 10% each month for the next three months. Assume also that the risk-free rate is 3% per annum continuously compounded.

Let $S_0 = \text{K}100$.

(i) Calculate the price of a vanilla European call option with maturity in three months' time and a strike price of K95. **(4)**

(ii) Calculate the price of a vanilla European put option with the same maturity and strike price as the contract in part (i). **(2)**

(iii) Assume the investor has a portfolio formed by a short position in the call option given in part (i) and a long position in the put option given in part (ii).

Determine how the value of the portfolio would differ if the possible change in the stock price was a fall of 20% instead of 10%. **(2)**

6. (a) List the six assumptions underlying the Black-Scholes model of option prices. **(3)**

(b) The current price of a non-dividend paying stock is K90 and its volatility is 30% per annum. The continuously compounded risk-free interest rate is 3% per annum.

Consider a European call option on this share with strike price K80 and expiry date in three months' time. Assume that the Black-Scholes model applies.

(i) Calculate the price of the call option. **(4)**

(ii) Define algebraically the delta of the call option. **(1)**

(iii) Calculate the value of the delta of the call option. **(1)**

(iv) Calculate the value of the delta of a European put option written on the same underlying, with the same strike and maturity as above. **(1)**

(c) A company's directors have decided to provide senior managers with a performance bonus scheme. The bonus scheme entitles the managers to a cash payment of K100,000 should the company share price have increased by more than 20% at the end of the next 6 months. In addition, the managers will be entitled to 5,000 free shares each, should the share price have increased by more than 10% at the end of the next 6 months. You are given the following data:

Current share price	K78.10
Continuously-compounded risk-free rate	5% pa
Share price volatility	25% pa
No dividends to be paid over the next 6 months.	

(i) By considering the terms of the Black-Scholes call option pricing formula, calculate the value of the bonus scheme to one manager. **(6)**

(ii) Explain two disadvantages of this bonus scheme as an incentive for managers to perform. **(2)**

(iii) Some shareholders are concerned that this scheme might cause an undesirable distortion to the managers' behaviour.

Suggest two modifications to the scheme that will ensure that the managers' aims coincide with the long-term objectives of the shareholders. **(2)**

End of Paper

STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-3.9	.00005	.00005	.00004	.00004	.00004	.00004	.00004	.00004	.00003	.00003
-3.8	.00007	.00007	.00007	.00006	.00006	.00006	.00006	.00005	.00005	.00005
-3.7	.00011	.00010	.00010	.00010	.00009	.00009	.00008	.00008	.00008	.00008
-3.6	.00016	.00015	.00015	.00014	.00014	.00013	.00013	.00012	.00012	.00011
-3.5	.00023	.00022	.00022	.00021	.00020	.00019	.00019	.00018	.00017	.00017
-3.4	.00034	.00032	.00031	.00030	.00029	.00028	.00027	.00026	.00025	.00024
-3.3	.00048	.00047	.00045	.00043	.00042	.00040	.00039	.00038	.00036	.00035
-3.2	.00069	.00066	.00064	.00062	.00060	.00058	.00056	.00054	.00052	.00050
-3.1	.00097	.00094	.00090	.00087	.00084	.00082	.00079	.00076	.00074	.00071
-3.0	.00135	.00131	.00126	.00122	.00118	.00114	.00111	.00107	.00104	.00100
-2.9	.00187	.00181	.00175	.00169	.00164	.00159	.00154	.00149	.00144	.00139
-2.8	.00256	.00248	.00240	.00233	.00226	.00219	.00212	.00205	.00199	.00193
-2.7	.00347	.00336	.00326	.00317	.00307	.00298	.00289	.00280	.00272	.00264
-2.6	.00466	.00453	.00440	.00427	.00415	.00402	.00391	.00379	.00368	.00357
-2.5	.00621	.00604	.00587	.00570	.00554	.00539	.00523	.00508	.00494	.00480
-2.4	.00820	.00798	.00776	.00755	.00734	.00714	.00695	.00676	.00657	.00639
-2.3	.01072	.01044	.01017	.00990	.00964	.00939	.00914	.00889	.00866	.00842
-2.2	.01390	.01355	.01321	.01287	.01255	.01222	.01191	.01160	.01130	.01101
-2.1	.01786	.01743	.01700	.01659	.01618	.01578	.01539	.01500	.01463	.01426
-2.0	.02275	.02222	.02169	.02118	.02068	.02018	.01970	.01923	.01876	.01831
-1.9	.02872	.02807	.02743	.02680	.02619	.02559	.02500	.02442	.02385	.02330
-1.8	.03593	.03515	.03438	.03362	.03288	.03216	.03144	.03074	.03005	.02938
-1.7	.04457	.04363	.04272	.04182	.04093	.04006	.03920	.03836	.03754	.03673
-1.6	.05480	.05370	.05262	.05155	.05050	.04947	.04846	.04746	.04648	.04551
-1.5	.06681	.06552	.06426	.06301	.06178	.06057	.05938	.05821	.05705	.05592
-1.4	.08076	.07927	.07780	.07636	.07493	.07353	.07215	.07078	.06944	.06811
-1.3	.09680	.09510	.09342	.09176	.09012	.08851	.08691	.08534	.08379	.08226
-1.2	.11507	.11314	.11123	.10935	.10749	.10565	.10383	.10204	.10027	.09853
-1.1	.13567	.13350	.13136	.12924	.12714	.12507	.12302	.12100	.11900	.11702
-1.0	.15866	.15625	.15386	.15151	.14917	.14686	.14457	.14231	.14007	.13786
-0.9	.18406	.18141	.17879	.17619	.17361	.17106	.16853	.16602	.16354	.16109
-0.8	.21186	.20897	.20611	.20327	.20045	.19766	.19489	.19215	.18943	.18673
-0.7	.24196	.23885	.23576	.23270	.22965	.22663	.22363	.22065	.21770	.21476
-0.6	.27425	.27093	.26763	.26435	.26109	.25785	.25463	.25143	.24825	.24510
-0.5	.30854	.30503	.30153	.29806	.29460	.29116	.28774	.28434	.28096	.27760
-0.4	.34458	.34090	.33724	.33360	.32997	.32636	.32276	.31918	.31561	.31207
-0.3	.38209	.37828	.37448	.37070	.36693	.36317	.35942	.35569	.35197	.34827
-0.2	.42074	.41683	.41294	.40905	.40517	.40129	.39743	.39358	.38974	.38591
-0.1	.46017	.45620	.45224	.44828	.44433	.44038	.43644	.43251	.42858	.42465
-0.0	.50000	.49601	.49202	.48803	.48405	.48006	.47608	.47210	.46812	.46414

STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
0.1	.53983	.54380	.54776	.55172	.55567	.55962	.56356	.56749	.57142	.57535
0.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
0.3	.61791	.62172	.62552	.62930	.63307	.63683	.64058	.64431	.64803	.65173
0.4	.65542	.65910	.66276	.66640	.67003	.67364	.67724	.68082	.68439	.68793
0.5	.69146	.69497	.69847	.70194	.70540	.70884	.71226	.71566	.71904	.72240
0.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.75490
0.7	.75804	.76115	.76424	.76730	.77035	.77337	.77637	.77935	.78230	.78524
0.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
0.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891
1.0	.84134	.84375	.84614	.84849	.85083	.85314	.85543	.85769	.85993	.86214
1.1	.86433	.86650	.86864	.87076	.87286	.87493	.87698	.87900	.88100	.88298
1.2	.88493	.88686	.88877	.89065	.89251	.89435	.89617	.89796	.89973	.90147
1.3	.90320	.90490	.90658	.90824	.90988	.91149	.91309	.91466	.91621	.91774
1.4	.91924	.92073	.92220	.92364	.92507	.92647	.92785	.92922	.93056	.93189
1.5	.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
1.6	.94520	.94630	.94738	.94845	.94950	.95053	.95154	.95254	.95352	.95449
1.7	.95543	.95637	.95728	.95818	.95907	.95994	.96080	.96164	.96246	.96327
1.8	.96407	.96485	.96562	.96638	.96712	.96784	.96856	.96926	.96995	.97062
1.9	.97128	.97193	.97257	.97320	.97381	.97441	.97500	.97558	.97615	.97670
2.0	.97725	.97778	.97831	.97882	.97932	.97982	.98030	.98077	.98124	.98169
2.1	.98214	.98257	.98300	.98341	.98382	.98422	.98461	.98500	.98537	.98574
2.2	.98610	.98645	.98679	.98713	.98745	.98778	.98809	.98840	.98870	.98899
2.3	.98928	.98956	.98983	.99010	.99036	.99061	.99086	.99111	.99134	.99158
2.4	.99180	.99202	.99224	.99245	.99266	.99286	.99305	.99324	.99343	.99361
2.5	.99379	.99396	.99413	.99430	.99446	.99461	.99477	.99492	.99506	.99520
2.6	.99534	.99547	.99560	.99573	.99585	.99598	.99609	.99621	.99632	.99643
2.7	.99653	.99664	.99674	.99683	.99693	.99702	.99711	.99720	.99728	.99736
2.8	.99744	.99752	.99760	.99767	.99774	.99781	.99788	.99795	.99801	.99807
2.9	.99813	.99819	.99825	.99831	.99836	.99841	.99846	.99851	.99856	.99861
3.0	.99865	.99869	.99874	.99878	.99882	.99886	.99889	.99893	.99896	.99900
3.1	.99903	.99906	.99910	.99913	.99916	.99918	.99921	.99924	.99926	.99929
3.2	.99931	.99934	.99936	.99938	.99940	.99942	.99944	.99946	.99948	.99950
3.3	.99952	.99953	.99955	.99957	.99958	.99960	.99961	.99962	.99964	.99965
3.4	.99966	.99968	.99969	.99970	.99971	.99972	.99973	.99974	.99975	.99976
3.5	.99977	.99978	.99978	.99979	.99980	.99981	.99981	.99982	.99983	.99983
3.6	.99984	.99985	.99985	.99986	.99986	.99987	.99987	.99988	.99988	.99989
3.7	.99989	.99990	.99990	.99990	.99991	.99991	.99992	.99992	.99992	.99992
3.8	.99993	.99993	.99993	.99994	.99994	.99994	.99994	.99995	.99995	.99995
3.9	.99995	.99995	.99996	.99996	.99996	.99996	.99996	.99996	.99997	.99997

The University of Zambia
School of Natural Sciences
Department of Mathematics & Statistics

2018/19 Academic Year
End of Year Final Examinations
MAT4119 - ENGINEERING MATHEMATICS III

TOTAL TIME ALLOWED: THREE (3) HOURS

INSTRUCTIONS ANSWER ANY FIVE (5) QUESTIONS.

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1. a. (i) Find the fifth Taylor polynomial, $P_5(x)$, of

$$f(x) = \sinh^{-1} x$$

about $x = 0$.

(ii) Find the value of x that can be used in $P_5(x)$ to estimate the value of $\ln 2$.

Hence, use four- digit rounding arithmetic and $P_5(x)$ to estimate $\ln 2$.

(iii) Find the upper bound of the error in part (ii). **(10 Marks)**

b. Use the quadratic formula that will give the most accurate approximation of the roots of

$$1.002x^2 - 11.01x + 0.01265 = 0$$

using four-digit rounding arithmetic. **(4 Marks)**

c. (i) Show that $p = 1.12$ is a root of multiplicity 2 of the equation

$$f(x) = x^4 - 2.24x^3 + 2.2544x^2 - 2.24x + 1.2544 = 0$$

(ii) Use modified Newton-Raphson method to approximate this root to 5 significant figures starting with $p_0 = 1$. **(6 Marks)**

2. a. Show graphically that the system of non- linear equations

$$x^2 + y = 11$$
$$y^2 + x = 7$$

has four possible solutions. Starting with $X^{(0)} = (3, 2.5)^t$, use Newton's method to approximate the solution to the system until

$$\frac{\left\| X^{(k)} - X^{(k-1)} \right\|_\infty}{\left\| X^{(k)} \right\|_\infty} < 5\times10^{-1}.$$ **(8 Marks)**

b. The following linear equations represent a system of pulleys where t_1 and t_2 (measured in pounds) are tensions in the ropes and a (measured in feet per second squared) is the acceleration:

$$t_1 - 2t_2 = 0$$
$$t_1 - 2a = 128$$
$$t_2 + a = 32$$

Find the solution to this system. **(5 Marks)**

c. The following system of linear equations represents the distribution of current flow in a circuit:

$$I_1 + I_2 - I_3 = 0$$
$$2I_1 - 4I_2 = 4$$
$$2I_1 + 5I_3 = 6,$$

where I_i, for each $i = 1,2,3,$ is current in amperes. Starting with $I^{(0)} = (1,0,1)'$, perform four iterations of the Gauss - Seidel method to approximate the solution to the system. **(7 Marks)**

3 a. The following table gives the current density (D) in amperes per square millimeter (A/mm^2) of insulated copper conductors for selected cross sectional area (A) in square millimeters.

A	2.0	3.0	5.0	5.5
D	0.015	0.009	0.006	0.0055

Estimate D for cross sectional area of 3.8 mm^2. **(6 Marks)**

b. Given the initial value problem $\frac{dy}{dx} = x + y^2, y(0) = 1$

Estimate $y\,(0.1)$ using the following Runge Kutta equations of order 4 and step size h= 0.1.

$$k_1 = hf(t_i, w_i),\ k_2 = hf\left(t_i + \frac{h}{2}, w_i + \frac{k_1}{2}\right),\ k_3 = hf\left(t_i + \frac{h}{2}, w_i + \frac{k_2}{2}\right),$$

$$k_4 = hf(t_{i+1}, w_i + k_3),\quad w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

(8 Marks)

c. Estimate the value of the integral $\int_0^1 \frac{x}{x^3+10}dx$ using the following Simpson's rule with eight subintervals :

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}[y_0 + 4(y_1 + \cdots + y_{n-1}) + 2(y_2 + \cdots + y_{n-2}) + y_n)]$$

(6 Marks)

4 a. Consider the integral $\int_{0.1}^{0.2} \frac{x^2}{cosx}\,dx$

Find the largest value of h if the above integral is approximated by composite Trapezoidal rule within an error of 10^{-6}.

The error in composite Trapezoidal rule : $E = -(b-a)\frac{h^2}{12}f''(\xi)$. **(6 Marks)**

b. The specific enthalpy (h) is an important element in thermodynamic processes. It is a function of temperature T. The following data was recorded from an experiment:

$T(°F)$	800	1000	1200	1400	1600
$h(Btu/lb)$	1305	1400	1585	1705	1825

(i) Estimate h at $T = 1100°F$ using Newton's forward difference polynomial.

(ii) For a process in which the pressure is constant, the specific heat capacity (C_p) equals the slope of the specific enthalpy (h) i.e $C_p = \frac{dh}{dT}$.

Estimate C_p $at\ T = 800°F$ using the polynomial derived in part (i). **(7 Marks)**

c. (i) Compute $f''(0.6)$ from the following table using the formula

$$f''(x_0) = \frac{f(x_0+h)-2f(x_0)+f(x_0-h)}{h^2} \quad with\ h = 0.4\ and\ 0.2.$$

x	0.2	0.4	0.5	0.6	0.7	0.8	1.0
f(x)	1.42	1.88	2.13	2.39	2.66	2.94	3.56

(ii) Given that the error involved in approximating second derivative by the above formula is of o(h^2), apply Richardson's extrapolation scheme to approximate $f''(0.6)$ with error o(h^4).

Richardson's extrapolation scheme : Suppose M is approximated by N(h) with error o(h^2), then improved estimate of M with error o(h^4) can be found from

$$M_{\text{improved}} = \left(\frac{4N\left(\frac{h}{2}\right)-N(h)}{3}\right).$$

(7 Marks)

5. a. Let Z be a complex variable. Given the function $f(z) = e^{(1+i)Z}$, apply Cauchy-Riemann equations to show that:

(i) $f(z)$ is analytic

(ii) $f'(z) = (1+i)e^{(1+i)z}$

(6 Marks)

b. (i) Find z such that $e^Z = 1$.

(ii) Find z such that $sin\ z=0$.

(8 Marks)

c. Evaluate the following integrals:

(i) $\int_c e^z \, dz$ where C is the line joining points (-1,1) and (1,1)

(ii) $\oint_C e^Z dz$, where C is a circle of radius 5 units centered at the point $z_0 = 2 - i$.

(6 Marks)

6. a. Evaluate $\oint_C \frac{z}{z^2-1} dz$ where C is a circle described as

(i) $|z - 1| = 1$
(ii) $|z + 1| = 1$
(iii) $|z - i| = 1$ **(9 Marks)**

b. Express *sin(2-i)* in the form *a+ib, a and b* are reals. **(4 Marks)**

c. Find the principal value of:
(i) $\ln(1 - i)$
(ii) $(1 + i)^{(-1+i)}$ **(7 Marks)**

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The University of Zambia
School of Natural Sciences
Department of Mathematics & Statistics

2018/19 Academic Year Final Examinations

MAT4300 – Elements of Functional Analysis

8th November, 2019

INSTRUCTIONS: (1) Write down your **Computer number** on each answer booklet used.

(2) There are six questions in this paper. **Answer any Five (5)** only.

(3) Write down the **question number** for each question attempted in the first column on the right side on the cover of the main answer booklet.

TIME ALLOWED: Three (3) hours.

Q1. (a) (i) Let X be a nonempty set. Define a metric on X. *(2 marks)*

(ii) Let (X, d) be a metric space. Define a closed sphere (ball) in X of radius r centered at a point $x_0 \in X$. *(2 marks)*

(b) Prove that the function defined by $d(x,y) = \max_{t\in[a,b]} |x(t) - y(t)|$ on the space $C[a,b]$ of all continuous functions on $[a,b]$ is a metric. *(8 marks)*

(c) Prove that in any metric space X, each closed sphere is a closed set. *(8 marks)*

Q2. (a) Give the definition of each of the following:

(i) Baire space *(2 marks)*

(ii) Compact subset K of a metric space (X, ρ). *(2 marks)*

(b) (i) State (without proof) the Cantor Intersection theorem. *(2 marks)*

(ii) Use the theorem in (i) to prove that a complete metric space is a Baire space. *(7 marks)*

(c) Let f be a continuous mapping from a compact metric space X to a metric space Y. Prove that the image $f[X] \subset Y$ is also compact. *(7 marks)*

Q3. (a) (i) Define a contraction mapping on a metric space X. *(2 marks)*

(ii) When is a metric space said to be complete? *(2 marks)*

(b) Show that if $T : X \to X$ is defined as $Tx = x^2$ where $X = [0, \frac{1}{3}]$, with the usual metric on $\mathbb{R}$, then T is a Contraction on $[0, \frac{1}{3}]$. *(3 marks)*

(c) Let (X, ρ) be a complete metric space and the mapping $T : X \to X$ be a contraction. Prove that T has exactly one fixed point. *(13 marks)*

Q4. (a) (i) Let $1 \leq p < \infty$. If $x = \{x_n\}_{n=1}^{\infty}$ and $x \in l^p$, define l^p. *(2 marks)*

(ii) Give the definition of a bounded linear transformation. *(2 marks)*

(b) (i) Show that $\left\{\frac{1}{\sqrt{n}}\right\}_1^{\infty} \notin l^2$. *(2 marks)*

(ii) Prove that $\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}$ is actually a norm in l^p space. *(4 marks)*

(c) (i) Prove that a linear transformation $T : X \to Y$ is continuous at the origin if and only if and only if there exists a real number such that $\|Tx\| \leq k\|x\|$, for every $x \in X$. *(4 marks)*

(ii) State and prove the Hahn-Banach theorem for normed linear spaces. *(6 marks)*

Q5. (a) (i) When is a normed linear space said to be a Banach space? *(2 marks)*

(ii) What is meant by a dual or conjugate space? *(2 marks)*

(b) Let $X = C[0,1]$ and $\boldsymbol{F} = \mathbb{R}$. Define $\Lambda_x(f) = f(x)$. Prove that $\Lambda_x \in X^*$. *(6 marks)*

(c) Let X and Y be normed linear spaces. Prove that if Y is a Banach space, then the set $\mathfrak{B}(X,Y)$ of bounded linear transformations $T : X \to Y$ is itself a Banach space. *(10 marks)*

Q6. (a) (i) Give the definition of a Hilbert space. *(1 marks)*

(ii) When is an orthonormal set in an inner product space said to be complete? *(2 marks)*

(b) (i) Prove that if M is a closed linear subspace of a Hilbert space $\mathcal{H}$ then $\mathcal{H} = M \oplus M^{\perp}$. *(4 marks)*

(ii) Let $\mathcal{H}$ be a Hilbert space and $\Lambda \in \mathcal{H}^*$. Prove that there exists a unique $y \in \mathcal{H}$ such that $\Lambda x = (x, y), \forall x \in \mathcal{H}$. *(5 marks)*

(c) (i) Let X be an inner product space and S an orthonormal set in X. Prove that there exists a complete orthonormal set S_0 in X such that $S \subset S_0$. *(3 marks)*

(ii) State and prove the Parseval's relation for Hilbert spaces. *(5 marks)*

END OF EXAMINATION

The University of Zambia

School of Natural Sciences

Department of Mathematics and Statistics

2018/19 Academic Year Final Examinations

MAT4615 - Nonparametric Statistics

Time allowed : 3 Hours

Instructions:

- There are **five (6)** questions in this examination paper. Attempt **any four (5)** questions. **All** questions carry **equal** marks.
- Indicate your **Computer Number** on all your answer booklets.
- You are required to show all **necessary** steps in your solutions.

*This paper consists of **4** pages of questions.*

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1. (a) (i) What is nonparametric statistics?
   (ii) State two advantages and two disadvantages of using nonparametric tests.
   (iii) State the four measurement scales and briefly explain about each of them.

   (b) Let $(X_i, Y_i), i = 1, 2, \ldots, n$ be a random sample, representing n paired cases. Assume $X_i \neq Y_i$ for $i = 1, 2, \ldots, n$. Let d_i denote the difference $X_i - Y_i$. Define a random variable $U_i, i = 1, 2, \ldots, n$ as $U_i = 1$ if $d_i > 0$ and $U_i = 0$ if $d_i < 0$. Let $T = \sum_{i=1}^{n} R_i U_i$, where R_i is the rank associated with $|d_i|$.

   (i) Find $E(T)$,
   (ii) Find the variance of T.

2. (a) Samples of cream from each of 10 dairies (A to J) are each divided into two portions. One portion from each is sent to Laboratory I, the other to Laboratory II, for bacterium counts. The counts (thousands bacteria m/l) are:

Diary	A	B	C	D	E	F	G	H	I	J
Lab I	11.7	12.1	13.3	15.1	15.9	15.3	11.9	16.2	15.1	13.6
Lab II	10.9	11.9	13.4	15.4	14.8	14.8	12.3	15.0	14.2	13.1

   Use the Wilcoxon signed-rank test to assess the evidence for any consistent difference between laboratories for subsamples from the same dairy. Use $\alpha = 0.05$ level of significance.

(b) A physical education teacher conducted an action research project to examine a strength and conditioning program. Using 12 male participants, she measures the number of curl ups they could do in 1 min. She measured their performance before the programs. Then, she measured their performance at 1 month intervals. The table below presents the performance results.

Table 1: Number of curl ups in one minute

Participants	**Baseline**	**Month 1**	**Month 2**
1	66	67	69
2	49	50	56
3	51	52	49
4	65	65	69
5	42	43	46
6	38	39	40
7	33	31	39
8	41	41	44
9	46	47	48
10	45	46	46
11	36	33	34
12	51	55	67

i. Use a Friedman test with $\alpha = 0.05$ to determine if one or more of the groups are significantly different.

ii. Which ones are different?

3. (a) Life expectancy showed a general tendency to increase during the nineteenth and twentieth centuries as standards of health care and hygiene improved. The table below gives the year of death and age at death for 13 males.

Year	1827	1884	1895	1908	1914	1918	1924	1928	1936
Age	13	83	34	1	11	16	68	13	77
Year	1941	1964	1965	1977					
Age	74	87	65	83					

(i) Calculate Spearman's ρ.

(ii) Is there an indication that life expectancy is increasing for this clan in more recent years? Use Spearman's ρ at $\alpha = 0.05$.

(iii) Calculate Kendall's τ.

(b) The data below shows assessment scores of two different classes who are being taught computer skills using two different methods.

Method 1	53	41	17	45	44	12	49	50
Method 2	91	18	14	21	23	99	16	10

Use Kolmogorov - Smirnov two-sample test to determine if there's a difference in methods of teaching computer skills. Use $\alpha = 0.05$.

4. (a) Lubischew (1962) gives measurements of maximum head width in units of 0.01 mm for three species of Chaetocnema. Part of his data is given below.

Species 1	53	50	52	50	49	47	54	51	52	57	
Species 2	49	49	47	54	43	51	49	51	50	46	49
Species 3	58	51	45	53	49	51	50	51			

Use a Kruskal - Wallis test to see if there is a species difference in head widths at 5% level of significance.

(b) Maxson (1977) studied the activity patterns of female birds. Using surveillance techniques he recorded the movements of seven female birds over a fixed period of time in hours. The percentage of time that these birds were in active movement are recorded : 12.8 12.9 13.3 13.4 13.7 13.8 14.5.
Test if it is okay to assume that these data come from a normal distribution with mean 14 and standard deviation 2 at 5% level of significance.

5. (a) (i) State any two advantages of the sign test over other tests.

(ii) On the day of the second round of the Open Golf Championship in 1987 before play started a television commentator said that conditions were such that the average scores of players were likely to be higher than those for the first round. For a random sample of 10 of the 77 players participating in both rounds the scores were:

Player	A	B	C	D	E	F	G	H	I	J
Round 1	73	73	74	66	71	73	68	72	73	72
Round 2	72	79	79	77	83	78	70	78	78	77

Use the sign test at 0.01 significance level to verify the commentator's claim.

(b) (i) A library has on its shelves 114 books on statistics. I take a random sample of 12 and want to test the hypothesis that the median number of pages, in all 114 books is 225. The numbers of pages in the sample of 12 books were:
126 142 156 228 245 246 370 419 433 454 478 503
At 0.05 significance level, test the hypothesis.

(ii) The measure of correlation as given by Spearman (1904) is usually designated by ρ(rho) and defined as

$$\rho = \frac{\sum_{i=1}^{n} R(X_i)R(Y_i) - n\left(\frac{n+1}{2}\right)^2}{\left(\sum_{i=1}^{n} R(X_i)^2 - n\left(\frac{n+1}{2}\right)^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n} R(Y_i)^2 - n\left(\frac{n+1}{2}\right)^2\right)^{\frac{1}{2}}}.$$

If there are no ties, show that

$$\rho = 1 - \frac{6\sum_{i=1}^{n}[R(X_i) - R(Y_i)]^2}{n(n^2-1)}.$$

6. (a) (i) State two advantages of using a binomial test.

(ii) It is known that at least 0.25 of a certain species of insect exhibit a particular characteristic A. Eighteen insects of that species are obtained from an unusual environment, and five of these have characteristic A. Test at 5% level of significance if it is reasonable to assume that insects from that environment have same probability of 0.25 of having characteristic A.

(iii) Lindsey, Herzberg and Watts (1987) give data for widths of first joint of the second tarsus for two species of the insect Chaetocnema. Do these indicate population differences between the width distributions for the two species?

Species A	131	134	137	127	128	118	134	129	131	115
Species B	107	122	144	131	108	118	122	127	125	124

Perform a Mann-Whitney hypothesis test at 5% level of significance.

(b) Let X and Y be independent random variables with distribution functions $F(x)$ and $G(y)$ respectively. Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_m$ be random samples from these distributions. Define

$$Z_{ij} = \begin{cases} 1, & \text{if } X_i < Y_j \\ 0, & \text{if } X_i > Y_j. \end{cases}$$

Find $E[U]$, for the statistic

$$U = \sum_{i=1}^{n} \sum_{j=1}^{m} Z_{ij},$$

assuming $X_i \neq Y_i$.

END OF EXAM

THE UNIVERSITY OF ZAMBIA

SCHOOL OF NATURAL SCIENCES

2019 ACADEMIC YEAR FINAL EXAMINATIONS

MAT 4622: CATEGORICAL DATA ANALYSIS

INSTRUCTIONS:

1. Answer any FIVE (5) questions.
2. Calculators are allowed.
3. You may use statistical tables or formulae provided if necessary.
4. Show all your work to earn full marks.

TIME: THREE (3) Hours

Question 1

A random sample of students is asked their opinion on a proposed core curriculum change. The results are presented below.

Class	**Opinion** Favouring	Opposing	Total
First year	120	80	200
Second year	70	130	200
Third year	60	70	130
Fourth year	40	60	100
Total	290	340	630

(a) In bullet format, list possible statistics you could use to analyse the association between *opinion* and *class grouping*.

(b) Test the hypothesis that the opinions are independent of the class grouping, at 5% level of significance using the G^2 statistic, i.e., the likelihood test.

Question 2

(a) (i) Briefly discuss the difference between nominal and ordinal variables.

(ii) Give reasons, in bullet form, why classifying variables correctly is important.

(b) A news publisher, trying to pinpoint his market's characteristics, wondered whether newspaper readership in the community is related to readers' educational achievement. A survey questioned adults in the area on the level of education and their frequency of readership. The results are shown in the following table.

Question 4

(a) Assume X has two categories 1 and 2, and the response variable Y also has two categories 1 and 2. The conditional distribution of Y given X is displayed in Table 1 (a) and Table 1 (b) shows the cross classification of X and Y for a sample size of n, when data are available.

		Y: 1	Y: 2	
X	1	$\pi_{1/1}$	$\pi_{2/1}$	1
	2	$\pi_{1/2}$	$\pi_{1/2}$	1

Table 1 (a)

		Y: 1	Y: 2	
X	1	n_{11}	n_{12}	n_{1+}
	2	n_{21}	n_{22}	n_{2+}

Table 1 (b).

Note: in Table 1 (a) $\Pr(Y = 1/X = 1) = \pi_{1/1}$ and $\Pr(Y = 1/X = 2) = \pi_{1/2}$

(i) If $n_{11} \sim Bin(n_{1+}, \pi_{1/1})$ and $n_{21} \sim Bin(n_{2+}, \pi_{1/2})$ and the two binomial distributions are independent, write down the joint distribution of (n_{11}, n_{21}).

(ii) If $\pi_{1/1} = \pi_{1/2} = \pi$, show that the joint distribution in (i) simplifies to $\binom{n_{1+}}{n_{11}}\binom{n_{2+}}{n_{21}}\pi^{n_{+1}}(1-\pi)^{n-n_{+1}}$, where $n_{+1} = n_{11} + n_{21}$

(iii) Find the maximum likelihood estimator of π in (ii).

(b) A food security program was carried out in Chipata in which 104 farmers were sponsored to practice conservation agriculture system while 405 were sponsored to carry out conventional agriculture system. At the end of the farming season the farmers were asked whether they felt food secure. The results of the study are shown below.

Farming type	Are you food secure? YES	NO	Total
Conservation	60	44	104
Conventional	186	219	405
Total	246	263	509

We would like to test the null hypothesis that the proportion of farmers feeling food secure is the same, i.e., $\pi_{1/1} = \pi_{1/2}$, using the Chi-square test given by:

$$\chi^2_{observed} = \sum_{i=1}^{2}\sum_{j=1}^{2}\frac{(n_{ij}-e_{ij})^2}{e_{ij}}.$$

(i) Briefly state why this test is not a test of independence.

(ii) Carry out the test at 5% level of significance.

Question 5

(a) Assume X has I categories and Y has J categories. A cross-classification of X and Y yields a table having IJ cells. If P(X= i, Y= j) = π_{ij}

(i) Briefly discuss what is meant by a saturated model.

(ii) Briefly discuss what is meant by an independence model.

(iii) Determine the difference in the degrees of freedom for the two models?

(b) Grades in a statistics course and an economics course taken during the same examination period for a group of students were as follows.

Statistics grade	Economics grade A	B	C	Other	
A	25	6	17	13	61
B	17	16	15	6	54
C	18	4	18	10	50
Other	10	8	11	20	49
	70	34	61	49	214

Carry out a test at 5% level of significance to determine whether grades in statistics and economics are independent using the standard or alternative formula.

Question 6

(a) The Gini's concentration measure of variation for a nominal response variable Y with J categories is defined by: $\text{Var(Y)} = 1 - \sum_{j=1}^{J} \pi_{+j}^2$

(i) State a constraint imposed on π_{+j}s.

(ii) By taking into consideration the constraint in (i) show that Var(Y) attaints its maximum value of $\frac{J-1}{J}$ when $\pi_{+j} = \frac{1}{J}$

(iii) What value does Var(Y) attain if $\pi_{+j} = 1$ for some j.

(b) An educator has the opinion that the grades high school students make are dependent on the amount of time they spend listening to music. A questionnaire was administered to a survey of 400 students with the following two questions: "How many hours per week do you listen to music?" and "What is the average grade for all your classes?" The data from the survey and the proportions are given in the tables below.

Raw data

HOURS SPENT LISTENING TO MUSIC	AVERAGE GRADE A	B	C	D	F	
Up to 10 hours	33	37	38	35	7	150
More than 10 hours	17	38	112	40	43	250
	50	75	150	75	50	400

Data in proportional form

HOURS SPENT LISTENING TO MUSIC	A	B	C	D	F	
	AVERAGE GRADE					
Up to 10 hours	0.08	0.09	0.09	0.09	0.02	0.37
More than 10 hours	0.04	0.10	0.28	0.10	0.11	0.63
	0.12	0.19	0.37	0.19	0.13	1

(i) Calculate the sample version of Gini's concentration given by:

$$\hat{\tau} = \frac{\sum_{i=1}^{I} \sum_{j=1}^{J} \frac{p_{ij}^2}{p_{i+}} - \sum_{j=1}^{J} p_{+j}^2}{1 - \sum_{j=1}^{J} p_{+j}^2}$$

(ii) Give an interpretation of the result in (i).

END OF EXAMINATION

THE UNIVERSITY OF ZAMBIA

SCHOOL OF NATURAL SCIENCES

2018/2019 ACADEMIC YEAR FINAL EXAMINATIONS

GES 3241: CLIMATOLOGY

TIME: Three Hours

INSTRUCTIONS: Answer question **1** and any other **two** questions. All questions carry equal marks. Candidates are encouraged to make use of illustrations wherever appropriate.

---

1. Write short explanatory notes on **ALL** of the following:

   (a) Latitudinal Variations (4 Marks)

   (b) Stratospheric ozone (4 Marks)

   (c) Colioris force (4 Marks)

   (d) Forms of heat transfer in the earth-atmosphere system (4 Marks)

   (e) Climate change mitigation (4 Marks)

2. Using specific examples, explain the classification and naming of air masses (20 Marks)

3. Explain the processes of exchange in the earth-atmosphere system during the day and during the night (20 Marks).

4. 'Climate change is influenced by both natural and anthropogenic factors'. Elucidate. (20 Marks)

5. With the aid of a diagram, describe the fundamental layers of the atmosphere and state their importance. (20 Marks)

END OF EXAMINATION

THE UNIVERSITY OF ZAMBIA

SCHOOL OF NATURAL SCIENCES

2018/19 ACADEMIC YEAR FINAL EXAMINATIONS

GES 3262: BIOGEOGRAPHY

TIME: **Three** hours

INSTRUCTIONS: Answer **any three** questions. All question carry equal marks. Candidates are encouraged to make use of illustrations wherever appropriate.

---

1. Write short explanatory notes on **ALL** of the following:
    - (a) Merits and demerits of the species concept
    - (b) Convergent versus divergent evolution
    - (c) Industrial melanism
    - (d) Fossil succession
    - (e) Importance of the quaternary period in biogeography.
2. Using an illustration, explain how an organism responds to changes in the environment.
3. Why are island flora and fauna vulnerable to invasion by non native species? Explain in terms of interspecific competition and competitive exclusion.
4. Outline the details of any five pieces of evidence that have been used as proof of past ice ages.
5. Burney and Robinson (2003) proposed the synergistic scenario. Describe the synergistic scenario and any three pieces of evidence used by these authors in support of their theory for the extinction of some species in Madagascar.

END OF EXAMINATION

THE UNIVERSITY OF ZAMBIA

SCHOOL OF NATURAL SCIENCES

2018/2019 ACADEMIC YEAR FINAL EXAMINATIONS

GES 3330: ENVIRONMENT AND DEVELOPMENT

TIME: Three Hours

INSTRUCTIONS: Answer one question from each section. All questions carry equal marks. Candidates are encouraged to use illustrations wherever appropriate.

Section A

1. On the 3rd of March 2017, Zambia launched a National Climate Change Policy aimed at lessening the impact of climate change in the country. Discuss how climate change is impacting Zambia's agricultural, water and health sectors addressed in this policy.

2. With reference to Zambia, discuss how sustainable development can be achieved.

Section B

3. 'Human development embodies a history of continuous redefinition to the human-environment relationship'. Elucidate.

4. 'The shift from modern to alternative agriculture is a mere return to traditional agricultural and indigenous knowledge practices'. Discuss.

Section C

5. 'Economic growth is good for ecosystems'. Discuss.
6. Write explanatory notes on all the following;
   (a) Gini Coefficient
   (b) Externalities and the environment
   (c) Pathways for invasive alien species
   (d) Resource curse hypothesis

END OF EXAMINATION

THE UNIVERSITY OF ZAMBIA

SCHOOL OF NATURAL SCIENCES

2018/2019 ACADEMIC YEAR FINAL EXAMINATIONS

GES 3352: SPATIAL DIMENSIONS OF DISEASE AND HEALTH CARE SERVICE PROVISION

TIME: Three Hours

INSTRUCTIONS: Answer questions **1** and any other two. Question 1 carries 40 marks and the rest of the questions carry 30 marks each. Candidates are encouraged to use illustrations wherever appropriate.

---

1. Write short explanatory notes on ALL of the following:

   a) Diffusion of disease.

   b) Disease surveillance

   c) Water borne diseases

   d) Market-oriented health care systems

   e) Development and disease

   f) Nutrition and health

   g) Geographical epidemiology.

   h) Disease prevention mechanism

2. Explain factors that influence the use of traditional medicine.

3. Explain reasons for inequalities in access to health care services in Zambia.

4. Outline Nutrition Transition and explain its implications in low and medium income countries.

5. Explain the process of disease mapping and its importance.

END OF EXAMINATION

THE UNIVERSITY OF ZAMBIA

SCHOOL OF NATURAL SCIENCES

2018/2019 ACADEMIC YEAR FINAL EXAMINATIONS

GES 4172: RURAL LAND USE AND LANDSCAPES

TIME: Three Hours

INSTRUCTIONS: Answer question **1** and any other **two** questions. All questions carry equal marks. Candidates are encouraged to use illustrations wherever appropriate.

---

1. Write short explanatory notes on ALL of the following:

   a) Vulnerability versus Resilience
   b) Gender and Natural Resources Management
   c) Direct-use and non-use values of Community Based Natural Resources Management (CBNRM) in livelihoods
   d) Dimensions of food security
   e) Role of rural industrialization in poverty reduction.

2. 'Responses to the impacts of climate change in Africa cannot be left to be the responsibility of small-scale rural producers alone'. Explain.
3. Discuss the four factors of production in relation to rural livelihoods in Zambia.
4. Discuss the three basic foundations on which Participatory Rural Appraisal (PRA) is based.
5. 'Rural landscapes have temporal and spatial heterogeneity'. Elucidate.

END OF EXAMINATION

THE UNIVERSITY OF ZAMBIA

SCHOOL OF NATURAL SCIENCES

2018/19 ACADEMIC YEAR EXAMINATION

GES 4192 : GEOGRAPHICAL DIMENSION OF SETTLEMENTS

TIME: Three hours

INSTRUCTIONS: Answer question 1 and any other two. All questions carry equal marks.

Candidates are encouraged to make use of illustrations wherever possible.

---

1. Write short explanatory notes on ALL of the following:
    a) Compact type rural settlement
    b) The concept of urban
    c) The peripheral urban structure model
    d) Population distribution versus population density
    e) Ebenezer Howard 'Garden City'.
2. Explain the main stages in the evolution of a village community.
3. Using examples of settlements, explain four (4) physical factors that cause areas to be sparsely populated.
4. 'Rural development is the process of improving the quality of life of the people living in a village'. Discuss.
5. 'No African was allowed to live outside the settlements designated by colonial authorities'. Elucidate.

END OF EXAMINATION

THE UNIVERSITY OF ZAMBIA

SCHOOL OF NATURAL SCIENCE

2018/2019 ACADEMIC YEAR FINAL EXAMINATIONS

GES 4342: ENVIRONMENT AND NATURAL RESOURCES MANAGEMENT

TIME: Three Hours

INSTRUCTIONS: Answer Questions **1** and any other **two**. Candidates are encouraged to use illustrations wherever appropriate.

---

1. Write short explanatory notes on ALL of the following:

    a) Survival value of wildlife (4 Marks)
    b) Challenges of managing open access to fisheries (4 Marks)
    c) Composting and sustainable waste management (4 Marks).
    d) Threats to wetlands in Zambia (4 Marks)
    e) Kabwe town as one of the top 10 most polluted city in the world (4 Marks)

2. 'The loss of key species can threaten the survival of all forms of life including human beings.' Discuss.

3. Using specific examples related to pollution;

    (a) explain the differences between a positive and a negative externality (10 Marks).
    (b) Policy instruments the government may use to control pollution (10 Marks)

4. Explain the three Es policy principles of integrated water resources management.

5. Discuss the strategies for sustainable wetland conservation and management (20 Marks).

END OF EXAMINATION

THE UNIVERSITY OF ZAMBIA
SCHOOL OF NATURAL SCIENCES

2018/19 ACADEMIC YEAR FINAL EXAMINATIONS

GES 4372: TOURISM, ENVIRONMENT AND DEVELOPMENT

TIME: Three Hours

INSTRUCTIONS: Answer **questions 1** and any other two. Question 1 carries **40 marks** and the rest carry **30 marks** each. Use of an approved calculator is allowed. Candidates are encouraged to use illustrations wherever appropriate.

---

1. Walusungu Bird Sanctuary covers an area of 185 hectares. It is home for over 1,200 bird species. In 2018, it received 15,250 visitors and of these, 60% were foreigners while 40% were locals who usually go for day trips, picnics and educational excursions. The Sanctuary has several walking trails. The average time of 1 hour is needed for a visitor to complete the trail. An average width of the trail is 1 metre per visitor and visitors use 1 square metre per person. The distance between groups of 10 with 1 tour guide is 100 metres. The Sanctuary is open from 08:00 hours to 17:00 hours every day apart from Sundays. It is open for 360 days in a year. The actual management capacity is 7 members of staff while the ideal management capacity is 22 members of staff. You have been asked by the owner to give advice on the ideal number of visitors the garden should host.

   Using the following correction factors;
   Precipitation = 1.02%; Erosion = 25.1% and Slope = 25.1%

   a) Calculate the Physical, Real and Effective carrying capacities giving both daily and annual visitor capacities (show all calculations).

   b) Explain the major criticisms of the carrying capacity method.

2. Discuss the three key areas in which tourism can benefit the environment.

3. Using tourism in Zambia as an example, discuss the possible social effects of the interactions between tourists and host communities.

4. Explain the eight different aspects which should be included in national tourism planning.

5. What are visitor surveys and how can they be used to enhance the tourism sector in Zambia?

END OF EXAMINATION

The University of Zambia

Physics Department

University Examinations 2018-19

PHY 1010: Introductory Physics

All questions carry equal marks. The marks are shown in brackets. Question 1 is compulsory. Attempt four (4) more questions. Clearly indicate on the answer script left column on the cover page the questions you have answered. Some useful equations are given at the back of the question paper.

Time : Three hours. **Maximum marks = 100.**

Do not forget to write your computer number clearly on the answer book as well as on the answer sheet for Question 1. Tie them together.

==

Wherever necessary use:

$g = 9.8 \text{m/s}^2$

$P_A = 1.01 \times 10^5$ N/m^2

1 cal. = 4.18 J

C_{water} = 4184 J/kg.°C

$R = 8314$ J/kmol.K

$K_B = 1.381 \times 10^{-23}$ J/K (Bolzmann's constant)

$\varepsilon_0 = 8.85 \times 10^{-12}$ C^2/Nm^2

$c = 3 \times 10^8$ m/s

$h = 6.63 \times 10^{-34}$ J-s

1 pascal = 1 N/m^2

1eV = 1.6×10^{-19} J

$e = 1.6 \times 10^{-19}$ C

$m_e = 9.11 \times 10^{-31}$ kg

$k = 9 \times 10^9$ Nm^2/C^2

$\mu_0 = 4\pi \times 10^{-7}$ N/A^2

ρ_{water} = 1000 kg/m^3

$G = 6.67 \times 10^{-11}$ Nm^2/kg^2

1 metric ton = 1000 kg

H_f = 80 cal/g-water

H_v = 539 cal/g-steam

1 litre = 1×10^{-3} m^3

Vol. sphere = $\frac{4}{3}\pi r^3$

8.314 J/K °C

N.B. Useful equations are on pages 7 and 8

1.0×10^{-12}

0.57652

Question 1: For each correct answer, 2 marks will be given. For each wrong answer, 0.67 will be deducted. For no answer, zero mark will be given. The minimum total mark for Question 1 is zero.

(A) In natural convection a heated portion of fluid moves because:

a) of molecular vibrations about the equilibrium
b) its density is less than that of the surrounding fluid.
c) of molecular collisions within it.
d) its molecular motions become aligned.

(B) A Carnot engine that operates between the absolute temperatures T_1 and T_2:

a) Has an efficiency of T_1/T_2.
b) Is 100% efficient.
c) Has an efficiency of a non-reversible engine.
d) Has the maximum efficiency possible for the given temperatures.

(C) Sound travels fastest in:

a) water
b) a vacuum
c) solids
d) air

(D) The stress on a wire supporting a load does not depend on:

a) wire's length
b) wire's diameter
c) the mass of the load
d) the acceleration of gravity

(E) Objects may acquire an excess or deficiency of charge electrostatically by:

a) grounding them
b) rubbing them together
c) shielding them
d) putting them together

(F) The rotational corresponding term to force in linear motion is:

a) angular momentum
b) moment of inertia
c) angular velocity
d) torque

(G) The amplitude of the motion of an object undergoing SHM is:

a) its minimum displacement on either side of the equilibrium position

b) its the maximum displacement on either side of the equilibrium position

c) its total range of motion

d) the number of cycles per second it undergoes

(H) In a perfectly elastic collision between two objects, their relative velocity after the collision is:

a) zero

b) less than their relative velocity before the collision

c) more than their relative velocity before the collision

d) equal to their relative speed before collision

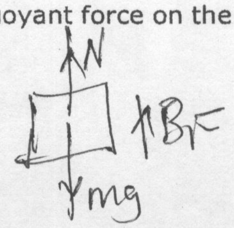
(I) A tablet of soap placed in water filled bathtub sinks. The buoyant force on the soap is:

a) more than its weight

b) equal to its weight

c) less than its weight

d) negligible



(J) Which of the following formulas expresses the relationship between the pressure and absolute temperature of a gas whose volume is fixed?

a) $\frac{P_1}{T_2} = \frac{P_2}{T_1}$

b) $\frac{P_1}{T_1} = \frac{P_2}{T_2}$

c) $\frac{P_1}{P_2} = \frac{T_2}{T_1}$

d) $\frac{P_1}{T_1} = \frac{T_2}{P_2}$

THE UNIVERSITY OF ZAMBIA

School of Natural Sciences

DEPARTMENT OF PHYSICS

PHY 2112: Atomic Physics and Magnetism in Matter

2018/19 Academic year FINAL EXAMINATION Academic Year

Instructions: Answer any **five (5)** questions only. They are of equal marks.
Time duration: Three (3) hours.

Some formulae and constants you may find useful:

$m_e = 9.11 \times 10^{-31}$ kg

$e = 1.6 \times 10^{-19}$C

$h = 6.625 \times 10^{-34}$ J.s

$\frac{mv^2}{r} = qvB$

$2d \sin\theta = m\lambda$

$\lambda_{max} T = 2.9 \times 10^{-3}$ mK

$\Delta\lambda = \lambda - \lambda_o = \frac{h}{m_o c}(1 - \cos\theta)$

$E = e\sigma T^4$

$I = I_o e^{-\mu x}$

$B = \mu_o(H + M)$

$\mu_o = 4\pi \times 10^{-7}$ A/m

$A = \lambda N$

$N = N_o e^{-\lambda t}$

$N_A = 6.02 \times 10^{23}$/mol

$\mu_m = (1 + \chi)$

$R_{sun} = 7.0 \times 10^8$ m

Surface temperature of the Sun = 6000K

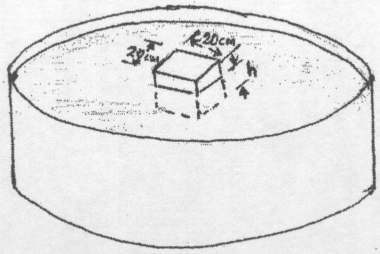
Attempt any four questions from the following:

Q2 (a) What is the minimum amount of ice at -15 ^{0}C that must be added to 0.60 kg water at 20 ^{0}C in order to bring the temperature of water down to 0 ^{0}C? Given c_{ice} = 2.09 kJ/kg ^{0}C, H_f = 335 kJ/kg, and c_{water} = 4.184 kJ/kg^{0}C. [11]

(b) A cube of wood with edge dimension 20 cm and density 650 kg floats on water as shown in the figure below.

i) What is the distance from the horizontal top surface of the cube to the water level? [6]

ii) What additional weight must be placed on the cube so that the top surface of the cube will be just level with the water surface? [3]

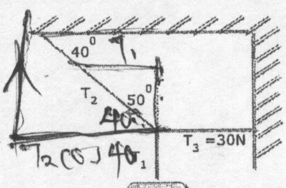


Q.3 (a) A sealed glass bulb is filled with pure nitrogen gas at a pressure of 2.0 atm. The volume of the container is 500 cm^3 and the temperature is 22 ^{0}C.

i) Calculate the number of moles and mass of the nitrogen gas (1 mole N_2 = 28 g) within the bulb. [5]

ii) If the nitrogen gas is heated to 100 ^{0}C, what will be the pressure? [5]

(b) In the figure below, the tension in the horizontal cord is 30N. Find the weight of W of the hanging object. [10]



Q.4 (a) An ideal refrigerator, which is a Carnot engine operating in reverse, operates between inside temperature of -4 ^{0}C and a room temperature of 27 ^{0}C. In a certain period, it absorbs 115 J from the interior. How much heat is rejected to the room? [9]

Some Useful Equations

Uniformly accelerated motion:

$x = \bar{v}t$ $\quad \bar{v} = \frac{1}{2}(v_f + v_i)$ $\quad v_f = v_i + at$ $\quad v_f^2 = v_i^2 + 2ax$ $\quad x = v_i t + \frac{1}{2}at^2$

Projectile motion:

$v_x = v_i \cos\theta_i$ = constant $\quad v_y = v_i \sin\theta_i - gt$ $\quad y = (v_i \sin\theta_i)t - \frac{1}{2}gt^2$

$$y = (\tan\theta_i)x - \left[\frac{g}{2v_i^2(\cos^2\theta_i)}\right]x^2 \qquad R = \frac{v_i^2}{g}\sin 2\theta \qquad t = \frac{2v_i \sin\theta}{g}$$

Force and motion:

$F = ma$ $\quad w = mg$ $\quad F_{AB} = -F_{BA}$ $\quad F_f = \mu F_N$

Work and Energy:

$PE = wh = mgh$ $\quad KE = \frac{1}{2}mv^2$ $\quad W = Fx\cos\theta$ $\quad P = \frac{W}{t} = Fv\cos\theta$

Linear momentum: $\quad p = mv$ $\quad F\Delta t = \Delta mv = m\ v_f - v_0$

Circular motion and gravitation:

$$T = \frac{2\pi r}{v} \qquad a_c = \frac{v^2}{r} \qquad F_c = \frac{mv^2}{r} \qquad F_{grav} = -G\frac{m_a m_b}{r^2} \qquad 1rev = 360^0 = 2\pi\, rad$$

$$v_T = \frac{(2\pi r)}{T} : \quad \tan\theta = \frac{v^2}{rg} :$$

Rotational motion and angular momentum:

$$\theta = \frac{s}{r} = \left(\frac{\omega_i + \omega_f}{2}\right)t \qquad \omega = \frac{\theta}{t} \qquad \theta = \omega_i t + \frac{1}{2}\alpha t^2 \qquad \omega_f = \omega_i + \alpha t \qquad v = \omega r$$

$$I = mk^2 \qquad \omega_f^2 = \omega_i^2 + 2\alpha\theta \qquad \alpha = \frac{\Delta\omega}{\Delta t} = \frac{a_T}{r} \qquad I = \sum mr^2 \qquad KE_{rot} = \frac{1}{2}I\omega^2$$

$$\tau = FL = I\alpha \qquad W = \tau\theta \qquad P = \tau\omega \qquad L = I\omega \qquad KE_{total} = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

Properties of matter:

$$\rho = \frac{m}{V} \qquad F = -kx \qquad \frac{\Delta L}{L_i} = \frac{1}{Y}\frac{F}{A} \qquad \phi = \frac{s}{d} = \frac{1}{s}\frac{F}{A} \qquad B = -\frac{\Delta P}{\Delta V / V_0}$$

$W_{app} = W\left(1 - \frac{\rho_{fluid}}{\rho}\right)$ $\quad F_B = \rho Vg$, (submerged object) $\quad F_B = Mg$ (floating M)

Thermal Properties of matter:

$PV = nRT$: $\Delta Q = mc\Delta T = nC\Delta T$: $\Delta L = \alpha L \Delta T$: $L_t = L_0(1+\alpha t)$: $\Delta V = \gamma V \Delta T$:

$W = P\Delta V$ $\quad (\Delta Q / \Delta t) = (kA\Delta T)/\Delta L$: $m = V \times \rho$ $\quad \frac{Q}{t} = eA\sigma T^4$ $\quad C_P = C_V + R$

$Q = n.C.\Delta T = \frac{3}{2} n.R.\Delta T$ for isobaric and iso-volumetric processes

Thermodynamics:

$\Delta Q = \Delta U + \Delta W$: $W = p.\Delta V$ $\quad PV = nRT$ $\quad n = \frac{m}{M}$ $\quad P_1V_1^{\gamma} = P_2V_2^{\gamma}$ $\quad T_1V_1^{\gamma-1} = T_2V_2^{\gamma-1}$

$COP_{ref} = \frac{Q_C}{W}$ $\quad COP_{heat\ pump} = \frac{Q_H}{W}$ $\quad W_{isothermal} = nRT \ln \frac{V_2}{V_1}$

$W_{adiabatic} = \frac{1}{\gamma - 1}(P_1V_1 - P_2V_2)$: $\quad COP_{\max-refr} = \frac{T_C}{W}$ $\quad COP_{\max-heat\ pump} = \frac{T_H}{W}$

$$e = 1 - \frac{T_c}{T_h} = \frac{work\ done}{input\ heat\ at\ high\ temp}$$

Waves and Sound:

$f = \frac{1}{\tau}$ $\quad v = \pm\sqrt{\frac{k}{m}(x_0^2 - x^2)}$ $\quad v = \sqrt{\frac{T}{m/L}}$ $\quad \tau = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}}$ $\quad a = -\left(\frac{k}{m}\right)x$ $\quad v = \sqrt{\frac{Y}{\rho}}$

$v = \sqrt{\frac{B}{\rho}}$ $\quad f' = f\,\frac{v \pm v_L}{v \mp v_S}$ $\quad (dB) = 10.\log\frac{I}{I_0}$

$\tau = \frac{2\pi x_0}{v_0} = 2\pi\left(\frac{x_0}{v_0}\right) = \frac{2\pi}{\omega}$ $\quad f = \frac{1}{2L}\sqrt{\frac{T}{m}}$ $\quad x = x_0\cos(\omega t)$ $\quad I_0 = 10^{-12}\ W/m^2$

Electric Field:

$\mathbf{F} \propto \frac{q_1 q_2}{r^2}$ or $F = k.\frac{q_1 q_2}{r^2}$; $k = 8.9874 \times 10^9$ N.m^2.C^{-2} or $k = (1/(4\pi\varepsilon_0))$ where ε_0 is the permittivity constant $= 8.85 \times 10^{-12}$ C^2.(N.m^2)$^{-1}$.

The University of Zambia

School of Natural Sciences

Department of Physics

2018/19 Academic Year

Mid-Year University Final Examinations

PHY 2611: Electricity and Magnetism

Instructions: Attempt *five* (5) questions only. They are all of equal marks. Time duration: Three(3) hours. Total Marks: 100

Some formulas you might find useful:

$m_e = 9.1 \times 10^{-31}$ Kg $\quad V = \int \frac{dq}{4\pi\varepsilon_o r} \quad C = \frac{Q}{V} \quad qE = qvB$

$1eV = 1.6 \times 10^{-19}$ J $\quad \int E.dS = \int \frac{dq}{\varepsilon_o} \quad V = \int \frac{dq}{4\pi\varepsilon_o r} \quad F = BIL$

$V = -\int E.dr \quad \frac{1}{2}mv^2 = eV \quad B = \mu_o nI \quad j = \sigma E$

$R = \rho \frac{L}{A} \quad \oint B.dl = \mu_o \int j.S \quad \mathcal{E} = -\frac{d\varphi}{dt} \quad \sin^2\theta = \frac{1-\cos 2\theta}{2}$

$E = \frac{V}{l} \quad I = nqAv_d \quad \varphi = B.A$

$P = VI \quad Y_{rms} = \sqrt{\frac{1}{T}\int_0^T Y_{max}^2 f(t)^2 dt}$

Q1 (a) A positive charge Q is uniformly distributed over a ring of radius R, as in Figure 1.

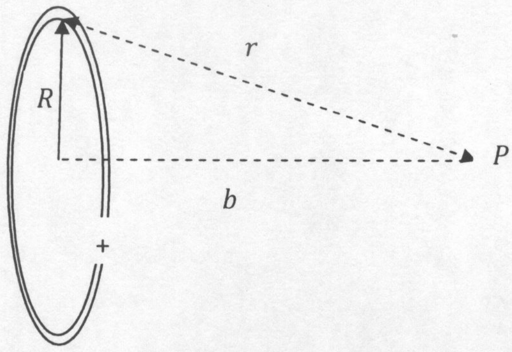


Figure 1

(i) Obtain an expression for the linear charge density on the ring [2]

(ii) Obtain an expression for the electric potential at an arbitrary point P located a distance b from the ring's center on its axis. [5]

(iii) Show that when $b \gg R$, the charge on the ring behaves like a point charge. [3]

(b) Repeat exercise (a) for the charge Q uniformly distributed on a disk of radius b, that is

(i) Obtain an expression for the surface charge density, σ [2]

(ii) Obtain an expression for the electric potential at an arbitrary point P located on the ring's axis a distance b from the center of the ring. [8]

Q2 (a) The potential difference between the plates of a parallel plate capacitor carrying a charge Q is V. If the surface charge density on the plates is σ, use Gauss's law to show that

$$C = \frac{\varepsilon_0 A}{d}$$

where C is the capacitance of the capacitor, A the area of the capacitor plates and ε_o the permittivity of free space. [5]

(b) Two equally charged particles, 3.2×10^{-3} m apart are released from rest.

Given that the acceleration of the first particle is 7.0 ms^{-2} and the acceleration of the second particle is 9.0 ms^{-2}, determine

(i) the mass of the second particle if the first particle has a mass of 6.3×10^{-7} kg. [4]

(ii) the common charge of the particles. [3]

(c) A solid insulating sphere of radius R has a non-uniform charge density ρ that varies with r according to the expression $\rho = Ar^2$, where A is a constant. Show that the electric field outside of the sphere is given by

$$E = \frac{AR^5}{5\varepsilon_o r^2}.$$ [6]

Draw the relevant spheres required in this derivation. [2]

Q3 (a) A certain wire has resistance 27.30 Ω when free from tension. What will its resistance become if tension is applied so that the length increases by 1% and the diameter increases by 0.4%. Given that its resistivity ρ remains the same? [6]

(b) A 25 m long coil of Nichrome wire has a diameter of 0.400 mm and carries a current of 0.500 A. Determine

(i) the electric field E in the wire [6]

(ii) the power delivered to it. [2]

$\rho_{Nichrome} = 1.5 \times 10^{-6}$ Ω.m

(c) A conducting spherical shell of inner and outer radius b and c respectively carries a charge $-Q$. Concentric with it is a solid conducting sphere of radius a, figure 2. Given that the outer part of the shell is earthed, determine the capacitance of this device. [8]

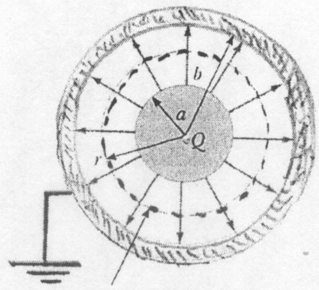


Figure 2

Q4 (a) (i) Find the equivalent capacitance of the network of capacitors in figure 3. [6]

(ii) If the potential difference between points a and b is 60 Volts, what charge is stored on C_3? [5]

Capacitor values: $C_1 = 5.00\ \mu\text{F}$, $C_2 = 10.0\mu\text{F}$ and $C_3 = 2.00\mu\text{F}$.

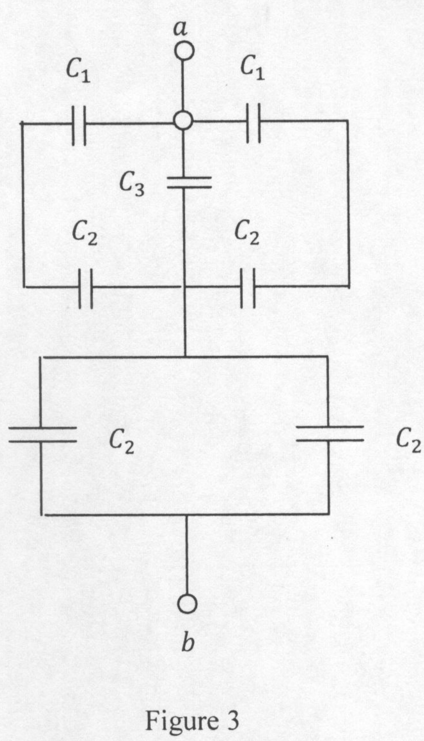


Figure 3

(b) A copper wire in a typical residential building has a cross-sectional area of 3.31×10^{-6} m^2. It carries a current of 10.0 Amp. Determine the drift speed v_d of the electrons. Note that each copper atom contributes one free electron to the current. Density of copper $\rho_{cu} = 8.95$ gcm^{-3}. Molar mass copper $M = 63.5$ g. [5]

(c) What is the required resistance of an immersion heater connected to a 110V power source that increases the temperature of 1.5 kg water from 10.0 °C to 50.0°C. in 10.0 min.?

[4]

Q5 (a) A rectangular loop carrying current I is placed in a uniform magnetic B field as shown in figure 4. Show that the resultant torque on the loop is given by

$$\tau = IAB \sin \varphi.$$

Where φ is the angle between the magnetic field lines and the normal vector to the plane of the loop and A is the geometrical area of the loop.

[6]

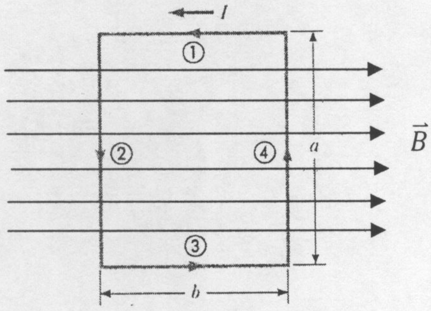


Figure 4

(b) A velocity selector consists of magnetic and electric fields directed in the y and z axes respectively. If the strength of the magnetic field is 0.0150 T, find the value of the electric field such that a 750 eV electron moving along the positive x axis can go through undeflected. [6]

(c) Figure 5 is a cross-sectional view of a co-axial cable. The center conductor is surrounded by a rubber layer, which in turn is surrounded by another rubber layer. In a particular application, the current in the inner conductor is 1.00 A directed out of the page and the current in the outer conductor is 3.00A into the page. Show that the **magnitude** and **direction** of the magnetic fields at points a and b are 200 μT towards the top of the page and 133.3 μT towards the bottom of the page respectively. Hint: Use Ampere's law. [8]

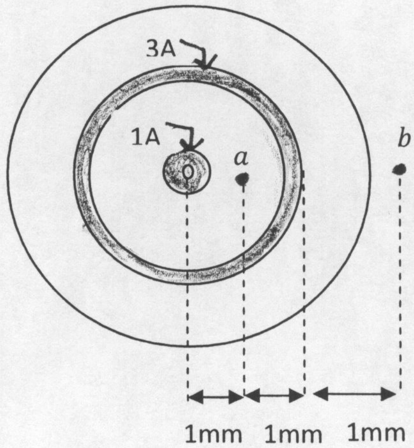


Figure 5

Q6 (a) A technician wearing a brass bracelet enclosing an area of 0.005 m^2 places her hand in a solenoid whose magnetic field is 5.00 T directed perpendicular to the cross-sectional area (or plane) of the bracelet. The electrical resistance around the circumference of the bracelet is 0.020 Ω. An unexpected power loss causes the field to drop to 1.5T in a time of 20×10^{-3} sec. Determine

(a) the flux linkage in the bracelet both before and after the power loss [2]
(b) The current induced in the bracelet and [3]
(c) The power delivered to the resistance of the bracelet. [2]

(b) A long solenoid has 400 turns per meter and carries a current that varies according to the equation $I = 30(1 - e^{-1.6t})$ A. Inside this solenoid and coaxial with it is a coil that has radius $R = 6$ cm and consists of 250 turns of fine wire, see figure below. What emf is induced in this smaller coil by the changing current? [10}

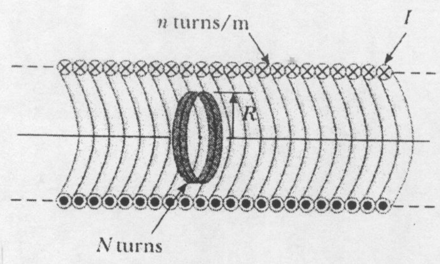


Figure 6

(c) What is back emf and why is it necessary that it exists in motor windings? [3]

Q7 (a) (i) The current, i in a certain circuit varies as $i = I_{max} \sin \omega t$, show that the I_{rms} of this current over a cycle, T where T is the period, is given by

$$I_{rms} = \frac{I_{max}}{\sqrt{2}}$$ [6]

(b) An RLC circuit consists of a 150Ω resistor, a $21.0\mu\text{F}$ capacitor and a 460 mH inductor all connected in series with a $120-\text{V}$, 60.0 Hz power supply.

(i) Determine the phase angle between the current and the applied voltage. [8]

(ii) Which of the two quantities, current or voltage leads the other? Explain how you arrive at the answer. [2]

(c) (i) Draw a simple diagram of a low pass filter utilizing an inductor L and a resistor R [2]

(ii) Draw a simple sketch of the transmission ratio, i.e. $\frac{v_o}{v_{in}}$ where v_o is the output signal and v_{in} is the input signal. [2]

THE UNIVERSITY OF ZAMBIA

School of Natural Sciences
Department of Physics
2018/2019 Academic Year
End of Year Examinations

PHY2712 – OPTICS

Duration: 3 hours — Maximum Marks: 100

Instructions

- This examination paper contains 7 questions. Attempt any 5 questions.
- Each question carries 20 marks. Marks allocated for each question are indicated in brackets [ ].
- Show all your working clearly. Omission of essential work will result in loss of marks.
- Follow carefully instructions written on the answer booklets.

CONSTANTS THAT MAY BE USEFUL

$e = 1.602 \times 10^{-19}$ C
$m_p = 1.67 \times 10^{-27}$ kg
$m_e = 9.11 \times 10^{-31}$ kg
$\varepsilon_o = 8.85 \times 10^{-12}$ $C^2N^{-1}m^{-2}$
$h = 6.63 \times 10^{-34}$ Js
$\mu_o = 4\pi \times 10^{-7}$ Ns^2C^{-2}
$c \cong 3.0 \times 10^8$ m/s
$n_{\text{water}} = 1.33$
$n_{\text{glass}} = 1.5$
$n_{\text{air}} = 1$

FORMULAE THAT MAY BE USEFUL

$n_1 \sin\theta_1 = n_2 \sin\theta_2$	$\frac{1}{f} = \frac{1}{u} + \frac{1}{v} = -\frac{2}{R}$	$\frac{n_1}{u} + \frac{n_2}{v} = \frac{n_2 - n_1}{R_1}$
$\frac{1}{f} = \left(\frac{n_2}{n_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$	$m_T = -\frac{h_2}{h_1} = -\frac{v}{u}$	$P = \frac{1}{f}$
$P = U + V = -2K$	$I = I_{\max} \cos^2\theta$	$\theta_{\min} = 1.22\frac{\lambda}{D}$
$2nt = \left(m + \frac{1}{2}\right)\lambda$, $m = 0, 1, 2, \ldots$	$2nt = m\lambda$, $m = 0, 1, 2, \ldots$	$a \sin\theta_{\text{dark}} = m\lambda$, $m = \pm1, \pm2, \pm3, \ldots$
$d \sin\theta_{\text{bright}} = m\lambda$, $m = 0, \pm1, \pm2, \pm3, \ldots$	$d \sin\theta_{\text{dark}} = \left(m + \frac{1}{2}\right)\lambda$, $m = 0, \pm1, \pm2, \pm3, \ldots$	$\tan\theta_p = \frac{n_2}{n_1}$

1. (a) State Fermats principle of least time. [2]

   (b) Use Fermats principle of least time to derive Snells law of refraction of light on a plane boundary. [7]

   (c) The radius of a spherical mirror is -18.0 cm. An object 4.0 cm high is located in front of the mirror at a distance of 45.0 cm. Find

      i. the focal length of the mirror, [1]

      ii. the power of the mirror, [1]

      iii. the image distance, and [2]

      iv. the magnification and size of image. [2]

   (d) Two thin lens having the following radii of curvature and refractive index are placed in contact. For the first lens, $R_1 = +12.0$ cm, $R_2 = -18.0$ cm and $n = 1.56$, and for the second lens $R_1 = -30.0$ cm, $R_2 = +20.0$ cm and $n = 1.65$. Find the combined powers and focal lengths of the optical system. [5]

---

2. (a) Derive the Gaussian form of the thin lens formula. [10]

   (b) Two thin lenses having focal lengths $f_1 = +9.0$ cm and $f_2 = -18.0$ cm are placed 3.0 cm apart. If an object 2.5 cm high is located 20.0 cm in front of the first lens, calculate the position and size of the final image. [10]

---

3. (a) Derive the Gaussian form of the mirror formula for a thin mirror. [10]

(b) A thin lens made up of glass of index 1.740 has two faces with radii $R_1 = +10.0$ cm and $R_2 = -25.0$ cm. The thin lens is then submerged in water of index 1.333 and a 3.5 cm object is located 30.0 cm in front of the lens. Calculate

i. the focal length of the submerged thin lens, [2]

ii. the power of the submerged thin lens, [2]

iii. the image distance in the water, [2]

iv. magnification of image formed in water, and [2]

v. size and nature of image formed in water. [2]

---

4. (a) Derive Bragg's Law of crystal diffraction and discuss briefly the method for crystal structure determination. [10]

(b) Why are X-rays used for crystal structure analysis? [2]

(c) In a typical experiment for X-ray production, X-rays are produced by acceleration of electrons through a potential difference V. If the potential difference is $V = 10$ kV, what is the wavelength of the produced X-rays? [2]

(d) Monochromatic light from a helium-neon laser ($\lambda = 632.8$ nm) is incident normally on a diffraction grating containing 6000 grooves per centimeter. Find the angles at which the first and second-order maxima are observed. [4]

(e) A telescope has an effective radius of 5.0 m. What is its limiting angle of resolution for 680 nm light? [2]

---

5. (a) What do you understand by the term interference of light? [1]

(b) A viewing screen is separated from a double slit by 4.80 m. The distance between the two slits is 0.0300 mm. Monochromatic light is directed toward the double slit and forms an interference pattern on the screen. The first dark fringe is 4.50 cm from the center line on the screen.

i. Determine the wavelength of light. [2]

ii. Calculate the distance between adjacent bright fringes. [2]

(c) A light source emits visible light of two wavelengths: $\lambda =$ 430 nm and $\lambda' = 510$ nm. The source is used in a double-slit interference experiment in which the distance between the slits and the viewing screen is 1.53 m and the slit separation distance is 0.0250 mm.

i. Find the separation distance between the third-order bright fringes for the two wavelengths. [5]

ii. Find the location on the screen where the bright fringes from the two wavelengthss overlap exactly. [5]
Use the small angle approximation

(d) Solar cells - devices that generate electricity when exposed to sunlight - are often coated with a transparent, thin film of silicon monoxide (SiO, $n = 1.45$) to minimize reflective losses from the surface. Suppose a silicon solar cell ($n = 3.5$) is coated with a thin film of silicon monoxide for this purpose. Determine the minimum film thickness that produces the least reflection at a wavelength of 550 nm, near the center of the visible spectrum. [5]

---

6. (a) What is meant by the phrase "direction of polarization of a light wave"? [2]

   (b) A given light wave can be unpolarized, partially polarized or completely polarized. Name any three kinds of polarization that you know. [3]

   (c) Suppose that two orthogonal optical disturbances are represented as

$$\vec{E}_x(z,t) = \hat{\imath}E_0\cos(kz - \omega t),$$

   and

$$\vec{E}_y(z,t) = \hat{\jmath}E_0\cos(kz - \omega t + \phi),$$

   where $\phi = \pi/2 + 2m\pi$, $m = 0, \pm 1, \pm 2, \ldots$. Find

      i. the resultant wave, and [4]

      ii. indicate the type of polarization of the resultant wave. [1]

   (d) Show that a linearly polarized wave can be synthesized from a right-circularly polarized wave and a left-circularly polarized wave. [5]

   (e) At a certain point in space, two orthogonal optical disturbances meet. One is given as

$$\vec{E}_x(z,t) = \hat{\imath}E_0\cos(kz - \omega t)$$

   and another one as

$$\vec{E}_y(z,t) = \hat{\jmath}E_0\cos(kz - \omega t + \phi),$$

   where the phase difference between the two is $-\pi/2$.

i. Write an expression representing the resultant optical disturbance. [2]

ii. What kind of polarization does the resultant optical disturbance depict? [1]

iii. If at $t = 0$, the resultant optical disturbance lies along a reference axis at an arbitrary point $z = z_0$, where will it lie at a later time, $t = \frac{kz_0}{\omega}$? [2]

---

7. (a) The most common technique for producing polarized light is to use two polarizing sheets (polarizer and analyzer) whose transmission axes make an angle θ with each other. Only a fraction of the polarized light incident on the analyzer is transmitted through it. Write

i. an expression that gives the intensity of the beam that is transmitted through the analyzer, and [2]

ii. the name of this expression. [2]

(b) At what angle should the transmission axis of the analyzer be positioned relative to the transmission axis of the polarizer so that

i. the intensity of the transmitted beam is maximum, and [2]

ii. complete absorption by the analyzer takes place. [2]

(c) In terms of the intensity of the unpolarized incoming light, what is the intensity of light as it passes through a single ideal polarizer? [2]

(d) Unpolarized light passes through two polaroid sheets. The axis of the first is vertical, and that of the second is at 30.0°

to the vertical. What fraction of the incident light is transmitted? [2]

(e) Plane-polarized light is incident on a single polarizing disk with the direction of $\vec{E}_0$ parallel to the direction of the transmission axis. Through what angle should the disk be rotated so that the intensity in the transmitted beam is reduced by a factor of

i. 3.00 [2]

ii. 5.00 [2]

(f) The angle of incidence of a light beam onto a reflecting surface is continuously variable. The reflected ray is found to be completely polarized when the angle of incidence is 48.0°. What is the index of refraction of the reflecting material? [2]

(g) How far above the horizon is the Moon when its image reflected in calm water is completely polarized? ($n_{\text{water}} = 1.33$) [2]

End of Examination

The University of Zambia

2018/19 Academic Year
University Examinations
PHY3032: Computational Physics I

This paper contains 6 questions. Answer <u>any four</u> questions. All questions carry equal marks. The marks are shown in brackets.

Time: Three (3) hours. **Maximum marks = 100.**

Write clearly your computer number on the answer book. Show your working clearly. Omission of essential work will lead to loss of marks.

SOME USEFUL PHYSICAL CONSTANTS

Acceleration due to gravity, $g = 9.81\ \text{ms}^{-2}$	Universal gas constant, $R = 8.314\ \text{Jmol}^{-1}\text{K}^{-1}$
Density of mercury, $\rho_{\text{mercury}} = 13600\ \text{kg/m}^3$	Density of water, $\rho_{\text{water}} = 1000\ \text{kg/m}^3$
Newton's Gravitation Constant, $G = 6.674\times10^{-11}\ \text{m}^3\text{kg}^{-1}\text{s}^{-2}$	Stefan Boltzmann constant, $\sigma = 5.672\times10^{-8}\ \text{Wm}^{-2}\text{K}^{-4}$
Planck's constant $h = 6.6261\times10^{-34}$ J.s	Boltzmann's constant $k = 1.38066\times10^{-23}$ J/K
Radius of the earth, $R = 6.4\times10^{6}$ m.	Mass of the Earth, $M_{\text{earth}} = 5.972\times10^{24}$ kg

Q1. **(a)** Express

(i) 38_{10} as a binary number, **[3]**

(ii) 11010110101101_2 it its Hexadecimal equivalent, **[4]**

(b) Consider the following Fortran 90 code:

```
IMPLICIT NONE
REAL :: y
REAL (KIND = 8):: h, p
DOUBLE PRECISION :: x = 2.5d0 , w
INTEGER :: J, k = 9, m = 3
INTEGER(Kind = 8) :: n = 2, L = 7, f, s
COMPLEX :: z
        w = n/K              ;        y = real(m)*real(k)
f = INT(y)          ;        J = mod(f,n)
h = NINT(x)         ;        z = cmplx(h,J)
p = n**10           ;        s = int(imag(z))
        PRINT*, w, y, f
                PRINT*, j, h, z
                PRINT*, p, s
END PROGRAM
```

What numerical values [including appropriate number of decimal points whenever necessary] for **w, y, f, j, h, z, p** and **s** [**in this order**] would be displayed on a computer screen after program execution? **[8]**

(c) A student modeled the time dependent alternating voltage $v(t)$ in a system as

$$v(t) = V_o \sin(100\pi t + \pi/3),$$

where t is time and $V_o = 350$ volts. In order to get numerical values of $v(t)$ against t correct to 3 decimal places in time steps of 0.001s for $0 \le t \le 10$s, the student wrote the following code. However, this Fortran 90 code could be compiled due to errors. Identify the 10 errors in the code and write correct code without changing the general format of the given code.

```
PROGRAM driver
        IMPLICIT NONE
        REAL (KIND = 4) :: t, tmax, dt=0.001
        OPEN(1,file="volt.txt")
                DO WHILE (t < tmax)
                        write(2,*) t, v(t)
                        t= t + 1
                ENDDO
1000  FORMAT (2f10.7)
END PROGRAM driver


REAL (KIND =4) FUNCTION v(t)
        IMPLICIT NONE
        REAL (KIND = 4)::       pi =4.0d0*atan(1.d0), Vo = 350.0d0
        REAL (KIND=4 ) ::       phase, t
                        V(t)= Vo*sin(wf*t + phase)
        RETURN
END FUNCTION v
```

[10]

Q2. **(a)** The following experimental data and were saved (together with headings) in a text file called "`input.txt`".

Table 1: Data collected for the energy exchange in a coupled pendulum

Distance $x\,[m]$	Average time, $t\,[s]$
0.17	6.94
0.18	5.19
0.19	4.21
0.20	3.63
0.21	2.95
0.22	2.70
0.23	2.12

Write a Fortran 90 code that

(i) uses the IOSTAT command in the READ statement, in a driver program, in order to automatically count the number n of data points in the file ***input.txt***, **[6]**

(ii) can identify an error in the data file [including the location of the error] if a non-numeric entry is included in ***input.txt***. **[2]**

(iii) can identify an **empty input file**. **[2]**

(b) What is meant by the following terms:

a. Byte, **[2]**
b. Kibibyte[KiB] , **[2]**
c. Kilobyte [KB], **[2]**
d. Computer. **[3]**

(c) Calculate the maximum and minimum values that can be stored in a 2-byte integer variable. **[6]**

Q3. When air is assumed to be an ideal gas, the velocity of sound v_s in dry air can be approximated by

$$v_s = \sqrt{\frac{\gamma k_B T}{m}},$$

where $\gamma = 1.4$, $k_B = 1.38064852 \times 10^{-23}\,\text{m}^2\text{kg s}^{-2}\text{K}^{-1}$, $m = 28.9628$ g/mol while T is temperature. Write a code, using Fortran 90/95, which has a driver and a double precision function with the following specifications:

a) The function should accept a value of temperature T and return the corresponding value of v_s. **[5]**

b) The driver program should

(i) use loop to generate double precision values of T for $273.15\text{K} \le T \le 323.15\text{K}$ in steps of 0.10K, **[5]**

(ii) save values of temperature T and corresponding velocity v_s, correct to 6 decimal places, to a file called "`velocity.txt`", **[4]**

(iii) create a `GNUPLOT` script called "`velocityplot.plt`" which can be used to plot the graph of v_s against T with approximate axis labels, **[8]**

(iv) send an instruction to the computer system to ran the script "`velocityplot.plt`" using `GNUPLOT` software. **[3]**

Q4. **(a)** A definite integral of a function $f(x)$ from lower limit $x_{\min}$ to upper limit $x_{\max}$ is expressed as

$$\int_{x_{\min}}^{x_{\max}} f(x)\,dx.$$

(i) If a definite integral is estimated as a single trapezoid whose coordinates are $(x_{\min},0)$, $(x_{\min}, f(x_{\min}))$, $(x_{\max},0)$ and $(x_{\max}, f(x_{\max}))$, show that the single segment trapezoidal approximation is

$$\int_{x_{\min}}^{x_{\max}} f(x)\,dx \approx \frac{dx}{2}\left[f(x_{\min}) + f(x_{\max})\right],$$

where $dx = x_{\max} - x_{\min}$ is the step size. **[4]**

(ii) Hence, or otherwise, show that if the domain $[x_{\min}, x_{\max}]$ is partitioned into N subintervals, then the composite trapezoidal approximation is

$$\int_{x_{\min}}^{x_{\max}} f(x)\,dx \approx \frac{dx}{2}\left[V + 2W\right],$$

where $dx = \dfrac{x_{\max} - x_{\min}}{N}$, $V = f(x_{\min}) + f(x_{\max})$ and $W = \sum_{i=1}^{N-1} f(x_i)$,

for $x_i = x_{\min} + i\,dx$ such that $i = 1, 2, 3, \ldots, N-1$. **[9]**

(b) Use the composite Trapezoidal algorithm with 10 subintervals to show, correct to 6 decimal places, that

$$\int_0^1 \exp(-x^3)\,dx \approx 0.756590.$$ **[12]**

Q5. **(a)** Consider a set of N-data points $[x_i, y_i]$; for $i = 1,2,3....N$. Assuming a straight line model of the form

$$y = a_0 + a_1 x,$$

is fitted to the data, use the Chi-square merit function

$$\chi^2 = \frac{1}{N}\sum_{i=1}^{N}[y_i - y]^2$$

to show that

$$a_0 = \frac{S_{xx}S_y - S_x S_{xy}}{NS_{xx} - S_x^2},$$

$$a_1 = \frac{NS_{xy} - S_x S_y}{NS_{xx} - S_x^2},$$

where $S_x = \sum_{i=1}^{N} x$, $S_y = \sum_{i=1}^{N} y_i$, $S_{xx} = \sum_{i=1}^{N} x_i^2$, $S_{xy} = \sum_{i=1}^{N} x_i y_i$ and $S_{yy} = \sum_{i=1}^{N} y_i^2$. **[13]**

(b) Write a code, using Fortran 90/95, that can read a set of N data $[x_i, y_i]$; for $i = 1,2,3....N$ from a file, called "***input.txt***" and implements linear least squares fitting to compute the intercept a_0 and gradient a_1. **[12]**

THE UNIVERSITY OF ZAMBIA

DEPARTMENT OF PHYSICS

SCHOOL OF NATURAL SCIENCE

2018 Mid-Year University Examinations

PHY3411: Analogue Electronics I

Instructions: Answer any **four** questions only. This examination paper has six questions. Each of the six questions carry equal marks. All essential working must be shown.

Time duration: three hours.

Some formulae that you might find useful:

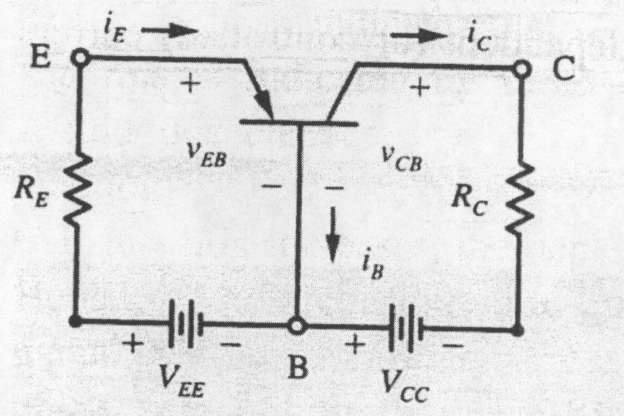
$np = n_i^2$	$v = \eta V_T \ln\left(\frac{i}{I_s}+1\right)$	$i = I_s\left(e^{v/\eta V_T}-1\right)$	$r = \eta \frac{V_T}{I_Q}$
$V_{eff} = V_{rms} = \sqrt{\frac{1}{T}\int_0^T v(t)^2\, dt}$	$V_{av} = \frac{1}{T}\int_0^T v(t)dt$	$T = \frac{1}{\sqrt{1+\left(\frac{1}{\omega RC}\right)^2}}$	$T = \frac{1}{\sqrt{1+(\omega RC)^2}}$
$I_E = I_C + I_B$	$I_C = \beta I_B$	$R = \rho \frac{L}{A}$	$f(E_C) = \frac{1}{1+e^{E_g/2K_BT}}$
$\frac{n}{N} = e^{-E_g/k_BT}$	$\sigma \approx n\mu_n q$	$\sigma \approx p\mu_p q$	$n \approx \frac{n_i^2}{N_A}$
$V_T = \frac{T}{11586}$	$\rho = \frac{1}{\sigma}$	$W = \frac{1}{2}LI^2$	$X_C = \frac{1}{\omega C}$
$f(E_C) = \frac{1}{1+e^{E_g/2K_BT}}$	$p \approx \frac{n_i^2}{N_D}$	$r = \eta \frac{V_T}{I_Q}$	$\sigma = \left(n\mu_e + p\mu_h\right)$

1. a) State Kirchhoff's current law. [1]

b) The circuit shown below has a silicon *pnp* bipolar junction transistor (BJT). Given that $R_E = 500\ \Omega$, $R_C = 1.2\ \text{k}\Omega$, $V_{EE} = 3$ V and $V_{CC} = 6$ V, analyse this circuit to confirm that the BJT is biased (i.e., operating) in the active region. The two approximations for the mode operation are $v_{EB} = 0.7$ V (i.e., the E-B junction is forward biased) while

$$i_C = \alpha i_E$$

where $\alpha \sim 1$ (i.e., the C-B junction is reverse biased).



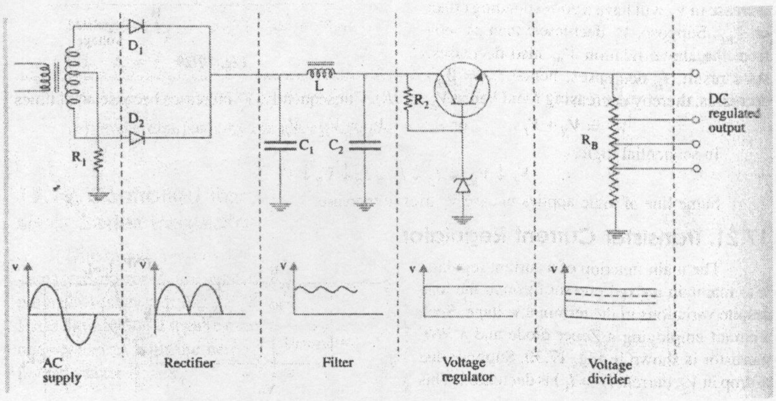
(i) Apply Kirchhoff's voltage law (KVL) around the loop on the left to show that $i_E = 4.6$ mA. [3]

(ii) Apply KVL around the loop on the right to find v_{CB}. [4]

(iii) Assuming that the cut-in voltage for typical silicon is $V\gamma = 0.5$ V, use the results in parts (i) and (ii) above to explain that they confirm this BJT as operating in active region. [2]

(iv) Draw the model circuit of the given circuit that shows the behavior of the BJT. [3]

c) The figure below shows a design of a complete power supply circuit.



(i) What is the type of rectifier in this circuit? [1]

(ii) Name the filter shown and also explain what each electric element (i.e., C_1, L and C_2) does to the rectified signal. [1 + 5]

(iii) Draw a typical *i-v* characteristic curve for a Zener diode and then discuss how it is used for voltage regulation in the given power supply circuit. [5]

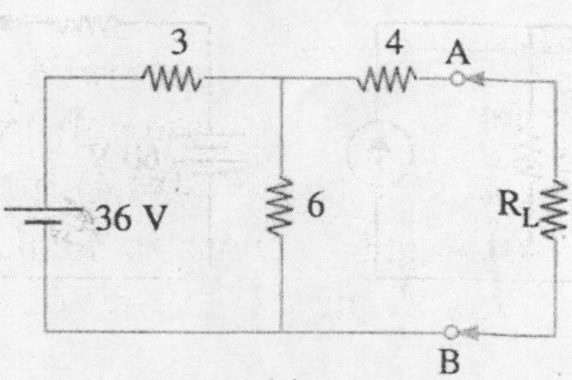
2. a) What are the other two configurations in which a transistor may be connected in a circuit besides that common-collector configuration? [2]

b) For the circuit given below,

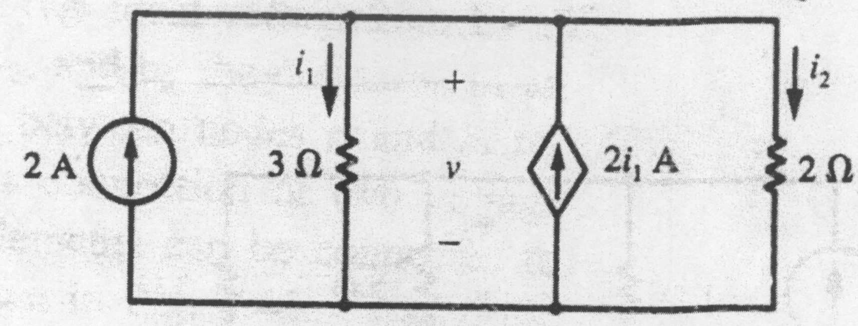
(i) draw a circuit that would enable you to perform the maximum power transfer theorem calculations, [1]

(ii) find the value of the load resistance (R_L) to be connected across terminals A and B which would extract maximum power from the circuit, and also [4]

(iii) evaluate the value of this maximum power transferred to the load resistance. [6]



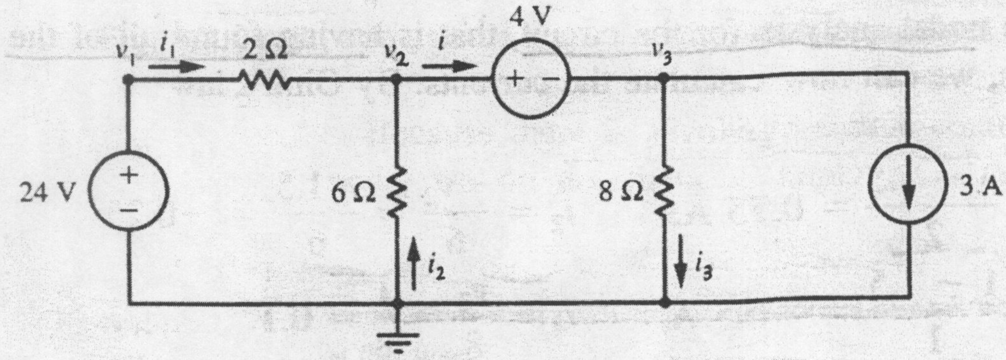
c) In the following circuit, the value of the dependent current source depends on the current i_1 through the 3-Ω resistor, the value of the dependent source is $2i_1$ with units of amperes. Find the currents i_1 and i_2, and also the value of the dependent current source. [8]



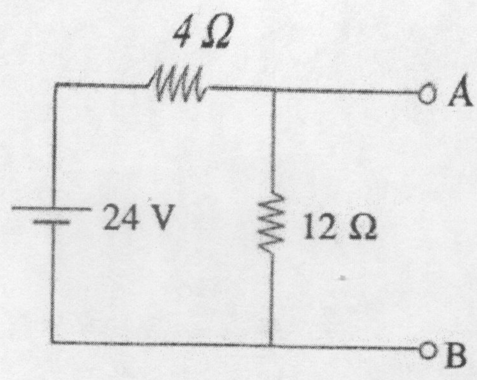
d) Draw the transistor symbols of the *npn* BJT and *pnp* BJT clearly showing the direction of the conventional current flows. [4]

3. a) State Norton's Theorem. [1]

b) For the circuit shown below, the 4 V source is connected between two nonreference nodes. Find the node voltages and the labelled currents. [19]



c) For the circuit shown below,



Determine

(i) the current I_N, and also [2]

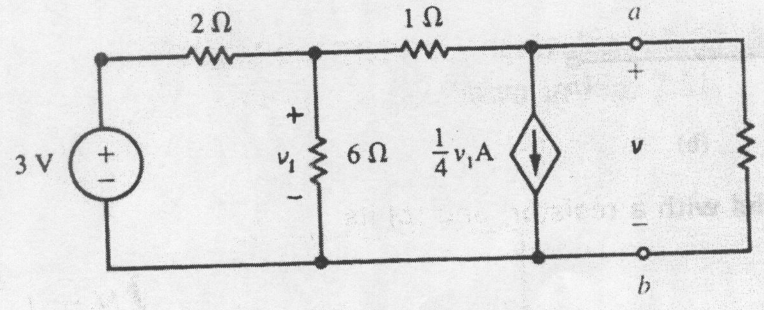
(ii) the resistance R_N [3]

using the Norton theorem.

4. a) State Kirchhoff's voltage law. [1]

b) At 300 K, the intrinsic concentration of silicon is 1.5 x 10^{16} m^{-3}, the free-electron mobility is 0.13 m^2/V-s and the mobility of holes is 0.05 m^2/V-s. Find the conductivity and resistivity of this semiconductor. [4]

c) For the circuit below, use Norton's Theorem to calculate the short-circuit current (i_{sc}) and the output resistance (R_o) as follows:



(i) redraw the given circuit replacing the 8-Ω resistor with a short circuit and also highlighting the label v_1 at the node connecting the three resistances. [1]

(ii) show that v_1 = 0.9 V by applying KCL at the node associated. [4]

(iii) apply KCL at an appropriate node to determine i_{sc} as equaling 0.675 A. [3]

(iv) redraw the given circuit into a diagram for the determination of R_o showing the new node v_l label clearly. [1]

(v) apply KCL at this new node to illustrate that this $v_l = 0.6v_o$. [4]

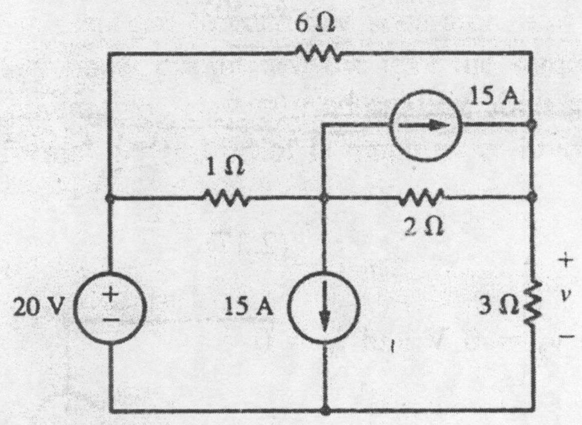
(vi) prove that the output current $(i_o) = 0.55v_o$ and then find R_o. [5]

(vii) calculate v_{oc} by effecting Thevenin's Theorem from the appropriate results above. [1]

(viii) draw both the Norton equivalent circuit and the Thevenin's equivalent circuit showing the values of each circuit element with the load resistor (R_L). [1]

5. a) State Thevenin's Theorem. [1]

b) Use the principle of superposition to analyse the following circuit.



(i) Compute the value of response to the top or upper independent current source of 15 A, labeling it v_{t15A}. [9]

(ii) Given that the response to the 20 V independent voltage source is v_{20V} = 12 V and of the bottom 15 A current source is i_{b15A} = -6 V, show that the total response voltage across the 3Ω resistor is v = 18 V by the effecting the superposition principle. [2]

(iii) Compare how you would treat both dependent sources and independent sources when determining the response due to either an independent current or voltage source of a given circuit as you apply the principle of superposition. [2]

c) Find the voltage v_0 for the circuit shown below where the germanium diode has a saturation current of 10 nA at 300 K as follows:

(i) Draw the *dc* equivalent circuit of the given figure.

(ii) Using the graphical method or numerical method $i_{dc} = I_Q = 7.96$ mA and $v_{dc} = V_Q = 0.704$ V by assuming that $v = 0.700$ V. Sketch the graphical analysis of the diode operation showing the *i-v* characteristic curve, load line, V_γ, and the quiescent point. By KVL, $1.5 = v_{dc} + 100 i_{dc}$ and also $i_{dc} = I_s\left(e^{v_{dc}/\eta V_T} - 1\right)$ are the two useful equations.

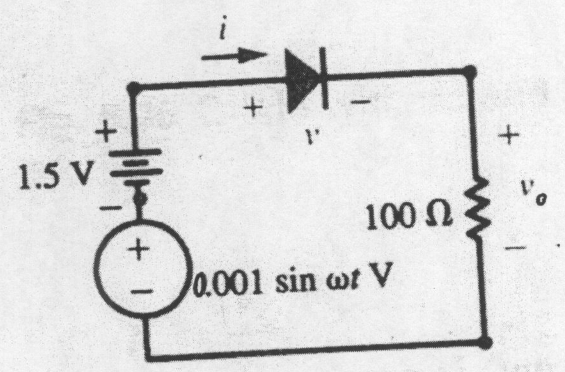
(iii) Show that, or Use the voltage divider formula to find, $v_{o(dc)} = 0.796$ V.

(iv) Draw the *ac* equivalent circuit for the given figure.

(v) Find r, the dynamic or *ac* resistance.

(vi) Find $v_{o(ac)}$.

(v) Write the expression for v_o. [11]



6. a) Why is a semiconductor referred to as bipolar while a metal is known to be unipolar? [2]

b) To compare the sine-wave current with a direct current, the joule heat produced in a resistor is considered. Given that *ac* current and voltage are

$$i = I_p \sin(wt) \text{ and}$$

$$v = V_p \sin(wt + \phi)$$

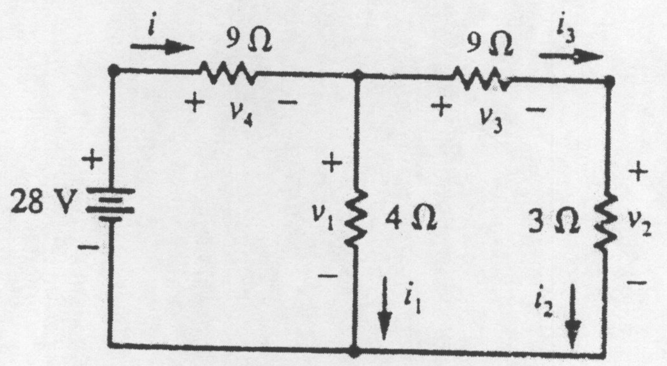
where I_p and V_p are the maximum or peak amplitude values, and ϕ is the phase difference, the average power calculated by averaging the IR losses over a complete cycle is

$$P_{av} = \frac{I_p^2 R}{2}.$$

Find the expression for the effective value, or *rms* value, of an alternating current by comparing the average power above with the joule heating in a resistor by *dc* I^2R.

[4]

c) For the circuit shown below,



(i) apply Kirchhoff's current law, Ohm's law, Kirchhoff's voltage law at respective circuit points to write necessary expression of the labelled current and voltages to show that $v_2 = 1.75$ V, [10]

(ii) use the voltage-divider formula to find v_1. [3]

(iii) calculate i_1, and [3]

(iv) apply the current-divider formula to find i. [3]

END

UNIVERSITY OF ZAMBIA
DEPARTMENT OF PHYSICS
UNIVERSITY MID-YEAR EXAMINATIONS
2019
PHY3531: INTRODUCTION TO QUANTUM MECHANICS

TIME: THREE HOURS
ANSWER: ANY FOUR QUESTIONS
ALL QUESTIONS CARRY EQUAL MARKS
TOTAL MARKS: 100

Useful information

$$j = \frac{\hbar}{2mi}(\Psi^* \frac{d}{dx}\Psi - \Psi \frac{d}{dx}\Psi^*), \qquad \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

$$[AB, C] = A[B, C] + [A, C]B$$

$$[A, BC] = B[A, C] + [A, B]C$$

$[L_i, L_j] = i\hbar L_k$ with i, j and k taken in cyclical order

$$\int \psi^* A\phi dV = \int (A\psi)^* \phi dV$$

$$dV = r^2 \sin\theta dr d\theta d\phi$$

---

1. (a) (i) Interpret the wave function $\Psi(\mathbf{r}, t)$ and hence explain why it must have the following properties: [2 marks]

(ii) It must be normalised, [2 marks]

(iii) It must be continuous, [2 marks]

(iv) Its first derivative must be continuous [2 marks]

(v) It must be single valued. [2 marks]

(vi) It must be bounded. [2 marks]

(b) Prove that if the potential acting on a particle is independent of time, the time-dependent Schroedinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\Psi(\mathbf{r}, t) + V(\mathbf{r})\Psi(\mathbf{r}, t) = i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r}, t)$$

separates into two equations. [6 marks]

(ii) Show that in such a case the wave function has the form

$$\Psi(\mathbf{r}, t) = \psi(\mathbf{r})e^{-i\omega t}$$

[3 marks]

(c) The force acting on a particle is given by

$$V(x) = -k/x^2$$

where k is a constant

(i) Write down the time-independent Schroedinger equation for the particle. [2 marks]

(ii) Write down the classical equation of motion for the particle and explain what the solution represents. [2 marks]

2. (a) A particle with energy E and mass m is trapped in a two-dimensional box of dimensions L_x and L_y so that the potential has the form

$$V(x, y) = \begin{cases} \infty & \text{for } x < 0 \text{ and } y < 0 \\ 0, & \text{for } 0 \le x \le L_x \text{ and } 0 \le y \le L_y \\ \infty & \text{for } x > L_x \text{ and } y > L_y \end{cases}$$

(i) Sketch the potential in the x and y directions. [2 marks]

(ii) Explain why inside the box, the Hamiltonian of the particle is

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2}$$

[2 marks]

(iii) Show that the normalised eigenfunctions of the particle are given by

$$\psi_{n_x n_y}(x,y) = \frac{2}{\sqrt{L_x L_y}} \sin\frac{n_x \pi x}{L_x} \sin\frac{n_y \pi y}{L_y} \quad \text{where } n_x, n_y = 1, 2, 3, ...$$

and that the energy eigenvalues are

$$E_{n_x n_y} = \frac{\pi^2 \hbar^2}{2m}\left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2}\right)$$

[16 marks]

(iv) Explain what is meant by degeneracy and deduce the degeneracy of the first and second excited states if $L_x = L_y$. [3 marks]

(v) Write down the eigenfunctions for the first and second excited states. [2 marks]

3. (a) (i) Prove that the operator

$$p_x = -i\hbar\frac{d}{dx}$$

is linear and Hermitian. Assume that the functions on which it acts vanish at $x = \pm\infty$. [6 marks]

(ii) Explain what this means for the eigenvalues of p_x. [2 marks]

(b) (ii) The Hamiltonian of a certain particle of mass m is

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2$$

where ω is a constant. One of its eigenfunctions is $\psi(x) = Ae^{-\alpha x^2}$.

(i) Show that the energy of the particle is

$$E = \frac{1}{2}\hbar\omega$$

and that value of α is

$$\alpha = \frac{m\omega}{2\hbar}$$

[13 marks]

(ii) Write down the normalised solution of the time-dependent Schroedinger equation for the particle. [4 marks]

4. **(a)** **(i)** Explain and prove the importance of the harmonic oscillator problem in physics. [3 marks]

(ii) The potential for the one-dimensional simple harmonic oscillator is $V(x) = \frac{1}{2}kx^2$. Show that the eigenfunctions of this oscillator must be of definite parity. [6 marks]

(b) The two-dimensional harmonic oscillator has the Hamiltonian

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{1}{2}k_x x^2 + \frac{1}{2}k_y y^2$$

(i) Starting from the time-independent Schroedinger equation for the oscillator

$$H(x,y)\psi(x,y) = E\psi(x,y)$$

show that the eigenfunctions of the oscillator are given by

$$\psi(x,y) = \psi_{n_x}(x)\psi_{n_y}(y)$$

where $\psi_{n_i}(\xi)$ (with $\xi = x$ or y) is a solution of the time-independent Schroedinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2}{d\xi^2}\psi_{n_i}(\xi) + \frac{1}{2}k_i\xi^2\psi_{n_i}(\xi) = E_{n_i}\psi_{n_i}(\xi)$$

and that energies are

$$E_{n_x n_y} = (n_1 + 1/2)\hbar\omega_1 + (n_2 + 1/2)\hbar\omega_2\,,\ \text{where } \omega_i = \sqrt{\frac{k_i}{m}} \text{ and } n_i = 0,1,2,....$$

[10 marks]

(ii) Show that when $k_x = k_y$ the excited states of this oscillator are in general degenerate. [2 marks]

(iii) Deduce the degeneracies and energies of the first and second excited states. [4 marks]

5.(a) A particle of mass m incident from the left encounters a potential well of the form

$$\begin{aligned} V(x) &= 0, \quad x \leq 0 \quad \text{(Region I)} \\ &= -V_0, \; 0 \leq x \leq a \quad \text{(Region II)} \\ &= 0, \quad x \geq a \quad \text{(Region III)} \end{aligned}$$

The energy of the particle is such that $E > 0$ and V_0 is positive.

(i) Sketch the potential. [2 marks]

(ii) Show that in the three regions the eigenfunction takes the following forms:

$$\psi_I(x) = Ae^{ikx} + Be^{-ikx}, \quad k = \sqrt{\frac{2mE}{\hbar^2}} \quad \text{(Region I)}$$

$$\psi_{II}(x) = Ce^{iqx} + De^{-iqx}, \quad q = \sqrt{\frac{2m\,(E + V_{01})}{\hbar^2}} \quad \text{(Region II)}$$

$$.\psi_{III}(x) = Fe^{ikx} \quad \text{(Region III)}$$

. [10 marks]

(iii) Interpret each of the terms in the expressions above. [3 marks]

(iv) Show that the reflection and transmission coefficients R and T are given respectively by

$$R = \left|\frac{B}{A}\right|^2 \quad \text{and} \quad T = \left|\frac{F}{A}\right|^2$$

. [4 marks]

(v) Justify and write down the equations that would be used to determine B and F in terms of A. Do not attempt to solve for B and F. [4 marks]

(vi) Explain how the behaviour of the particles differs from that in the classical case. [2 marks]

6. The angular momentum operator and its square frequently commute with the Hamiltonian of a system so that

$$[H, \mathbf{L}] = 0 \quad \text{and} \quad [H, L^2] = 0$$

(i) Explain why this is important. [3 marks]

(ii) Using one of the components of $\mathbf{L} = -i\hbar\mathbf{r} \times \nabla$ show that it commutes with L^2 so that

$$[L^2, L_i] = 0$$

and explain the meaning of this. [5 marks]

(iii) Keeping in mind that

$$[L_i, L_j] = i\hbar L_k \text{ with } (i, j, k) \text{ in cyclical order}$$

explain the implication of the result in **(ii)** for the treatment of angular momentum. [2 marks]

(b) In spherical polar coordinates, the operator for the z component of the angular momentum is

$$L_z = -i\hbar\frac{\partial}{\partial\phi}$$

Show that the normalised eigenfunctions of this operator are

$$\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}}e^{im\phi}, \quad m = 0, \pm 1, \pm 2, ...$$

[10 marks]

(c) The operator for the square of the angular momentum is

$$-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

Prove that as expected, $Y_{00}(\theta, \phi) = 1/\sqrt{4\pi}$ is an eigenfunction of both L_z and L^2, and give the eigenvalues. [5 marks]

END OF EXAMINATION

UNIVERSITY OF ZAMBIA
SCHOOL OF NATURAL SCIENCES
DEPARTMENT OF PHYSICS
MID-YEAR EXAMINATION
2018/2019 ACADEMIC YEAR

PHY4021
MATHEMATICAL METHODS FOR PHYSICS

DURATION: Three hours.

INSTRUCTIONS: Answer any **four** questions from the six given. *Each question carries 25 marks with the marks for parts of questions indicated.*

MAXIMUM MARKS: 100

DATE: Monday, 8th July 2019.

Formulae that may be needed:

1.
$$u_x = v_y, \quad u_y = -v_x, \quad u_r = \frac{v_\theta}{r}, \quad v_r = -\frac{u_\theta}{r}$$

2.
$$\int_C f(z)\, dz = \int_a^b f[z(t)]\dot{z}(t)\, dt.$$

3.
$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz, \qquad (n = 1, 2, \ldots)$$

4.
$$\cosh iz = \cos z, \quad \sinh iz = i\sin z, \quad \cos iz = \cosh z, \quad \sin iz = i\sinh z$$

5.
$$(\cosh z)' = \sinh z, \quad (\sinh z)' = \cosh z$$

6.
$$\left|\frac{z_{n+1}}{z_n}\right| \leq q < 1, \quad \text{for } n \text{ greater than some } N$$

7.
$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = L.$$

8.

$$\sqrt[n]{|z_n|} \leq q < 1, \quad \text{for } n \text{ greater than some } N$$

9.

$$\lim_{n\to\infty} \sqrt[n]{|z_n|} = L$$

10.

$$\begin{aligned} R &= \frac{1}{L^*}, \qquad L^* = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\ R &= \frac{1}{\tilde{L}}, \qquad \tilde{L} = \lim_{n\to\infty} \sqrt[n]{|a_n|} \\ R &= \frac{1}{\tilde{l}}, \qquad \tilde{l} = \text{largest limit of } \lim_{n\to\infty} \sqrt[n]{|a_n|} \end{aligned}$$

11.

$$\begin{aligned} \frac{1}{1-z} &= \sum_{m=0}^{\infty} z^m, \qquad |q| < 1; \qquad & e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \\ \cos z &= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}; \qquad & \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \\ \cosh z &= \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}; \qquad & \sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \end{aligned}$$

12.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

13.

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0} (z - z_0) f(z)$$

14.

$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

15.

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z\to z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)] \right\}$$

16.

$$\oint_C f(z)\, dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z)$$

17. Integrals of rational trigonometric functions of $\sin\theta$ and $\cos\theta$ (integration taken counterclockwise)

$$\int_C f(z) \frac{dz}{iz} = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} \left[\frac{f(z)}{iz}\right]_{z=z_0}, \qquad C: \; |z| = 1$$

where $f(z)$ is obtained from $f(\cos\theta, \sin\theta)$ by the substitutions

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right), \qquad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

18. Improper integrals of rational functions:

$$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum \operatorname{Res} f(z)$$

19.

$$y(x) = \sum_{m=0}^{\infty} a_m x^m$$

20.

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

21.

$$\int_a^b u\frac{dv}{dx}\,dx = [uv]_a^b - \int_a^b \frac{du}{dx}v\,dx$$

22. Inverse of 2×2 matrix:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

QUESTION 1

(a) Define a domain and define any terms used in its definition. (4 marks)

(b) Define a simple closed path and draw an example of such a path. Also draw two different examples of closed paths which are not simple. (4 marks)

(c) Draw and shade the region $1 < |z - 2 + i| < 2$ on the complex plane. (5 marks)

(d) Test whether or not the function $v = x^3 - 3xy^2$ is harmonic. If it is harmonic, find the harmonic conjugate. Write down the corresponding function in the form $f(z) = u(x,y) + iv(x,y)$, then confirm it can be written in the form $f(z) = iz^3 + c$, where c is a constant. (12 marks)

QUESTION 2

(a) From first principles, test whether or not the function

$$f(z) = z^2 + 2z + \overline{z}$$

is differentiable.

(12 marks)

(b) Integrate the following integral by the method of path

$$\int_C (2z^2 + z)\,dz,$$

where the path of integration C is $|z - 2 + i| = 2$. State the theorem which this result confirms. (13 marks)

QUESTION 3

(a) Integrate the function

$$g(z) = \frac{2z^2 + 3z + 2}{z^2 - 4}$$

around a unit circle with center

(i) $z = 2$, (4 marks)

(ii) $z = 1.5$, (4 marks)

(iii) $z = -2 - 0.5i$, and (4 marks)

(iv) $z = 0$, (4 marks)

using Cauchy's integral formula, the principle of deformation of path or Cauchy's integral theorem, whichever is appropriate. You must use the principle of deformation of path or Cauchy's integral theorem in preference to Cauchy's integral formula wherever they can be used.

(b) Use an appropriate test for convergence to show whether or not the series

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}$$

converges. (9 marks)

QUESTION 4

(a) Find the Laurent series of

(i)

$$f(z) = \frac{1}{z^3} e^{2z}, \qquad \text{and}$$

(6 marks)

(ii)

$$f(z) = \frac{1}{z^5} \cos \frac{1}{z}.$$

(6 marks)

(b) Integrate the following real integral using complex numbers:

$$\int_0^{2\pi} \frac{2}{8 - 6\cos\theta}\, d\theta.$$

(13 marks)

QUESTION 5

Diagonalise the following matrix:

$$A = \begin{bmatrix} 0 & 3 \\ 12 & 0 \end{bmatrix}.$$

(25 marks)

QUESTION 6

(a) Determine the rank of the matrix

$$A = \begin{bmatrix} 3 & -3 & 0 \\ 1 & 4 & 5 \\ 4 & 4 & 8 \end{bmatrix}.$$

(7 marks)

(b) Test whether or not the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is orthogonal. (6 marks)

(c) Use the power series method to solve the differential equation

$$(1-x)\frac{dy}{dx} = y.$$

Write the series solution at least up to $m = 5$. (12 marks)

END

THE UNIVERSITY OF ZAMBIA
SCHOOL OF NATURAL SCIENCES
Department of Physics
2018/2019 ACADEMIC YEAR
Mid Year University Examinations
PHY 4221 - Solid State Physics I

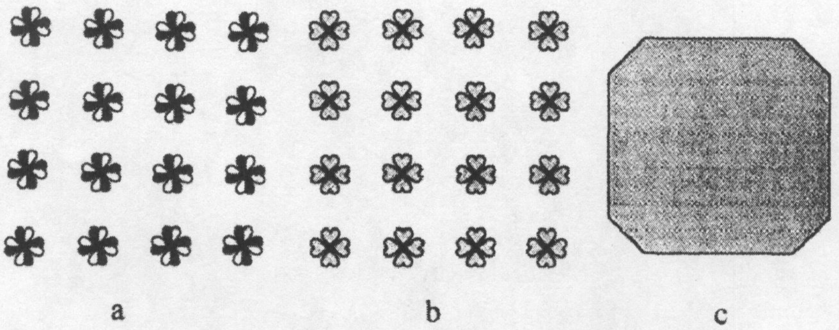
Duration : 3 Hours
Total marks: 100

INSTRUCTIONS

This question paper contains six questions.
Answer any four questions. Each question has 25 total marks.
Write clearly your computer number on the answer booklet provided.
All working should be shown clearly.
Omission of essential work will lead to loss of marks.
Additional information can be found at the end of this question paper.

QUESTION 1

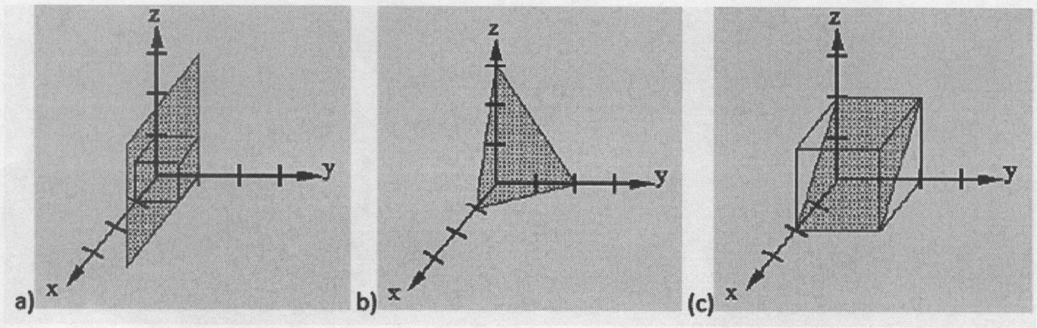
(a) In the figure below, two crystals (a and b) and a polygon (c) are shown. Identify the point group operations of the three objects assuming that the crystals are of infinite size. Show that the point groups of the two crystals are different, and one of them has the equivalent point group as the polygon. [7]



(b) A lattice can posses 2 fold, 3 fold 4 fold and 6 fold rotational symmeties but cannot posses a 5 fold rational symmetry. Prove this statement. [5]

(c) Derive the dispersion relationship for a one dimensional diatomic lattice. [7]

(d) Determine the Miller indices (hkl) of the shaded planes below.Show your work on each step to determine the plane.



[6]

QUESTION 2

(a) (i) Name the three most important kinds of probes used in diffraction experiments on crystals. [3]

(ii) Discuss the essential condition that must be satisfied if such probes are to be useful in understanding crytal structure. [2]

(b) In a quantitative model of bonding in ionic crystals, the total potential energy is assumed to be of the form (in SI units)

$$U_{tot}(R) = N\frac{A}{R^n} - N\frac{\alpha e^2}{4\pi\epsilon_0 R}$$

where N is the number of positive-negative ion pairs, R is the nearest neighbour distance in the material, α is the Madelung constant for the material, A and n are adjustable empirical parameters to fit the experimental data and ϵ_0 is the permitivity of free space.

(i) The first term in U_{tot} is a repulsive interaction while the second term in U_{tot} is an attractive interaction. Discuss the physical origin of the two terms. [4]

(ii) Sketch U_{tot} potential as a function of R. [3]

(iii) Calculate the equlibrium nearest-neighbour distance R_0 in terms of N, A, ϵ_0 and α. Also find a simple expression of the equilibrium bonding energy per ion pair. [8]

(iv) For an ionic salt NaCl, α=1.75, R_0=5.63 Å and the measured energy per ion pair is 7.95 eV. Use these numbers to estimate the value of the parameter n for NaCl. [5]

QUESTION 3

(a) State the four basic assumptions of the Drude model. [4]

(b) Deduce a value for the penetration length of the electron wave function outside the metal for electrons of the Fermi energy 5.26 eV, if the work function of the metal is typically 5.26 eV. [5]

(c) The atomic radius of potassium is 2.45 Å. Taking the Bravais lattice of potassium to be bcc, calculate its Fermi energy at 0 K. [5]

(d) Show that the number of energy states available for the electrons in a cubical box of 0.01 m side lying below an energy of 4 eV is 3.62×10^{24}. [6]

(e) The Fermi energy of copper is 7.24 eV at 0 K. Calculate its Fermi energy at 300 K. Comment on the variation of the Fermi energy as a function of temperature. [5]

QUESTION 4

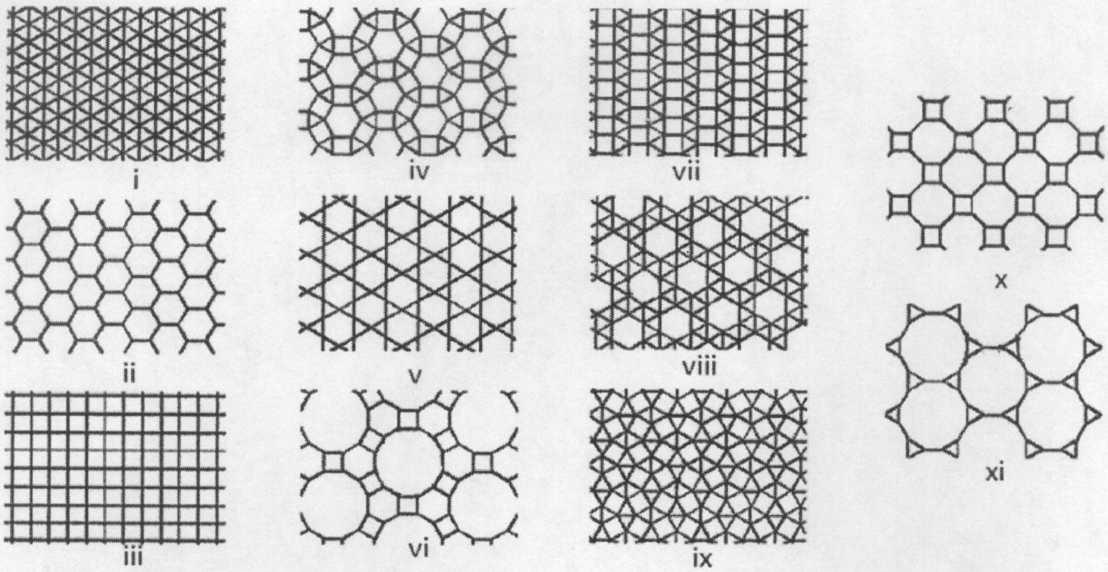
(a) Show that $\mathbf{c}/\mathbf{a} = (8/3)^{1/2}$ for hcp close packing of hard spheres. [5]

(b) Powder specimens of three different monoatomic cubic crystals are analyzed with a Debye-Scherrer camera. It is known that one sample is fcc, one is bcc, and one has the diamond structure. The approximate positions of the first four diffraction rings in each case

A	B	C
28.8°	42.2°	42.8°
41.05°	49.2°	73.15°
50.8°	72.2°	89°
59.6°	87.25°	115.2°

are shown in the table above. Identify the crystal structures of A, B and C. [12]

(c) Consider the two dimensional lattices below. Each of these can be



expressed as a Bravais lattice and a basis. Indicate how many basis elements are required in each case. [9]

QUESTION 5

(a) Derive the temperature dependence of the lattice specific heat capacity predicted by Debye model, and deduce its variation at low and high temperatures. [9]

(b) The Debye temperature and the Fermi energy for copper are 330 K and 8 eV, respectively, while the number of valence electrons per atom is 1. At what temperature are the contributions from the lattice and electronic specific heats equal? [5]

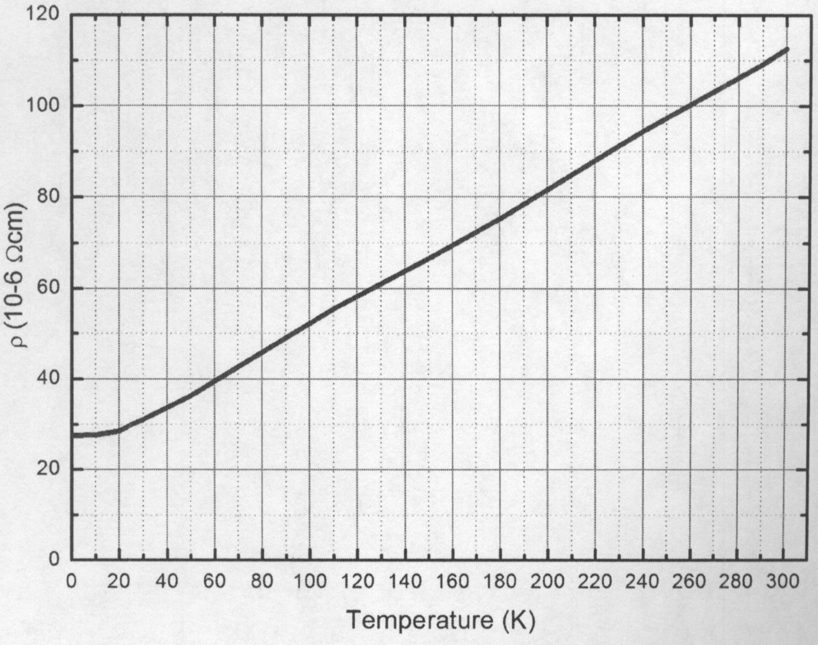
(c) Derive the density of states of a free electron gas in 3D, and sketch it at 0 K and at $T > 0$ K. [6]

(d) State the two relevant frequencies within the framework of the Drude model. Briefly describe the three spectral regimes that results from such a framework. [5]

QUESTION 6

(a) Show that taking the reciprocal of the reciprocal lattice gives the real space lattice again. Also, prove that $|\vec{a}_1.\vec{a}_2 \text{x} \vec{a}_3| = \text{V}$ (volume of a unit cell). [9]

(b) The figure below shows the electrical resistivity of $ErRhB_4$.



(i) From this figure, is this material ($ErRhB_4$) a metal or an insulator. Give a reason for your answer. [2]

(ii) Describe the physical processes that account for resistivity, and explain the temperature dependence of the resistivity at T very very near 0 K, T near 25 K and T near 300 K. [6]

(iii) Estimate the mean free path and the mean free time at T = 0 K and at T = 300 K [Useful numbers: $n = 10^{23}\text{cm}^{-3}$, $e = 5\text{x}10^{-10}\text{esu}$, $v_F = 10^8 cm/s$, $1(\Omega cm)^{-1} = 9x10^{11} esu$, $m = 10^{-27} g$]. [8]

———END OF EXAMINATIONS———

ADITIONAL INFORMATION

Electron charge e=1.602×10^{-19} C, electron mass $m_e = 9.11 \times 10^{-31}$ kg, proton mass $m_p = 1.67 \times 10^{-27}$ kg, neutron mass $m_n = 1.67 \times 10^{-27}$ kg, Avogadro's constant $N_A = 6.022 \times 10^{23}$ mol^{-1}, Boltzmann constant $k_B = 1.38 \times 10^{-23}$ JK^{-1}, Permitivity of free space $\epsilon_0 = 8.85 \times 10^{-12} s^2 m^{-2}$, permeability of free space $\mu_0 = 4\pi \times 10^{-7}$, Bohr magneton $\mu_B = 9.274 \times 10^{-24}$ Am^2, Planck's constant h = 6.62×10^{-34} J/s.

The first four diffraction peaks correspond to (110), (200), (112), and (220) for bcc; (110), (200), (220), and (311) for fcc; and (111),(220), (311), (400) for diamond structure.

$$\vec{b}_1 = 2\pi \frac{\vec{a}_2 * \vec{a}_3}{\vec{a}_1 . \vec{a}_2 * \vec{a}_3}; \vec{b}_2 = 2\pi \frac{\vec{a}_1 * \vec{a}_3}{\vec{a}_1 . \vec{a}_2 * \vec{a}_3}; \vec{b}_3 = 2\pi \frac{\vec{a}_1 * \vec{a}_2}{\vec{a}_1 . \vec{a}_2 * \vec{a}_3}; < n >= \frac{\sum_s s e^{(-s\hbar\omega/k_B T)}}{\sum_s e^{(-s\hbar\omega/k_B T)}};$$

$$\int_0^{+\infty} \frac{x}{e^x - 1} dx = 1.51; \int_0^{+\infty} \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15}; \int_{-\infty}^{+\infty} \frac{x^3 e^x}{(e^x + 1)^2} dx = \frac{\pi^2}{3}$$

Density of states $D(\omega) = \frac{L}{\pi} \frac{dk}{dw}$ in 1 dimension; $D(\omega) = \frac{V k^2}{2\pi^2} \frac{dk}{dw}$ in 3 dimensions.

THE UNIVERSITY OF ZAMBIA
SCHOOL OF NATURAL SCIENCES
Department of Physics
2018/2019 ACADEMIC YEAR
End of Year University Examinations
PHY 4222 - Solid State Physics II

Duration : 3 Hours
Total marks: 100
INSTRUCTIONS
This question paper contains six questions.
Answer any four questions. Each question carries 25 total marks.
Write clearly your computer number on the answer booklet provided.
All working should be shown clearly.
Omission of essential work will lead to loss of marks.
Additional information can be found at the end of this question paper.

QUESTION 1

(a) Begining from $n = \int_{Ec}^{\infty} D_e(E) f_e(E) dE$ (where D_e is the density of states, f_e is the Fermi distibution function), show that

$$n = 2\left(\frac{m_e k_B T}{2\pi\hbar^2}\right) exp\left[\frac{\mu - E_c}{k_B T}\right].$$

[10]

(b) With the aid of sketches, distinguish direct band gap semiconductors from indirect band gap semiconductors. [3]

(c) The electron and hole mobilities in a Si sample are 0.15 and 0.17 $m^2V^{-1}s^{-1}$, respectively.

(i) Determine the conductivity of intrinsic Si at 300 K if the intrinsic carrier concentration is $1.95 x 10^{20}$ atoms/m^3. [2]

(ii) The sample is then doped with $6 x 10^{23}$ antimony atoms/m^3. Determine the equilibrium electron and hole concentrations, the conductivity and the Fermi level relative to the intrinsic level. [6]

(iii) Hence find the resistance of a doped Si rod which is 1 cm long and 1 mm wide at 300K. [4]

....

QUESTION 2

....

(a) Explain the origin of the energy gap using two standing waves $\psi(+)$ and $\psi(-)$. [7]

(b) Derive the Clausius-Mossotti relation that expresses the relationship between the dielectric constant and atomic polarizability. [7]

(c) Using the Kronig-Penney model, show that for $P \ll 1$, the energy of the lowest energy band is

$$E = \frac{\hbar^2 P}{ma^2}.$$

[5]

(d) A sample of silicon is purified until it contains 10^{18} donors m^{-3}. Below what temperature does it cease to show intrinsic behaviour? ($E_G = 1.1$ eV and the intrinsic carrier concentration at 300 K is 2×10^{16} m^{-3}.) [6]

....

QUESTION 3

....

(a) Compare and contrast the parent compounds of electron doped cuprates to hole doped cuprates. [4]

(b) How is superconductivity without doping induced in the parent compounds of electron doped cuprates? [3]

(c) The critical fields at 16 K and 23 K of NbGe alloy are 18 T and 7.5 T, respectively. Determine the transition temperature and the critical field at 0 K. [10]

(d) Estimate the energy gap for Niobium (T_c=9.5 K) at T = 0 and show that the minimum photon wavelength needed to break the Cooper pair is $4.3x10^{-4}$ m. [6]

(e) Which compound holds the highest superconducting transition temperature? State this highest transition temperature recorded. [2]

....

QUESTION 4

....

(a) Derive the classical Langevin law of diamagnetism. [6]

(b) The most important contribution to the paramagnetism of $CuSO_4$ comes from Cu^{2+} ions for which the magnetic moment is due to single unpaired spin (L = 0, J = S = 1/2, g = 2).

(i) Write down the probabilities at temperature T that the moment lies parallel and antiparallel to the field, [2]

(ii) hence show that the magnetization for N ions per unit volume in a magnetic field B is

$$M = N\mu_B tanh(\mu_B B/K_B T).$$

[4]

(iii) Write down the internal energy and hence calculate the magnetic heat capacity C_B of ions in a constant field B. [3]

(iv) Deduce the limiting form of the specific heat at low temperatures. [2]

(c) A ferromagnetic crystal with $J=\frac{5}{2}$ and $g = 2$ has a Curie temperature of 125 K. Calculate the intrinsic flux density near 0 K. Also calculate the ratio of magnetization at 300 K in the presence of an external field of 50 Oe (50 G) to the spontaneous magnetization at 0 K? [8]

....

QUESTION 5

....

(a) With the aid of diagrams differentiate the three types of antiferromagnetism. [6]

(b) Briefly define ferroelectricity and piezoelectricity; and state the difference between insulators and dielectrics. [4]

(c) Find the frequency dependence of the electronic polarizability of an electron having the resonance frequency ω_0, treating the system as a simple harmonic oscillator. [4]

(d) The optical index of refraction and the dielectric constant of sodium chloride crystal are 1.7 and 8.2 respectively. Determine the percentage of electronic polarizability. [7]

(e) Explain the meaning and origin of piezoelectricity. Justify the statement "All ferroelectric crystals are piezoelectric, but all piezoelectric crystals are not ferroelectric". [4]

....

QUESTION 6

....

(a) A magnetic material has a magnetization of 2520 A m^{-1} and produces a flux density of 0.00423 Wb m^{-2}. Calculate the magnetizing force and the relative permeability of the material. [6]

(b) Use the London equation to show that the penetration of a parallel magnetic field into a superconducting film of thickness d in the xy plane is described by

$$B = B_e \frac{\cosh(z/\lambda)}{\cosh(d/\lambda)}$$

where B_e is the applied magnetic field and the centre of the film is at z=0. [9]

(c) The London penetration depth for a superconductor at 3 K and 7.09 K are 39.61 nm and 173 nm respectively. Show that the superconducting temperature of this superconductor is 7.2 K. [9]

(d) What are topological insulators? [1]

........................END OF EXAMINATION........................

ADDITIONAL INFORMATION

Planck's constant h = 6.62×10^{-34} Js, electron charge e = 1.602×10^{-19} C, electron mass $m_e = 9.11 \times 10^{-31}$ kg, proton mass $m_p = 1.67 \times 10^{-27}$ kg, neutron mass $m_n = 1.67 \times 10^{-27}$ kg, Avogadro's constant $A_v = 6.022 \times 10^{23}$ mol^{-1}, Boltzmann constant $k_B = 1.38 \times 10^{-23}$ JK^{-1}, Permitivity of free space $\epsilon_0 = 8.85 \times 10^{-12} s^2 m^{-2}$, permeability of free space $\mu_0 = 4\pi \times 10^{-7}$, Bohr magneton $\mu_B = 9.274 \times 10^{-24}$ Am^2

$$n_i = p_i = 2\left[\frac{2\pi k_B T}{h^2}\right]^{\frac{3}{2}} (m_n^* m_p^*)^{\frac{3}{4}} exp\left(\frac{E_V - E_C}{2k_B T}\right);$$

Or

$$n = n_i exp\left(\frac{E_f - E_i}{k_B T}\right)$$

$$P\frac{\sin \alpha a}{\alpha a} + \cos \alpha a = \cos ka$$

$$M = Ng\mu_B J B_J(y)$$

where

$$y = \left(\frac{g\mu_B J B}{k_B T}\right); B_J(y) \approx \frac{y(J+1)}{3J}$$

$$\frac{\lambda(0)^2}{\lambda(T)^2} = 1 - \left(\frac{T}{T_c}\right)^2$$

$$\frac{\lambda(0)^2}{\lambda(T)^2} = 1 - \left(\frac{T}{T_c}\right)^4$$

$$H_c(T) = H_c(0)\left(1 - \left(\frac{T}{T_c}\right)^2\right)$$

$$\chi = \frac{\mu_0 N \mu_B^2}{3k_B T} g^2 J(J+1)$$

$$L = \sum m_l$$

$$\int_{-\infty}^{+\infty} \frac{x^3 e^x}{(e^x+1)^2} dx = \frac{\pi^2}{3}$$

$$\int_0^{+\infty} \frac{x}{e^x - 1} dx = 1.51$$

$$\int_0^{+\infty} \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15}$$

$$g = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}$$

$$\int_0^{\infty} x^{\frac{1}{2}} e^{-x} dx = \frac{\pi^{\frac{1}{2}}}{2}$$

$$\int_0^{\infty} x^4 e^{-x} dx = 24$$

$$\lambda_k = \frac{\hbar^2 k^2}{2m}$$

$$\alpha = \frac{2mE}{\hbar^2}$$

$$\cos(2\Theta) = \cos^2\Theta - \sin^2\Theta$$

$$\sin(2\Theta) = 2\sin\Theta\cos\Theta$$

$$\tan(2\Theta) = \frac{2\tan\Theta}{1 - \tan^2\Theta}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

THE UNIVERSITY OF ZAMBIA
SCHOOL OF NATURAL SCIENCES
DEPARTMENT OF PHYSICS

2019 ACADEMIC YEAR FINAL EXAMINATION

PHY4442: DIGITAL ELECTRONICS II

TIME: THREE HOURS

MAXIMUM MARKS : 100

Attempt ANY FOUR questions
All questions carry equal marks.
The marks are shown in brackets.

THE UNIVERSITY OF ZAMBIA
SCHOOL OF NATURAL SCIENCES
Department of Physics
2018/2019 ACADEMIC YEAR
Mid Year University Examinations
PHY 4221 - Solid State Physics I

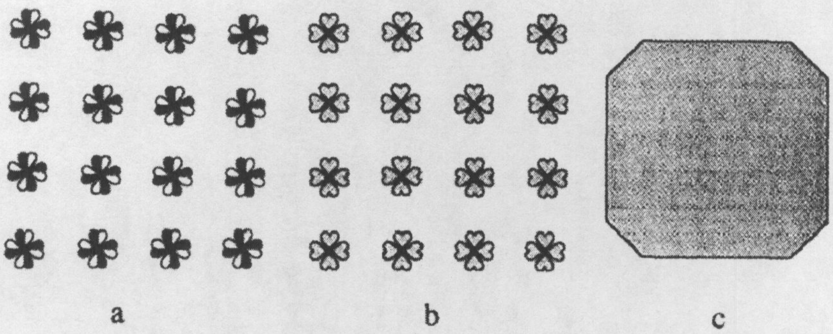
Duration : 3 Hours
Total marks: 100

INSTRUCTIONS

This question paper contains six questions.
Answer any four questions. Each question has 25 total marks.
Write clearly your computer number on the answer booklet provided.
All working should be shown clearly.
Omission of essential work will lead to loss of marks.
Additional information can be found at the end of this question paper.

QUESTION 1

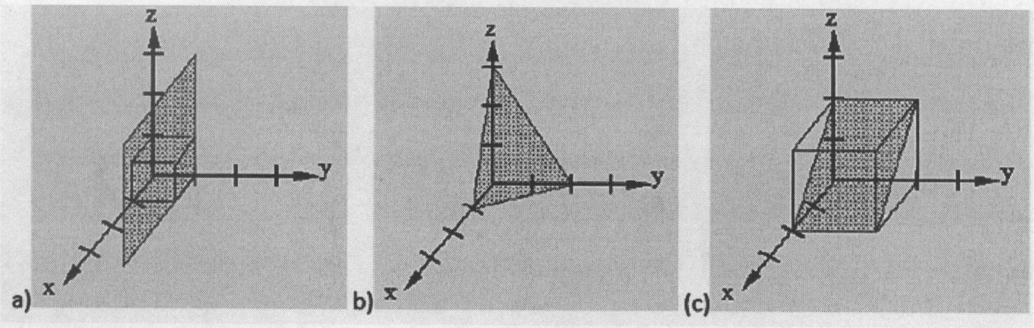
(a) In the figure below, two crystals (a and b) and a polygon (c) are shown. Identify the point group operations of the three objects assuming that the crystals are of infinite size. Show that the point groups of the two crystals are different, and one of them has the equivalent point group as the polygon. [7]



(b) A lattice can posses 2 fold, 3 fold 4 fold and 6 fold rotational symmeties but cannot posses a 5 fold rational symmetry. Prove this statement. [5]

(c) Derive the dispersion relationship for a one dimensional diatomic lattice. [7]

(d) Determine the Miller indices (hkl) of the shaded planes below.Show your work on each step to determine the plane.



[6]

QUESTION 2

(a) (i) Name the three most important kinds of probes used in diffraction experiments on crystals. [3]

(ii) Discuss the essential condition that must be satisfied if such probes are to be useful in understanding crytal structure. [2]

(b) In a quantitative model of bonding in ionic crystals, the total potential energy is assumed to be of the form (in SI units)

$$U_{tot}(R) = N\frac{A}{R^n} - N\frac{\alpha e^2}{4\pi\epsilon_0 R}$$

where N is the number of positive-negative ion pairs, R is the nearest neighbour distance in the material, α is the Madelung constant for the material, A and n are adjustable empirical parameters to fit the experimental data and ϵ_0 is the permitivity of free space.

(i) The first term in U_{tot} is a repulsive interaction while the second term in U_{tot} is an attractive interaction. Discuss the physical origin of the two terms. [4]

(ii) Sketch U_{tot} potential as a function of R. [3]

(iii) Calculate the equlibrium nearest-neighbour distance R_0 in terms of N, A, ϵ_0 and α. Also find a simple expression of the equilibrium bonding energy per ion pair. [8]

(iv) For an ionic salt NaCl, α=1.75, R_0=5.63 Å and the measured energy per ion pair is 7.95 eV. Use these numbers to estimate the value of the parameter n for NaCl. [5]

QUESTION 3

(a) State the four basic assumptions of the Drude model. [4]

(b) Deduce a value for the penetration length of the electron wave function outside the metal for electrons of the Fermi energy 5.26 eV, if the work function of the metal is typically 5.26 eV. [5]

(c) The atomic radius of potassium is 2.45 Å. Taking the Bravais lattice of potassium to be bcc, calculate its Fermi energy at 0 K. [5]

(d) Show that the number of energy states available for the electrons in a cubical box of 0.01 m side lying below an energy of 4 eV is 3.62×10^{24}. [6]

(e) The Fermi energy of copper is 7.24 eV at 0 K. Calculate its Fermi energy at 300 K. Comment on the variation of the Fermi energy as a function of temperature. [5]

QUESTION 4

(a) Show that $\mathbf{c}/\mathbf{a} = (8/3)^{1/2}$ for hcp close packing of hard spheres. [5]

(b) Powder specimens of three different monoatomic cubic crystals are analyzed with a Debye-Scherrer camera. It is known that one sample is fcc, one is bcc, and one has the diamond structure. The approximate positions of the first four diffraction rings in each case

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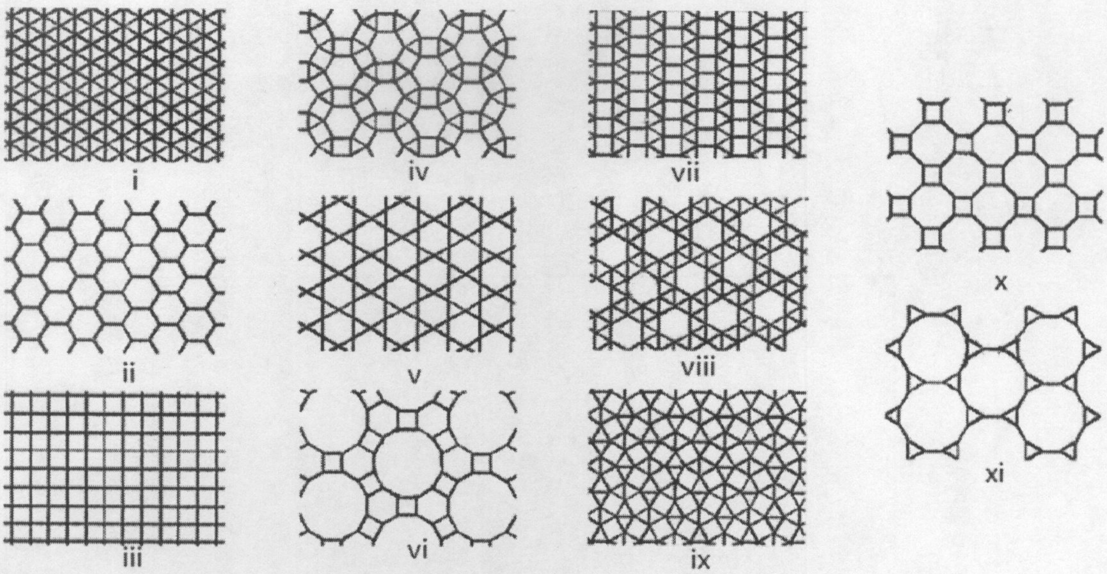
are shown in the table above. Identify the crystal structures of A, B and C. [12]

(c) Consider the two dimensional lattices below. Each of these can be



expressed as a Bravais lattice and a basis. Indicate how many basis elements are required in each case. [9]

QUESTION 5

(a) Derive the temperature dependence of the lattice specific heat capacity predicted by Debye model, and deduce its variation at low and high temperatures. [9]

(b) The Debye temperature and the Fermi energy for copper are 330 K and 8 eV, respectively, while the number of valence electrons per atom is 1. At what temperature are the contributions from the lattice and electronic specific heats equal? [5]

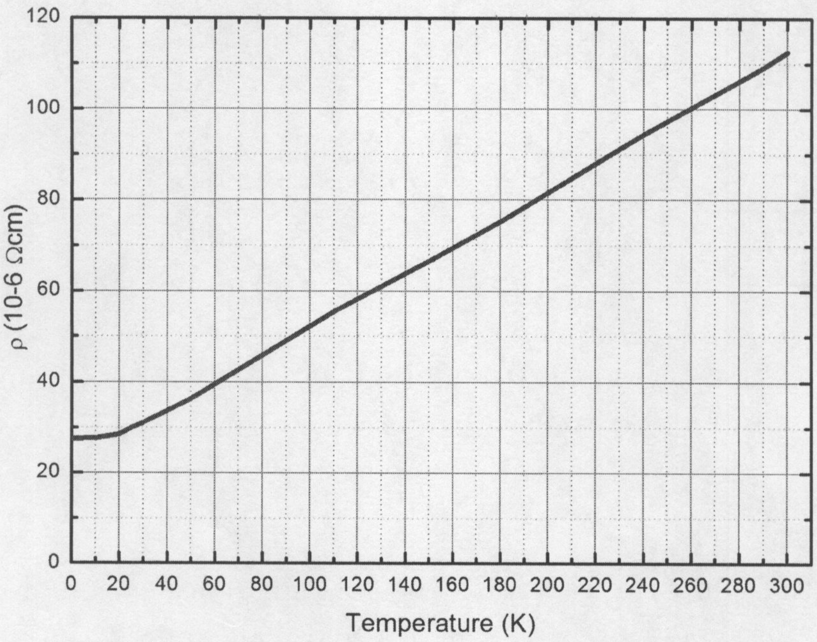
(c) Derive the density of states of a free electron gas in 3D, and sketch it at 0 K and at T > 0 K. [6]

(d) State the two relevant frequencies within the framework of the Drude model. Briefly describe the three spectral regimes that results from such a framework. [5]

QUESTION 6

(a) Show that taking the reciprocal of the reciprocal lattice gives the real space lattice again. Also, prove that $|\vec{a}_1 . \vec{a}_2 x \vec{a}_3| = V$ (volume of a unit cell). [9]

(b) The figure below shows the electrical resistivity of $ErRhB_4$.



(i) From this figure, is this material ($ErRhB_4$) a metal or an insulator. Give a reason for your answer. [2]

(ii) Describe the physical processes that account for resistivity, and explain the temperature dependence of the resistivity at T very very near 0 K, T near 25 K and T near 300 K. [6]

(iii) Estimate the mean free path and the mean free time at T = 0 K and at T = 300 K [Useful numbers: $n = 10^{23}cm^{-3}$, $e = 5x10^{-10}$esu, $v_F=10^8$cm/s, $1(\Omega cm)^{-1}=9x10^{11}$esu, $m=10^{-27}$g]. [8]

———END OF EXAMINATIONS———

ADITIONAL INFORMATION

Electron charge e=1.602x10^{-19} C, electron mass m_e = 9.11x10^{-31} kg, proton mass m_p = 1.67x10^{-27} kg, neutron mass m_n = 1.67x10^{-27} kg, Avogadro's constant N_A = 6.022x10^{23} mol^{-1}, Boltzmann constant k_B = 1.38x10^{-23} JK^{-1}, Permitivity of free space ϵ_0 = 8.85x10^{-12}s^{2}m^{-2}, permeability of free space $\mu_0 = 4\pi$x10^{-7}, Bohr magneton μ_B = 9.274x10^{-24} Am^2, Planck's constant h = 6.62x10^{-34} J/s.

The first four diffraction peaks correspond to (110), (200), (112), and (220) for bcc; (110), (200), (220), and (311) for fcc; and (111),(220), (311), (400) for diamond structure.

$$\vec{b}_1 = 2\pi\frac{\vec{a}_2 * \vec{a}_3}{\vec{a}_1.\vec{a}_2 * \vec{a}_3}; \vec{b}_2 = 2\pi\frac{\vec{a}_1 * \vec{a}_3}{\vec{a}_1.\vec{a}_2 * \vec{a}_3}; \vec{b}_3 = 2\pi\frac{\vec{a}_1 * \vec{a}_2}{\vec{a}_1.\vec{a}_2 * \vec{a}_3}; < n >= \frac{\sum_s se^{(-s\hbar\omega/k_BT)}}{\sum_s e^{(-s\hbar\omega/k_BT)}};$$

$$\int_0^{+\infty} \frac{x}{e^x - 1}dx = 1.51; \int_0^{+\infty} \frac{x^3}{e^x - 1}dx = \frac{\pi^4}{15}; \int_{-\infty}^{+\infty} \frac{x^3 e^x}{(e^x + 1)^2}dx = \frac{\pi^2}{3}$$

Density of states $D(\omega) = \frac{L}{\pi}\frac{dk}{dw}$ in 1 dimension; $D(\omega) = \frac{Vk^2}{2\pi^2}\frac{dk}{dw}$ in 3 dimensions.

8085 / 8080A Instruction summary by Functional Groups

DATA TRANSFER (COPY)

Hex	Mnemonic	Hex	Mnemonic	Hex	Mnemonic	Hex	Mnemonic
40	MOV B,B	58	MOV E,B	70	MOV M,B	1A	LDAX D
41	MOV B,C	59	MOV E,C	71	MOV M,C	2A	LHLD
42	MOV B,D	5A	MOV E,D	72	MOV M,D	3A	LDA
43	MOV B,E	5B	MOV E,E	73	MOV M,E	02	STAX B
44	MOV B,H	5C	MOV E,H	74	MOV M,H	12	STAX D
45	MOV B,L	5D	MOV E,L	75	MOV M,L	22	SHLD
46	MOV B,M	5E	MOV E,M	77	MOV M,A	32	STA
47	MOV B,A	5F	MOV E,A	78	MOV A,B	01	LXI B
48	MOV C,B	60	MOV H,B	79	MOV A,C	11	LXI D
49	MOV C,C	61	MOV H,C	7A	MOV A,D	21	LXI H
4A	MOV C,D	62	MOV H,D	7B	MOV A,E	31	LXI SP
4B	MOV C,E	63	MOV H,E	7C	MOV A,H	F9	SPHL
4C	MOV C,H	64	MOV H,H	7D	MOV A,L	E3	XTHL
4D	MOV C,L	65	MOV H,L	7E	MOV A,M	EB	XCHG
4E	MOV C,M	66	MOV H,M	7F	MOV A,A	D3	OUT
4F	MOV C,A	67	MOV H,A	06	MVI B	DB	IN
50	MOV D,B	68	MOV L,B	0E	MVI C	C5	PUSH B
51	MOV D,C	69	MOV L,C	16	MVI D	D5	PUSH D
52	MOV D,D	6A	MOV L,D	1E	MVI E	E5	PUSH H
53	MOV D,E	6B	MOV L,E	26	MVI H	F5	PUSH PSW
54	MOV D,H	6C	MOV L,H	2E	MVI L	C1	POP B
55	MOV D,L	6D	MOV L,L	36	MVI M	D1	POP D
56	MOV D,M	6E	MOV L,M	3E	MVI A	E1	POP H
57	MOV D,A	6F	MOV L,A	0A	LDAX B	F1	POP PSW

ARITHMETIC

Hex	Mnemonic	Hex	Mnemonic	Hex	Mnemonic	Hex	Mnemonic
80	ADD B	CE	ACI	D6	SUI	23	INX H
81	ADD C	90	SUB B	DE	SBI	33	INX SP
82	ADD D	91	SUB C	09	DAD B	05	DCR B
83	ADD E	92	SUB D	19	DAD D	0D	DCRC
84	ADD H	93	SUB E	29	DAD H	15	DCR D
85	ADD L	94	SUB H	39	DAD SP	1D	DCR E
86	ADD M	95	SUB L	27	DAA	25	DCR H
87	ADD A	96	SUB M	04	INR B	2D	DCR L
88	ADC B	97	SUB A	0C	INR C	35	DCR M
89	ADC C	98	SBB B	14	INR D	3D	DCR A
8A	ADC D	99	SBB C	1C	INR E	0B	DCX B
8B	ADC E	9A	SBB D	24	INR H	1B	DCX D
8C	ADC H	9B	SBB E	2C	INR L	2B	DCX H
8D	ADC L	9C	SBB H	34	INR M	3B	DCX SP
8E	ADC M	9D	SBB L	3C	INR A		
8F	ADC A	9E	SBB M	03	INX B		
C6	ADI	9F	SBB A	13	INX D		

LOGICAL

Hex	Mnemonic	Hex	Mnemonic	Hex	Mnemonic	Hex	Mnemonic
37	STC	A9	XRA C	B3	ORA E	BD	CMP L
A0	ANA B	AA	XRA D	B4	ORA H	BE	CMP M
A1	ANA C	AB	XRA E	B5	ORA L	BF	CMP A
A2	ANA D	AC	XRA H	B6	ORA M	FE	CPI
A3	ANA E	AD	XRA L	B7	ORA A	07	RLC
A4	ANA H	AE	XRA M	F6	ORI	0F	RRC
A5	ANA L	AF	XRA A	B8	CMP B	17	RAL
A6	ANA M	EE	XRI	B9	CMP C	1F	RAR
A7	ANA A	B0	ORA B	BA	CMP D	2F	CMA
E6	ANI	B1	ORA C	BB	CMP E	3F	CMC
A8	XRA B	B2	ORA D	BC	CMP H		

BRANCHING

Hex	Mnemonic	Hex	Mnemonic	Hex	Mnemonic
C3	JMP	D7	RST 2	EC	CPE
C2	JNZ	DF	RST 3	F4	CP
CA	JZ	E7	RST 4	FC	CM
D2	JNC	EF	RST 5	C9	RET
DA	JC	F7	RST 6	C0	RNZ
E2	JPO	FF	RST 7	C8	RZ
EA	JPE	CD	CALL	D0	RNC
F2	JP	C4	CNZ	D8	RC
FA	JM	CC	CZ	E0	RPO
E9	PCHL	D4	CNC	E8	RPE
C7	RST 0	DC	CC	F0	RP
CF	RST 1	E4	CPO	F8	RM

CONTROL

Hex	Mnemonic
00	NOP
76	HLT
F3	DI
FB	EI
20	RIM
30	SIM

THE UNIVERSITY OF ZAMBIA
PHYSICS DEPARTMENT
SECOND SEMESTER EXAMINATIONS 2019
PHY4535 QUANTUM MECHANICS II

TIME: THREE HOURS
ANSWER: ANY FOUR QUESTIONS
MAXIMUM MARKS: 100

The Hamiltonian of the harmonic oscillator is

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}kx^2$$

$$\int_{-\infty}^{\infty} e^{-\beta x^2} dx = \sqrt{\frac{\pi}{\beta}} \text{ and } \int_{-\infty}^{\infty} x^2 e^{-\beta x^2} dx = \frac{1}{2\beta}\sqrt{\frac{\pi}{\beta}}$$

$$\int \cot\theta d\theta = \ln|\sin\theta| + C$$

$$\int_0^{\infty} x^n e^{-\mu x} dx = n!\mu^{-n-1}$$

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1. **(a)** **(i)** Show that the expression for the first-order energy correction in non-degenerate time-independent perturbation theory is

$$E_n^{(1)} = \left\langle \psi_n^{(0)} \left| H' \right| \psi_n^{(0)} \right\rangle$$

where H' is the perturbation. [8]

(ii) A particle moving in a one-dimensional infinite potential well with walls at $x = -a$ and $x = a$ is acted upon by the perturbation $H' = \varepsilon \left| x \right|$, where ε is a very small parameter. This perturbation has the effect of modifying the bottom of the potential. Find the first-order correction to the energy of the ground state. [9]

The even-parity eigenfunctions are given by

$$\psi_n(x) = \sqrt{\frac{1}{a}} \cos \frac{n\pi x}{2a}, \qquad n = 1, 3, 5, ...$$

Note that $|x| = x$ for $x > 0$ while $|x| = -x$ for $x < 0$.

(b) If a system is initially in the state $\psi_a^{(0)}$ of the Hamiltonian H_0 and if the perturbation $\lambda H'(t)$ acts from t_0 to t on the system, transitions will occur to other states $\psi_b^{(0)}$ of H_0 with transition probability amplitudes given to first order by

$$c_{ba}^{(1)} = (i\hbar)^{-1} \int_0^t H'_{ba}(t') \exp(i\omega_{ba} t') dt'$$

where

$$H'_{ba}(t) = \left\langle \psi_b^{(0)} \left| H'(t) \right| \psi_a^{(0)} \right\rangle \text{ and } \omega_{ba} = \frac{E_b^{(0)} - E_a^{(0)}}{\hbar}.$$

Show that if in fact the perturbation is constant and acts during the time interval $0 \leq t \leq t_0$ then the first-order change in the energy of any state is the same as is predicted by time-independent perturbation theory. [8]

2.(a) When a time-independent perturbation H' acts on a system, the first order change $E^{(1)}$ in the energy of an α-degenerate level is obtained from the secular equation

$$\begin{vmatrix} H'_{11} - E^{(1)} & H'_{12} & \dots & H'_{1\alpha} \\ H'_{21} & H'_{22} - E^{(1)} & \dots & H'_{2\alpha} \\ \dots & \dots & \dots & \dots \\ H'_{\alpha 1} & H'_{\alpha 2} & \dots & H'_{\alpha\alpha} - E^{(1)} \end{vmatrix} = 0$$

where

$$H'_{ij} = \left\langle \psi_i^{(0)} \left| H' \right| \psi_j^{(0)} \right\rangle$$