Spectrum-preserving maps on Banach algebras

by

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DECLARATION BY CANDIDATE

I hereby declare that this dissertation is my own work and effort; and that it has not been submitted anywhere for any award. Where other sources of information have been used, they have been acknowledged.

Name : ....

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CERTIFICATE OF APPROVAL BY PROJECT SUPERVISOR

I hereby declare that this dissertation is from Isaac Phiri's own work and effort and all other sources of information used have been acknowledged. This dissertation has been submitted with my approval.

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CERTIFICATE OF APPROVAL BY EXAMINERS

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Abstract

The classical Gleason-Kahane-Żelazko's Theorem gave rise to the so-called Kaplansky's problem. This theorem states that a linear functional ϕ on a complex Banach algebra A is multiplicative (and non-zero) if and only if $\phi(\mathrm{a}) \in \mathrm{sp(a)}$ $(\mathrm{a} \in \mathrm{A})$, where sp(a) denotes the spectrum of a. The so-called Kaplansky's problem is concerned with identifying the Jordan homomorphisms among all linear maps $\Phi : \mathrm{A} \to \mathrm{B}$ between complex Banach algebras A and B in terms of spectra. To this end Kaplansky suggested to translate $\phi(\mathrm{a}) \in \mathrm{sp(a)}$ $(\mathrm{a} \in \mathrm{A})$ such that ϕ is a linear functional for a linear map $\Phi : \mathrm{A} \to \mathrm{B}$ into the property of shrinking the spectrum, that is $\mathrm{sp}\big(\Phi(\mathrm{a})\big) \subset \mathrm{sp(a)}$ $(\mathrm{a} \in \mathrm{A})$, in the case of Banach algebras A and B. In the case of A and B being semisimple, let $\Phi : \mathrm{A} \to \mathrm{B}$ be a surjective linear map with the property of preserving the spectrum, that is $\mathrm{sp}\big(\Phi(\mathrm{a})\big) = \mathrm{sp(a)}$ $(\mathrm{a} \in \mathrm{A})$, it is still an open problem as to whether or not Φ is a Jordan isomorphism from A onto B. There are a number of partial results in the literature regarding spectrum preserving maps that are surjective with respect to semisimple Banach algebras. In this dissertation, we explore four of these results. The first result shows that a surjective spectrum-preserving map between Von Neumann algebras is a Jordan homomorphism. The second result shows that a surjective spectra isometry between finite dimensional semisimple Banach algebras is a Jordan homomorphism. The third result shows that a bijective spectrum-preserving map $\Phi : \mathrm{A} \to \mathrm{B}$ such that A= $\mathcal{B}(\mathrm{X})$ and B= $\mathcal{B}(\mathrm{Y})$ being algebras of all bounded maps defined on Banach spaces X and Y respectively is a Jordan homomorphism. The fourth result shows that a bijective almost multiplicative or anti-multiplicative map $\Phi : \mathrm{A} \to \mathrm{B}$ such that it almost preserves the spectra, where A= $\mathcal{B}(\mathrm{X})$ and B= $\mathcal{B}(\mathrm{Y})$, being algebras of all bounded maps defined on superreflexive Banach spaces X and Y respectively, is a Jordan homomorphism. None of them answers the so-called Kaplansky's problem completely, but they are all motivating results with a view of accelerating the search for an eventual solution. However result four is stronger in implications than the rest.

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Dedication

To Precious Mano, for all your support to me.

CONTENTS

Chapter 1

Introduction

Linear preserver problems have become an area of lively interest in many parts of mathematics. Some of the most popular linear preserver problems are those concerned with describing the linear maps preserving properties related to invertibility.

This subject goes back to 1897 with the seminal work by G. Frobenius describing the linear maps between matrix algebras with the property of preserving the determinant. Since then, a lot of related results have been collected. We refer our readers to surveys [4,5,10,14,18,24].

This dissertation looks at the so-called Kaplansky's problem, a linear preserver problem concerned with maps that preserve properties related to invertibility. In this dissertation, we aimed at giving a comprehensive exploration of the so-called Kaplansky's problem in which four different results have been given and none of them answers the problem in full. The significance of these results is to motivate researchers worldwide with a view of quickening the search for an eventual solution.

Let $\Phi : \mathrm{A} \to \mathrm{B}$ be a linear map from Banach algebra A into Banach algebra B. Then the map will be called a Jordan homomorphism if it satisfies equation $\Phi(\mathrm{a}^2) = \Phi(\mathrm{a})^2$ for all $\mathrm{a} \in \mathrm{A}$. Let A be a unital Banach algebra with 1 as its unit and let $\mathrm{inv}(\mathrm{A})$ be the set of all invertible elements of A. Then the spectrum of element a of Banach algebra A is the set $\mathrm{sp}(\mathrm{a}) = \left\{\lambda \in \mathbb{C} : \mathrm{a} - \lambda 1 \notin \mathrm{inv}(\mathrm{A})\right\}$.

In 1970, Kaplansky gave us the following task which we are referring to as the so-called Kaplansky's problem in this dissertation.

Let A and B be complex unital Banach algebras. Furthermore, let $\Phi : \mathrm{A} \to \mathrm{B}$ be a linear map with the property that $\mathrm{sp}\big(\Phi(\mathrm{a})\big) \subset \mathrm{sp}(\mathrm{a})$ $(\mathrm{a} \in \mathrm{A})$. Is it true that Φ is a Jordan homomorphism?

With the above question in mind, the four possible answers this dissertation has given are aimed at identifying Jordan homomorphisms among all linear maps between complex Banach algebras in terms of spectra.

The formulation of the so-called Kaplansky's problem was aided by the classical Gleason-Kahane-Żelazko's Theorem. This theorem states that a linear functional ϕ on a complex Banach algebra A is multiplicative (and non-zero) if and only if $\phi(\mathrm{a}) \in \mathrm{sp}(\mathrm{a})$ $(\mathrm{a} \in \mathrm{A})$.

Kaplansky translated the condition $\phi(\mathrm{a}) \in \mathrm{sp}(\mathrm{a})$ $(\mathrm{a} \in \mathrm{A})$ for a linear functional ϕ in the above theorem to imply condition $\mathrm{sp}\big(\Phi(\mathrm{a})\big) \subset \mathrm{sp}(\mathrm{a})$ $(\mathrm{a} \in \mathrm{A})$, the property of shrinking the spectrum, for a linear map $\Phi : \mathrm{A} \to \mathrm{B}$ in the case of Banach algebras A and B.

Attempts to answer Kaplansky in affirmative, in the case of $\Phi : \mathrm{A} \to \mathrm{B}$ being unital and invertibility preserving by various scholars, has proved a failure. The answer to this question may be negative in the case when Φ is not surjective or the Banach algebras A and B are not semisimple. We refer our readers to surveys [18,24].

In 2000, Bernard Aupetit pioneered another school of thought on the so-called Kaplansky's problem by narrowing it to Banach algebras that are only semisimple. Survey [5] gives us the Bernard Aupetit's question as follows:

Let $\Phi : \mathrm{A} \to \mathrm{B}$ be a surjective linear map between semisimple Banach algebras with the property of preserving the spectrum, that is $\mathrm{sp}\big(\Phi(\mathrm{a})\big) = \mathrm{sp}(\mathrm{a})$ for every a in A. Is it true that Φ is a Jordan isomorphism from A onto B?

There has been active research on the above open problem formulated by Bernard Aupetit and results gathered so far still show that the open problem is still not solved.

The four results given in this dissertation to answer the so-called Kaplansky's problem are all based on the Bernard Aupetit's open problem stated above and none of them answers the so-called Kaplansky's problem in full.

It is worth noting that the open problem by Bernard Aupetit is merely a tool which this dissertation has used to solve the so-called Kaplansky's problem. Other open problems that may deal with the so-called Kaplansky's problem have not been explored in this dissertation.

The first result (result 3.2.7) shows that a surjective spectrum preserving map between Von Neumann algebras is a Jordan homomorphism. The second result (result 3.3.12) shows that a surjective spectra isometry between finite dimensional semisimple Banach algebras is a Jordan homomorphism. The third result (result 3.4.14) shows that a bijective spectrum-preserving map $\Phi : \mathrm{A} \to \mathrm{B}$, where $\mathrm{A}= \mathcal{B}(\mathrm{X})$ and $\mathrm{B}= \mathcal{B}(\mathrm{Y})$, being algebras of all bounded maps defined on Banach spaces X and Y respectively, is a Jordan homomorphism. The fourth result (Theorems 5.2.3, 5.2.6 , 5.2.7) shows that a bijective almost multiplicative or anti-multiplicative map $\Phi : A \rightarrow B$ such that it almost preserves the spectra, where A= $\mathcal{B}$(X) and B= $\mathcal{B}$(Y), being algebras of all bounded maps defined on superreflexive Banach spaces X and Y respectively, is a Jordan homomorphism.

None of them answers the so-called Kaplansky's problem completely, but they are all motivating results with a view of accelerating the search for an eventual solution. However result four is stronger in implications than the rest.

Below is a summary of what is expected in the remainder of this dissertation.

Chapter 2 is a collection of basic materials which will be useful in presenting the rest of the dissertation. Some of the materials included are definitions, symbols and theorems . An assumption has been made that readers will have some basic knowledge on ring theory, matrix theory, real analysis, functional analysis, topology and complex analysis to enhance the understanding of the contents of this dissertation.

Section 1 of Chapter 3 introduces the fundamental question that Kaplansky asked (item 3.1.1), which has already been outlined above, in what is referred to as the so-called Kaplansky's problem. In this section, we have outlined some of the setbacks to the problem discovered by scholars (Examples 3.1.5, 3.1.6 and 3.1.7). This section ends with the Bernard Aupetit's open problem (item 3.1.8) stated above, an open problem without the setbacks raised in the examples. This is the tool which the four results of this dissertation have been achieved in solving the so-called Kaplansky's problem.

The remainder of Chapter 3 presents three possible answers to the so-called Kaplansky's problem. The first solution (result 3.2.7), in section 1, is on Von Neumann algebras. The second one (result 3.3.12), in section 2, is on spectra isometries. The third one (result 3.4.14), in section 3, is on the invertibility of spectrum-preserving bijective maps. None of them answers the problem completely.

Let A be a unital Banach algebra and $a \in A$. Let $\varepsilon > 0$ be given such that $0 < \varepsilon < 1$. Then the ε-condition spectrum of element $a \in A$ is the set $\sigma_\varepsilon(a) = \left\{\lambda \in \mathbb{C} : \| \lambda 1 - a \| \ \| (\lambda 1 - a)^{-1} \| \geq \frac{1}{\varepsilon}\right\}$ with the convention that $\| \lambda 1 - a \| \ \| (\lambda 1 - a)^{-1} \| = \infty$ if $\lambda \in \mathrm{sp}(a)$.

Let A be a unital Banach algebra and $a \in A$. Given $\varepsilon > 0$, the ε-pseudospectrum is defined to be the set $\mathrm{sp}_\varepsilon(a) = \{\lambda \in \mathbb{C} : \| (a - \lambda 1)^{-1} \| > \varepsilon^{-1}\}$ with the convention $\| (a - \lambda 1)^{-1} \| = \infty$ if $\lambda \in \mathrm{sp}(a)$.

Section 1 of Chapter 4 is devoted to the development of an approximate version of the classical Gleason-Kahane-Żelazko's Theorem for commutative Banach algebras arising from the ε-condition spectrum.

Section 2 of Chapter 4 is devoted to the development of another approximate version of the classical Gleason-Kahane-Żelazko's Theorem for commutative Banach algebras arising from the ε-pseudospectrum.

As mentioned earlier, the classical Gleason-Kahane-Żelazko's Theorem gave rise to the so-called Kaplansky's problem. In particular, the modification done on the classical Gleason-Kahane-Żelazko's Theorem arising from the ε-pseudospectrum is what this dissertation has chosen to present the approximate spectra versions of Chapter 5.

Chapter 5 is devoted to the spectral analysis arising from the ε-pseudospectrum$'s$ approximate classical Gleason-Kahane-Żelazko's Theorem, for commutative Banach algebras, into what we are calling the approximate spectra versions to the so-called Kaplansky's problem. The approximate spectra version of Chapter 5 is an attempt to give an alternative form to the last solution in Chapter 3, result 3.4.14, on the invertibility of spectrum-preserving bijective maps.

This dissertation has given a great deal of attention to the classical Gleason-Kahane-Żelazko's Theorem because this theorem naturally answers the so-called Kaplansky's problem for commutative Banach algebras. By looking at its intrinsic features, we are able to discover salient features of the so-called Kaplansky's problem such as a linear map being non-zero. Besides the above fact, this theorem has directly influenced the development of an approximate version described in Chapter 4 arising from the ε-pseudospectrum, the main tool in the exploration of Chapter 5, the approximate spectra version of the so-called Kaplansky's problem. The approximate version is a more recent approach to the so-called Kaplansky's problem and it is currently offering a new impetus into the problem.

Chapter 6 is the conclusion of this dissertation. In the conclusion, we have once again mentioned the so-called Kaplansky's problem, which is our research question. We have also indicated that because of the so-called Kaplansky's problem being too general, as it may be negative if the linear map involved is not surjective or the Banach algebras involved are not semisimple, the Bernard Aupetit's open question, which does not have this outlined problem, has been used as a tool to achieve the four solutions of this dissertation. We have indicated that the four solutions given in this dissertation are a mere motivation for the search of an eventual solution as themselves are too specific as their solutions are only applicable under the special features attached to their respective Banach algebras.

Chapter 2

Preliminaries

This chapter is a collection of tools to use in this dissertation.

2.1 Banach algebras

This section is a collection of classes of Banach algebras this dissertation has used.

2.1.1 Definition ([24], p. 1)

A complex algebra is a vector space A over the complex field $\mathbb{C}$ together with a mapping $(x, y) \to xy$ of $A \times A$ into A that satisfies the following axioms :

(i) $x(yz) = (xy)z$ for all $x, y, z \in A$.

(ii) $x(y + z) = xy + xz$, $(x + y)z = xz + yz$ for all $x, y, z \in A$.

(iii) $(\alpha x)y = \alpha(xy) = x(\alpha y)$ for all $x, y \in A$ and $\alpha \in \mathbb{C}$.

If in addition, A is a Banach space with norm $\| . \|$ satisfying $\| ab \| \leq \| a \| \| b \|$ for all $a, b \in A$, then A is a complex Banach algebra.

2.1.2 Notation

If A is unital in the sense that it has a multiplicative identity 1_A , we also require that $\| 1_A \| = 1$. Where confusion is unlikely to occur for identity 1_A, the standard practice of writing 1 for this multiplicative identity will be adopted.

2.1.3 Notation

For the space of all bounded linear maps on Banach space X , which itself is a Banach algebra, the symbol $\mathcal{B}(X)$ will be used. The multiplicative identity on $\mathcal{B}(X)$ will be I . In this dissertation, all vector spaces are over the complex field $\mathbb{C}$; and that rings and Banach algebras will be assumed unital.

2.1.4 Definition ([18], p. 3)

A map $a \to a^*$ on Banach algebra A will be called an involution if the following hold:

(i) $(a^*)^* = a$ for all $a \in A$;

(ii) $(a+b)^* = a^* + b^*$ for all $a, b \in A$;

(iii) $(ba)^* = a^* b^*$ for all $a, b \in A$;

(iv) $(\lambda a)^* = \bar{\lambda} a^*$ for all $a \in A$ and for all $\lambda \in \mathbb{C}$.

If $L \subseteq A$ has the property that $a^* \in L$ whenever $a \in L$, then L is self-adjoint.

2.1.5 Definition ([15], p. 108)

Let A be a Banach algebra equipped with an involution satisfying equation

$$\| a^* a \| = \| a \|^2$$

for all $a \in A$. Then A is called a $C^* -$ algebra.

2.1.6 Definition ([15], p. 47)

Let (T_n) be a sequence in $\mathcal{B}(H)$, where H is a Hilbert space. Then (T_n) converges to T, a member of $\mathcal{B}(H)$, in the strong operator topology if for each $x \in H$, we have that $\| T_n x - Tx \| \to 0$ as $n \to \infty$.

2.1.7 Definition ([15], p. 47)

Let (T_n) be a sequence in $\mathcal{B}(H)$, where H is a Hilbert space. Then (T_n) converges to T, a member of $\mathcal{B}(H)$, in the weak operator topology if
$\langle T_n x , y \rangle \to \langle Tx , y \rangle$ for each fixed $x, y \in H$.

2.1.8 Definition ([18], p. 4)

Let H be a Hilbert space. Then a Von Neumann algebra on H is a self-adjoint unital subalgebra A of $\mathcal{B}(H)$ that is closed in the weak operator topology.

2.1.9 Definition ([24], p. 3)

(a) Let A and B be Banach algebras. A map $\Phi : A \to B$ is called an (algebra) homomorphism if the following conditions hold for all $a, b \in A$ and $\alpha \in \mathbb{C}$:
(i) $\Phi(a+b) = \Phi(a) + \Phi(b)$;
(ii) $\Phi(\alpha a) = \alpha \Phi(a)$;
(iii) $\Phi(ab) = \Phi(a)\Phi(b)$;
(iv) $\Phi(1) = 1$.

(b) Let A and B be Banach algebras. A map $\Phi : A \to B$ is called an (algebra) anti-homomorphism if the following four conditions hold for all a,b $\in$ A and $\alpha \in \mathbb{C}$:
(i) $\Phi(a + b) = \Phi(a) + \Phi(b)$;
(ii) $\Phi(\alpha a) = \alpha\Phi(a)$;
(iii) $\Phi(ab) = \Phi(b)\Phi(a)$;
(iv) $\Phi(1) = 1$.

2.1.10 Definition

Definition 2.1.9(a) can be broken into the following categories: monomorphism, epimorphism, isomorphism and automorphism. We now give their definitions in full.

(i) An injective (one-one) homomorphism is called a monomorphism ;

(ii) A surjective (onto) homomorphism is called an epimorphism ;

(iii) Let A and B be Banach algebras. A map $\Phi : A \to B$ is called an (algebra) isomorphism if Φ is a bijective homomorphism. The term bijective means both injective (one-to-one) and surjective (onto);

(iv) Let $\Phi : A \to A$ be an isomorphism of a Banach algebra A onto itself. Then Φ will be called a Banach algebra automorphism.

2.1.11 Definition

Definition 2.1.9(b) can be broken into the following categories: anti-isomorphism and anti-epimorphism. We now give their definitions in full.

(i) A surjective (onto) anti-homomorphism is called an anti-epimorphism;

(ii) The map $\Phi : A \to B$ is called an (algebra) anti-isomorphism if Φ is a bijective anti-homomorphism.

2.2 Superreflexive Spaces and Standard Operator Algebras

This section introduces the use of standard operator algebras in superreflexive Banach spaces. This idea will be used in the proofs of Theorem 5.2.3 and Theorem 5.2.6 in identifying the maps that preserve the spectrum with respect to the ε-pseudospectrum.

2.2.1 Definition ([2], p. 37)

Let $\mathbb{I}$ be an infinite set. Choose a non-empty family, say $\mathcal{F}$, of subsets of $\mathbb{I}$ such that the following hold :

(i) $\emptyset \notin \mathcal{F}$;

(ii) $\mathcal{F}$ is closed by finite intersection of its members;

(iii) Let $A \in \mathcal{F}$ and $A \subset B$ where B is arbitrary in $\mathbb{I}$; then $B \in \mathcal{F}$.

Such a set $\mathcal{F}$ satisfying conditions (i), (ii) and (iii) will be called a filter on $\mathbb{I}$.

A filter that is maximal among all filters by set inclusion will be called an ultrafilter. Throughout this thesis, the symbol $\mathcal{U}$ will denote an ultrafilter. All ultrafilters on $\mathbb{N}$ will be free ultrafilters. This is equivalent to saying that the complement of each ultrafilter will be a finite set in $\mathbb{N}$.

2.2.2 Definition ([2], p. 38)

Let X be a topological space. Let (x_n) be a sequence in X such that $\lim_{\mathcal{U}} x_n = l$ in X, then l is the $\mathcal{U}$- limit of (x_n) if for every neighborhood V of l , the set $\{ n \in \mathbb{N} : x_n \in V \}$ is in the ultrafilter $\mathcal{U}$.

One usage of ultrafilters is in the creation of ultrapower sets. In the creation of ultrapower sets, we first consider Banach space $\mathcal{L}^\infty(X)$ of all bounded sequences (x_n) with $x_n \in X$ $(n \in \mathbb{N})$ equipped with the norm $\| (x_n) \| = \sup \| x_n \|$.

2.2.3 Definition ([4], p. 237)

Let $\mathcal{U}$ be an ultrafilter on $\mathcal{L}^\infty(X)$. Then we define the set $\mathcal{N}_{\mathcal{U}}(X)$ as

$$\left\{ (x_n) \in \mathcal{L}^\infty(X) : \lim_{\mathcal{U}} \| x_n \| = 0 \right\} .$$

If it is clear to which $\mathcal{N}_{\mathcal{U}}(X)$ we are referring to, we do not include the underlining Banach space in our notation. We write it as $\mathcal{N}_{\mathcal{U}}$.

2.2.4 Definition ([4], p. 237)

The quotient Banach space $\mathcal{L}^\infty(X)/\mathcal{N}_{\mathcal{U}} = \left\{ \mathbf{y} = (y_n) + \mathcal{N}_{\mathcal{U}} : (y_n) \in \mathcal{L}^\infty(X) \right\}$ is called the ultrapower of X with respect to $\mathcal{U}$. We denote it by $X^{\mathcal{U}}$.

The norm on $X^{\mathcal{U}}$ will be given by $\lim_{\mathcal{U}} \| x_n \| = \| \mathbf{x} \|$ where $\mathbf{x} = (x_n) + \mathcal{N}_{\mathcal{U}}$. Every bounded sequence of elements in X which is defined only almost everywhere on $\mathbb{N}$ can be thought of (without any confusion) an element of $X^{\mathcal{U}}$.

2.2.5 Theorem ([2], Theorem 1.36)

Let X be a Banach space. Then $X^{\mathcal{U}}$ contains X as a closed vector space.

2.2.6 Notation

If X is a vector space, then a vector space consisting of all linear functionals defined on X will be denoted by X^*. The set X^* will be referred to as the dual of X. We will denote the dual of X^* with symbol X^{**} if it exists.

We now give some three facts arising from duality theory in Banach spaces as given by [12] on page 107 :

(i) For fixed $x \in X$ and for all $f \in X^*$, there exist a $g_x \in X^{**}$ such that $g_x(f) = f(x)$;

(ii) To each $x \in X$, there exists a canonical map C related to a corresponding $g_x \in X^{**}$ by formula $Cx = g_x$;

(iii) If the range of the canonical map C is the whole of Banach space X^{**}, then X is reflexive.

2.2.7 Definition ([21], p. 235)

A superreflexive Banach space is a Banach space X such that any Y finitely representable in X is reflexive.

2.2.8 Theorem ([21], Theorem 2.3)

Let X be a Banach space and X^* its dual. Then $(X^*)^{\mathcal{U}} = (X^{\mathcal{U}})^*$ if and only if X is *superreflexive*.

2.2.9 Definition ([2], p. 124)

An operator $\Phi : X \to Y$ between vector spaces is said to be a rank one operator if its range is one dimensional.

From this definition, we see that if $y \neq 0$ in Y (the range of Φ), then there exists a uniquely determined linear functional $f \in X^*$ on X satisfying $\Phi(x) = f(x)y$ for each $x \in X$.

2.2.10 Notation

We use the following notation $\big(f \otimes y\big)(x)$ for $f(x)y$. Thus $\Phi = f \otimes y$ whenever $\Phi(x) = \big(f \otimes y\big)x = f(x)y$.

2.2.11 Definition ([2], p. 124)

An operator $\Phi : X \to Y$ between two vector spaces is said to be a finite rank operator if its range is finite dimensional.

Thus if $\Phi : X \to Y$ is a finite rank operator and $y_1, \ldots\ldots, y_k$ is a basis in the range $\Phi(X)$, then there exists (uniquely determined) linear functionals $f_1, \ldots\ldots, f_k$ on X such that $\Phi = \sum_{i=1}^{k} f_i \otimes y_i$ implies $\Phi(x) = \sum_{i=1}^{k} f_i(x) y_i$.

2.2.12 Definition ([24], p. 2)

Let I be a non empty subset of a complex Banach algebra A. Then I is called a left ideal of A if the following conditions hold :

(i) If $a, b \in I$, then $a + b \in I$.

(ii) If $a \in I$ and $\alpha \in \mathbb{C}$, then $\alpha a \in I$.

(iii) If $a \in I$ and $r \in A$, then $ra \in I$.

Similarly, a non empty subset I of a complex Banach algebra A is called a right ideal of A if the following conditions hold :

(i) If $a, b \in I$, then $a + b \in I$.

(ii) If $a \in I$ and $\alpha \in \mathbb{C}$, then $\alpha a \in I$.

(iii) If $a \in I$ and $r \in A$, then $ar \in I$.

We will use the word ideal to imply that the ideal is a left ideal as well as a right ideal.

2.2.13 Definition ([18], p. 7)

Let X be a Banach space. By a standard operator algebra on Banach space X, we mean any closed unital sub algebra A of $\mathcal{B}(X)$ that contains the ideal $\mathcal{F}(X)$ of finite rank operators .

The set $F_n(X)$ will be a collection of all operators whose rank is atmost n for some $n \in \mathbb{N}$.

2.2.14 Lemma ([4], Lemma 2.2)

Let X be a Superreflexive Banach space. Then the Banach algebras $\mathcal{B}(X)^{\mathcal{U}}$ and $\{\mathbf{T}^* : \mathbf{T} \in \mathcal{B}(X)^{\mathcal{U}}\}$ are unital standard operator algebras on $X^{\mathcal{U}}$ and $(X^{\mathcal{U}})^*$ respectively .

Proof

Let $\mathbf{T} \in \mathcal{B}(X)^{\mathcal{U}}$ be with finite rank. Then we can write

$$\mathbf{T} = \sum_{j=1}^{J} \mathbf{x}_j \otimes \mathbf{f}_j = \mathbf{x}_1 \otimes \mathbf{f}_1 + \mathbf{x}_2 \otimes \mathbf{f}_2 + \ldots\ldots + \mathbf{x}_J \otimes \mathbf{f}_J$$

with $\mathbf{x}_j = (x_n^j) \in X^{\mathcal{U}}$ and $\mathbf{f}_j \in (X^{\mathcal{U}})^* (j = 1, 2, \ldots.., J)$. Since $(X^{\mathcal{U}})^* = (X^*)^{\mathcal{U}}$ by Theorem 2.2.8 , it follows that $\mathbf{f}_j = (f_n^j) \in (X^*)^{\mathcal{U}}$ $(j = 1, 2, \ldots.., J)$ and therefore

$$\mathbf{T} = \sum_{j=1}^{J} x_n^j \otimes f_n^j = x_n^1 \otimes f_n^1 + x_n^2 \otimes f_n^2 + \ldots\ldots\ldots. + x_n^J \otimes f_n^J \in \mathcal{B}(X)^{\mathcal{U}}$$

Let $\mathbf{S} \in \mathcal{B}(\mathrm{X}^{\mathcal{U}})^*$ be with finite rank . Then we can write $\mathbf{S} = \sum_{j=1}^{J} \mathbf{f}_j \otimes \mathbf{F}_j$ with $(\mathrm{f}_n^j) \in (\mathrm{X}^{\mathcal{U}})^*$ and $\mathbf{F}_j \in (\mathrm{X}^{\mathcal{U}})^{**}(j = 1, 2, ..., J)$. Since $(\mathrm{X}^{\mathcal{U}})^* = (\mathrm{X}^*)^{\mathcal{U}}$, then $(\mathrm{f}_n^j) \in (\mathrm{X}^*)^{\mathcal{U}}$ $(j = 1, 2, ..., J)$. Let ι be the natural embedding of $\mathrm{X}^{\mathcal{U}}$ into $(\mathrm{X}^{\mathcal{U}})^{**}$. Since X is superreflexive, then there exist a $\mathrm{Y} \subseteq \mathrm{X}$ with the property that it is finitely representable in X (see Definition 2.2.7) . Since $\mathrm{X} \subseteq \mathrm{X}^{\mathcal{U}}$ by Theorem 2.2.5 and $\mathrm{Y} \subseteq \mathrm{X}$, it follows that $\mathrm{Y} \subseteq \mathrm{X}^{\mathcal{U}}$. This confirms that $\mathrm{X}^{\mathcal{U}}$ is reflexive. We can write $\mathbf{F}_j = \iota(\mathbf{x}_j)$ for some $\mathbf{x}_1 = (\mathrm{x}_n^1), ..., \mathrm{x}_J = (\mathrm{x}_n^J) \in \mathrm{X}^{\mathcal{U}}$.

Let $\mathbf{T} = \sum_{j=1}^{J} \left(\mathrm{x}_j \otimes \mathrm{f}_j\right) \in \mathcal{B}(\mathrm{X})^{\mathcal{U}}$. Then $\mathbf{S} = \mathbf{T}^*$.

2.3 Surjectivity modulus

This section introduces the Open Mapping Theorem and its surjectivity modulus. For the case of bounded operators, the operator norm is the required surjectivity modulus. The ultimate goal is to use this constant in the proofs of Theorem 5.2.3 and Theorem 5.2.6 .

2.3.1 Notation ([4], p. 239)

We use notation $\mathbb{B}_{\mathrm{X}}$ to mean a unit ball on set X . For a unit ball on $\mathbb{C}$, we use the symbol $\mathbb{D}$.

2.3.2 Definition ([12], p. 247)

A subset M of a metric space X is said to be

(a) rare (or nowhere dense) in X if its closure $\overline{\mathrm{M}}$ has no interior points.

(b) meager (or of the first category) in X if M is the union of countably many sets each of which is rare in X,

(c) non-meager (or of the second category) in X if M is not meager in X.

2.3.3 Definition ([23], p. 41)

Let M be a complete normed linear space. Let Φ be a continuous mapping of M onto a normed space Y . Let Y be non-meager in itself, then Φ is open and Y is also complete.

2.3.4 Theorem ([12], p. 286)

A bounded linear operator from a Banach space X onto a Banach space Y is an open mapping.
Hence if Φ is bijective , Φ^{-1} is continuous and thus bounded.

For the proof of the Open Mapping Theorem, we refer our readers to [12]. A consequence of the Open Mapping Theorem is the set inclusion

$$\varrho\, \mathbb{B}_{\mathrm{Y}} \subset \mathrm{T}\big(\mathbb{B}_{\mathrm{X}}\big) \quad \text{for} \quad \text{some} \quad \varrho > 0\ ,$$

a result we need in the proof of Lemma 2.3.8

2.3.5 Definition ([4] p. 239 , [16] p. 1)

Let X and Y be Banach spaces and let $\mathrm{T} \in \mathcal{B}(\mathrm{X}, \mathrm{Y})$. Then the surjectivity modulus of T is defined by

$$\kappa(\mathrm{T}) \ = \ \sup\Big\{\delta \ \geq \ 0 :\ \delta \mathbb{B}_{\mathrm{Y}} \subset \mathrm{T}\big(\mathbb{B}_{\mathrm{X}}\big)\Big\}\ .$$

2.3.6 Example

First and foremost, Definition 2.3.5 entails us that we need a unit ball on the domain and another one on the range . Furthermore, $\mathrm{T} \in \mathcal{B}(\mathrm{X}, \mathrm{Y})$ should be surjective with respect to Banach spaces X and Y . In this case, the operator norm $\|\ \mathrm{T}\ \|$ is the required surjectivity modulus.

2.3.7 Proposition ([4], p. 239)

Let $\mathrm{T} \in \mathcal{B}(\mathrm{X}, \mathrm{Y})$. If T is invertible, then $\kappa(\mathrm{T}) \ = \ \|\ \mathrm{T}\ \| \ = \ \|\ \mathrm{T}^{-1}\ \|^{-1}$.

This statement is given without proof in [4]. We give a proof here.

Proof

Since the operation $\mathcal{B}(\mathrm{X}, \mathrm{Y}) \ \rightarrow \ \mathcal{B}(\mathrm{Y}, \mathrm{X})$ defined by $\mathrm{T} \ \rightarrow \ \mathrm{T}^{-1}$ is continuous with respect to the norm $\|\ .\ \|$, it follows that $\|\ \mathrm{T}\ \| \ = \ \|\ (\mathrm{T}^{-1})^{-1}\ \| \ = \ \|\ \mathrm{T}^{-1}\ \|^{-1}$.

2.3.8 Lemma ([4], p. 239)

Let X and Y be complex Banach spaces and let $\mathbf{T} \ = \ (\mathrm{T}_{\mathrm{n}}) \ \in \ \mathcal{B}(\mathrm{X}, \mathrm{Y})^{\mathcal{U}} \ \subset \ \mathcal{B}(\mathrm{X}^{\mathcal{U}}\ ,\ \mathrm{Y}^{\mathcal{U}})$. Then $\kappa(\mathbf{T}) \ = \ \lim_{\mathcal{U}} \kappa(\mathrm{T}_{\mathrm{n}})$.

Proof

Pick an $\mathrm{S} \in \mathcal{B}(\mathrm{Y})$ with $\|\ \mathrm{S}\ \| \ = \ 1$. Then for fixed $\mathrm{T} \in \mathcal{B}(\mathrm{Y})$, it follows that $\|\ \mathrm{ST}\ \| \ \geq \ \|\ \mathrm{T}\ \|$.

From the above fact, we can say that

$$\kappa(\mathrm{T}) \ = \ \inf\Big\{\ \|\ \mathrm{ST}\ \| \ :\ \mathrm{S} \in \mathcal{B}(\mathrm{Y}),\ \|\ \mathrm{S}\ \| \ = \ 1\Big\} \tag{2.1}$$

Let $\varepsilon \ > \ 0$. On account of (2.1), for each $\mathrm{n} \in \mathbb{N}$, there exist $\mathrm{S}_{\mathrm{n}} \in \mathcal{B}(\mathrm{Y})$ with $\|\ \mathrm{S}_{\mathrm{n}}\ \| \ = \ 1$ and

$$\|\ \mathrm{S}_{\mathrm{n}}\mathrm{T}_{\mathrm{n}}\ \| \ < \ \kappa(\mathrm{T}_{\mathrm{n}}) \ + \ \varepsilon\ , \quad \text{by definition of infimum .}$$

Let $\mathbf{S} = (S_n) \in \mathcal{B}(Y)^{\mathcal{U}}$. Then $\| \mathbf{S} \| = 1$ and therefore (2.1) yields

$$\kappa(\mathbf{T}) \leq \| \mathbf{ST} \| = \lim_{\mathcal{U}} \| S_n T_n \| \leq \lim_{\mathcal{U}} \kappa(T_n) + \varepsilon.$$

This gives

$$\kappa(\mathbf{T}) \leq \lim_{\mathcal{U}} \kappa(T_n) \tag{2.2}$$

Conversely, if $\lim_{\mathcal{U}} \kappa(T_n) = 0$, then $\kappa(\mathbf{T}) = 0$ and such an inequality holds. We now assume that $0 < \lim_{\mathcal{U}} \kappa(T_n)$. Let $0 < \varrho < \lim_{\mathcal{U}} \kappa(T_n)$. Then $\varrho < \kappa(T_n)$. Since $\varrho < \kappa(T_n)$, then Theorem 2.3.4 shows us that $\varrho \mathbb{B}_Y \subset T_n\big(\mathbb{B}_X\big)$. Let

$$\mathbf{y} = (y_n) \in \mathbb{B}_{Y^{\mathcal{U}}} \text{ and } 0 < r < 1 .$$

Then $\lim_{\mathcal{U}} \| y_n \| \leq 1$ and therefore $r\, y_n \in \mathbb{B}_Y$. Hence, for each $n \in \mathbb{N}$, there is an $x_n \in \mathbb{B}_X$ with $r\varrho y_n = T_n(x_n)$. Then

$\mathbf{x} = (x_n) \in \mathbb{B}_{X^{\mathcal{U}}}$ satisfies $r\varrho \mathbf{y} = \mathbf{T}(\mathbf{x})$. Hence $r\varrho \leq \kappa(\mathbf{T})$. Letting $r \to 1$ first, then we have $\varrho \leq \kappa(\mathbf{T})$. Secondly letting $\lim_{\mathcal{U}} \kappa(T_n) \to \varrho$, it follows that

$$\lim_{\mathcal{U}} \kappa(T_n) \leq \kappa(\mathbf{T}) \tag{2.3}$$

Combining (2.2) and (2.3), we have

$$\kappa(\mathbf{T}) = \lim_{\mathcal{U}} \kappa(T_n) \quad \text{as required.}$$

2.4 Complex analysis

In this section, we look at some of the results from Complex analysis that have a bearing on this dissertation. Of particular importance, to which Complex analysis has been applied, are the two approximate versions of the classical Gleason-Kahane-Żelazko's Theorem for commutative Banach algebras used in this dissertation. These are the ε-condition spectrum′s approximate classical Gleason-Kahane-Żelazko's Theorem for commutative Banach algebras and the ε-pseudospectrum′s approximate classical Gleason-Kahane-Żelazko's Theorem for commutative Banach algebras of Chapter 4. In both versions, the process leading to the modification of the Gleason-Kahane-Żelazko's Theorem involves the usage of analytic functions of the form of Theorem 2.4.3 whose genus (Definition 2.4.5) is of finite order. In this section, we have also included the Jensen's Theorem (Theorem 2.4.9), a theorem essential to justify that the linear functionals involved within the ε- condition spectrum and the ε-pseudospectrum environments are δ-almost multiplicative. The approximate spectra versions of Chapter 5 are a consequence of Chapter 4, in particular, the ε-pseudospectrum′s approximate classical Gleason-Kahane-Żelazko's Theorem for commutative Banach algebras.

2.4.1 Lemma ([6] p. 94 , [8] Lemma III.3.7)

Let $f : \mathbb{E} \to \mathbb{C}$ be an analytic function such that $f(0) = 0$ where $\mathbb{E}$ is an open unit ball with $| f(z) | < 1$ for all $z \in \mathbb{E}$. Then

(a) $| f'(0) | \leq 1$

(b) $| f(z) | \leq | z |$ for all $z \in \mathbb{E}$.

Moreover if $| f(z) | = | z |$ for some non-zero $z \in \mathbb{E}$ or $| f'(0) | = 1$, then $f(z) = \exp(i\theta z)$ for some $\theta \in \mathbb{R}$ and all $z \in \mathbb{E}$.
This is the classical Schwarz's Lemma.

2.4.2 Theorem ([3], p. 183 ; [26], Theorem 3.1)

Let $f(z)$ be an entire function with no zeros. Then $f(z) = \exp\Big(g(z)\Big)$ for some entire function g .

2.4.3 Theorem ([26], Theorem 3.2)

Let (α_n) be a sequence of non-zero complex numbers such that $| \alpha_n | \to \infty$. Then there is a sequence (m_n) of non-negative integers such that the infinite product $f(z) = z^k \exp\big(g(z)\big)\,(1 - \frac{z}{\alpha_n}) \exp\Big(\frac{z}{\alpha_1} + \frac{1}{2}(\frac{z}{\alpha_2})^2 + ... + \frac{1}{m_n}(\frac{z}{\alpha_n})^{m_n}\Big)$ defines an entire function f.

2.4.4 Definition ([8], p. 212)

Let $(\alpha_n) \in \mathbb{C}$ be a sequence of non-zero complex numbers such that $| \alpha_n | \to \infty$, (m_n) be a sequence of non-negative integers and $z \in \mathbb{C}$. Let function f be defined as in Theorem 2.4.3 . Then the rank of f is defined as the smallest integer (m_n) for which $\sum_{n=1}^{\infty}\Big(\frac{|z|}{|\alpha_n|}\Big)^{m_n+1} < \infty$.

2.4.5 Definition ([26], p. 10)

Let $f(z) = z^k \exp\big(g(z)\big)\,(1 - \frac{z}{\alpha_n}) \exp\Big(\frac{z}{\alpha_1} + \frac{1}{2}(\frac{z}{\alpha_2})^2 + ... + \frac{1}{m_n}(\frac{z}{\alpha_n})^{m_n}\Big)$ be defined as in 2.4.3 . The maximum between the rank of f (Definition 2.4.4) and the degree of $g(z)$ is called the genus of f .

2.4.6 Definition ([3], p. 208)

Let $M(r)$ be the maximum of $| f(z) |$ on $| z | = r$. The order of an entire function $f(z)$ is the smallest λ such that inequality $M(r) \leq \exp\big(r^{\lambda + \varepsilon}\big)$ holds as long as $r \to \infty$ for any $\varepsilon > 0$. Clearly, this definition is equivalent to equation $\limsup_{|z|\to\infty} \frac{\ln\Big(\ln M(r)\Big)}{\ln |z|} = \lambda$.

2.4.7 Examples

(i) The function $f(z) = z$ is of order 0.

(ii) The function $f(z) = \exp(z) =$ is of order 1.

2.4.8 Theorem ([26], Theorem 3.5)

If f is an entire function with finite order λ, then its genus cannot exceed λ in Definition 2.4.6

This theorem is known as the Hadamard's Factorization Theorem. We will apply this theorem in Theorem 4.1.1

2.4.9 Theorem ([3], p. 208)

Suppose $f(z)$ is analytic inside and on circle $|z| = r$ except for zeros at $a_1, a_2, \ldots, a_n$ repeated according to multiplicity and suppose $f(0)$ is finite and different from zero. Then

$$\frac{1}{2\pi}\int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta = \log |f(0)| + \sum_{k=1}^{n} \log\left(\frac{r}{|a_k|}\right).$$

This is the Jensen's Theorem.

2.5 Some useful spectral properties

This section introduces some useful spectral properties required in this dissertation. The spectral mapping theorem is specifically meant to support the proof of Lemma 3.2.2 of Chapter 3. The ε-pseudospectral properties are meant to support most of the arguments of Chapter 5.

2.5.1 Definition ([18], p. 4)

Let A be a unital Banach algebra. Denote $inv(A)$, the set of all invertible elements of A. The spectrum of element a of Banach algebra A is the set

$$sp(a) = \{\lambda \in \mathbb{C} : a - \lambda 1 \notin inv(A)\},$$

while the spectra radius of a is the set

$$r(a) = \sup\{ |\lambda| : \lambda \in sp(a)\}.$$

2.5.2 Theorem ([24] Theorem 1.5.26 (iv))

Let $H(\Omega)$ be the set of all analytic functions on an open set Ω of $\mathbb{C}$. Let A be a Banach algebra and $a \in A$. Additionally, if Ω contains $sp(a)$ and an arbitrary smooth contour c of Ω surrounding $sp(a)$, then the mapping

$$\phi \longrightarrow \phi(a) = \frac{1}{2\pi i}\oint_c \frac{\phi(\lambda)}{\lambda - a} \, d\lambda$$

from $H(\Omega)$ into A satisfies equation $\mathrm{sp}\,\big(\phi(\mathrm{a})\big) = \phi\,\big(\mathrm{sp}(\mathrm{a})\big)$. This theorem is referred to as the Spectral Mapping Theorem .

We remind our readers that in Complex analysis,

$$\phi(\mathrm{a}) = \frac{1}{2\pi \mathrm{i}} \oint_{\mathrm{c}} \frac{\phi(\lambda)}{\lambda - \mathrm{a}}\, \mathrm{d}\lambda$$

is the famous Cauchy integral formula. We refer our readers to Theorem 2.6.1 of [1] and its proof for a recap.

2.5.3 Definition ([4] p. 236, [22] p. 13)

Let A be a unital Banach algebra. We define ε-pseudospectrum of $\mathrm{a} \in \mathrm{A}$ as the set

$$\mathrm{sp}_\varepsilon(\mathrm{a}) = \{\lambda \in \mathbb{C} \;:\; \|\, (\mathrm{a} - \lambda 1)^{-1} \,\| > \varepsilon^{-1}\} \text{ given } \varepsilon > 0 \,.$$

On the other hand if $\lambda \in \mathrm{sp}(\mathrm{a})$, then $\|\, (\mathrm{a} - \lambda 1)^{-1} \,\| = \infty$.

2.5.4 Lemma

Let A be a unital Banach algebra. Then $\mathrm{sp}(\mathrm{a}) \subset \mathrm{sp}_\varepsilon(\mathrm{a})$ $(\mathrm{a} \in \mathrm{A}\,,\ \varepsilon > 0)$.

Proof

If $\lambda \notin \mathrm{sp}_\varepsilon(\mathrm{a})$, then $\|\, (\mathrm{a} - \lambda 1)^{-1} \,\| \leq \varepsilon^{-1}$. This implies $\lambda \notin \mathrm{sp}(\mathrm{a})$, since $\lambda \in \mathrm{sp}(\mathrm{a})$ implies $\|\, (\mathrm{a} - \lambda 1)^{-1} \,\| = \infty$. Thus $\mathrm{sp}(\mathrm{a}) \subset \mathrm{sp}_\varepsilon(\mathrm{a})$.

2.5.5 Lemma ([4], Lemma 2.1(1))

Let A be a unital Banach algebra. $\mathrm{sp}_\epsilon(\mathrm{a})$ is open $(\mathrm{a} \in \mathrm{A}, \varepsilon > 0)$.

Proof

If $\mathrm{sp}_\varepsilon(\mathrm{a})$ contains a boundary point λ , then there is a sequence (λ_n) in the complement of $\mathrm{sp}_\varepsilon(\mathrm{a})$ which converges to λ . But this would imply that $\|\, (\mathrm{a} - \lambda_\mathrm{n} 1)^{-1} \,\| \leq \varepsilon^{-1}$ for all n, whereas $\|\, (\mathrm{a} - \lambda 1)^{-1} \,\| > \varepsilon^{-1}$. This contradicts the continuity of the norm.

2.5.6 Lemma ([4], Lemma 2.1(3))

Let A be a unital Banach algebra. $\mathrm{sp}_{\varepsilon_1}(\mathrm{a}) \subset \mathrm{sp}_{\varepsilon_2}(\mathrm{a})$ $(\mathrm{a} \in \mathrm{A}\,,\ 0 < \varepsilon_1 < \varepsilon_2)$.

Proof

$0 < \varepsilon_1 < \varepsilon_2$ implies that ${\varepsilon_2}^{-1} < {\varepsilon_1}^{-1}$ and the result follows immediately.

2.5.7 Lemma ([4], Lemma 2.1(2))

Let A be a unital Banach algebra. $\mathrm{sp(a)} = \bigcap_{\varepsilon>0} \mathrm{sp}_\varepsilon(\mathrm{a})$ $(\mathrm{a} \in \mathrm{A}\ ,\ \varepsilon > 0)$.

Proof

By Lemma 2.5.4 , we have $\mathrm{sp(a)} \subseteq \mathrm{sp}_\varepsilon(\mathrm{a})$. It follows that $\mathrm{sp(a)} \subseteq \bigcap_{\varepsilon>0} \mathrm{sp}_\varepsilon(\mathrm{a})$. Conversely, suppose $\lambda \in \bigcap_{\varepsilon>0} \mathrm{sp}_\varepsilon(\mathrm{a})$, then $\|\ (\mathrm{a} - \lambda 1)^{-1}\ \| > \varepsilon^{-1}$ for all $\varepsilon > 0$. Taking $\varepsilon = \mathrm{dist}\big(\lambda\ ,\ \mathrm{sp(a)}\ \big)$, it follows that $\lambda \in \mathrm{sp(a)}$.

2.5.8 Lemma ([4], Lemma 2.1(4))

Let A be a unital Banach algebra. $\mathrm{sp}_{|\alpha|\varepsilon}(\alpha \mathrm{a}) = \alpha\ \mathrm{sp}_\varepsilon(\mathrm{a})$ $(\mathrm{a} \in \mathrm{A}\ ,\ \varepsilon > 0\ ,\ \alpha \neq 0)$.

Proof

Suppose $\lambda \in \mathrm{sp}_{|\alpha|\varepsilon}(\alpha \mathrm{a})$. Then $\|\ (\alpha \mathrm{a} - \lambda 1)^{-1}\ \| > |\ \alpha\ |^{-1}\varepsilon^{-1}$ by Definition 2.5.3 . It follows that $\|\ \big(\mathrm{a} - \frac{\lambda}{\alpha}\ 1\ \big)^{-1}\ \| > \varepsilon^{-1}$. Thus $\frac{\lambda}{\alpha} \in \mathrm{sp}_\varepsilon(\mathrm{a})$. This implies that $\lambda \in \alpha\ \mathrm{sp}_\varepsilon(\mathrm{a})$.

Conversely, suppose that $\lambda \in \alpha\ \mathrm{sp}_\varepsilon(\mathrm{a})$. Then we have $\frac{\lambda}{\alpha} \in \mathrm{sp}_\varepsilon(\mathrm{a})$. This implies that $\|\ \big(\mathrm{a} - \frac{\lambda}{\alpha}\ 1\ \big)^{-1}\ \| > \varepsilon^{-1}$. It follows that $\|\ (\alpha \mathrm{a} - \lambda 1)^{-1}\ \| > |\ \alpha\ |^{-1}\varepsilon^{-1}$. Hence $\lambda \in \mathrm{sp}_{|\alpha|\varepsilon}(\alpha \mathrm{a})$ by Definition 2.5.3 .

2.5.9 Lemma ([4], Lemma 2.1(5))

Let A be a unital Banach algebra , $\mathrm{a} \in \mathrm{A}$ and $\varepsilon > 0$.
$\lambda \in \mathrm{sp}_\varepsilon(\mathrm{a}) \implies |\ \lambda\ | < \|\ \mathrm{a}\ \| + \varepsilon$.

Proof

$\sup \{\ |\ \lambda - \beta\ |\ : \beta \in \mathrm{sp(a)}\} = \mathrm{dist}\big(\lambda, \mathrm{sp(a)}\big) > \|\ (\mathrm{a} - \lambda 1)^{-1}\ \|^{-1}$.
From $\sup \{\ |\ \lambda\ |\ : \lambda \in \mathrm{sp(a)}\} = \mathrm{r(a)} < \|\ \mathrm{a}\ \|$, it follows that
$|\ \lambda\ | - \|\ \mathrm{a}\ \| < \|\ (\mathrm{a} - \lambda 1)^{-1}\ \|^{-1} < \varepsilon$.

2.5.10 Lemma ([4], Lemma 2.1(6))

Let A be a unital Banach algebra . $\mathrm{sp(a)} + \varepsilon\mathbb{D} \subset \mathrm{sp}_\varepsilon(\mathrm{a})$ $(\mathrm{a} \in \mathrm{A}\ ,\ \varepsilon > 0)$ where $\mathbb{D} = \{\lambda \in \mathbb{C}\ :\ |\ \lambda\ | \leq 1\}$.

Proof

Suppose $\beta \notin \mathrm{sp}_\varepsilon(\mathrm{a})$. By Lemma 2.5.9, we have $|\ \beta\ | \geq \|\ \mathrm{a}\ \| + \varepsilon$. This means that $\beta \notin \mathrm{sp(a)} + \varepsilon\mathbb{D}$. Hence $\mathrm{sp(a)} + \varepsilon\mathbb{D} \subset \mathrm{sp}_\varepsilon(\mathrm{a})$.

2.5.11 Lemma ([4], Lemma 2.1(7))

Let a and b be elements of a unital Banach algebra A such that $\|\ \mathrm{a}\ \| + \varepsilon > 0$. Then $\mathrm{sp}_\varepsilon(\mathrm{a} + \mathrm{b}) \subset \mathrm{sp}_{\varepsilon + \|\mathrm{a}\|}(\mathrm{b})$.

Proof

Suppose $\lambda \notin \mathrm{sp}_{\varepsilon+\|a\|}(b)$. Then by Lemma 2.5.9 , we have

$$|\,\lambda\,| \;\geq\; \|\,a\,\| + \varepsilon + \|\,b\,\| \;=\; \|\,a\,\| + \|\,b\,\| + \varepsilon \quad \text{given} \quad \varepsilon > 0 \,.$$

Since

$$|\,\lambda\,| \;\geq\; \|\,a\,\| \;+ \|\,b\,\| \;+\; \varepsilon \;\geq\; \|\,a + b\,\| \;+\; \varepsilon \quad \text{given} \quad \varepsilon > 0 \,,$$

it follows that $|\,\lambda\,| \geq \|\,a + b\,\| + \varepsilon$ given $\varepsilon > 0$. Hence $\lambda \notin \mathrm{sp}_{\varepsilon}(a+b)$. Therefore $\mathrm{sp}_{\varepsilon}(a+b) \subset \mathrm{sp}_{\varepsilon+\|a\|}(b)$ as required.

2.5.12 Lemma ([4], Lemma 2.1(8))

Let a and b be elements of a unital Banach algebra A such that $\|\,b\,\| < \varepsilon$. Then $\mathrm{sp}(a+b) \subset \mathrm{sp}_{\varepsilon}(a)$.

Proof

Let $\|\,b\,\| < \varepsilon$ given $\varepsilon > 0$. Then if $\lambda \notin \mathrm{sp}_{\varepsilon}(a)$, we have $|\,\lambda\,| \geq \varepsilon + \|\,a\,\|$. Since $\|\,b\,\| < \varepsilon$, it follows that $|\,\lambda\,| > \|\,b\,\| + \|\,a\,\|$. This means that $\lambda \notin \mathrm{sp}(a+b)$. Hence the assertion holds.

2.5.13 Lemma ([4], Lemma 2.3)

Let X be a Banach space and let $\mathbf{S} = (S_n), \mathbf{T} = (T_n) \in \mathcal{B}(X)^{\mathcal{U}} \subset \mathcal{B}(X^{\mathcal{U}})$. Suppose that there are bounded sequences of positive numbers (ε_n) and (δ_n) such that $\mathrm{sp}_{\varepsilon_n}(S_n) \subset \mathrm{sp}_{\delta_n}(T_n)$ almost everywhere on $\mathbb{N}$. Then $\mathrm{sp}_{\varepsilon}(\mathbf{S}) \subset \mathrm{sp}_{\delta}(\mathbf{T})$ whenever $\varepsilon, \delta > 0$ are such that $\varepsilon < \lim_{\mathcal{U}} \varepsilon_n$ and $\delta > \lim_{\mathcal{U}} \delta_n$.

Proof

Let $0 < \varepsilon < \varepsilon' < \lim_{\mathcal{U}} \varepsilon_n$ and $\lim_{\mathcal{U}} \delta_n < \delta' < \delta$. Let $z \in \mathrm{sp}_{\varepsilon}(S)$. Since $z = \lim_{\mathcal{U}} z_n$, it follows that $z_n \in \mathrm{sp}_{\varepsilon'}(S_n)$ almost everywhere. Since $\varepsilon' < \varepsilon_n$ and $\delta_n < \delta' < \delta$ almost everywhere, then by Lemma 2.5.6 and our hypothesis, we have $\mathrm{sp}_{\varepsilon'}(S_n) \subset \mathrm{sp}_{\varepsilon_n}(S_n) \subset \mathrm{sp}_{\delta_n}(T_n) \subset \mathrm{sp}_{\delta'}(T_n)$ almost everywhere on $\mathbb{N}$. Then by Lemma 2.5.6, it follows that $\mathrm{sp}_{\varepsilon}(S) \subset \mathrm{sp}_{\delta}(T)$ as required.

2.5.14 Lemma ([4], Lemma 2.4)

Let X be a Banach space and let $\mathbf{S} = (S_n),\ \mathbf{T} = (T_n) \in \mathcal{B}(X)^{\mathcal{U}} \subset \mathcal{B}(X^{\mathcal{U}})$.

Suppose that $\mathrm{sp}(\mathbf{S}) \subset \mathrm{sp}(\mathbf{T})$. Then for each $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $\mathrm{sp}(S_n) \subset \mathrm{sp}_{\varepsilon}(T_n)$.

Proof

Suppose the assertion is false. Then there exist an $\varepsilon > 0$ and a sequence (z_n) of complex numbers such that $z_n \in \mathrm{sp}(S_n) \,/\, \mathrm{sp}_{\varepsilon}(T_n)$ ($n \in \mathbb{N}$). Let $z = \lim_{\mathcal{U}} z_n$. It follows that $z \in \mathrm{sp}(\mathbf{S})$. On the other hand $\mathbf{T} - z\mathbf{I} = (T_n - z_n I)$. Since

$z_n \notin sp_\varepsilon(T_n)$, it follows that $\| (T_n - z_n I)^{-1} \| \le \frac{1}{\varepsilon}$ for each $n \in \mathbb{N}$. This means that $\mathbf{T} - z\mathbf{I}$ is an invertible operator which contradicts the fact that $z \in sp(\mathbf{T})$.

2.6 δ-almost multiplicative maps

This section aims at incorporating the δ- almost multiplicative maps into $\delta-$condition spectrum analysis.

2.6.1 Definition ([13], p. 93)

Let A be a unital Banach algebra and let $\delta > 0$ be given. A linear functional $\phi : A \to \mathbb{C}$, is said to be δ-almost multiplicative if $| \phi(ab) - \phi(a)\phi(b) | \le \delta \| a \| \| b \|$ for all $a, b \in A$.

2.6.2 Lemma

Let A be a complex Banach algebra and $\phi : A \to \mathbb{C}$ a linear functional. If $\delta = \| \phi \| + \| \phi \|^2$, then ϕ is δ-almost multiplicative.

Proof

Suppose $a, b \in A$. Then $| \phi(ab) - \phi(a)\phi(b) | \le | \phi(ab) | + | \phi(a)\phi(b) | \le \| \phi(ab) \| + \| \phi(a)\phi(b) \| \le \| \phi \| \| a \| \| b \| + \| \phi \|^2 \| a \| \| b \| \le (\| \phi \| + \| \phi \|^2) \| a \| \| b \|$.

Set $\delta = \| \phi \| + \| \phi \|^2$ and we are done.

2.6.3 Definition ([13], p. 94)

Let A be a Banach algebra. For $0 < \varepsilon < 1$, the ε-condition spectrum of an element a in A is the set

$$\sigma_\varepsilon(a) = \{\lambda \in \mathbb{C} : \| \lambda 1 - a \| \; \| (\lambda 1 - a)^{-1} \| \ge \frac{1}{\varepsilon}\}$$

with the convention that $\| \lambda 1 - a \| \; \| (\lambda 1 - a)^{-1} \| = \infty$

when $\lambda 1 - a$ is not invertible.

2.6.4 Theorem ([13], p. 93)

Let A be a complex Banach algebra with unit 1 and let ϕ be a δ-almost multiplicative linear functional on A with $\phi(1) = 1$. Then $\phi(a) \in \sigma_\delta(a)$ for every element a in A.

Proof

Let $a \in A$ and $\phi(a) = \lambda$. If $\lambda 1 - a$ is not invertible, then

$$\| \lambda 1 - a \| \; \| (\lambda 1 - a)^{-1} \| = \infty$$

so that $\lambda = \phi(\mathrm{a}) \in \sigma_\delta(\mathrm{a})$ by Definition 2.6.3. Next assume that $\lambda 1 - \mathrm{a}$ is invertible. Then $1 = |\phi(1)| = |\phi(\lambda 1 - \mathrm{a})\phi((\lambda 1 - \mathrm{a})^{-1})| \leq \delta \| \lambda 1 - \mathrm{a} \| \| (\lambda 1 - \mathrm{a})^{-1} \|$. That is,

$$\| \lambda 1 - \mathrm{a} \| \| (\lambda 1 - \mathrm{a})^{-1} \| \geq \frac{1}{\delta}$$

which implies $\phi(\mathrm{a}) \in \sigma_\delta(\mathrm{a})$.

2.6.5 Lemma ([13], p. 94)

Let A be a complex Banach algebra with unit 1. Then for every $\delta > 0$ such that $0 < \delta < 1$, $\mathrm{sp}(\mathrm{a}) \subseteq \sigma_\delta(\mathrm{a})$ for every element a in A.

Proof

If $\lambda \in \mathrm{sp}(\mathrm{a})$, then $\lambda 1 - \mathrm{a}$ is not invertible and so $\| \lambda 1 - \mathrm{a} \| \, \| (\lambda 1 - \mathrm{a})^{-1} \| = \infty$. The result then follows from Definition 2.6.3.

2.6.6 Lemma ([13], p. 95)

Let $\lambda \in \sigma_\varepsilon(\mathrm{a})$, then $|\lambda| \leq \frac{1+\varepsilon}{1-\varepsilon} \| \mathrm{a} \|$ for every ε such that $0 < \varepsilon < 1$.

Proof

If $|\lambda| \leq \| \mathrm{a} \|$, then we have $|\lambda| \leq \| \mathrm{a} \| < \frac{1+\varepsilon}{1-\varepsilon} \| \mathrm{a} \|$. Suppose that $|\lambda| > \| \mathrm{a} \|$. Then $\lambda 1 - \mathrm{a}$ is invertible and $\big(|\lambda| - \| \mathrm{a} \| \big)^{-1} \geq \| (\mathrm{a} - \lambda 1)^{-1} \|$. Thus

$$\begin{aligned} 1 &\leq \varepsilon \| \lambda 1 - \mathrm{a} \| \, \| (\lambda 1 - \mathrm{a})^{-1} \| \leq \varepsilon \| \lambda 1 - \mathrm{a} \| \big(|\lambda| - \| \mathrm{a} \| \big)^{-1} \\ &\leq \varepsilon\big(|\lambda| + \| \mathrm{a} \|\big)\big(|\lambda| - \| \mathrm{a} \| \big)^{-1}. \end{aligned}$$

This implies that $|\lambda| - \| \mathrm{a} \| \leq \varepsilon\big(|\lambda| + \| \mathrm{a} \|\big)$ so that $|\lambda| \leq \frac{1+\varepsilon}{1-\varepsilon} \| \mathrm{a} \|$.

2.6.7 Lemma

Let a and b be elements of commutative Banach algebra A. Let $\phi : \mathrm{A} \to \mathbb{C}$ be a linear functional defined on A. Then the identity

$$4\Big(\phi(\mathrm{ab}) - \phi(\mathrm{a})\phi(\mathrm{b})\Big) = \phi\Big((\mathrm{a}+\mathrm{b})^2\Big) - \Big(\phi(\mathrm{a}+\mathrm{b})\Big)^2 - \phi\Big((\mathrm{a}-\mathrm{b})^2\Big) + \Big(\phi(\mathrm{a}-\mathrm{b})\Big)^2$$

holds for all $\mathrm{a}, \mathrm{b} \in \mathrm{A}$.

Proof

We now simplify the right hand side of the identity. Observe that

$$\phi\Big((\mathrm{a}+\mathrm{b})^2\Big) = \phi\Big(\mathrm{a}^2 + \mathrm{ab} + \mathrm{ba} + \mathrm{b}^2\Big) = \phi\Big(\mathrm{a}^2 + 2\mathrm{ab} + \mathrm{b}^2\Big) = \phi(\mathrm{a}^2) + 2\phi(\mathrm{ab}) + \phi(\mathrm{b}^2),$$

$$-\Big(\phi(a+b)\Big)^2 \;=\; -\Big(\phi(a)+\phi(b)\Big)^2 \;=\; -(\phi(a))^2 - 2\phi(a)\phi(b) - (\phi(b))^2,$$

$$-\phi\Big((a-b)^2\Big) = -\phi\Big(a^2 - ab - ba + b^2\Big) = -\phi\Big(a^2 - 2ab + b^2\Big) = -\phi(a^2) + 2\phi(ab) - \phi(b^2)$$

and

$$\Big(\phi(a-b)\Big)^2 \;=\; \Big(\phi(a)-\phi(b)\Big)^2 \;=\; (\phi(a))^2 - 2\phi(a)\phi(b) + (\phi(b))^2 \;.$$

Summing up these four parts gives us the left hand side of the identity.

2.6.8 Lemma ([13], Lemma 4)

Let A be a commutative Banach algebra and $\phi : A \to \mathbb{C}$ be a linear functional.
If $|\ \phi(a^2) - (\phi(a))^2\ | \ \le\ \delta$ for all $a \in A$ with $\|\ a\ \| \ =\ 1$, then
ϕ is 2δ-almost multiplicative.

Proof

Replacing a non-zero a by $\frac{a}{\|a\|}$ in inequality $|\ \phi(a^2) - (\phi(a))^2\ | \ \le\ \delta$,
we obtain $\frac{1}{\|a\|^2}\ |\ \phi(a^2) - (\phi(a))^2\ | \ \le\ \delta$ which further implies that
$|\ \phi(a^2) - (\phi(a))^2\ | \ \le\ \delta\|\ a\ \|^2 \ =\ \delta$ since $\|\ a\ \| \ =\ 1$. It now follows that
$|\ \phi(a^2) - (\phi(a))^2\ | \ \le\ 2\delta \ =\ 2\delta\ \|\ a\ \|\|\ a\ \|$ for all $a \in A$. Hence for arbitrary $a, b \in A$
with $\|\ a\ \| \ =\ \|\ b\ \| \ =\ 1$, we have

$$|\ \phi((a+b)^2) - (\phi(a+b))^2\ | \ \le\ \delta\ \|\ a+b\ \|\|\ a+b\ \| \ \le\ 2\delta\Big(\|\ a\ \| + \|\ b\ \|\Big) \ =\ 4\delta \ \text{ and}$$

$$|\ \phi((a-b)^2) - (\phi(a-b))^2\ | \ \le\ \delta\ \|\ a-b\ \|\|\ a-b\ \| \ \le\ 2\delta\Big(\|\ a\ \| + \|\ b\ \|\Big) \ =\ 4\delta.$$

From Lemma 2.6.7, it follows that

$$|\ \phi(ab) - \phi(a)\phi(b)\ | \ \le\ \frac{1}{4}\ |\ \phi((a+b)^2) - (\phi(a+b))^2\ | \ +\ \frac{1}{4}\ |\ \phi((a-b)^2)\ -\ (\phi(a-b))^2\ |$$

$$\le\ \frac{1}{4}\big(4\delta + 4\delta\big) \ =\ 2\delta.$$

For any $a,b \in A$, we get $|\ \phi(ab) - \phi(a)\phi(b)\ | \ \le\ 2\delta\ \|\ a\ \|\|\ b\ \|$ as required.

2.7 Non-zero multiplicative linear functionals (Characters)

The formulation of the so-called Kaplansky's problem was aided by the classical Gleason-Kahane-Żelazko's Theorem. This theorem states that a linear functional ϕ on a complex Banach algebra A is multiplicative (and non-zero) if and only if
$\phi(a) \in sp(a) \ \ (a \in A)$.

Kaplansky translated the condition $\phi(\mathrm{a}) \in \mathrm{sp}(\mathrm{a})$ $(\mathrm{a} \in \mathrm{A})$ for a linear functional ϕ in the above theorem to imply condition $\mathrm{sp}\big(\Phi(\mathrm{a})\big) \subset \mathrm{sp}(\mathrm{a})$ $(\mathrm{a} \in \mathrm{A})$, the property of shrinking the spectrum, for a linear map $\Phi : \mathrm{A} \to \mathrm{B}$ in the case of Banach algebras A and B.

This section introduces the non-zero multiplicative linear functionals. The word character will be used to denote a non-zero multiplicative linear functional.

The ultimate goal of introducing these characters is to lead us to item 2.7.8, the Gleason-Kahane -Żelazko's Theorem.

2.7.1 Definition ([7], p. 46)

Let A be a unital complex Banach algebra and ϕ a non-zero multiplicative linear functional defined on A. Then ϕ is called a character. We shall denote the set of characters defined on A with symbol ΔA.

2.7.2 Lemma ([25], Lemma 2.3.2)

If A is a complex unital Banach algebra and ϕ is a character on A, then $\phi(1) = 1$ and $\phi(\mathrm{a}) \neq 0$ for all $\mathrm{a} \in \mathrm{inv(A)}$.

Proof

We know that $1^2 = 1$. Therefore $\phi(1) = \phi(1^2) = \phi(1)\phi(1)$. This implies that $\phi(1)\Big(1 - \phi(1)\Big) = 0$. Hence $\phi(1) = 0$ or $\phi(1) = 1$. Then for all $\mathrm{a} \in \mathrm{A}$ and using $\phi(\mathrm{a}) = \phi(\mathrm{a}1) = \phi(\mathrm{a})\phi(1)$, it follows that $\phi(\mathrm{a}) = 0$ with respect to $\phi(1) = 0$. This cannot occur unless $\phi \equiv 0$. It follows that $\phi(1) = 1$. Furthermore, $1 = \phi(1) = \phi(\mathrm{aa}^{-1}) = \phi(\mathrm{a})\phi(\mathrm{a}^{-1})$ for all $\mathrm{a} \in \mathrm{inv(A)}$ so that $\phi(\mathrm{a}) \neq 0$.

2.7.3 Lemma

If A is a complex unital Banach algebra and ϕ is a character on A, then $\phi(\mathrm{a}) \in \mathrm{sp}(\mathrm{a})$ for all $\mathrm{a} \in \mathrm{A}$.

Proof

If $\lambda \in \mathrm{sp}(\mathrm{a})$, then $\mathrm{a} - \lambda 1 \notin \mathrm{inv(A)}$ and so $\phi(\mathrm{a} - \lambda 1) = \phi(\mathrm{a}) - \lambda$ is not in $\mathrm{inv}(\mathbb{C}) = \mathbb{C}/\{0\}$ by Lemma 2.7.2 . It follows that $\lambda - \phi(\mathrm{a}) = 0$ or $\lambda = \phi(\mathrm{a})$.

2.7.4 Lemma ([7], Lemma 2.1.5)

Let A be a commutative unital Banach algebra. Every $\phi \in \Delta\mathrm{A}$ is a bounded linear functional on A and $|\ \phi(\mathrm{a})\ | \leq \mathrm{r(a)}$ holds for all $\mathrm{a} \in \mathrm{A}$. In particular, $\|\ \phi\ \| \leq 1$ and $\|\ \phi\ \| = 1$ if A is unital.

2.7.5 Corollary

If A is a complex unital Banach algebra and ϕ is a character on A, then $\mathrm{sp}\Big(\phi(a)\Big) \subset \mathrm{sp}(a)$ for all $a \in A$.

Proof

Suppose $\lambda \notin \mathrm{sp}(a)$. By Lemma 2.7.3 , $\phi(a) - \lambda$ is invertible. This means that $\lambda \notin \mathrm{sp}\Big(\phi(a)\Big)$. Hence $\mathrm{sp}\Big(\phi(a)\Big) \subset \mathrm{sp}(a)$ for all $a \in A$ as required.

2.7.6 Definition ([9], Definition 2.1)

Let $\Phi : A \to B$ be a linear map between unital complex Banach algebras A and B. By a Jordan homomorphism, we mean a multiplicative and additive map satisfying equation $\Phi(a^2) = \big(\Phi(a)\big)^2$ for all $a \in A$.

2.7.7 Theorem ([7], Theorem 2.1.2)

Let A be a commutative complex unital Banach algebra and ϕ a character on A. If $\phi(a) = 0$ for all $a \notin \mathrm{inv}(A)$, then ϕ is a Jordan homomorphism.

Proof

By Lemma 2.7.2 , we have $\phi(1) = 1$. Let $\phi(a) = 0$, $n \geq 2$ and denote by P the polynomial $P(\lambda) = \phi((\lambda 1 - a)^n)$ of degree n. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the roots of P. Since $0 = P(\lambda_i) = \phi((\lambda_i 1 - a)^n)$ and ϕ is non-zero, then by Lemma 2.7.2 , we have $(\lambda_i 1 - a)^n \notin \mathrm{inv}(A)$ and so $\lambda_i 1 - a \notin \mathrm{inv}(A)$. It follows that $\lambda_i \in \mathrm{sp}(a)$ for each i. Define

$$\sup_{1 \leq i \leq n} \mid \lambda_i \mid = r(a) \tag{2.4}$$

Then $\mid \lambda_i \mid \leq r(a)$ for each i. Since λ_i are the roots of polynomial P, it follows that $\lambda - \lambda_i = 0$ for each i. We can write

$$0 = (\lambda - \lambda_1)(\lambda - \lambda_2)....(\lambda - \lambda_n) =$$

$$\lambda^n - \lambda^{n-1}(\textstyle\sum_{i=1}^n \lambda_i) + \lambda^{n-2}(\sum_{\substack{i=1 \\ i<j}}^{n} \lambda_i \lambda_j) - .... = \phi((\lambda 1 - a)^n) =$$

$$\lambda^n - \lambda^{n-1} n \phi(a) + \lambda^{n-2} \frac{n(n-1)}{2} \phi(a^2) - ....$$

Thus we have $\sum_{i=1}^n \lambda_i = n\phi(a) = 0$ and $\sum_{\substack{i=1 \\ i<j}}^{n} \lambda_i \lambda_j = \frac{n(n-1)}{2} \phi(a^2)$. We have

$$\big(\textstyle\sum_{i=1}^n \lambda_i\big)^2 = \sum_{i=1}^n (\lambda_i)^2 + 2 \sum_{\substack{i=1 \\ i<j}}^{n} \lambda_i \lambda_j \tag{2.5}$$

Observe that

$$\big(\textstyle\sum_{i=1}^n \lambda_i\big)^2 = n^2 \phi(a)^2 \tag{2.6}$$

and

$$2\sum_{\substack{i=1\\i<j}}^{n} \lambda_i\lambda_j = n(n-1)\phi(a^2) \tag{2.7}$$

Putting (2.5), (2.6) and (2.7) together, we produce equation

$$n^2\phi(a)^2 \;=\; \sum_{i=1}^{n}(\lambda_i)^2 + 2\Big[\frac{n(n-1)}{2}\phi(a^2)\Big] \tag{2.8}$$

From (2.8), it follows that

$$n^2 \mid \phi(a)^2 - \phi(a^2) \mid \;=\; \mid \sum_{i=1}^{n}(\lambda_i)^2 - n\phi(a^2) \mid \;\leq\; n(r(a))^2 \;+\; n \mid \phi(a^2) \mid$$

by an application of triangle inequality and (2.4) . We end up with inequality

$$\mid \phi(a)^2 - \phi(a^2) \mid \;\leq\; \frac{1}{n}(r(a))^2 \;+\; \frac{1}{n} \mid \phi(a^2) \mid .$$

Hence $\mid \phi(a)^2 - \phi(a^2) \mid \to 0$ as $n \to \infty$ so that $\phi(a^2) = (\phi(a))^2$.

2.7.8 Theorem ([4], p. 248)

Let A be a complex unital Banach algebra and $\phi : A \to \mathbb{C}$ a linear functional. Then ϕ is a character if and only if $\phi(a) \in sp(a)$ $(a \in A)$. This is the classical Gleason-Kahane-Żelazko's Theorem (GKZ).

2.8 Semisimple algebras

This section deals with semisimple algebras. It was mentioned earlier, in Chapter 1, that the concept of semisimple improves Kaplansky's open problem. The improvement of this open problem is item 3.1.8 in this dissertation, a suggestion given by Bernard Aupetit in [5]. It is still an open problem regarding the search for a much more reasonable open problem due to the fact that the Bernard Aupetit's open problem also lacks generality. It only works for certain classes of Banach algebras.

2.8.1 Definition ([19], p. 263)

Let J be an ideal of a complex Banach algebra A. Then J is a proper ideal of A if $J \neq A$.

2.8.2 Lemma ([25], Lemma 2.4.2)

Let ϕ be a character on a unital complex Banach algebra A.

The set $\ker(\phi) = \{a \in A : \phi(a) = 0\}$ is a proper ideal of A.

Proof

Let $a, b \in \ker(\phi)$ and $x \in A$. Then it follows that $\phi(a) = \phi(b) = 0$ implying that $\phi(a - b) = 0$. This shows that $a - b \in \ker(\phi)$. Hence $\ker(\phi)$ is a vector space. Then from multiplicative equation $\phi(ax) = \phi(a)\phi(x)$, we have that $\phi(ax) = 0$ showing that $ax \in \ker(\phi)$. We also know that $1 \in A$ cannot belong to $\ker(\phi)$ because $\phi(1) \neq 0$. Thus $\ker(\phi) \neq A$. Then by Definition 2.8.1, $\ker(\phi)$ is a proper ideal.

2.8.3 Theorem ([24], Theorem 1.2.4)

If A is a complex Banach algebra, then the following sets are identical :

(i) the intersection of all maximal left ideals of A,

(ii) the intersection of all maximal right ideals of A.

2.8.4 Definition ([24], p. 3)

The radical of complex Banach algebra A, denoted by rad(A), is defined to be the two identical sets given in Theorem 2.8.3.

2.8.5 Definition ([24] p. 3 ; [18] p. 5)

Let A be a complex Banach algebra. We say that A is semisimple if $\mathrm{rad}(A) = \{0\}$.

2.8.6 Proposition ([18], Proposition 1.11(c))

Let A be a unital Banach algebra and $x \in A$. Then if $x \in \mathrm{rad}(A)$, then $1_A - xy \in \mathrm{inv}(A)$ for all $y \in A$. Similarly, $1_A - yx \in \mathrm{inv}(A)$ for all $y \in A$.

Proof

If $x \in \mathrm{rad}(A)$, then $r_A(xy) = 0$ for all $y \in A$. This implies that $1_A - xy \in \mathrm{inv}(A)$ for all $y \in A$. Similarly, $1_A - yx \in \mathrm{inv}(A)$ for all $y \in A$.

2.8.7 Definition ([18], p. 9)

Let $a \in A$ where A is a unital Banach algebra. Then we shall say that a is nilpotent if $a^n = 0$ for some $n \in \mathbb{N}$.

2.8.8 Definition ([18] p. 9 , [23] p. 11)

Let A be a unital Banach algebra. An element $a \in A$ is called quasinilpotent if $\mathrm{sp}(a) = \{0\}$.

2.8.9 Theorem ([24], Theorem 1.5.11)

If X is a Banach space, then $\mathcal{B}(X)$ is semisimple.

2.8.10 Theorem ([24], Theorem 1.5.13)

Every unital standard operator algebra A on a Banach space X is semisimple.

2.8.11 Theorem ([18], Theorem 1.17)

Let A be a Banach algebra. Then the following properties are equivalent.

(i) $a \in \mathrm{rad}(A)$.

(iii) $r(x) = r(a + x)$ for all $x \in A$.

(iii) $r(a + x) = 0$ for all x that are quasinilpotent in A.

(iv) There exists $C \geq 0$ such that $r(x) \leq C \| x - a \|$ for all x in a neighborhood of $a \in A$.

2.9 Extreme spectra state

This section deals with spectra states. A spectra state deals with a character in a unit ball. This section lays the ground for spectra isometries on finite dimensional semisimple Banach algebras in Chapter 3. It should be noted that chapter 3 has the first set of solutions to the so-called Kaplansky's problem. So, one of them is on spectra isometry characterisation via spectra states.

2.9.1 Definition ([19], p. 70)

Let E be a convex set and $x_1, x_2 \in E$. Then we say $x \in E$ is an extreme point of E if $x = x_1 t + (1 - t)x_2 \in E$ and $t \in (0, 1)$, implies that $x = x_1 = x_2$.

We use symbol ex(E) to denote the set of all extreme points of E.

The theorem that follows is based on a locally convex space. We also note that in what proceeds, symbol co(E) will mean the convex hull of set E.

2.9.2 Theorem ([7], p. 323)

Let X be a locally convex topological vector space and let E be a non-empty convex subset of X. If $E \subseteq X$ is compact, then E is the closed convex hull of the set of its extreme points. That is

$$E = \overline{\mathrm{co}\Big(\mathrm{ex}(E)\Big)} .$$

This theorem is the Krein-Milman's Theorem.

2.9.3 Definition ([18], p. 60)

Let A be a commutative Banach algebra. A spectra state on A is a linear functional $\phi : A \rightarrow \mathbb{C}$ such that $|\phi(a)| \leq r(a)$ for all $a \in A$ and $\phi(1) = 1$. The set of spectra states on A is denoted by $\sum_A$.

We note the following arising from an analysis of this definition.

(i) ϕ is multiplicative by Lemma 2.7.4 .

(ii) Since $|\phi(a)| \leq r(a) \leq \| a \|$ for all $a \in A$, it follows that $\phi(a) \in sp(a)$.

2.9.4 Lemma ([18], Lemma 5.2.5)

Let A be a commutative unital Banach algebra and ϕ be a linear functional on A.

(i) ϕ is a spectra state on A if $\phi(a) \in co\Big(sp(a)\Big)$ for all $a \in A$

(ii) If ϕ is an extreme spectra state, then ϕ is multiplicative.

Proof

(i) Suppose ϕ is a spectra state. Then $\phi(a) \in sp(a)$ for all $a \in A$.

Since $sp(a) \subseteq co\Big(sp(a)\Big)$ (the convex hull of the spectrum of a),

it follows that $\phi(a) \in co\Big(sp(a)\Big)$.

(ii) ϕ being an extreme spectra state implies that $\phi(a) \in \overline{co(\ sp(a)\ )}$ if and only if $\phi(a) \in sp(a)$ for all $a \in A$. Thus ϕ is multiplicative.

2.10 Linear preservers

This section introduces the terminologies this dissertation has used with respect to the concept of preservation of maps. Basing on these terminologies and referring back to Chapter 1, it is clear that Kaplansky's conjecture incorporates item 2.10.2 while Bernard Aupetit's conjecture incorporates item 2.10.1 .

2.10.1 Definition ([18], p. 12)

Let A and B be complex unital Banach algebras. A linear map $\Phi : A \rightarrow B$ is said to be spectrum preserving if for all $x \in A$, we have $sp\Big(\Phi(x)\Big) = sp(x)$.

2.10.2 Definition ([18], p. 12)

Let A and B be complex unital Banach algebras. A linear map $\Phi : A \rightarrow B$ is said to be spectrum compressing if for all $x \in A$, we have $sp\Big(\Phi(x)\Big) \subseteq sp(x)$.

2.10.3 Definition ([18], p. 12)

Let A and B be complex unital Banach algebras. A linear map $\Phi : A \to B$ is said to be a spectra isometry if for all $x \in A$, we have $r\Big(\Phi(x)\Big) = r(x)$.

2.10.4 Lemma ([18], Lemma 1.25)

Let A and B be complex unital Banach algebras and let linear map $\Phi : A \to B$ be a spectra isometry. Then $\ker(\Phi) \subseteq \mathrm{rad}(A)$.

Proof

Let $x \in \ker(\Phi)$, then for any $y \in A$, we have that

$$r(y) = r\Big(\Phi(y)\Big) = r\Big(\Phi(x+y)\Big) = r(x+y) .$$

In particular, if $y \in A$ is quasinilpotent, then

$$0 = r(y) = r\Big(\Phi(y)\Big) = r\Big(\Phi(x+y)\Big) = r(x+y) .$$

Then by Theorem 2.8.11, we have that $x \in \mathrm{rad}(A)$. Hence $\ker(\Phi) \subseteq \mathrm{rad}(A)$ as required.

2.10.5 Definition

Let X be a topological vector space. X will be called a Hausdorff space if distinct elements in X have disjointed neighborhoods.

We will use the symbol $\mathrm{dist}_H(x , y)$ to denote the Hausdorff distance between a pair of elements x and y of Hausdorff space X.

An example of a topological vector space that is Hausdorff is the real vector space $\mathbb{R}$ with the usual metric.

2.10.6 Definition ([4], p. 255)

Let A and B be complex unital Banach algebras and $\Phi : A \to B$ be a linear map. The approximately spectrum compressing condition is given by
$\mathrm{sp}\big(\Phi(a)\big) \subset \mathrm{sp}_\varepsilon(a)$ $(a \in A, \ \| a \| = 1)$ for some $\varepsilon > 0$.

The equivalence to this definition is the statement $\mathrm{dist}_H\big(\mathrm{sp}\big(\Phi(a)\big), \mathrm{sp}(a)\big) < \varepsilon$ for some $\varepsilon > 0$.

2.10.7 Definition ([4], p. 255)

Let X be a Banach space. Let A be a complex unital Banach algebra and let $\Phi : X \to A$ be a linear map. We say that Φ is spectrally dominated by a function $\phi : X \to [0, \infty)$ if $r\big(\Phi(x)\big) \le \phi(x)$ $(x \in X)$.

2.10.8 Definition ([4], p. 257)

A linear map $\Phi : X \to B$ from a linear subspace X of a complex unital Banach algebra A into a complex unital Banach algebra B is said to be spectrally bounded if there exist a constant $M \geq 0$ such that $r\big(\Phi(x)\big) \leq M\, r(x)$ for each $x \in X$.

2.10.9 Definition ([18], p. 44)

Let X and Y be locally convex spaces. Let Ω be a subspace of $\mathcal{B}(X)$ that contains $F(X)$ (finite rank operators on X) , and let $\Phi : \Omega \to \mathcal{B}(Y)$ be a linear map. We call Φ rank reducing if $\mathrm{rank}\big(\Phi(T)\big) \leq \mathrm{rank}(T)$ for every $T \in F(X)$.

2.11 Ultraprime Banach algebras

This section is the conclusion to the preliminaries. In this section, Banach algebras that have a Jordan multiplicative structure (Definition 2.11.2) are introduced. We further show that by using a Jordan structure given a surjective linear functional ϕ with $\mathrm{jmult}(\phi) < \delta$ $(\delta > 0)$, ϕ is continuous.

One reason why $\mathrm{jmult}(\phi) < \delta$ is needed is to deal with natural law $\mathrm{jmult}(\phi) = \min\{\ \mathrm{mult}(\phi), \mathrm{amult}(\phi)\ \}$ in Theorem 2.11.8 . This theorem is also used in the proof of Theorem 2.11.11 . Theorem 2.11.11 is gateway to the understanding of some portions of Chapter 5. An example to this effect is Theorem 5.2.3 .

A pre-requisite to Theorem 2.11.11 is a Banach algebra that is ultraprime. So ultraprimes are introduced in order to deal with Theorem 2.11.11. The incorporation of δ-almost multiplicative maps in Banach algebras that are Almost Multiplicative Near Multiplicative and their opposite algebras (the Almost Anti-Multiplicative Near Anti-Multiplicative) is aimed at the approximate spectra versions of Chapter 5.

2.11.1 Definition ([4], p. 240)

Let A and B be Banach algebras and let $\Phi : A \to B$ be a linear map.

We measure the multiplicativity of Φ by using the constant

$\mathrm{mult}(\Phi) = \sup\Big\{ \| \Phi(ab) - \Phi(a)\Phi(b) \| \ : \ a, b \in A, \ \| a \| = \| b \| = 1 \Big\}$.

For the anti-multiplicativity of Φ , we use the constant

$\mathrm{amult}(\Phi) = \sup\Big\{ \| \Phi(ab) - \Phi(b)\Phi(a) \| \ : \ a, b \in A, \ \| a \| = \| b \| = 1 \Big\}$.

For the Jordan multiplicativity of Φ , we use the constant

$\mathrm{jmult}(\Phi) = \sup\Big\{ \| \Phi(a^2) - \Phi(a)^2 \| \ : \ a \in A, \ \| a \| = 1 \Big\}$.

2.11.2 Definition ([17], p. 3)

A Jordan algebra consists of a real vector space equipped with a bilinear product (.)

$$\mathrm{x} \,.\, \mathrm{y} \;=\; \frac{1}{2}(\, \mathrm{xy} \;+\; \mathrm{yx}\,)$$

satisfying commutative law $\mathrm{x} \,.\, \mathrm{y} = \mathrm{y} \,.\, \mathrm{x}$ and the Jordan identity

$$(\mathrm{x}^2 \,.\, \mathrm{y}) \,.\, \mathrm{x} \;=\; \mathrm{x}^2 \,.\, (\mathrm{y} \,.\, \mathrm{x})$$

2.11.3 Definition

If it is required that the multiplication in Banach algebra A be Jordan, that is to say all elements satisfy Definition 2.11.2, then such an algebra will be referred to as a Jordan-Banach algebra.

A Banach algebra inclusive of items under Definition 2.11.1 will have a multiplication defined by Definition 2.11.2 .

2.11.4 Proposition ([4], Proposition 3.1)

Let A be a Banach algebra and let B be a semisimple Banach algebra. Let $\Phi : \mathrm{A} \to \mathrm{B}$ be a surjective linear map. Then Φ is continuous in any of the following cases:

(1) $\mathrm{mult}(\Phi) < \infty$

(2) $\mathrm{amult}(\Phi) < \infty$

(3) $\mathrm{jmult}(\Phi) < \infty$

The proof of this proposition is on page 241 of [4] . We mention here that constant $\| \Phi \| + \| \Phi \|^2$ of Lemma 2.6.2 is crucial in the proof of this proposition . Since $\mathrm{jmult}(\Phi) = \| \Phi \| + \| \Phi \|^2$, $\mathrm{mult}(\Phi) = \| \Phi \| + \| \Phi \|^2$ and $\mathrm{amult}(\Phi) = \| \Phi \| + \| \Phi \|^2$ for $\delta-$almost multiplicative maps, then these constants are comparable under inequality

$$\mathrm{jmult}(\Phi) \;\leq\; \min\{\mathrm{mult}(\Phi), \mathrm{amult}(\Phi)\} \;.$$

We will use this inequality later in Corollary 2.11.17 .

2.11.5 Corollary ([4], Corollary 3.2)

Let ϕ be a linear functional on a Banach algebra A with $\mathrm{jmult}(\phi) < \infty$. Then ϕ is continuous and $\| \phi \| \leq \frac{1+\sqrt{1+4\mathrm{jmult}(\phi)}}{2}$.

Proof

Let $\phi \neq 0$. It follows that ϕ is surjective. The range (algebra $\mathbb{C}$) is automatically semisimple. Therefore, Proposition 2.11.4 imply that ϕ is continuous. To complete the proof, we observe that

$$|\ \phi(\mathrm{a}^2) - \phi(\mathrm{a})^2\ | \ \leq \ \|\ \phi(\mathrm{a}^2) - \phi(\mathrm{a})^2\ \| \ \leq \ \mathrm{jmult}(\phi) \tag{2.9}$$

for each $\mathrm{a} \in \mathrm{A}$ with $\|\ \mathrm{a}\ \| = 1$. Once again, for each $\mathrm{a} \in \mathrm{A}$ with $\|\ \mathrm{a}\ \| = 1$, we have

$$|\ \phi(\mathrm{a}^2)\ | \leq \|\ \phi\ \|\|\ \mathrm{a}^2\ \| \leq \ \|\ \phi\ \|\|\ \mathrm{a}\ \|^2 = \|\ \phi\ \|\|\ \mathrm{a}\ \|\|\ \mathrm{a}\ \| = \|\ \phi\ \| \tag{2.10}$$

and

$$|\ \phi(\mathrm{a})\ |^2 \ = \ \Big(\frac{|\ \phi(\mathrm{a})\ |}{\|\ \mathrm{a}\ \|}\Big)\Big(\frac{|\ \phi(\mathrm{a})\ |}{\|\ \mathrm{a}\ \|}\Big) \ = \ \|\ \phi\ \|^2 \tag{2.11}$$

Combining (2.9) (2.10) and (2.11) we arrive at

$$|\ \phi(\mathrm{a})\ |^2 \leq \ |\ \phi(\mathrm{a})^2 - \phi(\mathrm{a}^2)\ | \ + \ |\ \phi(\mathrm{a}^2)\ | \ \leq \ \mathrm{jmult}(\phi) + \|\ \phi\ \|.$$

It follows that $\|\ \phi\ \|^2 - \|\ \phi\ \| \ \leq \ \mathrm{jmult}(\phi)$ if and only if

$(\|\ \phi\ \| - \frac{1}{2})^2 \leq \ \frac{1}{4} + \mathrm{jmult}(\phi)$. Hence $\|\ \phi\ \| \ \leq \ \frac{1+\sqrt{1+4\ \mathrm{jmult}(\phi)}}{2}$ as required.

2.11.6 Definition ([24], p.7)

We shall say subset A of Banach algebra B is a prime ideal if the following conditions hold:

(i) A is an ideal

(ii) If A_1 and A_2 are two ideals contained in A , then $\mathrm{A}_1\mathrm{A}_2 \ \subset \ \mathrm{A}$.

(iii) $\mathrm{A}_1\mathrm{A}_2 \ \subset \ \mathrm{A}$ implies that $\mathrm{A}_1 \ \subset \ \mathrm{A}$ or $\mathrm{A}_2 \ \subset \ \mathrm{A}$.

2.11.7 Definition ([24], p.6)

An ideal A of Banach algebra B is Primitive if it is kernel of an irreducible representation of B.

2.11.8 Theorem ([4], p. 244)

Let A and B be Banach algebras. Let $\Phi \ : \ \mathrm{A} \to \mathrm{B}$ be a surjective map. Then every Jordan homomorphism $\Phi \ : \ \mathrm{A} \to \mathrm{B}$ is either a homomorphism or an anti-homomorphism whenever B is a prime algebra. This theorem is due to Herstein.

(i) A particular case of Theorem 2.11.8 is when

$$\mathrm{jmult}(\Phi) \ = \ 0 \ \text{ implies that } \ \min\ \{\ \mathrm{mult}(\ \Phi)\ ,\ \mathrm{amult}(\Phi)\ \} \ = \ 0 \ .$$

(ii) The approximate version of Herstein' s Theorem used in this paper dwells on the concept of $\mathrm{jmult}(\Phi)$ being small enough to simultaneously imply that $\min\{\ \mathrm{mult}(\ \Phi)\ ,\ \mathrm{amult}(\Phi)\ \}$ is also small.

(iii) To this end, we become involved with the so-called ultraprime instead of just the mere prime.

2.11.9 Definition ([4], p. 245)

A Banach algebra A is ultraprime if the corresponding ultrapower $\mathrm{A}^{\mathcal{U}}$ with respect to some, hence every (countably incomplete) ultrafilter $\mathcal{U}$ is a prime Banach algebra.

This is equivalent to a property that there is a $\mathrm{K} > 0$ such that

$$\sup\{\|\ \mathrm{axb}\ \| :\ \mathrm{x} \in \mathrm{A}, \|\ \mathrm{x}\ \| = 1\}\ \geq\ \mathrm{K}\|\ \mathrm{a}\ \|\|\ \mathrm{b}\ \| \quad (\mathrm{a}, \mathrm{b} \in \mathrm{A})\ .$$

We now give an example of an ultraprime space as given by [4] on page 245. For each Banach space X, $\mathcal{B}(\mathrm{X})$ and more generally, all closed subalgebras of $\mathcal{B}(\mathrm{X})$ containing the finite rank operators.

2.11.10 Lemma ([4], Lemma 3.4)

Let A and B be Banach algebras and let $\mathbf{\Phi} = (\Phi_\mathrm{n}) \in \mathcal{B}(\mathrm{A}, \mathrm{B})^{\mathcal{U}} \subset \mathcal{B}(\mathrm{A}^{\mathcal{U}}, \mathrm{B}^{\mathcal{U}})$.
Then the following identities hold:

$$\mathrm{mult}(\mathbf{\Phi}) = \lim_{\mathcal{U}} \mathrm{mult}\ (\Phi_\mathrm{n})$$

$$\mathrm{amult}(\mathbf{\Phi}) = \lim_{\mathcal{U}} \mathrm{amult}\ (\Phi_\mathrm{n})$$

$$\mathrm{jmult}(\mathbf{\Phi}) = \lim_{\mathcal{U}} \mathrm{jmult}\ (\Phi_\mathrm{n})$$

Proof

We will only show that $\mathrm{mult}(\mathbf{\Phi}) = \lim_{\mathcal{U}} \mathrm{mult}\ (\Phi_\mathrm{n})$ holds. In the same way we can prove the identities corresponding to $\mathrm{amult}(\mathbf{\Phi})$ and $\mathrm{jmult}(\mathbf{\Phi})$.

Given $\mathbf{a} = (\mathrm{a_n}), \mathbf{b} = (\mathrm{b_n}) \in \mathrm{A}^{\mathcal{U}}$, we have

$$\begin{aligned}
\|\ \mathbf{\Phi}(\mathbf{ab}) - \mathbf{\Phi}(\mathbf{a})\mathbf{\Phi}(\mathbf{b})\ \| \ &= \ \lim_{\mathcal{U}} \|\ \Phi_\mathrm{n}(\mathrm{a_n b_n}) - \Phi_\mathrm{n}(\mathrm{a_n})\Phi_\mathrm{n}(\mathrm{b_n})\ \| \\
&= \lim_{\mathcal{U}} \left\{ \frac{\|\ \Phi_\mathrm{n}(\mathrm{a_n b_n}) - \Phi_\mathrm{n}(\mathrm{a_n})\Phi_\mathrm{n}(\mathrm{b_n})\ \|}{\|\ \mathrm{a_n}\ \|\|\ \mathrm{b_n}\ \|} \right\} \|\ \mathrm{a_n}\ \|\|\ \mathrm{b_n}\ \| \\
&\leq \lim_{\mathcal{U}} \sup \left\{ \frac{\|\ \Phi_\mathrm{n}(\mathrm{a_n b_n}) - \Phi_\mathrm{n}(\mathrm{a_n})\Phi_\mathrm{n}(\mathrm{b_n})\ \|}{\|\ \mathrm{a_n}\ \|\|\ \mathrm{b_n}\ \|} \right\} \|\ \mathrm{a_n}\ \|\|\ \mathrm{b_n}\ \| \\
&= \lim_{\mathcal{U}}\ \mathrm{mult}(\Phi_\mathrm{n}) \|\ \mathrm{a_n}\ \|\|\ \mathrm{b_n}\ \| \ = \ \lim_{\mathcal{U}}\ \mathrm{mult}(\Phi_\mathrm{n}) \|\ \mathbf{a}\ \|\|\ \mathbf{b}\ \|\ .
\end{aligned}$$

Thus

$$\|\ \mathbf{\Phi}(\mathbf{ab}) - \mathbf{\Phi}(\mathbf{a})\mathbf{\Phi}(\mathbf{b})\ \| \ \leq\ \lim_{\mathcal{U}}\ \mathrm{mult}(\Phi_\mathrm{n}) \|\ \mathbf{a}\ \|\|\ \mathbf{b}\ \|\ . \tag{2.12}$$

From (2.12), we obtain

$$\frac{\| \mathbf{\Phi}(\mathbf{ab}) - \mathbf{\Phi}(\mathbf{a})\mathbf{\Phi}(\mathbf{b}) \|}{\| \mathbf{a} \| \| \mathbf{b} \|} \leq \lim_{\mathcal{U}} \mathrm{mult}(\Phi_n)$$

which shows that

$$\mathrm{mult}(\mathbf{\Phi}) \leq \lim_{\mathcal{U}} \mathrm{mult}(\Phi_n) \ . \tag{2.13}$$

Conversely, we set $\varepsilon > 0$ and for each $n \in \mathbb{N}$, we pick $a_n, b_n \in A$ with

$\| a_n \| = \| b_n \| = 1$ and $\mathrm{mult}(\Phi_n) - \varepsilon < \| \Phi_n(a_n b_n) - \Phi_n(a_n)\Phi_n(b_n) \|$.

By taking the limit through $\mathcal{U}$ we arrive at

$$\lim_{\mathcal{U}} \mathrm{mult}(\Phi_n) - \varepsilon \leq \lim_{\mathcal{U}} \| \Phi_n(a_n b_n) - \Phi_n(a_n)\Phi_n(b_n) \| = \| \mathbf{\Phi}(\mathbf{ab}) - \mathbf{\Phi}(\mathbf{a})\Phi(\mathbf{b}) \| \leq \mathrm{mult}(\mathbf{\Phi})$$

where $\mathbf{a} = (a_n), \mathbf{b} = (b_n) \in A^{\mathcal{U}}$ and $\| \mathbf{a} \| = \| \mathbf{b} \| = 1$. We see that

$$\lim_{\mathcal{U}} \mathrm{mult}(\Phi_n) \leq \mathrm{mult}(\mathbf{\Phi}) \tag{2.14}$$

Combining (2.13) and (2.14), we have $\lim_{\mathcal{U}} \mathrm{mult}(\Phi_n) = \mathrm{mult}(\mathbf{\Phi})$ as required.

2.11.11 Theorem ([4], Theorem 3.5)

Let A be a Banach algebra and let B be an ultraprime Banach algebra. Then for each k, K, ε there is $\delta > 0$ such that if $\Phi \in \mathcal{B}(A, B)$ is such that $\mathrm{jmult}(\Phi) < \delta$, $k < \kappa(\Phi)$, and $\| \Phi \| < K$, then $\min\{\mathrm{mult}(\Phi), \mathrm{amult}(\Phi)\} < \varepsilon$.

Proof

Suppose the assertion of the theorem is false. Then there exist $\tau > 0$ and a sequence (Φ_n) in $\mathcal{B}(A, B)$ such that

$$k < \kappa(\Phi_n) \tag{2.15}$$

$$\| \Phi_n \| < K \tag{2.16}$$

$$\mathrm{jmult}(\Phi_n) \leq \frac{1}{n} \tag{2.17}$$

and

$$\min\{\mathrm{mult}(\Phi_n), \mathrm{amult}(\Phi_n)\} \geq \tau \tag{2.18}$$

Let us consider $\mathbf{\Phi} = (\Phi_n) \in \mathcal{B}(A^{\mathcal{U}}, B^{\mathcal{U}})$. On account of (2.15) and by Lemma 2.3.8 we have $k(\mathbf{\Phi}) = \lim_{\mathcal{U}} \kappa(\Phi_n) \geq k$ and so $\mathbf{\Phi}$ is surjective. Moreover, by (2.17) and Lemma 2.11.10, $\mathrm{jmult}(\mathbf{\Phi}) = \lim_{\mathcal{U}} \mathrm{jmult}(\Phi_n) = 0$, and hence $\mathbf{\Phi}$ is a Jordan homomorphism.

Since $B^{\mathcal{U}}$ is prime, $\mathbf{\Phi}$ is surjective and $\mathrm{jmult}(\mathbf{\Phi}) = 0$, then this map satisfies Theorem 2.11.8 (Herstein's Theorem). This means that map $\mathbf{\Phi}$ is either a homomorphism or an anti-homomorphism. It follows that $\min\{\mathrm{mult}(\mathbf{\Phi}), \mathrm{amult}(\mathbf{\Phi})\} = 0$

contradicting (2.18).

2.11.12 Corollary ([4], Corollary 3.6)

Let A be a Banach algebra. Then for each $\varepsilon > 0$ there is a $\delta > 0$ such that if ϕ is a linear functional on A with $\mathrm{jmult}(\phi) < \delta$, then $\mathrm{mult}(\phi) < \varepsilon$.

Proof

Let $\varepsilon > 0$. Take $\mathrm{k} > 0$ such that $\mathrm{k} + \mathrm{k}^2 < \varepsilon$ and $\mathrm{K} = 2$. Let $\delta > 0$ be given by Theorem 2.11.11 . Of course, we can assume that $\delta < 1$. Suppose ϕ is a linear functional on A with $\mathrm{jmult}(\phi) < \delta$, then $\mathrm{jmult}(\phi) < \infty$. It follows that ϕ satisfies Corollary 2.11.5. This implies that ϕ is continuous and

$$\| \phi \| \leq \frac{1 + \sqrt{1 + 4\mathrm{jmult}(\phi)}}{2}.$$

Since $\mathrm{jmult}(\phi) < \delta < 1$, then it follows that $\| \phi \| \leq \frac{1+\sqrt{1+4\delta}}{2} < 2$. We also point out that $\kappa(\phi) = \| \phi \|$. Consequently if $\mathrm{k}(\phi) > \mathrm{k}$ for some $\mathrm{k} > 0$, then we can conclude that $\mathrm{mult}(\phi) < \varepsilon$ by Theorem 2.11.11. Finally, if $\mathrm{k}(\phi) \leq \mathrm{k}$, then

$$\mathrm{mult}(\phi) \leq \| \phi \| + \| \phi \|^2 \leq \mathrm{k} + \mathrm{k}^2 < \varepsilon.$$

2.11.13 Definition ([4], p. 247)

Let A and B be Banach algebras. The pair (A, B) is said to be

Almost Multiplicative Near Multiplicative (AMNM) if to each $\mathrm{K}, \varepsilon > 0$ there is a $\delta > 0$ such that if $\Phi \in \mathcal{B}(\mathrm{A}, \mathrm{B})$ with $\| \Phi \| < \mathrm{K}$ and $\mathrm{mult}(\Phi) < \delta$ then $\| \Phi - \Psi \| < \varepsilon$ for some continuous homomorphism $\Psi : \mathrm{A} \to \mathrm{B}$.

For a general discussion on AMNM, we refer our readers to [11].

2.11.14 Definition ([4], p. 247)

Let A and B be Banach algebras. We call (A, B) an Almost Anti-Multiplicative

Near Anti-Multiplicative (AAMNAM) pair if for each $\mathrm{K}, \varepsilon > 0$ there is a

$\delta > 0$ such that if $\Phi \in \mathcal{B}(\mathrm{A}, \mathrm{B})$ with $\| \Phi \| < \mathrm{K}$ and $\min\{\mathrm{mult}(\Phi) , \mathrm{amult}(\Phi)\} < \delta$ then $\| \Phi - \Psi \| < \varepsilon$ for some continuous homomorphism or anti-homomorphism

$\Psi : \mathrm{A} \to \mathrm{B}$.

We note that the general implication of this definition is that it automatically incorporates Definition 2.11.13 on the AMNM. The only addition to this definition is that we include a discussion on the opposite algebra (the algebra of all anti-multiplicative maps) alongside the algebra of multiplicative maps. So, an example that may include a discussion on AAMNAM is where Banach algebras A and B, in this definition, are finite dimensional.

For a general discussion on AAMNAM, we refer our readers to [11].

2.11.15 Corollary ([4], Corollary 3.9)

Let (A,B) be an AAMNAM pair with B ultraprime. Then for each $\mathrm{k},\mathrm{K},\varepsilon > 0$ there is $\delta > 0$ such that if $\Phi \in \mathcal{B}(\mathrm{A},\mathrm{B})$ with $\mathrm{jmult}(\Phi) < \delta$, $\mathrm{k} < \kappa(\Phi)$ and $\| \Phi \| < \mathrm{K}$, then $\| \Phi - \Psi \| < \varepsilon$ for some epimorphism or anti-epimorphism $\Psi : \mathrm{A} \to \mathrm{B}$.

Proof

By Theorem 2.11.11 , there exist $\rho > 0$ such that $\min\{\mathrm{mult}(\Phi), \mathrm{amult}(\Phi)\} < \rho$. We are now reduced to proving statement $\| \Phi - \Psi \| < \varepsilon$ in the proposition. Clearly, Definition 2.11.14 concludes our proof.

2.11.16 Proposition ([4], Proposition 3.8)

Let H be a separable Hilbert space. Then $\big(\mathcal{B}(\mathrm{H}), \mathcal{B}(\mathrm{H})\big)$ is an AAMNAM pair.

2.11.17 Corollary ([4], Corollary 3.10)

Let H be a separable Hilbert space. Then for each $\mathrm{k},\mathrm{K},\varepsilon > 0$ there is $\delta > 0$ such that if $\Phi : \mathcal{B}(\mathrm{H}) \to \mathcal{B}(\mathrm{H})$ is a linear map with $\mathrm{jmult}(\Phi) < \delta$, $\mathrm{k} < \kappa(\Phi)$ and $\| \Phi \| < \mathrm{K}$, then $\| \Phi - \Psi \| < \varepsilon$ for some epimorphism or anti-epimorphism $\Psi : \mathcal{B}(\mathrm{H}) \to \mathcal{B}(\mathrm{H})$.

Proof

On account of Proposition 2.11.16, $\big(\mathcal{B}(\mathrm{H}), \mathcal{B}(\mathrm{H})\big)$ is an AAMNAM pair. Since $\mathrm{jmult}(\Phi) < \delta$, it follows that $\mathrm{jmult}(\Phi) < \infty$. Then by Proposition 2.11.4 , we have that

$$\mathrm{jmult}(\Phi) \;\leq\; \min\{\mathrm{mult}(\Phi), \mathrm{amult}(\Phi)\} \; .$$

$\mathcal{B}(\mathrm{H})$ is an ultraprime Banach algebra (see example under Definition 2.11.9) and $\mathrm{jmult}(\Phi) < \delta$ (in the hypothesis) , it follows that Theorem 2.11.11 is satisfied. For this reason, there exist $\rho > 0$ such that $\min\{\mathrm{mult}(\Phi), \mathrm{amult}(\Phi)\} < \rho$. Once again, we are reduced to proving statement $\| \Phi - \Psi \| < \varepsilon$ in the proposition, just as we did in Proposition 2.11.15. Clearly, Definition 2.11.14 concludes our proof.

We will continue being silent on the continuity of Φ because such maps are automatically continuous whenever Φ is non-zero. In order to justify our claim, we narrow ourselves to Corollary 2.11.5 . Since $\mathrm{k} < \kappa(\Phi)$, it follows that $\| \Phi \| \neq 0$. This shows that $\Phi \neq 0$. By Corollary 2.11.5 , Φ is surjective. Since $\mathrm{jmult}(\Phi) < \infty$, then by Corollary 2.11.5 again, it follows that Φ is continuous.

Chapter 3

Results related to the so-called Kaplansky's problem

3.1 The so-called Kaplansky's problem

The aim of this section is to state the so-called Kaplansky's problem and then provide results and examples that show that the problem in this form has some shortcomings, which lead to consideration of the Bernard Aupetit's open problem, which does not have some of the problems associated with the so-called Kaplansky's problem.

3.1.1 The actual question ([4], p. 234)

The actual question involving the so-called Kaplansky's problem was formulated as follows in terms of spectra:

Let A and B be complex Banach algebras. Suppose $\Phi : A \rightarrow B$ is a unital invertibility preserving map such that $\mathrm{sp}\big(\Phi(a)\big) \subset \mathrm{sp}(a)$. Is it true that Φ is a Jordan homomorphism?

3.1.2 Lemma ([9], Lemma 2.1)

Let A be a Banach algebra which need not be commutative and B be a commutative Banach algebra. Then each Jordan homomorphism $\Phi : A \rightarrow B$ is a homomorphism.

Proof

Since $\Phi(x^2) = \big(\Phi(x)\big)^2$ for all $x \in A$, it follows that

$$\Phi\big((x+y)^2\big) = \big(\Phi(x+y)\big)^2 \tag{3.1}$$

by replacing x with x+y . Expanding on both sides of the above Jordan homomorphism equation (3.1) , we end up with

$$\Phi(xy + yx) = \Phi(x)\Phi(y) + \Phi(y)\Phi(x) . \tag{3.2}$$

Since Banach algebra B is commutative by assumption, then (3.2) becomes

$$\Phi(xy + yx) = 2\,\Phi(x)\Phi(y)\ . \tag{3.3}$$

Replacing x with x^2, then (3.3) becomes

$$\Phi(x^2y + yx^2) = 2\,\Phi(x^2)\Phi(y) = 2\,\big(\Phi(x)\big)^2\Phi(y)\ . \tag{3.4}$$

Replacing y with xy + yx in (3.3), we have

$$\Phi\big(x(xy + yx) + (xy + yx)x\big) = \Phi\big(x^2y + 2xyx + yx^2\big) = 4\,\big(\Phi(x)\big)^2\Phi(y)\ . \tag{3.5}$$

Subtracting (3.4) from (3.5), we arrive at $\Phi(xyx) = \big(\Phi(x)\big)^2\Phi(y)$.

Fix $a \in A$ and $b \in A$ arbitrary; and put

$$2t = \Phi(ab - ba)\ . \tag{3.6}$$

Fix $a \in A$ and $b \in A$ arbitrary in (3.3), then (3.3) becomes

$$\Phi(ab + ba) = 2\,\Phi(a)\Phi(b)\ . \tag{3.7}$$

(3.6) and (3.7) imply that

$$\Phi(ab) = \Phi(a)\Phi(b) + t \quad \text{and} \tag{3.8}$$

$$\Phi(ba) = \Phi(b)\Phi(a) - t\ . \tag{3.9}$$

We get the squares of (3.6), (3.8) and (3.9) separately. Then the sum of the squares of (3.8) and (3.9) subtracted from (3.6) makes us arrive at $t = 0$.

3.1.3 Proposition ([20], Proposition 1.3)

Let R and R′ be two rings with identities 1 and 1′ respectively such that in R′, the equation $2x = 0$ implies that $x = 0$. If $\Phi : R \to R'$ is a Jordan homomorphism whose range contains 1′, then Φ preserves invertibility.

Proof

Let $\Phi : R \to R'$ be a Jordan homomorphism between rings R and R′. Define an operation $\Phi(x \circ y) = \Phi(x) \circ \Phi(y)$ such that $x \circ y = xy + yx$ and $\Phi(x) \circ \Phi(y) = \Phi(x)\Phi(y) + \Phi(y)\Phi(x)$ for all $x, y \in R$. Observe that with this notation in place, we are able to derive equation

$$\Phi(xy + yx) = \Phi(x)\Phi(y) + \Phi(y)\Phi(x)\ ,$$

the standard Jordan homomorphism equation.

First we show that $\Phi(1) = 1'$. By assumption, $1' = \Phi(u)$ for some $u \in R$; and so

$$2.1' = \Phi(u + u) = \Phi(1u + u1) = \Phi(1 \circ u) = \Phi(1) \circ \Phi(u) =$$

$$\Phi(1)\Phi(\mathrm{u}) + \Phi(\mathrm{u})\Phi(1) \;=\; \Phi(1).1' \;+\; 1'.\Phi(1) \;=\; 2\;\Phi(1).$$

Thus we arrive at $2.1' \;=\; 2\;\Phi(1)$ implying $1' \;=\; \Phi(1)$. We conclude that Φ is unital. Since Φ is a Jordan homomorphism, by Lemma 3.1.2, it follows that

$$\Phi(\mathrm{xyx}) \;=\; \Phi(\mathrm{x})\Phi(\mathrm{y})\Phi(\mathrm{x})$$

for all $\mathrm{x}, \mathrm{y} \in \mathrm{R}$. If $\mathrm{a} \in \mathrm{R}$ is invertible, then

$$\Phi(\mathrm{a}) \;=\; \Phi(\mathrm{a}\mathrm{a}^{-1}\mathrm{a}) \;=\; \Phi(\mathrm{a})\Phi(\mathrm{a}^{-1})\Phi(\mathrm{a})\;. \tag{3.10}$$

Let

$$\mathrm{p}_1 \;=\; \Phi(\mathrm{a})\Phi(\mathrm{a}^{-1}) \quad \text{and} \quad \mathrm{p}_2 \;=\; \Phi(\mathrm{a}^{-1})\Phi(\mathrm{a})\;. \tag{3.11}$$

From (3.10) and (3.11), we have

$$\mathrm{p}_1{}^2 \;=\; \Big(\Phi(\mathrm{a})\Phi(\mathrm{a}^{-1})\Phi(\mathrm{a})\Big)\Phi(\mathrm{a}^{-1}) \;=\; \Phi(\mathrm{a})\Phi(\mathrm{a}^{-1}) \;=\; \mathrm{p}_1 \tag{3.12}$$

and

$$\mathrm{p}_2{}^2 \;=\; \Phi(\mathrm{a}^{-1})\Big(\Phi(\mathrm{a})\Phi(\mathrm{a}^{-1})\Phi(\mathrm{a})\Big) \;=\; \Phi(\mathrm{a}^{-1})\Phi(\mathrm{a}) \;=\; \mathrm{p}_2\;. \tag{3.13}$$

On the other hand $\mathrm{p}_1 + \mathrm{p}_2 \;=\; \Phi(\mathrm{a})\circ\Phi(\mathrm{a}^{-1}) \;=\; \Phi(\mathrm{a}\circ\mathrm{a}^{-1}) \;=$

$$\Phi(\mathrm{a}\mathrm{a}^{-1}) \;+\; \Phi(\mathrm{a}^{-1}\mathrm{a}) \;=\; \Phi(1+1) \;=\; 2.\Phi(1) \;=\; 2.1'$$

implying that

$$\mathrm{p}_2 \;=\; 2.1' \;-\; \mathrm{p}_1\;. \tag{3.14}$$

Replacing (3.14) in (3.13), we have $(2.1' - \mathrm{p}_1)^2 \;= 2.1' - \mathrm{p}_1$, which leads to $2(\mathrm{p}_1 - 1') \;=\; 0$ and so $\mathrm{p}_1 \;=\; 1'$. Replacing this value in (3.11), we conclude that

$$\Phi(\mathrm{a}^{-1}) \;=\; \Big(\Phi(\mathrm{a})\Big)^{-1}.$$

3.1.4 Theorem ([18], Theorem 2.2)

Let $\Phi : \mathrm{A} \to \mathrm{B}$ be a unital, invertibility preserving map from a Banach algebra A into a semisimple commutative Banach algebra B. Then Φ is a homomorphism.

Proof

Let $\tau : \mathrm{B} \to \mathbb{C}$ where B is semisimple. Let $\Phi : \mathrm{A} \to \mathrm{B}$. Then $\tau \circ \Phi \in \Delta\mathrm{A}$ where $\Delta\mathrm{A}$ is the set of all characters (non-zero multiplicative linear functionals) on A.

We begin by proving that τ is continuous. Let $(x_n) \in A$, $n \in \mathbb{N}$, be with properties $x_n \to 0$ and $\Phi(x_n) \to \Phi(x)$ for some $\Phi(x) \in B$. If $\Phi(x)$ were non-zero and the fact that B is semisimple, then there exist a multiplicative linear functional τ on B such that $\tau(\Phi(x)) \neq 0$. Since $\tau(\Phi(x)) \neq 0$, we can assume, at this stage, without loss of generality, that we are dealing with a unital Banach algebra where $\tau(\Phi(1)) = 1$. Therefore, τ is invertible. This further implies that τ is continuous at 0 so that $(\tau \circ \Phi)(x_n) \to (\tau \circ \Phi)(0) = 0$. This shows that $\Phi(x) = 0$ contradicting our assumption that $\Phi(x) \neq 0$. Therefore Φ is a closed map. Therefore Φ is continuous for all $x \in A$. We now prove that Φ is a homomorphism. If this were false, then $\Phi(xy) \neq \Phi(x)\Phi(y)$ for some $x, y \in A$. We know that any multiplicative linear functional on B separates points of B. Then there is multiplicative linear functional $\tau \in \Delta B$ such that $(\tau \circ \Phi)(xy) \neq (\tau \circ \Phi)(x)(\tau \circ \Phi)(y)$. Clearly $\tau \circ \Phi$ is unital and invertibility preserving. So we are reduced to proving that a map with these conditions is a homomorphism. Let $\tau \circ \Phi = \phi$. Then we show that $\phi : A \longrightarrow \mathbb{C}$ is a Jordan homomorphism. We do it by showing that $\phi(x^2) = (\phi(x))^2$ for any $x \in A$. Let $x \in A$ be fixed. Consider the function $\Psi : \mathbb{C} \to \mathbb{C}$ defined by $\Psi(z) = \phi(e^{zx})$ for each $x \in A$, where e is the exponential function of variable z. By hypothesis, ϕ is invertible. This shows that Ψ has no zeros on $\mathbb{C}$. This means that ϕ is analytic everywhere on $\mathbb{C}$. So, there is an entire function f with $\Psi(z) = e^{f(z)}$, for every $z \in \mathbb{C}$, such that $f(0) = 0$ and $\mathrm{Re}\,(f(z)) \leq | f ||| z \|$ for all $z \in \mathbb{C}$. By

Lemma 2.4.1 (Schwarz′s lemma), $f(z) = \beta z$ for some $\beta \in \mathbb{C}$. Observe that we have

$$\Psi(z) = \phi(e^{zx}) = e^{f(z)} = e^{\beta z}.$$

Expanding both sides of equation $\phi(e^{zx}) = e^{\beta z}$, we have

$$\phi[\sum_{n=0}^{\infty} \frac{(zx)^n}{n!}] = \sum_{n=0}^{\infty} \frac{(\beta z)^n}{n!}$$

if and only if

$$\phi\Big[\sum_{n=0}^{\infty} \Big(x^n\Big)\Big(\frac{z^n}{n!}\Big)\Big] = \sum_{n=0}^{\infty} \Big(\beta^n\Big)\Big(\frac{z^n}{n!}\Big).$$

We therefore have $1 + z\phi(x) + \frac{\phi(x^2)z^2}{2} + ... = 1 + \beta z + \frac{\beta^2 z^2}{2} + ...$ implying that $z\phi(x) = \beta z$, $\frac{\phi(x^2)z^2}{2} = \frac{\beta^2 z^2}{2}, ...$ We therefore have $\phi(x) = \beta$, $\phi(x^2) = \beta^2, ...$ We now work with equations

$$\phi(x) = \beta \tag{3.15}$$

and

$$\phi(x^2) = \beta^2 \tag{3.16}$$

Squaring both sides of equation (3.15) and comparing it with (3.16), we have

$$\phi(x^2) = (\phi(x))^2$$

which we had set to prove. So ϕ is a Jordan homomorphism. By Lemma 3.1.2, we conclude that $\phi(xy) = \phi(x)\phi(y)$.

3.1.5 Example ([18], Example 2.3)

Let H be a Hilbert space and $\Phi : \mathcal{B}(\mathrm{H}) \to \mathcal{B}(\mathrm{H})$ be a linear map such that $\Phi(\mathrm{I}) = 0$. Consider the map $\Psi : \mathcal{B}(\mathrm{H}) \to \mathcal{B}(\mathrm{H} \oplus \mathrm{H})$ defined by

$$\Psi(\mathrm{T}) = \begin{pmatrix} \mathrm{T} & \Phi(\mathrm{T}) \\ 0 & \mathrm{T} \end{pmatrix}$$

where $\mathrm{T} \in \mathcal{B}(\mathrm{H})$. Then Ψ is a unital invertibility preserving linear map that may not be a Jordan homomorphism.

3.1.6 Example

This example follows from the example given by [10] on page 6 .

If $\mathcal{A}$ is the algebra of upper triangular 3×3 matrices, the linear mapping

$$\Phi \begin{pmatrix} \mathrm{a}_{11} & \mathrm{a}_{12} & \mathrm{a}_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & \mathrm{a}_{33} \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} a_{11} & a_{23} & a_{13} \\ 0 & a_{22} & a_{12} \\ 0 & 0 & a_{33} \end{pmatrix}$$

is clearly unital and preserves invertibility.

$$\text{If } \mathrm{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} ,$$

then it follows that $\Phi(\mathrm{A}^2) = \mathrm{A}^2$ but $\Phi(\mathrm{A})^2 = \Phi(\mathrm{A})$.

This shows that $\Phi(\mathrm{A}^2) - \Phi(\mathrm{A})^2$ is in general non-zero.

These two examples tempt us to give the following example.

3.1.7 Example ([18], Example 2.4)

There exists a Banach algebra A, which is not semisimple, having the property that there exists a surjective linear spectrum preserving map $\mathrm{T} : \mathrm{A} \to \mathrm{A}$ which is not a Jordan homomorphism.

Proof

Let A be the subalgebra of $\mathrm{M}_4(\mathbb{C})$ consisting of matrices of the form $\begin{pmatrix} \mathrm{a} & \mathrm{b} \\ 0 & \mathrm{c} \end{pmatrix}$ with $\mathrm{a}, \mathrm{b}, \mathrm{c} \in \mathrm{M}_2(\mathbb{C})$. Define a linear mapping

T from A onto A by $\mathrm{T}\begin{pmatrix} \mathrm{a} & \mathrm{b} \\ 0 & \mathrm{c} \end{pmatrix} = \begin{pmatrix} \mathrm{a} & \mathrm{b} \\ 0 & \mathrm{c}^{\mathrm{t}} \end{pmatrix}$ where c^{t} is the transpose of c. Clearly, T is linear, bijective and unital. Furthermore, T is invertibility preserving. However

$$\mathrm{T}\left(\begin{pmatrix} \mathrm{a} & \mathrm{b} \\ 0 & \mathrm{c} \end{pmatrix}^2\right) - \left(\mathrm{T}\begin{pmatrix} \mathrm{a} & \mathrm{b} \\ 0 & \mathrm{c} \end{pmatrix}\right)^2 = \begin{pmatrix} 0 & \mathrm{bc} - \mathrm{bc}^{\mathrm{t}} \\ 0 & 0 \end{pmatrix} \in \mathrm{rad}(\mathrm{A})$$

which is in general not zero. This implies that in order to obtain a reasonable open problem, we have to require that A be semisimple.

3.1.8 Bernard Aupetit's open problem ([4], p. 234)

Let A and B be semisimple Banach algebras and let $\Phi : A \to B$ be a surjective map such that

$$\mathrm{sp}(\Phi(a)) = \mathrm{sp}(a) \ (a \in A).$$

Is it true that Φ is a Jordan isomorphism? This is still an open question.

3.2 First solution by way of idempotents in Von Neumann algebras

The main result of this section is Theorem 3.2.6 , which is obtained via idempotent characterisation in Von Neumann algebras. A Von Neumann algebra is a special kind of Banach algebra (see Definition 2.1.8). One of the key results required to obtain Theorem 3.2.6 is Theorem 3.2.3 , which states that the spectrum of every idempotent is contained in the set $\{ 0, 1 \}$.

3.2.1 Proposition ([5], Proposition 2.1)

Let A and B be two semisimple Banach algebras. Suppose that Φ is a spectrum-preserving linear mapping from A into B.

(i) Then Φ is injective.

(ii) If Φ is surjective (onto), then $\Phi(1) = 1$.

(iii) If Φ is (surjective) onto, then Φ is continuous. Consequently there exist two positive constants α and β such that $\alpha \| x \| \leq \| \Phi(x) \| \leq \beta \| x \|$ for every $x \in A$.

Proof

(i) Suppose that $\Phi(a) = 0$. Then $\mathrm{sp}(a+x) = \mathrm{sp}\left(\Phi(a+x)\right) = \mathrm{sp}\left(\Phi(x)\right) = \mathrm{sp}(x)$ for every $x \in A$. It follows from Theorem 2.8.11 that $a \in \mathrm{rad}\, A$. Since A is semisimple, this means that $\mathrm{rad}\, A = \{0\}$ and so $a = 0$.

(ii) We have $\mathrm{sp}\left(\Phi(1)\right) = \mathrm{sp}(1) = \{1\}$, so $\Phi(1) = 1 + q$, where q is quasinilpotent. For x arbitrary, we have

$\mathrm{sp}\left(\Phi(1+x)\right) = \mathrm{sp}\left(1 + q + \Phi(x)\right) = 1 + \mathrm{sp}\left(q + \Phi(x)\right) = 1 + \mathrm{sp}(x).$

Hence $\mathrm{sp}\big(q + \Phi(x)\big) = \mathrm{sp}(x) = \mathrm{sp}\left(\Phi(x)\right)$. Since Φ is surjective, $q \in \mathrm{Rad}\, B = \{0\}$, so $\Phi(1) = 1$.

(iii) From (ii), condition $\Phi(1) = 1$ implies that Φ is continuous. By hypothesis, Φ is spectrum preserving. Then we have $\mathrm{r}\big(\Phi(\mathrm{x})\big) = \mathrm{r}(\mathrm{x})$, for every $\mathrm{x} \in \mathrm{A}$. It follows that either $\| \Phi(\mathrm{x}) \| \leq \| \mathrm{x} \|$ or $\| \mathrm{x} \| \leq \| \Phi(\mathrm{x})\|$. We conclude that there exist two positive constants α and β such that

$$\alpha \| \mathrm{x} \| \leq \| \Phi(\mathrm{x}) \| \leq \beta \| \mathrm{x} \|$$

for every $\mathrm{x} \in \mathrm{A}$.

3.2.2 Lemma ([5], Lemma 2.2)

Let A be a Banach algebra and $\mathrm{e} \in \mathrm{A}$ be an idempotent. For every $\mathrm{a} \in \mathrm{A}$, we have $\mathrm{sp(a)} \subseteq \overline{\mathrm{D}}\Big(0, (\| \mathrm{e} \| + \| 1 - \mathrm{e} \|) \| \mathrm{a} - \mathrm{e} \| \Big) \bigcup \overline{\mathrm{D}}\Big(1, (\| \mathrm{e} \| + \| 1 - \mathrm{e} \|) \| \mathrm{a} - \mathrm{e} \| \Big)$.

Proof

If $\mathrm{e} = 0$ or $\mathrm{e} = 1$, then the result is obvious. Suppose that e is a non-trivial idempotent element; that is $\mathrm{e} \neq 0$, $\mathrm{e} \neq 1$ and $\mathrm{sp(e)} = \{0, 1\}$. Then from identity

$$1 = (\lambda - \mathrm{e})(\lambda - \mathrm{e})^{-1} = (\lambda - \mathrm{e})\Big((\lambda - 1)^{-1}\mathrm{e} + \lambda^{-1}(1 - \mathrm{e})\Big)$$

$$= \Big((\lambda - 1)^{-1}\mathrm{e} + \lambda^{-1}(1 - \mathrm{e})\Big)(\lambda - \mathrm{e}) = (\lambda - \mathrm{e})^{-1}(\lambda - \mathrm{e}) = 1$$

for $\lambda \in \mathbb{C}$ such that $\lambda \notin \{ 0, 1 \}$, it follows that

$$(\lambda - \mathrm{e})^{-1} = (\lambda - 1)^{-1}\mathrm{e} + \lambda^{-1}(1 - \mathrm{e})$$

and

$$(\lambda - \mathrm{e}) = (\lambda - 1)\mathrm{e} + \lambda(1 - \mathrm{e}) .$$

By Theorem 2.5.2 (the Spectral Mapping Theorem), it follows that

$$\mathrm{sp}\Big((\lambda - \mathrm{e})^{-1}\Big) = \mathrm{sp}\Big((\lambda - 1)^{-1}\mathrm{e} + \lambda^{-1}(1 - \mathrm{e})\Big) = \Big\{ (\lambda - 1)^{-1}, \lambda^{-1} \Big\}.$$

Consequently

$$\mathrm{r}\Big((\lambda - \mathrm{e})^{-1}\Big) = \max\Big\{ | \lambda - 1 |^{-1}, | \lambda|^{-1} \Big\} = \frac{1}{\mathrm{dist}(\lambda , \mathrm{sp(e)})} .$$

Then we have $\| (\lambda - \mathrm{e})^{-1} \| = \| (\lambda - 1)^{-1}\mathrm{e} + \lambda^{-1}(1 - \mathrm{e}) \|$

$$\leq | \lambda - 1 |^{-1} \| \mathrm{e} \| + | \lambda^{-1} | \| 1 - \mathrm{e} \| \leq \Big(\| \mathrm{e} \| + \| 1 - \mathrm{e} \| \Big) \mathrm{r}\Big((\lambda - \mathrm{e})^{-1}\Big)$$

which implies that

$$\frac{1}{\mathrm{r}\big((\lambda - \mathrm{e})^{-1}\big)} \leq \frac{\| \mathrm{e} \| + \| 1 - \mathrm{e} \|}{\| (\lambda - \mathrm{e})^{-1} \|} .$$

Hence

$$\text{dist}(\lambda\ ,\ \text{sp(e)}) \leq \frac{\| e \| + \| 1 - e \|}{\| (\lambda - e)^{-1} \|} . \tag{3.17}$$

Suppose that the result is false ? By hypothesis, there exist $a \in A$ and $\lambda \in \text{sp}(a)$ such that

$$\lambda \notin \overline{D}\Big(0, (\| e \| + \| 1 - e \|) \| a - e \| \Big) \bigcup \overline{D}\Big(1, (\| e \| + \| 1 - e \|) \| a - e \| \Big).$$

It follows that

$$\text{dist}(\lambda\ ,\ \text{sp(e)}) > (\| e \| + \| 1 - e \|) \| a - e \| \tag{3.18}$$

for every $\lambda \neq 0\ ,\ 1$. Combining (3.17) and (3.18) , we have

$$(\| e \| + \| 1 - e \|) \| a - e \| < \frac{\| e \| + \| 1 - e \|}{\| (\lambda - e)^{-1} \|}$$

which further implies that

$$\| a - e \| < \frac{1}{\| (\lambda - e)^{-1} \|} .$$

Hence

$$\| (a - e) (\lambda - e)^{-1} \| < 1 .$$

It follows that

$$\lambda - a = (\lambda - e) \Big[1 - (a - e) (\lambda - e)^{-1}\Big]$$

is invertible. This is a contradiction.

3.2.3 Theorem ([18], Theorem 3.4)

An element e of a semisimple Banach algebra A is an idempotent if and only if $\text{sp}(e) \subseteq \{ 0, 1 \}$ and there exist $C\ ,\ r > 0$ such that $\text{sp}(a) \subseteq \text{sp}(e) + C \| a - e \|$ for every $a \in A$ with $\| a - e \| < r$. For the proof of this theorem, we refer our readers to Theorem 5.2.7 of [24] on page 78.

3.2.4 Theorem ([18], Theorem 3.5)

Let A and B be semisimple Banach algebras, and let $\Phi : A \to B$ be a surjective spectrum-preserving map. Then Φ preserves idempotents.

Proof

Since Φ is a surjective spectrum-preserving map, then we have $\text{sp}(\Phi(x)) = \text{sp}(x)$ for all $x \in A$. It follows that $\Phi(1) = 1$. Since A and B are semisimple, we have

$\Phi(0) = 0$. Hence $\mathrm{sp(e)} \subseteq \{ 0,1 \}$. This implies that $\mathrm{sp}(\Phi(\mathrm{e})) \subseteq \{ 0,1 \}$ since Φ is spectrum preserving. Then by Theorem 3.2.3, there exist C , $\mathrm{r} > 0$ such that $\mathrm{sp(a)} \subseteq \mathrm{sp(e)} + \mathrm{C} \| \mathrm{a} - \mathrm{e} \|$ for every $\mathrm{a} \in \mathrm{A}$ with $\| \mathrm{a} - \mathrm{e} \| < \mathrm{r}$. Therefore, for some $\alpha > 0$, we have $\mathrm{sp}(\Phi(\mathrm{a}) \subseteq \mathrm{sp}(\Phi(\mathrm{e})) + \frac{\mathrm{C}}{\alpha} \| \Phi(\mathrm{a}) - \Phi(\mathrm{e}) \|$ for every $\mathrm{a} \in \mathrm{A}$ with $\| \Phi(\mathrm{a}) - \Phi(\mathrm{e}) \| < \alpha\mathrm{r}$. Since Φ is surjective, then for $\mathrm{y} \in \mathrm{B}$, there exist an $\mathrm{a} \in \mathrm{A}$ such that $\Phi(\mathrm{a}) = \mathrm{y}$. Hence we have

$$\mathrm{sp}\ (\mathrm{y}) \subseteq \mathrm{sp}(\Phi(\mathrm{e})) + \frac{\mathrm{C}}{\alpha} \| \mathrm{y} - \Phi(\mathrm{e}) \|$$

for every $\mathrm{a} \in \mathrm{A}$ with $\| \mathrm{y} - \Phi(\mathrm{e}) \| < \alpha\mathrm{r}$. This shows that $\Phi(\mathrm{e})$ is an idempotent in B by an application of Theorem 3.2.3 again.

3.2.5 Theorem ([5], Theorem 1.2)

Let A, B be two semisimple Banach (or Jordan-Banach) algebras and let Φ be a surjective spectrum-preserving linear mapping from A into B. Then Φ transforms a set of orthogonal idempotents of A into a set of orthogonal idempotents of B.

Proof

By Theorem 3.2.4 , $\Phi : \mathrm{A} \to \mathrm{B}$ is idempotent preserving . We only need to show that orthogonality is preserved. Let $\mathrm{x}, \mathrm{y} \in \mathrm{A}$ be idempotents such that

$$\mathrm{xy} = \mathrm{yx} = 0\ . \tag{3.19}$$

By an application of (3.19), it follows that

$$(\mathrm{x} + \mathrm{y})^2 = \mathrm{x}^2 + \mathrm{xy} + \mathrm{yx} + \mathrm{y}^2 = \mathrm{x} + \mathrm{y}\ .$$

Hence $\mathrm{x} + \mathrm{y}$ is an idempotent in A. Since $\Phi : \mathrm{A} \to \mathrm{B}$ is idempotent preserving by Theorem 3.2.4, it follows that

$$\Phi(\mathrm{x}),\ \Phi(\mathrm{y}),\ \Phi(\mathrm{x} + \mathrm{y}) = \Phi(\mathrm{x}) + \Phi(\mathrm{y})$$

are idempotents in B. Then the equality $(\Phi(\mathrm{x}) + \Phi(\mathrm{y}))^2 = \Phi(\mathrm{x}) + \Phi(\mathrm{y})$ reduces to proving

$$\Phi(\mathrm{x})\Phi(\mathrm{y}) + \Phi(\mathrm{y})\Phi(\mathrm{x}) = 0\ . \tag{3.20}$$

Multiplying (3.20) on the left by $\Phi(\mathrm{x})$, we have

$$\Phi(\mathrm{x})\Phi(\mathrm{y}) + \Phi(\mathrm{x})\Phi(\mathrm{y})\Phi(\mathrm{x}) = 0\ . \tag{3.21}$$

Multiplying (3.20) on the right by $\Phi(\mathrm{x})$, we have

$$\Phi(\mathrm{x})\Phi(\mathrm{y})\Phi(\mathrm{x}) + \Phi(\mathrm{y})\Phi(\mathrm{x}) = 0\ . \tag{3.22}$$

Subtracting (3.22) from (3.21) , we obtain

$$\Phi(\mathrm{x})\Phi(\mathrm{y}) - \Phi(\mathrm{y})\Phi(\mathrm{x}) = 0\ . \tag{3.23}$$

Results of (3.20) and (3.23) show that (3.20) = (3.23) if and only if

$$\Phi(\mathrm{x})\Phi(\mathrm{y}) = \Phi(\mathrm{y})\Phi(\mathrm{x}) = 0\ .$$

Therefore orthogonality is preserved.

3.2.6 Theorem ([18], Theorem 3.6)

Let A be a Von Neumann algebra and let B be a semisimple Banach algebra.

Let $\Phi : A \to B$ be a continuous linear map that preserves idempotents.

Then Φ is a Jordan homomorphism.

Proof

Let A_{sa} be the set of self-adjoint elements of A. Take any $h \in A_{sa}$ with finite spectrum. Then $h = \sum_{k=1}^{n} t_k p_k$, where $\{t_k\}_{k=1}^{n}$ are real numbers and $\{p_k\}_{k=1}^{n}$ is an orthogonal family of projections of idempotents (i.e self-adjoint idempotents). By Theorem 3.2.5, then $\{\Phi(p_k)\}_{k=1}^{n}$ is an orthogonal family of idempotents of B. It follows that

$$\Phi(h) = \Phi\left(\sum_{k=1}^{n} t_k p_k\right) = t_1\Phi(p_1) + ... + t_n\Phi(p_n) .$$

$$\text{Hence} \qquad \Phi(h)^2 = {t_1}^2\Phi(p_1)^2 + ... + {t_n}^2\Phi(p_1)^2$$

$$\text{Also} \qquad \Phi(h^2) = \Phi({t_1}^2{p_1}^2 + ... + {t_n}^2{p_n}^2) .$$

It follows that

$$\Phi(h)^2 = \Phi(h^2) \tag{3.24}$$

for every $h \in A_{sa}$. Now replacing h by $h_1 + h_2$, in (3.24), for arbitrary $h_1, h_2 \in A_{sa}$, we get

$$\Phi(h_1h_2 + h_2h_1) = \Phi(h_1)\Phi(h_2) + \Phi(h_2)\Phi(h_1) .$$

Finally, an arbitrary element $a \in A$ can be written as $a = h_1 + i\,h_2$ with $h_1, h_2 \in A_{sa}$. Then

$$a^2 = ({h_1}^2 - {h_2}^2) + i(h_1h_2 + h_2h_1) .$$

It follows readily that $\Phi(a^2) = \Phi(a)^2$. Hence Φ is a Jordan homomorphism.

3.2.7 Theorem ([18], Theorem 3.1)

Let A and B be Von Neumann algebras. Let $\Phi : A \to B$ be a surjective spectrum-preserving linear map. Then Φ is a Jordan isomorphism.

Proof

By Theorem 3.2.4, Φ preserves idempotents, since Von Neumann algebras are semisimple. Let $\Phi(x) = 0$ for some $x \in A$. Since B is semisimple (Recall Definition 2.1.8 and Theorem 2.8.9), then $sp\big(\Phi(x)\big) = 0$. Since $\Phi : A \to B$ is spectrum-preserving, then $0 = sp\big(\Phi(x)\big) = sp(x)$. Since A is also semisimple, it follows that $x = 0$. Thus $\Phi(0) = 0$ implying that Φ is continuous. Continuity at 0 implies that Φ is injective. Since Φ is surjective by hypothesis, it follows that Φ is bijective. Then by Theorem 3.2.6, Φ is a Jordan homomorphism.

3.3 Second solution by way of spectra isometries

The main result in this section, Theorem 3.3.12 , is obtained via spectra isometries on finite dimensional semisimple Banach algebras. A key result to establish Theorem 3.3.12 is Nagasawa's Theorem (Theorem 3.3.2), which describes conditions under which a spectra state between commutative Banach algebras becomes an isomorphism.

3.3.1 Question to Discuss ([18], p. 59)

We base our arguments on the reformulated question of Kaplansky. Let A and B be semisimple unital Banach algebras. Let $\Phi : A \rightarrow B$ be a unital surjective spectra isometry. Must Φ be a Jordan isomorphism?

3.3.2 Nagasawa's Theorem ([18], Theorem 5.1)

Let A and B be commutative semisimple Banach algebras. If $\Phi : A \rightarrow B$ is a unital surjective spectra isometry from A into B, then Φ is an isomorphism.

Proof

Assume $\Phi : A \rightarrow B$ is a unital surjective spectra isometry from A into B. Let $\sum_B$ be a set of spectra states of B. Let $f \in \sum_B$ be an extreme point. Set $g = f \circ \Phi$. Since Φ is a spectra isometry, it follows that $r\big(\Phi(x)\big) = r(x)$ for all $x \in A$. Since f is a spectra state, it follows that

$$| f\big(\Phi(x)\big) | \leq r\big(\Phi(x)\big) = r(x) .$$

Since $g = f \circ \Phi$, it follows that

$$| g(x) | \leq r\big(\Phi(x)\big) = r(x)$$

implying that g is a spectra state. As f is extreme, g is also extreme. Since $g = f \circ \Phi$ and f are extreme spectra states on $\sum_A$ and $\sum_B$ respectively, they are multiplicative by Lemma 2.9.4 (ii).

$$\text{This implies that } f\big(\Phi(xy)\big) = g(xy) = g(x)g(y) = \big(f \circ \Phi(x)\big)\big(f \circ \Phi(y)\big)$$
$$= f\big(\Phi(x)\big)f\big(\Phi(y)\big) = f\big(\Phi(x)\Phi(y)\big) .$$

It follows that $f\big(\Phi(xy)\big) = f\big(\Phi(x)\Phi(y)\big)$. Since A and B are both semisimple, then Φ and Φ^{-1} are continuous. Thus Φ is injective. Injectivity of Φ implies injectivity of f . It follows that $\Phi(xy) = \Phi(x)\Phi(y)$ for all $x, y \in A$.

3.3.3 Definition ([18], p. 1)

Let A be a unital Banach algebra. Let $x \in A$ be such that $xy = yx$ for all $y \in A$. Then $x \in A$ is a centre in A. The set containing all centers of A will be denoted by $\mathcal{Z}(A)$.

Note that some algebras may have more than one center. For example every element in a commutative algebra is a center.

3.3.4 Corollary ([18], Corollary 5.4)

Let A and B be semisimple Banach algebras. If $\Phi : A \to B$ is a unital surjective spectra isometry from A into B, then $\Phi : \mathcal{Z}(A) \to \mathcal{Z}(B)$ is an isomorphism.

Proof

If A is a semisimple Banach algebra, so is $\mathcal{Z}(A)$. Equally, if B is a semisimple Banach algebra, so is $\mathcal{Z}(B)$. By definition of a center of an algebra, we know that elements of $\mathcal{Z}(A)$ and $\mathcal{Z}(B)$ are commutative in their respective sets. Instead of dealing with $\Phi : A \to B$, we become involved with restriction map $\phi : \mathcal{Z}(A) \to \mathcal{Z}(B)$ which satisfies by inheritance all the properties of Φ . By hypothesis, ϕ is a unital spectra isometry on commutative semisimple Banach algebras. It follows that Theorem 3.3.2 is satisfied. Since ϕ satisfies Theorem 3.3.2, then Φ also satisfies Theorem 3.3.2. Hence the proof.

3.3.5 Lemma ([18], Lemma 5.5)

Let X and Y be semisimple Banach algebras and $\Phi : X \to Y$ be a unital surjective spectra isometry. Let $a \in \mathcal{Z}(X)$ be a central idempotent and let $\Phi(a) = b$. Then $(1 - b)\Phi(aX) = \{0\}$. Furthermore, the map $\Phi : aX \to bY$ given by $\Phi_a(ax) = b\Phi(ax)$, where $x \in X$, is a unital surjective spectra isometry.

Proof

Since X and Y are semisimple Banach algebras and $\Phi : X \to Y$ is a unital surjective spectra isometry, then Theorem 3.3.2 is satisfied so that Φ is an isomorphism. Hence Φ is a non-zero multiplicative linear map. Then for some $a \in \mathcal{Z}(X)$ and the surjectivity of Φ , it follows that $\Phi(a) = b$. By Corollary 3.3.4, $\Phi(a) = b$ is a central idempotent. Since Φ is multiplicative, it follows that $\Phi(ax) = \Phi(a)\Phi(x)$ for all $x \in X$. Since $\Phi(a) = b$ is an idempotent, it follows that $b^2 = \Phi(a)^2 = \Phi(a) = b$. Therefore, we get

$$(1 - b)\Phi(ax) = (1 - b)\big(\Phi(a)\Phi(x)\big) = (b - b^2)\Phi(x) = 0 \times \Phi(x) = \{0\}$$

for each $x \in X$. Since Φ is surjective, then for each $y \in Y$, there is an $x \in X$ such that $\Phi(x) = y$. It follows that

$$by = b\Phi(x) = b^2\big(\Phi(x)\big) = b\big(b\Phi(x)\big) = b\big(\Phi(a)\Phi(x)\big) = b\Phi(ax) = \Phi_a(ax).$$

3.3.6 Corollary ([18], Corollary 5.7)

Let M_i $(i = 1, 2)$ be semisimple Banach algebras with the property that every unital surjective spectra isometry from M_i onto a semisimple Banach algebra is a Jordan isomorphism. Then $M_1 \oplus M_2$ has the same property.

Proof

Consider the semisimple Banach algebra $X = M_1 \oplus M_2$, and let $\Phi : X \to Y$ be a unital surjective spectra isometry. Let $a = (1,0) \in X$ so that $aX = M_1$ and $(1 - a)X = M_2$. Clearly both a and $1 - a = (0,1)$ are central idempotents of X. The maps

$$\Phi : M_1 \to \Phi(a)Y \quad \text{and} \quad \Phi : M_2 \to \big(1 - \Phi(a)\big)Y$$

are unital spectra isometries. Since Φ is multiplicative by Lemma 3.3.5 , it follows that for the map $\Phi : M_1 \to \Phi(a)Y$, we have

$$\Phi\Big((ax)^2\Big) = \Phi\Big((ax)(ax)\Big) = \Phi(ax)\Phi(ax) = \Phi(a)\Phi(x)\Phi(a)\Phi(x) =$$

$$\Phi(a)^2\Phi(x)^2 = \Big(\Phi(a)\Phi(x)\Big)^2 = \Big(\Phi(ax)\Big)^2$$

and for the map $\Phi : M_2 \to \big(1 - \Phi(a)\big)Y$, we have

$$\Phi\Big(\big((1 - a)x\big)^2\Big) = \Big(\Phi\big((1 - a)x\big)\Big)^2$$

implying that both maps are Jordan isomorphisms. Indeed for any $x \in X$, write $x = (m_1, m_2)$ where $m_i \in M_i$ for $i = 1, 2$. Then since $\Phi(m_1) \in \Phi(a)Y$ and $\Phi(m_2) \in \big(1 - \Phi(a)\big)Y$, we have

$$\Phi(x^2) = \Phi\big((m_1{}^2 \ , \ m_2{}^2)\big) = \Phi(m_1)^2 + \Phi(m_2)^2 = \Phi(x)^2$$

as desired.

3.3.7 Notation

Let $\mathcal{S}I_n$ be a space on $n \times n$ complex matrices with trace zero. Let $M_n(\mathbb{C})$ be the direct sum of $\mathbb{C}I$ and $\mathcal{S}I_n$, that is

$$M_n(\mathbb{C}) = \mathbb{C}I \oplus \mathcal{S}I_n \ .$$

We mention here that $\mathcal{S}I_n$ is a linear span of the set $\mathcal{N}_n$ of nilpotent elements of $M_n(\mathbb{C})$. Finally, let GL_n be the group of invertible elements of $M_n(\mathbb{C})$.

3.3.8 Theorem ([18], Theorem 5.9)

Fix $n \geq 1$. Let $\Phi : \mathcal{S}I_n \to \mathcal{S}I_n$ be an invertible linear map such that $\Phi(\mathcal{N}_n) \subseteq \mathcal{N}_n$. Then there exist $\alpha \in \mathbb{C}^*$ and $GL_n \in M_n(\mathbb{C})$ such that either

$$\Phi(X) = \alpha\, A\, X\, A^{-1} \qquad \text{or} \qquad \Phi(X) = \alpha\, A\, X^t\, A^{-1} \qquad (\, A \in GL_n \ , \ X \in \mathcal{S}I_n) \ .$$

3.3.9 Proposition ([24], Proposition p. 5)

Let X and Y be Banach spaces. If $\Phi : \mathcal{B}(X) \to \mathcal{B}(Y)$ is a linear mapping defined as $\Phi(a) = b\, a\, b^{-1}$ for some invertible linear operator $b : X \to Y$, then Φ is an isomorphism. If $\Phi(a) = c\, a^t\, c^{-1}$ for some invertible linear operator $c : X \to Y$, then Φ is an anti-isomorphism.

Proof

Suppose $\Phi(a) = b\, a\, b^{-1}$. Then Φ possesses the following properties:

(i) For any $\alpha \in \mathbb{C}$, we have that

$$\Phi(\alpha\, a) = b\, \alpha\, a\, b^{-1} = \alpha\, (b\, a\, b^{-1}) = \alpha\, \Phi(a)$$

(ii) For any $a, b \in X$, we have that

$$\Phi(\, a + x\,) = b\, (\, a + x\,)\, b^{-1} = b\, a\, b^{-1} + b\, x\, b^{-1} = \Phi(\, a\,) + \Phi(\, x\,)$$

(iii) For invertible Φ, we have that

$$\Big(\Phi(a)\Big)^{-1} = \big(b\, a\, b^{-1}\big)^{-1} = b\, a^{-1}\, b^{-1} = \Phi(a^{-1})$$

These three conditions imply that Φ is an isomorphism. A similar argument will show that Φ is an anti-isomorphism when the above three conditions are applied to $\Phi(a) = c\, a^t\, c^{-1}$.

(iv) Additionally, $\Phi(a^2) = b\, a^2\, b^{-1} = b\, a\, b^{-1} b\, a\, b^{-1} = \Phi(a)\Phi(a) = \Phi(a)^2$ which is just the Jordan homomorphism equation. A similar argument will show that Φ is a Jordan anti-homomorphism when this condition is applied to $\Phi(a) = c\, a^t\, c^{-1}$.

3.3.10 Lemma ([18], Lemma 5.10)

For $n \geq 1$, if $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a unital surjective spectra isometry, then Φ is a Jordan automorphism of $M_n(\mathbb{C})$.

Proof

We consider a quasinilpotent element in $M_n(\mathbb{C})$. This is so because every quasinilpotent element of $M_n(\mathbb{C})$ is nilpotent. Then for this chosen $X \in \mathcal{S}I_n$, we conclude that there exists spectra isometry Φ preserving nilpotent operators of $M_n(\mathbb{C})$. By Theorem 3.3.8, there exist $\alpha \in \mathbb{C}^*$ and $GL_n \in M_n(\mathbb{C})$ such that either

$$\Phi(X) = \alpha\, A\, X\, A^{-1} \qquad \text{or} \qquad \Phi(X) = \alpha\, A\, X^t\, A^{-1} \qquad (\, A \in GL_n \;,\; X \in \mathcal{S}I_n\,).$$

We now deal with the former case $\Phi(X) = \alpha\, A\, X\, A^{-1}$. Since Φ is a spectra isometry, then $\alpha = 1$. Therefore, for any $\lambda \in \mathbb{C}$ and $X \in \mathcal{S}I_n$, we have

$$\Phi(\lambda I + X) = A\, (\lambda I + X)\, A^{-1} = \lambda I + A\, X\, A^{-1}.$$

Hence $\Phi(Y) = A\, Y\, A^{-1}$ for all $Y \in M_n(\mathbb{C})$. Similarly, the later case implies that $\Phi(Y) = A\, Y^t\, A^{-1}$ for all $Y \in M_n(\mathbb{C})$. In either case, Φ is a Jordan isomorphism of $M_n(\mathbb{C})$ by Proposition 3.3.9.

3.3.11 Theorem ([18], Theorem 5.8)

Let A be a finite dimensional semisimple algebra over $\mathbb{C}$. Then there exist integers $\{n_i : 1 \leq i \leq k\} \subset \mathbb{N}$ so that $A \cong M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus .... \oplus M_{n_k}(\mathbb{C})$.

3.3.12 Theorem([18], Theorem 5.11)

Let X and Y be finite dimensional semisimple Banach algebras. If $\Phi : X \to Y$ is a unital surjective spectra isometry, then Φ is a Jordan isomorphism.

Proof

By Theorem 3.3.11 , we can write

$$X \cong M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus .... \oplus M_{n_k}(\mathbb{C}).$$

$$Y \cong M_{m_1}(\mathbb{C}) \oplus M_{m_2}(\mathbb{C}) \oplus .... \oplus M_{m_p}(\mathbb{C}).$$

Let $\{a_1, a_2, ..., a_n\}$ and $\{b_1, b_2, ..., b_p\}$ be the families of minimal idempotents in X and Y respectively. Note that $a_jX \cong M_{n_j}(\mathbb{C})$ and $b_jY \cong M_{m_j}(\mathbb{C})$ for each index j , and so

$$\mathcal{Z}(X) = \mathbb{C}_{a_1} \oplus .... \oplus \mathbb{C}_{a_k} \quad \text{and} \quad \mathcal{Z}(Y) = \mathbb{C}_{b_1} \oplus .... \oplus \mathbb{C}_{b_p} .$$

Since $\Phi : \mathcal{Z}(X) \longrightarrow \mathcal{Z}(Y)$ is an algebraic isomorphism by Corollary 3.3.4 , then idempotents are mapped bijectively. Hence $k = p$. We may now assume $\Phi(a_j) = b_j$ for all $1 \leq j \leq k$. By Corollary 3.3.6 , map $\Phi : a_iX \longrightarrow b_jY$ is well defined. We conclude that Φ is a Jordan isomorphism as required.

3.4 Third solution by way of invertibility spectrum-preserving bijective maps

This section provides results for the so-called Kaplansky's problem in terms of invertibility spectrum-preserving bijective maps. A characterisation of bijective maps on locally convex topological vector spaces is obtained. This characterisation invokes the use of rank one operators, which involve standard operator algebras. The main result is Theorem 3.4.14, which is established via the characterisation of rank one operators.

3.4.1

In this section, we will use the symbol $w \otimes g^*$ to stand for the rank one operator on locally convex topological vector space W such that $w \in W$ and $g^* \in W^*$. Our readers are referred to Definition 2.2.9 for a recap. The action of the operator $w \otimes g^*$ on W will be given by equation $(w \otimes g^*)y = g^*(y)w$ for every $y \in W$. Our readers are referred to Notation 2.2.10 for a recap. It should be noted that both $w \in W$ and $g^* \in W^*$ are non-zero in order for the operator to be necessarily a rank one.

3.4.2 Notation

Let W be a locally convex topological space. We use the symbol $L(\mathrm{W})$ to denote the algebra of all continuous linear operators on a locally convex topological vector space W.

3.4.3 Proposition ([18], Proposition 4.3)

Let $\mathrm{g} \in \mathrm{W}$, $\mathrm{A} \in \mathcal{B}(\mathrm{W})$, $\mathrm{g}^* \in \mathrm{W}^*$ and $\mathrm{g}^*\Big((\lambda - \mathrm{A})^{-1}\mathrm{g}\Big) = 1$. Then for $\lambda \notin \mathrm{sp}(\mathrm{A})$, we have that $\lambda \in \mathrm{sp}\Big(\mathrm{A} + (\mathrm{g} \otimes \mathrm{g}^*)\Big)$.

Proof

Assume that $\mathrm{g}^*\Big((\lambda - \mathrm{A})^{-1}\mathrm{g}\Big) = 1$. Then for $0 \neq \mathrm{g} \in \mathrm{W}$, we have that $\Big[\mathrm{g}^*\Big((\lambda - \mathrm{A})^{-1}\mathrm{g}\Big)\Big]\mathrm{g} = \mathrm{g}$ implying that $1 \in \mathrm{sp}\Big(\mathrm{g}^*((\lambda - \mathrm{A})^{-1}\mathrm{g})\Big)$ is an eigenvalue.

Now $\Big(\mathrm{A} + (\mathrm{g} \otimes \mathrm{g}^*)\Big)\Big((\lambda - \mathrm{A})^{-1}\mathrm{g}\Big) = \mathrm{A}\Big((\lambda - \mathrm{A})^{-1}\mathrm{g}\Big) + (\mathrm{g} \otimes \mathrm{g}^*)\Big((\lambda - \mathrm{A})^{-1}\mathrm{g}\Big) =$

$$\mathrm{A}\Big((\lambda - \mathrm{A})^{-1}\mathrm{g}\Big) + \mathrm{Ig} = \mathrm{A}\Big((\lambda - \mathrm{A})^{-1}\mathrm{g}\Big) + (\lambda - \mathrm{A})\Big((\lambda - \mathrm{A})^{-1}\mathrm{g}\Big) =$$

$$(\mathrm{A} + \lambda - \mathrm{A})\Big((\lambda - \mathrm{A})^{-1}\mathrm{g}\Big) = \lambda\Big((\lambda - \mathrm{A})^{-1}\mathrm{g}\Big) .$$

$$\text{Thus} \quad \Big(\mathrm{A} + (\mathrm{g} \otimes \mathrm{g}^*)\Big)\Big((\lambda - \mathrm{A})^{-1}\mathrm{g}\Big) = \lambda\Big((\lambda - \mathrm{A})^{-1}\mathrm{g}\Big)$$

implies that

$$\lambda \in \mathrm{sp}\Big(\mathrm{A} + (\mathrm{g} \otimes \mathrm{g}^*)\Big) .$$

3.4.4 Corollary

Let $\mathrm{g} \in \mathrm{W}$, $\mathrm{A} \in \mathcal{B}(\mathrm{W})$, $\mathrm{g}^* \in \mathrm{W}^*$ and $\lambda \notin \mathrm{sp}(\mathrm{A})$. Then $\lambda \in \mathrm{sp}\Big(\mathrm{A} + (\mathrm{g} \otimes \mathrm{g}^*)\Big)$ if and only if $\mathrm{g}^*\Big((\lambda - \mathrm{A})^{-1}\mathrm{g}\Big) = 1$.

Proof

Assume that $\mathrm{g}^*\Big((\lambda - \mathrm{A})^{-1}\mathrm{g}\Big) = 1$. If $\lambda \notin \mathrm{sp}(\mathrm{A})$, then $\lambda \in \mathrm{sp}\Big(\mathrm{A} + (\mathrm{g} \otimes \mathrm{g}^*)\Big)$ by Proposition 3.4.3 . Conversely, suppose that $\lambda \in \mathrm{sp}\Big(\mathrm{A} + (\mathrm{g} \otimes \mathrm{g}^*)\Big)$, then there exists a non-zero vector g such that $\Big(\mathrm{A} + (\mathrm{g} \otimes \mathrm{g}^*)\Big)\mathrm{g} = \lambda\mathrm{g}$. This implies that $\mathrm{Ag} + (\mathrm{g} \otimes \mathrm{g}^*)\mathrm{g} = \lambda\mathrm{g}$. This implies that $(\mathrm{g} \otimes \mathrm{g}^*)\mathrm{g} = (\lambda - \mathrm{A})\mathrm{g}$. This implies that $(\lambda - \mathrm{A})^{-1}(\mathrm{g} \otimes \mathrm{g}^*)\mathrm{g} = \mathrm{g}$. This implies that

$$(\lambda - \mathrm{A})^{-1}\mathrm{g} = \frac{\mathrm{g}}{\mathrm{g} \otimes \mathrm{g}^*} .$$

Hence $\mathrm{g}^*\Big((\lambda - \mathrm{A})^{-1}\mathrm{g}\Big) = 1$ as required.

3.4.5 Theorem ([18], Theorem 4.4)

Let W be a locally convex space and $A \in \mathcal{B}(W)$.

The following are equivalent:

(a) $\mathrm{Rank}(A) \leq 1$

(b) $sp(T+\alpha A) \bigcap sp(T+\beta A) \subseteq sp(T)$ for all $T \in \mathcal{B}(W)$ and $\alpha \neq \beta$ $\big(\alpha,\beta \in \mathbb{C}\big)$.

(c) $sp(T+\alpha A) \bigcap sp(T+\beta A) \subseteq sp(T)$ for all $T \in \mathcal{F}_2(W)$ and $\alpha \neq \beta$ $\big(\alpha,\beta \in \mathbb{C}\big)$.

Proof

If $A = 0$, then (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c)

We now prove the case when $A \neq 0$. Suppose $\mathrm{Rank}(A) = 1$. Then there exist $g \in W$ and $g^* \in W^*$ such that $A = g \otimes g^*$. Let $T \in \mathcal{B}(W)$ and $\lambda \notin sp(T)$.

Corollary 3.4.4 then implies that

$$\lambda \in sp\big(T + (g \otimes g^*)\big) \text{ if and only if } g^*\Big((\lambda - T)^{-1} g\Big) = 1 \text{ . Hence}$$

there does not exist more than one scalar $\alpha \in \mathbb{C}$ such that $\lambda \in sp(T+\alpha A)$. This amounts to proving (b). Since $\mathcal{B}(W) \supseteq \mathcal{F}_2(W)$, (b) $\Longrightarrow$ (c) is trivial. We prove that (c) $\Longrightarrow$ (a). We assume that (c) holds and assume that $\mathrm{Rank}(A) \geq 2$. Suppose $A = kI$ for some $k \neq 0$. Let

$$sp(T) \subseteq \{0, k\}$$

and $T \in \mathcal{F}_2(W)$, then

$$sp(T+A) \bigcap sp(T+2A) = \{2k\} \neq sp(T) \text{ for all } k \neq 0 \text{ .}$$

We obtain a contradiction in this setting. Therefore, every A such that $\mathrm{Rank}(A) \geq 2$ is not a multiple of the identity I.

We now consider two cases.

case 1: There is a vector $g \in W$ such that $\{ g, Ag, A^2g\}$ is linearly independent. Set $U = \mathrm{span}\{ g, Ag, A^2g\}$ and choose any (closed) complement V of U in W.

Define $N \in \mathcal{B}(W)$ by

$$N(g) = g - Ag \text{ ,}$$
$$N(Ag) = Ag - 2A^2g \text{ ,}$$
$$N(A^2g) = -\frac{1}{2}g + \frac{3}{2}Ag - 2A^2g$$

and $N(v) = 0$ for all $v \in V$. Clearly $N \in \mathcal{F}_2(W)$ and $N^3 = 0$. Condition $N(g) = g - Ag$ implies that $(N+A)g = g$. Condition $N(Ag) = Ag - 2A^2g$ implies that $(N+2A)Ag = Ag$. Therefore $1 \in sp(N+A) \bigcap sp(N+2A)$ but

$\mathrm{sp(N)} = \{0\}$. Hence

$$\mathrm{sp(N + A)} \bigcap \mathrm{sp(N + kA)} \neq \mathrm{sp(N)} \ .$$

We obtain a contradiction.

case 2: The set $\{ \mathrm{w}, \ \mathrm{Aw}, \ \mathrm{A^2w}\}$ is linearly dependent for every $\mathrm{w} \in \mathrm{W}$ such that A satisfies a quadratic polynomial equation $\mathrm{p(A)} = 0$. By considering four subcases according as $\mathrm{p(t)} = (\mathrm{t} - \alpha)(\mathrm{t} - \beta)$, $\mathrm{p(t)} = (\mathrm{t} - \alpha)^2$, $\mathrm{p(t)} = \mathrm{t}(\mathrm{t} - \alpha)$ and $\mathrm{p(t)} = \mathrm{t}^2 \quad \alpha \neq \beta \neq 0$, and by the standard decomposition of algebraic operators, we see that A has a finite dimensional invariant subspace $\mathcal{M}$ of W such that $\mathrm{A}|\mathcal{M}$ has one of the following four matrix representation for a suitable basis of $\mathrm{A}|\mathcal{M}$.

$$(\mathrm{a}) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

$$(\mathrm{b}) = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$$

$$(\mathrm{c}) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(\mathrm{d}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

By considering a complement $\mathcal{S}$ of $\mathcal{M}$ in W and an operator $\mathrm{N} \in F_2(\mathrm{W})$ such that $\mathrm{N}|\mathcal{S} = 0$ while $\mathrm{N}|\mathcal{M}$ has matrix representation

$$(\mathrm{a}') = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$(\mathrm{b}') = \begin{pmatrix} 0 & 0 \\ \alpha^2 & 0 \end{pmatrix}$$

$$(\mathrm{c}') = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2\alpha & 2\alpha \\ 0 & -2\alpha & -2\alpha \end{pmatrix} \quad \text{or}$$

$$(\mathrm{d}') = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

respectively. One observes that $N^2 = 0$ and $sp(N) = \{0\}$. However, by letting $k = \alpha\beta^{-1}$, 4, 2 or 2 respectively, we conclude that $sp(N + A) \bigcap sp(N + kA)$ contains a non-zero scalar, namely α, 2α, 2α, $\sqrt{2}$ respectively implying that

$$sp(N + A) \bigcap sp(N + kA) \neq sp(N) .$$

We obtain a contradiction.

3.4.6 Theorem ([18], Theorem 4.5)

Let W be a locally convex space and $A \in \mathcal{B}(W)$.

The following are equivalent:

(i) $\mathrm{Rank}(A) \leq 1$

(ii) For all $T \in F_2(W)$, there exist a compact subset $K_T \subset \mathbb{C}$ such that $sp(T + \alpha A) \bigcap sp(T + \beta A) \subseteq K_T$ for all $\alpha \neq \beta$ $\big(\alpha, \beta \in \mathbb{C}\big)$.

Proof

(i) $\Longrightarrow$ (ii). This follows from $\big[(a) \Longrightarrow (c)\big]$ of Theorem 3.4.5 and the fact that finite rank operators have finite spectrum. We assume (ii) and show that $\mathrm{Rank}(A) \leq 1$. We begin by showing that $sp(A)$ does not contain more than one non-zero complex number. Suppose to the contrary that $\alpha, \beta \in sp(A)$ with $\alpha \neq \beta$ and both are non-zero. Putting $T = 0$ in (ii), we get

$$sp(\alpha A) \bigcap sp(\beta A) \subseteq K_T$$

for all $\alpha \neq \beta$. But note that

$$1 \in sp(\alpha^{-1}A) \bigcap sp(\beta^{-1}A)$$

implies that

$$\lambda \in sp(\lambda\alpha^{-1}A) \bigcap sp(\lambda\beta^{-1}A) \subseteq K_T$$

for all $\lambda \in \mathbb{C}$. But then $\mathbb{C} \subset K_T$ is absurd. So we can fix a non-zero scalar, say k such that $sp(A) \subseteq \{0, k\}$. Take a non-zero $g \in W$ and $g^* \in W^*$. Set $T = g \otimes g^*$ and define the analytic function

$$\Phi : \big[\mathbb{C} - \{\frac{1}{k}\}\big] \to \mathbb{C}$$

by

$$\Phi(\lambda) = g^*\Big((I - \lambda A)^{-1}g\Big) .$$

Observe that the only singularities of our function are $\frac{1}{k}$ and ∞. We shall prove that they are at worst poles. First take any $M > 0$ such that $K_T \subseteq \overline{D}(0, M)$. We show that for any $|\mu| > M$, the equation $\Phi(\lambda) = \mu$ cannot have more than one solution. To verify this claim, we show that $\Phi(\lambda) = \mu$ if and only if $\mu \in \mathrm{sp}(T + \mu\lambda A)$. By Corollary 3.4.4, for $\mu = \Phi(\lambda)$, it follows that $\mu = g^*\Big((I - \lambda A)^{-1} g\Big)$ if and only if $\langle\, g^*\Big((I - \lambda A)^{-1}\mu^{-1} g\Big) = 1$ if and only if $\mu \in \mathrm{sp}(\mu\lambda A + T)$. As a consequence, if $\Phi(\lambda_1) = \Phi(\lambda_2) = \mu$ for some $\lambda_1 \neq \lambda_2$, then

$$\mu \in \mathrm{sp}(T + \mu\lambda_1 A) \cap \mathrm{sp}(T + \mu\lambda_2 A) \subseteq K_T,$$

contradicting the choice of μ. Let analytic function Φ have poles (or removable singularities) at $\frac{1}{k}$ and ∞. Picard's Big Theorem, ensures that each of these values force the function to take infinitely many solutions. We see that Φ is a rational function, that is

$$\Phi(\lambda) = \frac{P(\lambda)}{Q(\lambda)}$$

where P and Q are polynomials with $P(k^{-1}) \neq 0$ and

$$Q(\lambda) = (k\lambda - 1)^n$$

for some integer $n \geq 0$. We will show that $\deg P \leq 1$ and $\deg Q \leq 1$.

For all μ with $|\,\mu\,| > M$, we know that the equation

$$P(\lambda) - \mu Q(\lambda) = 0$$

has atmost one solution. If the degree of $P - \mu Q$ is greater than 1, then any λ satisfying $P(\lambda) - \mu Q(\lambda) = 0$, being a multiple root, must also satisfy equation

$$P'(\lambda) - \mu Q'(\lambda) = 0.$$

It follows that for any such μ, the corresponding solution λ satisfies the polynomial equation

$$Q'(\lambda)P(\lambda) - Q(\lambda)P'(\lambda) = 0.$$

Observe that if $Q'P - P'Q$ is identically zero, then the differential equation $\frac{P'}{P} = \frac{Q'}{Q}$ transforms to $\left(\frac{P}{Q}\right)' = \frac{P}{Q}$ so that $\Phi = \frac{P}{Q}$ is a constant and we are done. On the other hand, if $Q'P - PQ'$ is not identically zero, then the equation $P'(\lambda) - \mu Q'(\lambda) = 0$ above has only a finite number of solutions. Consequently, the degree of $P - \mu Q$ is more than 1 only finitely many μ for which $|\,\mu\,| > M$, a contradiction. Therefore it follows that $P - \mu Q$ has degree atmost 1 for all such μ. This implies that $\deg P \leq 1$ and $\deg Q \leq 1$ also. We conclude that $\Phi(\lambda) = \frac{a\lambda + b}{k\lambda - 1}$ or $\Phi(\lambda) = a\lambda + b$ for some constants $a, b \in \mathbb{C}$. The next step is to use this information to show that A satisfies either $A^2 = kA$ or $A^2 = 0$. Note that Φ satisfies either $\Phi''(\lambda) = \frac{2k}{1 - k\lambda}\Phi'(\lambda)$ or $\Phi''(\lambda) = 0$. It follows that

$$\Phi''(0) = 2k\,\Phi'(0)\,. \tag{3.25}$$

$$\Phi''(0) = 0\,. \tag{3.26}$$

By direct calculation from formula

$$\Phi(\lambda) \;=\; \langle\, (\mathrm{I} \;-\; \lambda \mathrm{A})^{-1}\mathrm{g}\;,\; \mathrm{g}^*\rangle,$$

we get

$$\Phi'(0) \;=\; \langle \mathrm{Ag}\;,\; \mathrm{g}^*\rangle\;. \tag{3.27}$$

and

$$\Phi''(0) \;=\; 2\langle \mathrm{A}^2\mathrm{g}\;,\; \mathrm{g}^*\rangle\;. \tag{3.28}$$

Substituting (3.27) into (3.25) , we obtain

$$\Phi''(0) \;=\; 2\mathrm{k}\;\langle \mathrm{Ag}\;,\; \mathrm{g}^*\rangle\;. \tag{3.29}$$

For all $\mathrm{g} \in \mathrm{W}$ and $\mathrm{g}^* \in \mathrm{W}^*$, (3.28) $-$ (3.29) implies $\langle(\mathrm{A}^2 - \mathrm{kA})\mathrm{g}\;,\;\mathrm{g}^*\rangle = 0$ and (3.28) $-$ (3.26) implies $\langle \mathrm{A}^2\mathrm{g}\;,\;\mathrm{g}^*\rangle = 0$. We show that either $\mathrm{A}^2 = \mathrm{kA}$ or $\mathrm{A}^2 = 0$. First for fixed $\mathrm{g} \in \mathrm{W}$, W^* is a union of two subspaces

$$\Big\{\mathrm{g}^* \in \mathrm{W}^* : \Big\langle(\mathrm{A}^2 \;-\; \mathrm{kA})\mathrm{g}\;,\; \mathrm{g}^*\Big\rangle \;=\; 0\;\Big\}$$

and

$$\Big\{\mathrm{g}^* \in \mathrm{W}^* : \Big\langle \mathrm{A}^2\mathrm{g}\;,\; \mathrm{g}^*\Big\rangle \;=\; 0\Big\}.$$

So one of the subspaces must be all of W^* . Then W in turn is a union of two subspaces $\mathrm{g} \in \mathrm{W}$ such that $\big(\mathrm{A}^2 - \mathrm{kA}\big)\mathrm{g} = 0$ or $\mathrm{A}^2\mathrm{g} = 0$. It follows that $\mathrm{A}^2 - \mathrm{kA} = 0$ or $\mathrm{A}^2 = 0$.

Finally we argue that the assumption $\operatorname{rank} \mathrm{A} > 1$ must lead to a contradiction.

There are two cases to consider.

(a) Assume that $\mathrm{A}^2 = \mathrm{kA}$. Let $\mathrm{x}, \mathrm{y} \in \operatorname{ran}(\mathrm{A})$ be linearly independent vectors. Then of course $\mathrm{Ax} = \mathrm{kx}$ and $\mathrm{Ay} = \mathrm{ky}$. Let $\mathrm{g}^* \in \mathrm{W}^*$ be such that $\mathrm{g}^*(\mathrm{x}) = 0$ and $\mathrm{g}^*(\mathrm{y}) = 1$ and set $\mathrm{T} = \mathrm{y} \otimes \mathrm{g}^*$. Take any $\lambda \in \mathbb{C}$ and observe that

$$\big(\mathrm{T} + \mathrm{k}^{-1}\lambda\mathrm{A}\big)\mathrm{x} \;=\; 0 + \lambda\mathrm{x} \;=\; \lambda\mathrm{x} \quad \text{and}$$

$$\Big(\mathrm{T} + \mathrm{k}^{-1}(\lambda \;-\; 1)\mathrm{A}\Big)\mathrm{y} \;=\; \mathrm{y} + (\lambda \;-\; 1)\mathrm{y} \;=\; \lambda\mathrm{y}.$$

This implies that

$$\lambda \;\in\; \operatorname{sp}\big(\mathrm{T} + \mathrm{k}^{-1}\lambda\mathrm{A}\big) \bigcap \operatorname{sp}\Big(\mathrm{T} + \mathrm{k}^{-1}(\lambda \;-\; 1)\mathrm{A}\Big) \;\subseteq\; \mathrm{K}_{\mathrm{T}} \quad \text{for all} \quad \lambda \;\in\; \mathbb{C}\;.$$

This situation violates the compactness of K_{T}.

(b) Assume that $A^2 = 0$. Let $x, y \in \operatorname{ran}(A)$ be linearly independent vectors. If $x = Au$ and $y = Av$ for some $u, v \in W$, then $\{x, y, u, v\}$ is a linearly independent set. Then set $\mathcal{M} = \operatorname{span}\{x, y, u, v\}$. Then $\mathcal{M}$ is a subspace of W which is clearly invariant for A. Furthermore, the matrix representation of map A restricted to subspace $\mathcal{M}$ is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

relative to the ordered basis $\{u, x, v, y\}$. Now choose $g^*, h^* \in W^*$ such that $g^*(x) = 1$, $g^*(u) = g^*(v) = g^*(y) = 0$ while $h^*(y) = 1$, $h^*(u) = h^*(v) = h^*(x) = 0$. Then set $T = (u \otimes g^*) + 4\,(v \otimes h^*)$ a rank-two operator for which $\mathcal{M}$ is also invariant. Note that T restricted to $\mathcal{M}$ has matrix representation

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

relative to the basis $\{u, x, v, y\}$. We see that every complex number λ implies that

$$\lambda \in \operatorname{sp}\big(T + \lambda^2 A\big) \bigcap \operatorname{sp}\big(T + \frac{\lambda^2}{4} A\big).$$

As in case (a), this is also absurd. Thus in both cases, we must have $\operatorname{rank}(A) \leq 1$.

3.4.7 Lemma ([18], Lemma 4.6)

Let X and Y be locally convex spaces. Let Ω be a unital subspace of $\mathcal{B}(X)$ containing all finite rank operators on X. If $\Phi : \Omega \to \mathcal{B}(Y)$ preserves invertibility and $\Phi(T) \in F_2(Y)$ for every $T \in \Omega$, then $\operatorname{rank}\big(\Phi(A)\big) \leq 1$ for every $A \in F_1(X)$.

Proof

Define $\Psi : \Omega \to \mathcal{B}(Y)$ by $\Psi(T) = \Phi(I)^{-1}\Phi(T)$. Then $\Psi(I) = I$. Moreover Ψ preserves invertibility. If $A \in F_1(X)$, then for every $T \in \Omega$ and $\alpha \neq \beta$ in $\mathbb{C}$, we have

$$\operatorname{sp}(T + \alpha A) \bigcap \operatorname{sp}(T + \beta A) \subseteq \operatorname{sp}(T)$$

by Theorem 3.4.5 . Since $\Psi(I) = I$ and Ψ preserves invertibility, then

$$\operatorname{sp}\big(\Psi(T)\big) \subseteq \operatorname{sp}(T)$$

for all $T \in \Omega$. Thus we get

$$\operatorname{sp}\Big(\Psi(T) + \alpha\Psi(A)\Big) \bigcap \operatorname{sp}\Big(\Psi(T) + \beta\Psi(A)\Big) \subseteq \operatorname{sp}\big(\Psi(T)\big) \subseteq \operatorname{sp}(T)$$

for all scalars $\alpha \neq \beta$. Since $\mathrm{sp(T)}$ is a compact subset of $\mathbb{C}$ and $\Psi(\mathrm{T}) \in F_2(\mathrm{Y})$, then Theorem 3.4.6, is satisfied. We conclude that $\mathrm{rank}\Big(\Psi(\mathrm{A})\Big) \leq 1$.

3.4.8 Lemma ([18], Lemma 4.7)

Let X be a locally convex space. For every $\mathrm{x} \in \mathrm{X}$ and $\mathrm{x}^* \in \mathrm{X}^*$, define the sets

$$\mathrm{L_x} = \big\{\mathrm{x} \otimes \mathrm{g}^* : \mathrm{g}^* \in \mathrm{X}^*\big\}$$

and

$$\mathrm{R_{x^*}} = \big\{\mathrm{g} \otimes \mathrm{x}^* : \mathrm{g} \in \mathrm{X}\big\}.$$

Let S be a non-trivial subspace of $\mathcal{B}(\mathrm{X})$ consisting of rank one operators and take $\mathrm{s} = \mathrm{x} \otimes \mathrm{x}^* \in \mathrm{S}$ be non-zero.

(i) If $\dim \mathrm{S} = 1$, then $\mathrm{S} = \mathrm{L_x} \bigcap \mathrm{R_{x^*}}$.

(ii) If $\dim \mathrm{S} > 1$, then either $\mathrm{S} \subseteq \mathrm{L_x}$ or $\mathrm{S} \subseteq \mathrm{R_{x^*}}$ but not both.

Proof

Both $\mathrm{L_x}$ and $\mathrm{R_{x^*}}$ are maximal subspaces of $\mathcal{B}(\mathrm{X})$ that are contained in $F_1(\mathrm{X})$. Since $\mathrm{S} \in F_1(\mathrm{X})$, then $\mathrm{S} \subset \mathrm{L_x}$ and $\mathrm{S} \subset \mathrm{R_{x^*}}$. It follows that

$$\mathrm{S} = \mathrm{L_x} \bigcap \mathrm{R_{x^*}}.$$

To prove (ii), take any $\mathrm{y} \otimes \mathrm{y}^* \in \mathrm{S}$. Since S is a subspace, then

$$\big(\mathrm{x} \otimes \mathrm{x}^*\big) + \big(\mathrm{y} \otimes \mathrm{y}^*\big) \in \mathrm{S}$$

is of rank one and either

$$\mathrm{y} \in \mathbb{C}\mathrm{x} \quad \text{or} \quad \mathrm{y}^* \in \mathbb{C}\mathrm{x}^* .$$

Accordingly, either $\mathrm{y} \otimes \mathrm{y}^* \in \mathrm{L_x}$ or $\mathrm{y} \otimes \mathrm{y}^* \in \mathrm{R_{x^*}}$. This means that

$$\mathrm{S} \subseteq \mathrm{L_x} \bigcup \mathrm{R_{x^*}} .$$

Since S , $\mathrm{L_x}$ and $\mathrm{R_{x^*}}$ are subspaces, it follows that

$$\mathrm{S} \subseteq \mathrm{L_x} \text{ or } \mathrm{S} \subseteq \mathrm{R_{x^*}} .$$

As $\dim\Big(\mathrm{L_x} \bigcap \mathrm{R_{x^*}}\Big) = 1$, we cannot have both $\mathrm{S} \subseteq \mathrm{L_x}$ and $\mathrm{S} \subseteq \mathrm{R_{x^*}}$.

3.4.9 Lemma ([18], Lemma 4.8)

Let X and Y be locally convex spaces. If $\Phi : \mathcal{F}(X) \to \mathcal{B}(Y)$ is a rank-reducing linear map, then at least one of the following is true:

(i) For every $x \in X$, $\Phi(L_x) \subseteq L_{y_x}$ for some $y_x \in Y$

(ii) For every $x \in X$, $\Phi(L_x) \subseteq R_{y_x^*}$ for some $y_x{}^* \in Y^*$.

Proof

For each $x \in X$, $\Phi(L_x)$ is a subspace of $\mathcal{B}(Y)$ consisting of rank 1 operators. Hence by Lemma 3.4.8, either $\Phi(L_x) \subseteq L_{y_x}$ for some $y_x \in Y$ or $\Phi(L_x) \subseteq R_{y_x^*}$ for some $y_x{}^* \in Y^*$. If $\dim \Phi(L_x) \leq 1$ for every $x \in X$ then both (i) and (ii) are true. So we assume that there exist an $x_0 \in X$ with the property $\dim \Phi(L_{x_0}) \geq 2$. Then either $\Phi(L_{x_0}) \subseteq L_{y_{x_0}}$ for some $y_{x_0} \in Y$ or $\Phi(L_{x_0}) \subseteq R_{y_{x_0}{}^*}$ for some $y_{x_0}{}^* \in Y^*$. We first suppose that $\Phi(L_{x_0}) \subseteq L_{y_{x_0}}$ for some $y_{x_0} \in Y$ and fix $x \in X$. We are done if $\Phi(L_x) \subseteq L_{y_x}$ for some $y_x \in Y$. Otherwise suppose that $\Phi(L_{x_0}) \subseteq R_{y_{x_0}{}^*}$ for some $y_{x_0}{}^* \in Y^*$. Then there exist linear transformations $A : X^* \to Y^*$ and $B : X^* \to Y$ such that

$$\Phi(x_0 \otimes g^*) = y_0 \otimes Ag^*$$

and

$$\Phi(x \otimes g^*) = Bg^* \otimes y_x{}^*$$

for all $g^* \in X^*$. Their sum

$$\Big(y_0 \otimes Ag^*\Big) + \Big(Bg^* \otimes y_x{}^*\Big) = \Phi\Big((x_0 + x) \otimes g^*\Big)$$

is a rank 1 operator. Hence for each $g^* \in X^*$,

$$\text{either} \quad \{Ag^*,\ y_x{}^*\} \quad \text{or} \quad \{y_0,\ Bg^*\}$$

is a linearly dependent set. In other words,

$$X^* = \{g^* : Ag^* \in \mathbb{C}y_x{}^*\} \bigcup \{g^* : Bg^* \in \mathbb{C}y_0\},$$

expressing X^* as a union of two subspaces. One of these subspaces must be all of X^*. But $\operatorname{rank}(A) = \dim \Phi(L_{x_1}) \geq 2$, and so the latter case prevails so that $\Phi(L_x) \subseteq \mathbb{C}(y_0 \otimes y_x{}^*) \subseteq L_{y_0}$. Summarising, if $\dim \Phi(L_{x_1}) \geq 2$ and $\Phi(L_{x_1}) \subseteq L_{y_{x_1}}$, then for each $x \in X$, $\Phi(L_x) \subseteq L_{y_x}$ for some $y_x \in Y$. Similarly, if $\dim \Phi(L_{x_1}) \geq 2$ and $\Phi(L_{x_1}) \subseteq R_{y_{x_1}{}^*}$, then for each $x \in X$, $\Phi(L_x) \subseteq R_{y_x^*}$ for some $y_x{}^* \in Y^*$.

3.4.10 Lemma ([18], Lemma 10)

Let X and Y be locally convex spaces and let $\Phi : \mathcal{F}(X) \to \mathcal{B}(Y)$ be a linear map.

(i) Suppose that for each $x \in X$, there exist a $y_x \in Y$ such that $\Phi(L_x) \subseteq L_{y_x}$; and there exist $x_1, x_2 \in X$, such that y_{x_1} and y_{x_2} are linearly independent.

Then there exist linear transformations $A : X \to Y$

and $B : X^* \to Y^*$, such that

$$\Phi(x \otimes x^*) = Ax \otimes Bx^*$$

for all $x \in X$ and $x^* \in X^*$.

(ii) Suppose that for each $x \in X$, there exist a $y_x{}^* \in Y^*$ such that $\Phi(L_x) \subseteq R_{y_x{}^*}$; and there exists $x_1, x_2 \in X$, such that $y_{x_1}{}^*$ and $y_{x_2}{}^*$ are linearly independent.

Then there exist linear transformations $C : X^* \to Y$ and

$D : X \to Y^*$ such that

$$\Phi(x \otimes x^*) = Cx^* \otimes Dx$$

for all $x \in X$ and $x^* \in X^*$.

Proof

We prove (i) only. The proof for (ii) is similar. Assume $\Phi(L_x) \subseteq L_{y_x}$ for all $x \in X$ and that there exist $x_1, x_2 \in X$ such that y_{x_1} and y_{x_2} are linearly independent. With condition $\Phi(L_x) \subseteq L_{y_x}$ for all $x \in X$, we insist that

$$y_x = 0 \quad \text{if and only if} \quad \Phi(L_x) = 0.$$

For each $x \in X$, by Lemma 3.4.9, there is a linear transformation

$$B_x : X^* \to Y^* \quad \text{such that} \quad \Phi(x \otimes x^*) = y_x \otimes B_x\, x^*$$

for all $x^* \in X^*$. When $y_x = 0$, we choose $B_x = 0$. On the other hand, if $y_x \neq 0$, then necessarily $B_x \neq 0$. Let us write $B = B_{(x_1+x_2)}$. Note that for all $x^* \in X^*$,

$$y_{(x_1 + x_2)} \otimes B\, x^* = \Phi\Big((x_1 + x_2) \otimes x^*\Big) = \big(y_{x_1} \otimes B_{x_1} x^*\big) + \big(y_{x_2} \otimes B_{x_2} x^*\big).$$

As y_{x_1} and y_{x_2} are linearly independent, by our assumption, it follows that the set $\{B_{x_1} x^*, B_{x_2} x^*\} \subseteq \mathbb{C}B\, x^*$ for all $x^* \in X^*$. So, by Lemma 3.4.9, there are nonzero scalars $\alpha, \beta \in \mathbb{C}$ such that

$$B_{x_1} = \alpha B \quad , \quad B_{x_2} = \beta B .$$

In particular, B_{x_1}, B_{x_2} are linearly dependent. Now let $x \in X$. If $y_x = 0$, then $B_x = 0B = 0$. If $y_x \neq 0$, then either $\{y_x , y_{x_1}\}$ or $\{y_x , y_{x_2}\}$ must be linearly independent. We conclude that for each $x \in X$, $B_x = \alpha_x B$ for some $\alpha_x \in \mathbb{C}$, then

$$\Phi(x \otimes x^*) = y_x \otimes B_x x^* = y_x \otimes \alpha_x B x^* = \alpha_x y_x \otimes Bx^*$$

for all $x \in X$ and for all $x^* \in X^*$. Define $A : X \to Y$ by $Ax = \alpha_x y_x$. Then

$$\Phi(x \otimes x^*) = Ax \otimes Bx^*$$

for all $x \in X$ and for all $x^* \in X^*$. Finally, as $B \neq 0$, it follows that A is linear.

3.4.11 Lemma ([18], Lemma 11)

Let X and Y be locally convex spaces. Let $\Phi : \mathcal{F}(X) \to \mathcal{B}(Y)$ be a rank reducing linear map. Then one of the following holds:

(i) There exist linear transformations $A : X \to Y$ and $B : X^* \to Y^*$ such that $\Phi(x \otimes x^*) = Ax \otimes Bx^*$ for all $x \in X$ and for all $x^* \in X^*$.

(ii) There exist linear transformations $C : X^* \to Y$ and $D : X \to Y^*$ such that $\Phi(x \otimes x^*) = Cx^* \otimes Dx$ for all $x \in X$ and for all $x^* \in X^*$.

(iii) There exist ${y_0}^* \in Y^*$ and a bilinear map $\Psi : X \times X^* \to Y$ such that

$\Phi(x \otimes x^*) = \Psi(x , x^*) \otimes {y_0}^*$ for all $x \in X$ and for all $x^* \in X^*$.

(iv) There exist $y_0 \in Y$ and a bilinear map $\pi : X \times X^* \to Y^*$ such that

$\Phi(x \otimes x^*) = y_0 \otimes \pi(x , x^*)$ for all $x \in X$ and for all $x^* \in X^*$.

Proof

We may assume that $\Phi \neq 0$. Then one of the following is true.

Case I : For every $x \in X$, there exist a $y_x \in Y$ such that $\Phi(L_x) \subseteq L_{y_x}$.

Case II : For every $x \in X$, there exist a ${y_x}^* \in Y^*$ such that $\Phi(L_x) \subseteq R_{{y_x}^*}$.

We treat *Case I* first. We insist that $y_x = 0$ if and only if $\Phi(L_x) = 0$.

Consider the following mutually exclusive subcases.

Case I (a) : there exist $x_1, x_2 \in X$ such that y_{x_1} and y_{x_2} are linearly independent.

Case I (b) : there exist $0 \neq y_0 \in Y$ such that $\Phi(L_x) \subseteq L_{y_0}$ for all $x \in X$.

In case of *Case I (a)* : Lemma 3.4.10 shows that Φ gives part (i) of lemma.

In case of *Case I (b)* : there is a function $\pi : X \times X^* \to Y^*$ such that

$\Phi(x \otimes x^*) = y_0 \otimes \pi(x , x^*)$ for all $x \in X$ and for all $x^* \in X^*$. Since $y_0 \neq 0$ and $(x , x^*) \to \Phi(x \otimes x^*)$ is bilinear, it follows that π is also bilinear. Hence this subcase leads to the form (iv).

It remains to consider *Case II* . Once again there are two subcases.

Case II (a) : there exist $x_1, x_2 \in X$ such that ${y_{x_1}}^*$ and ${y_{x_2}}^*$ are linearly independent.

Case II (b) : there exist $0 \neq {y_0}^* \in Y^*$ such that $\Phi(L_x) \subseteq R_{{y_0}^*}$ for all $x \in X$.

As above *Case II (a)* implies that Φ takes the form (ii) where as *Case II (b)* takes the form (iii).

3.4.12 Lemma ([18], Lemma 4.2)

Let V be a vector space and $v \in V$. Let g^* be a linear functional on V and $A \in L(V)$ be an invertible operator, where $L(V)$ is the algebra of all continuous linear operators on locally convex topological vector space V. Then the operator $A - (v \otimes g^*)$ is invertible if and only if $g^*(A^{-1}v) \neq 1$. If $g^*(A^{-1}v) \neq 1$, then $A - (v \otimes g^*)$ is invertible in every unital subalgebra of $L(V)$ that contains

$$\Big\{ A,\ A^{-1},\ A - (v \otimes g^*) \Big\}.$$

3.4.13 Theorem ([18], Theorem 4.13)

Let X be a Banach space and let Y be a locally convex space. Let Ω be a standard operator algebra on X and let $\Phi : \Omega \to \mathcal{B}(Y)$ be a unital, invertibility preserving map whose range contains $F(Y)$. Then either $\Phi / F(X) = 0$ or Φ is injective. If Φ is injective, then either

(i) there is a weak-weak continuous bijective linear transformation $P : Y \to X$ such that $\Phi(T) = P^{-1}TP$ for all $T \in \Omega$, or

(ii) there is a weak-weak* continuous bijective linear transformation $P : Y \to X^*$ such that $\Phi(T) = P^{-1}T^*P$ for all $T \in \Omega$.

Proof

Since $\Phi(I) = I$ in either (i) or (ii), we see that $\operatorname{rank} \Phi(I) \leq \operatorname{rank} I$. For any $x \in X$ and $x^* \in X^*$, we may write

$$\Phi(x \otimes x^*) = y_{x,x^*} \otimes {y_{x,x^*}}^*,$$

where $y_{x,x^*} \in Y$ and ${y_{x,x^*}}^* \in Y^*$. Fix an operator $0 \neq E \in \Omega$ and let $\lambda \in \mathbb{C}$. If $|\lambda| < \| E \|^{-1}$, then $I - \lambda E$ is invertible in Ω. Since Φ preserves invertibility, $I - \lambda\Phi(E)$ is invertible in $\mathcal{B}(Y)$. For any $x \in X$ and $x^* \in X^*$, consider the analytic functions

$$\Upsilon_{x,x^*} : D\Big(0,\ \| E \|^{-1}\Big) \to \mathbb{C} \quad \text{and} \quad \Gamma_{x,x^*} : D\Big(0,\ \| E \|^{-1}\Big) \to \mathbb{C}$$

defined by

$$\Upsilon_{x,x^*}(z) = \Big\langle (I - zE)^{-1}x\ ,\ x^* \Big\rangle = \operatorname{Tr}\Big[\Big(I - zE\Big)^{-1}(x \otimes x^*)\Big]$$

and

$$\Gamma_{x,x^*}(z) = \Big\langle \Big(I - z\,\Phi(E)\Big)^{-1} y_{x,x^*}\ ,\ {y_{x,x^*}}^* \Big\rangle = \operatorname{Tr}\Big[\Big(I - z\,\Phi(E)\Big)^{-1}\Phi(x \otimes x^*)\Big]$$

If $\Upsilon_{x,x^*}(\lambda) \neq 1$, then by Lemma 3.4.12, we see that

$$I - \lambda E - (x \otimes x^*)$$

is invertible in Ω. Since Φ preserves invertibility, it follows that

$$I - \lambda\,\Phi(E) - (y_{x,x^*} \otimes {y_{x,x^*}}^*)$$

is invertible in $\mathcal{B}(\mathrm{Y})$ implying that $\Gamma_{x,x^*}(\lambda) \neq 1$. Equivalently, if $\Gamma_{x,x^*}(\lambda) = 1$, we get $\Upsilon_{x,x^*}(\lambda) = 1$. Since both Υ_{x,x^*} and Γ_{x,x^*} are linear in x and x^*, we see that $\Gamma_{x,x^*}(\lambda) = \mu \neq 0$, then $\Upsilon_{x,x^*}(\lambda) = \mu$ also. Since both Υ_{x,x^*} and Γ_{x,x^*} are analytic, we see that if $\Gamma_{x,x^*}(\lambda) \neq 0$, then $\Upsilon_{x,x^*} \equiv \Gamma_{x,x^*}$. Indeed, if $\Gamma_{v,v^*} \not\equiv 0$ for some $v \in X$ and $v^* \in X^*$, we can write X^* as a union of two subspaces, $\mathcal{J}$ and $\mathcal{K}$, where

$$\mathcal{J} = \left\{x^* \in X^* : \Gamma_{v,x^*} \equiv 0\right\}$$

and

$$\mathcal{K} = \left\{x^* \in X^* : \Upsilon_{v,x^*} \equiv \Gamma_{v,x^*}\right\}.$$

So one of the two subspaces must be the whole of X^*. Since $\Gamma_{v,v^*} \not\equiv 0$ by our assumption, we see that $\mathcal{K} = X^*$. A similar argument applied to v^* shows that $\Upsilon_{x^*,v} \equiv \Gamma_{x^*,v}$ for all $x \in X$. Fix $w \in X$ and $w^* \in X^*$. If $\Gamma_{w,w^*} \not\equiv 0$, then from the above, we have $\Upsilon_{w,w^*} \equiv \Gamma_{w,w^*}$. Suppose that $\Gamma_{w,w^*} \equiv 0$. For any $k \in \mathbb{C}$, consider

$$\Gamma_{w+kv,w^*+kv^*} \equiv k\Big(\Gamma_{v,w^*} + \Gamma_{w,v^*}\Big) + k^2\,\Gamma_{v,v^*}.$$

Since $\Gamma_{v,v^*} \not\equiv 0$, we must have $\Gamma_{w+kv,w^*+kv^*} \not\equiv 0$ for some $k \neq 0$. Thus

$$\Upsilon_{w+kv,w^*+kv^*} \equiv \Gamma_{w+kv,w^*+kv^*}$$

implying that $\Upsilon_{w,w^*} \equiv \Gamma_{w,w^*}$. So we have shown that for all $0 \neq E \in \Omega$, either $\Upsilon_{x,x^*} \equiv \Gamma_{x,x^*}$ for all $x \in X$ and $x^* \in X^*$ or $\Gamma_{x,x^*} \equiv 0$ for all $x \in X$ and $x^* \in X^*$. Taking derivatives at $z = 0$, we get that for all $E \in \Omega$, either

$$\big\langle Ex\,,\ x^*\big\rangle = \mathrm{Tr}\Big[\ \Phi(E)\ \Phi(x \otimes x^*)\ \Big] \tag{3.30}$$

for all $x \in X$ and $x^* \in X^*$ or

$$\mathrm{Tr}\Big[\ \Phi(E)\ \Phi(x \otimes x^*)\ \Big] = 0 \tag{3.31}$$

for all $x \in X$ and $x^* \in X^*$. Thus Ω itself is a union of two subspaces corresponding to equations (3.30) and (3.31). Therefore, either (3.30) or (3.31) holds for every $E \in \Omega$. Suppose (3.31) holds, there exist $E_{u,u^*} \in \Omega$ such that $\Phi(E_{u,u^*}) = u \otimes u^*$. Then we find that

$$\big\langle \Phi\big(x \otimes x^*\big)u\,,\ u^*\big\rangle = \mathrm{Tr}\Big[\ \big(u \otimes u^*\big)\ \Phi\big(x \otimes x^*\big)\ \Big] = 0$$

for all $x \in X$, $x^* \in X^*$, $u \in X$ and $u^* \in X^*$. This implies that

$$\Phi\big(x \otimes x^*\big) = 0$$

for all $x \in X$ and $x^* \in X^*$. Therefore, $\Phi(E) = 0$ for all $\mathcal{F}(X)$. It is clear that if (3.30) holds, then Φ is injective. By Lemma 3.4.11, there are four possibilities for Φ.

(a) There exist linear transformations $A : X \to Y$ and $B : X^* \to Y^*$ such that $\Phi(x \otimes x^*) = Ax \otimes Bx^*$ for all $x \in X$ and for all $x^* \in X^*$.

(b) There exist linear transformations $C : X^* \to Y$ and $D : X \to Y^*$ such that $\Phi(x \otimes x^*) = Cx^* \otimes Dx$ for all $x \in X$ and for all $x^* \in X^*$.

(c) There exist $y_0{}^* \in Y^*$ and a bilinear map $\Psi : X \times X^* \to Y$ such that $\Phi(x \otimes x^*) = \Psi(x , x^*) \otimes y_0{}^*$ for all $x \in X$ and for all $x^* \in X^*$.

(d) There exist $y_0 \in Y$ and a bilinear map $\pi : X \times X^* \to Y^*$ such that $\Phi(x \otimes x^*) = y_0 \otimes \pi(x , x^*)$ for all $x \in X$ and for all $x^* \in X^*$.

If $\dim Y = 1$, then $\mathcal{B}(Y) = \mathbb{C}I$. Since $\Phi : \Omega \to \mathbb{C}I$ is unital and injective, it follows that $\dim \mathcal{F}(X) = \dim \Omega = 1$ forcing $\dim X = 1$. Thus $\Phi(T) = P^{-1}TP$ for all $T \in \Omega$, where $P : Y \to X$ is any isomorphism.

From now on, we assume that $\dim Y > 1$. We argue that cases (c) and (d) cannot occur. Then (3.30) becomes

$$\langle Ex , x^* \rangle = \langle \Phi(E)\, \Psi(x, x^*) , y_0{}^* \rangle .$$

Since $y_0{}^* \neq 0$ and $\dim Y \geq 2$, there exist $y_1 \neq y_2 \in Y$ such that

$$\langle y_1, y_0{}^* \rangle = \langle y_2, y_0{}^* \rangle = 1.$$

Take any $0 \neq y^* \in Y^*$. Choose standard operators $M_1, M_2 \in \Omega$ such that

$$\Phi(M_1) = y_1 \otimes y^*, \qquad \Phi(M_2) = y_2 \otimes y^* .$$

It follows that $M_1 \neq M_2$. However, for all $x \in X$, $x^* \in X^*$ and $i = 1, 2$, we have

$$\langle M_i x , x^* \rangle = \langle \Phi(M_i)\, \Psi(x, x^*) , y_0{}^* \rangle = \langle y_i , y_0{}^* \rangle \langle \Psi(x, x^*) , y^* \rangle = \langle \Psi(x, x^*) , y^* \rangle .$$

This implies that $M_1 = M_2$ which is absurd. A similar argument shows that (d) does not occur. Next, assume that (a) holds. Then (3.30) becomes

$$\langle Ex , x^* \rangle = \langle \Phi(E)Ax , Bx^* \rangle . \tag{3.32}$$

In particular, if $A \neq 0$, pick $x_0 \in X$ and $y_0{}^* \in Y^*$ such that $\langle Ax_0 , y_0{}^* \rangle = 1$. For each $y \in Y$, there is a unique $E_y \in \Omega$ such that $\Phi(E_y) = y \otimes y_0{}^*$. Now (3.32) implies that

$$\langle E_y x_0 , x^* \rangle = \langle \Phi(E_y)Ax_0 , Bx^* \rangle = \langle y , Bx^* \rangle .$$

Define $P : Y \to X$ by $Py = E_y x_0$. Clearly P is linear, and

$$\langle Py , x^* \rangle = \langle y , Bx^* \rangle . \tag{3.33}$$

(3.32) = (3.33) implies that for all $x \in X$ and $x^* \in X^*$, we have that

$$\langle Ex , x^* \rangle = \langle P\Phi(E)Ax , x^* \rangle \tag{3.34}$$

or equivalently, $E = P\Phi(E)A$. Putting $E = I$ gives $I = PA$, so that P is surjective. To show that P is injective, take any $0 \neq y \in Y$. Fix $0 \neq y^* \in Y^*$, and take $E \in \Omega$ such that $\Phi(E) = y \otimes y^*$. Then (3.34) gives

$0 \neq E = P(y \otimes y^*)A = (Py \otimes y^*)A$, proving that $Py \neq 0$. Now that P is bijective, we also have $A = P^{-1}$. Then by (3.34), it follows that

$$\Phi(E) = P^{-1}EP.$$

Finally, suppose that case (*b*) holds, then (3.30) becomes

$$\langle Ex\ ,\ x^* \rangle = \langle \Phi(E)Cx^*\ ,\ Dx \rangle\ . \tag{3.35}$$

In particular, $C \neq 0$. Pick $x_1{}^* \in X^*$ and for all $y_1{}^* \in Y^*$ such that $\langle Cx_1{}^*\ ,\ y_1{}^* \rangle \neq 0$. Given $y \in Y$, there is a unique $E_y \in \Omega$ such that $\Phi(E_y) = y \otimes y_1{}^*$ and

$$\langle x\ ,\ E^*{}_y x_1{}^* \rangle = \langle E_y x\ ,\ x_1{}^* \rangle = \langle \Phi(E_y)Cx_1{}^*\ ,\ Dx \rangle = \langle y,\ Dx \rangle\ .$$

Define $P : Y \to X^*$ by $Py = E^*{}_y x_1{}^*$ which is linear. Then

$$\langle x\ ,\ Py \rangle = \langle y\ ,\ Dx \rangle\ . \tag{3.36}$$

It follows that (3.35) is equivalent to statement

$$\langle Ex\ ,\ x^* \rangle = \langle x\ ,\ P\Phi(E)Cx^* \rangle$$

or $E^* = P\Phi(E)C$. We can now proceed as in case (a) to prove that P is bijective, $C = P^{-1}$ and $\Phi(E) = P^{-1}E^*P$.

The next result, the main result of this section, answers the so-called Kaplansky's problem in affirmative by way of equivalence of the four parts of this theorem (Theorem 3.4.13).

3.4.14 Theorem ([18], Theorem 4.13)

Let X and Y be Banach spaces and suppose that $\Phi : \mathcal{B}(X) \to \mathcal{B}(Y)$ is a unital bijective linear map. The following are equivalent:

(a) Φ preserves invertibility

(b) Φ is a Jordan isomorphism

(c) Φ is either an isomorphism or an anti-isomorphism

(d) either Y is isomorphic to X and $\Phi(T) = P^{-1}TP$ for all $T \in \mathcal{B}(X)$ with $P \in \mathcal{B}(Y, X)$ an isomorphism or Y is isomorphic to X^* and

$\Phi(T) = P^{-1}T^*P$ for all $T \in \mathcal{B}(X)$ with $P \in \mathcal{B}(Y, X^*)$ an isomorphism.

Proof

That (b) implies (a) is exactly Proposition 3.1.3. That (a) implies (d) is exactly Theorem 3.4.13. That (d) implies (c) followed by (c) implies (b) is obvious.

Chapter 4

Approximately Gleason-Kahane-Żelazko's Theorem

The aim of this chapter is to present an approximate version of the classical Gleason-Kahane-Żelazko's Theorem for commutative Banach algebras arising from the ε-pseudospectrum discussed in section 2.5 of Chapter 2. This approximate version is the main result needed for the material in Chapter 5.

The classical Gleason-Kahane-Żelazko's Theorem gave rise to the so-called Kaplansky's problem (problem 3.1.1). This theorem states that a linear functional ϕ on a complex Banach algebra A is multiplicative (and non-zero) if and only if $\phi(\mathrm{a}) \in \mathrm{sp}(\mathrm{a}) \quad (\mathrm{a} \in \mathrm{A})$.

In the case where A is commutative, our concern is the natural task of replacing the spectra condition $\phi(\mathrm{a}) \in \mathrm{sp}(\mathrm{a}) \quad (\mathrm{a} \in \mathrm{A})$, in the classical Gleason-Kahane-Żelazko's Theorem, with ε-pseudospectrum condition $\phi(\mathrm{a}) \in \mathrm{sp}_\varepsilon(\mathrm{a}) \quad (\mathrm{a} \in \mathrm{A})$ where $\parallel \mathrm{a} \parallel = 1$ for some $\varepsilon > 0$.

4.1 ε-condition spectrum's approximate classical Gleason-Kahane-Żelazko's Theorem

The theorem below is an approximate version of the classical Gleason-Kahane-Żelazko's Theorem for commutative Banach algebras arising by way of the ε-condition spectrum. This theorem is the main result of [13].

4.1.1 Theorem ([13], Theorem 3)

Let A be a commutative Banach algebra with unit 1, $0 < \varepsilon < \frac{1}{3}$ and $\phi : \mathrm{A} \to \mathbb{C}$ be a linear functional. If $\phi(\mathrm{a}) \in \sigma_\varepsilon(\mathrm{a})$ for every a in A, then ϕ is δ-almost multiplicative, where

$$\delta = \tfrac{4}{\ln(1/\varepsilon)}(1 + \tfrac{2}{(\ln(2)/3)^2}).$$

Proof

We mention here that the ε-condition spectrum is a special case of the Ransford spectrum. The most important property in the Ransford spectrum which is applied in the ε-condition spectrum is property $\sigma_\varepsilon(1) = \{1\}$. We do not give a reference here but interested readers on the Ransford spectrum are referred to [13] for further referencing.

Since $\sigma_\varepsilon(1) = \{1\}$, we have $\phi(1) = 1$. Next, we replace spectra value λ with spectral value $\phi(a)$ in Lemma 2.6.6 . It follows that $\| \phi \| \leq \frac{1+\varepsilon}{1-\varepsilon}$. Hence norm of ϕ is bounded and therefore continuous. Next, let $a \in A$ with $\| a \| = 1$. Define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = \phi(\exp(za)), \text{ for all } z \in \mathbb{C}. \tag{4.1}$$

Then f is an entire function. Also, since for all $z \in \mathbb{C}$,

$$| f(z) | \leq \| \phi \| \| \exp(za) \| \leq (\tfrac{1+\varepsilon}{1-\varepsilon}) \exp(| z |) ,$$

the function f is of exponential type of order less than or equal to one.
From equation (4.1), by linearity and continuity of ϕ, f also has the form

$$f(z) = \sum_{n=0}^{\infty} \frac{\phi(a^n)z^n}{n!} \tag{4.2}$$

Let α_j, $j = 1, 2, ...$, denote the zeros of f arranged in such a way that
$| \alpha_1 | \leq | \alpha_2 | \leq .....$

CLAIM.

$$\phi(a^2) - \left(\phi(a)\right)^2 = -\sum_{j=1} \frac{1}{(\alpha_j)^2} . \tag{4.3}$$

The right hand side of the above becomes a finite sum if the number of zeros of f is finite, and reduces to 0 if f has no zero.

CASE 1 : Genus of $f(z)$ is 1. By the Hadamard's Factorization Theorem (Theorem 2.4.8), there exist a polynomial g of degree less than or equal to 1 such that

$$f(z) = \phi(\exp(za)) = \exp(g(z))\prod(1 - \frac{z}{\alpha_j})\exp(\frac{z}{\alpha_j}), \tag{4.4}$$

for all $z \in \mathbb{C}$. Since $f(0) = \phi(1) = 1$, we have $g(0) = 0$. Hence we may assume $g(z) = \beta z$ for some $\beta \in \mathbb{C}$. Next, each term in the product can be written as

$$(1 - \frac{z}{\alpha_j})\exp(\frac{z}{\alpha_j}) = (1 - \frac{z}{\alpha_j})(1 + \frac{z}{\alpha_j} + \frac{1}{2}\frac{z^2}{(\alpha_j)^2} + ...) = 1 - \frac{1}{2}\frac{z^2}{(\alpha_j)^2} + ...$$

Thus,

$$f(z) = (1 + \beta z + \beta^2\frac{z^2}{2} + ...)\prod(1 - \frac{1}{2}\frac{z^2}{(\alpha_j)^2} + ...) . \tag{4.5}$$

Comparing the coefficients of z and z^2 in the two expressions (4.2) and (4.5) of $f(z)$, we get

$$\phi(\mathrm{a}) = \beta\ ,\ \tfrac{1}{2}\phi(\mathrm{a}^2) = \tfrac{1}{2}\beta^2 - \tfrac{1}{2}\textstyle\sum_{\mathrm{j}=1} \frac{1}{(\alpha_{\mathrm{j}})^2}\ .$$

Thus

$$\phi(\mathrm{a}^2) - \big(\phi(\mathrm{a})\big)^2 = -\Big(\sum_{\mathrm{j}=1} \frac{1}{\alpha^2{}_{\mathrm{j}}}\Big)\ . \tag{4.6}$$

CASE 2 : Genus of f(z) is 0. By the Hadamard's Factorization Theorem (Theorem 2.4.8), there exists a polynomial g of degree zero such that

$$\mathrm{f(z)} = \exp(\mathrm{g(z)})\prod(1 - \frac{\mathrm{z}}{\alpha_{\mathrm{j}}}) \tag{4.7}$$

for all $\mathrm{z} \in \mathbb{C}$. Since $\mathrm{f}(0) = \phi(1) = 1$, we have $\mathrm{g} = 0$. It follows that

$$\mathrm{f(z)} = \prod(1 - \frac{\mathrm{z}}{\alpha_{\mathrm{j}}}) = (1 - \frac{\mathrm{z}}{\alpha_1})(1 - \frac{\mathrm{z}}{\alpha_2}).........$$

Using $\mathrm{f(z)} = \phi(\exp(\mathrm{za}))$ in (4.2) again, we have that
$\mathrm{f(z)} = \phi(\exp(\mathrm{za})) = 1 + \mathrm{z}\phi(\mathrm{a}) + \frac{\mathrm{z}^2}{2}\phi(\mathrm{a}^2) + ...$
These two situations of f(z) give us $\phi(\mathrm{a}) = -\sum_{\mathrm{j}=1} \frac{1}{\alpha_{\mathrm{j}}}$
and $\frac{1}{2}\phi(\mathrm{a}^2) = \sum_{\mathrm{i}<\mathrm{j}} \frac{1}{\alpha_{\mathrm{i}}\alpha_{\mathrm{j}}}$. It follows that
$(\phi(\mathrm{a}))^2 = (\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + ...)(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + ...) = \phi(\mathrm{a}^2) + \sum_{\mathrm{j}=1} \frac{1}{(\alpha_{\mathrm{j}})^2}$.
Thus we end up with the same expression as (4.6) and that proves the claim.
To get the bound for the right hand side of (4.3), we estimate $\mid \alpha_{\mathrm{j}} \mid$ in two ways. First, since $0 = \mathrm{f}(\alpha_{\mathrm{j}}) = \phi(\exp(\alpha_{\mathrm{j}}\mathrm{a}))$, we have $0 \in \sigma_\varepsilon(\exp(\alpha_{\mathrm{j}}\mathrm{a}))$.
Hence

$$\tfrac{1}{\varepsilon} \leq \ \parallel \exp(\alpha_{\mathrm{j}}\mathrm{a}) \parallel\parallel \exp(-\alpha_{\mathrm{j}}\mathrm{a}) \parallel \ \leq \exp(2 \mid \alpha_{\mathrm{j}} \mid\parallel \mathrm{a} \parallel)\ .$$

Since $\parallel \mathrm{a} \parallel = 1$, we obtain $\mid \alpha_{\mathrm{j}} \mid \ \geq \frac{1}{2}\ln(\frac{1}{\varepsilon})$.
The second estimate is obtained from Jensen's formula (Theorem 2.4.9) . For this, let $\mathrm{r} > 0$, n(r) denote the number of zeros of f in the closed disc with center at the origin and radius r, and

$$\mathrm{M(r)} := \sup\{\mid \mathrm{f(r}\exp(\mathrm{i}\theta)) \mid :\ 0 \leq \theta \leq 2\pi\} \leq \tfrac{1+\varepsilon}{1-\varepsilon}\exp(\mathrm{r})\ .$$

Then by the Jensen's formula (Theorem 2.4.9),

$$\mathrm{n(r)}\ln(2) \leq \ln(\mathrm{M(2r)}) \leq \ln(\tfrac{1+\varepsilon}{1-\varepsilon}) + 2\mathrm{r}\ .$$

Putting $\mathrm{r} = \mid \alpha_{\mathrm{j}} \mid$ in the above inequality, we get

$$\mathrm{j}\ \ln(2) \leq \ln(\tfrac{1+\varepsilon}{1-\varepsilon}) + 2 \mid \alpha_{\mathrm{j}} \mid\ .$$

Suppose ε is such that $\ln(\frac{1+\varepsilon}{1-\varepsilon}) \leq \frac{1}{2}\ln(\frac{1}{\varepsilon})$. (This is satisfied if $0 < \varepsilon < \frac{1}{3}$).
It follows that $\mid \alpha_j \mid \geq \frac{1}{2}\ln(\frac{1}{\varepsilon}) \geq \ln(\frac{1+\varepsilon}{1-\varepsilon})$, so that $j \ln(2) \leq 3 \mid \alpha_j \mid$. Hence, $\mid \alpha_j \mid \geq (\ln(2)/3)j$. Let $\gamma := \frac{1}{2}\ln(\frac{1}{\varepsilon})$ and $\eta := (\ln(2)/3)$. Then $\mid \alpha_j \mid \geq \gamma$ as well as $\mid \alpha_j \mid \geq \eta j$ for all j. Consider $k = [\gamma]$, the integral part of γ. Then $k \leq \gamma \leq k+1$. Recall $\phi(a^2) - \big(\phi(a)\big)^2 = -\sum_{j=1} \frac{1}{(\alpha_j)^2}$.

To estimate $\mid \phi(a^2) - \big(\phi(a)\big)^2 \mid$, we split the right hand side sum into two parts and use the first inequality for $1 \leq j \leq k$ and the second inequality for $j > k$. Hence

$$\mid \phi(a^2) - \big(\phi(a)\big)^2 \mid \leq$$

$$\sum_{j=1}^{k} \mid \frac{1}{(\alpha_j)^2} \mid +$$

$$\sum_{j=k+1}^{\infty} \mid \frac{1}{(\alpha_j)^2} \mid$$

$$\leq \frac{k}{\gamma^2} + \frac{1}{\eta^2}\sum_{j=k+1}^{\infty}(\frac{1}{j^2})$$

$$\leq \frac{1}{\gamma} + \frac{1}{\eta^2}\Big(\frac{1}{(k+1)^2} + \int_{j=k+1}^{\infty} (\frac{1}{x^2})dx\Big)$$

$$\leq \frac{1}{\gamma} + \frac{1}{\eta^2}\Big(\frac{1}{(k+1)^2} + \frac{1}{(k+1)}\Big)$$

$$\leq \frac{1}{\gamma} + \frac{1}{\eta^2}(\frac{1}{\gamma^2} + \frac{1}{\gamma}) \leq \frac{1}{\gamma}(1 + \frac{2}{\eta^2}).$$

We have proved that $\mid \phi(a^2) - \big(\phi(a)\big)^2 \mid \leq \delta_1$ for all $a \in A$ with $\parallel a \parallel = 1$, where

$$\delta_1 := \frac{1}{\gamma}(1 + \frac{2}{\eta^2}).$$

Thus the conclusion follows from Lemma 2.6.8 with

$$\delta := 2\delta_1 = (\tfrac{2}{\gamma})(1 + \tfrac{1}{\eta^2}) = \tfrac{4}{\ln(1/\varepsilon)}\Big(1 + \frac{2}{\big(\ln(2)/3\big)^2}\Big).$$

Using this theorem we have just proved, we can deduce the classical Gleason-Kahane-Żelazko's Theorem for commutative Banach algebras.

4.1.2 Corollary ([13], Corollary 6)

Let A be a complex commutative unital Banach algebra and $\phi : \mathrm{A} \to \mathbb{C}$ be a linear functional. If $\phi(\mathrm{a}) \in \mathrm{sp}(\mathrm{a})$ for every a in A, then ϕ is multiplicative.

Proof

Since $\mathrm{sp}(\mathrm{a}) \subseteq \sigma_\varepsilon(\mathrm{a})$ for every $0 < \varepsilon < 1$ by Lemma 2.6.5, then $\phi(\mathrm{a}) \in \sigma_\varepsilon(\mathrm{a})$, for all $\mathrm{a} \in \mathrm{A}$, $0 < \varepsilon < 1$. It follows that Theorem 4.1.1 is satisfied. Therefore, ϕ is δ-almost multiplicative, where

$$\delta = \frac{4}{\ln(1/\varepsilon)}\Big(1 + \frac{2}{\big(\ln(2)/3\big)^2}\Big)$$

by Theorem 4.1.1 . It follows that $\delta \to 0^+$ as $\varepsilon \to 0^+$. Hence ϕ is multiplicative.

4.2 ε-pseudospectrum's approximate classical Gleason-Kahane-Żelazko's Theorem

Theorem 4.2.2, in this section, is the approximate version of the classical Gleason-Kahane-Żelazko's Theorem, for commutative Banach algebras, this dissertation has used for the approximate spectra of Chapter 5. This approximate version of the classical Gleason-Kahane-Żelazko's Theorem, for commutative Banach algebras, is developed via the ε-pseudospectrum (Definition 2.5.3) .

The tools that we have used in the development of Theorem 4.1.1, the main result of [13], in the last section, are the same tools we are using in the development of Theorem 4.2.2 in this section. The main difference lies in the use of Definition 2.5.3 (ε-pseudospectrum) instead of Definition 2.6.3 (ε-condition spectrum).

4.2.1 Lemma ([4], Lemma 4.1)

Let A be a Banach algebra and let ϕ be a linear functional on A with the property that $\phi(\mathrm{a}) \in \mathrm{sp}_\varepsilon(\mathrm{a})$ for each $\mathrm{a} \in \mathrm{A}$ with $\| \mathrm{a} \| = 1$ for some $\varepsilon > 0$. Then $\| \phi \| \leq 1 + \varepsilon$ and ϕ is continuous.

Proof

Let $\varepsilon > 0$ be given. Then for each $\mathrm{a} \in \mathrm{A}$ with $\| \mathrm{a} \| = 1$, we have $| \phi(\mathrm{a}) | < 1 + \varepsilon$ since $\phi(\mathrm{a}) \in \mathrm{sp}_\varepsilon(\mathrm{a})$. It follows that

$$\| \phi \| = \sup \left\{ \frac{|\phi(\mathrm{a})|}{\| \mathrm{a} \|} : \| \mathrm{a} \| = 1 \right\} \leq \left[\frac{1+\varepsilon}{\| \mathrm{a} \|}\right] = 1 + \varepsilon$$

implying that $\| \phi \| \leq 1 + \varepsilon$. We conclude that ϕ is continuous.

4.2.2 Theorem ([4], Theorem 4.2)

Let A be a Banach algebra. Then the following assertions hold.

(i) For each $\varepsilon > 0$ and $0 < \mathrm{v} < 1$ there is $\delta > 0$ such that if ϕ is a linear functional on A with $\phi(\mathrm{a}) \in \mathrm{sp}_\delta(\mathrm{a})$ $(\mathrm{a} \in \mathrm{A}, \| \mathrm{a} \| = 1)$, then $\mathrm{mult}(\phi) < \varepsilon$ and $\| \phi \| > \mathrm{v}$.

(ii) For each $\varepsilon > 0$ and $0 < \mathrm{v} < 1$, there is $\delta > 0$ such that if ϕ is a linear functional on A with $\mathrm{mult}(\phi) < \delta$ and $\| \phi \| > \mathrm{v}$, then $\phi(\mathrm{a}) \in \mathrm{sp}_\varepsilon(\mathrm{a})$ $(\mathrm{a} \in \mathrm{A}, \| \mathrm{a} \| = 1)$.

Proof

In order to prove the first assertion in the theorem, let us define $\Gamma : (0,1) \to (0,\infty)$ by

$$\Gamma(\mathrm{t}) = \frac{4(1+\mathrm{t})}{\log(1/\mathrm{t})}\left(1 + \frac{18}{(\log\ 2)^2}\right) + \frac{\mathrm{t}(1+\mathrm{t})^2}{1-\mathrm{t}}\ \big(\mathrm{t} \in (0,1)\big).$$

We claim that $\mathrm{jmult}(\phi) \leq \Gamma(\delta)$ whenever ϕ is a linear functional on A such that $\phi(\mathrm{a}) \in \mathrm{sp}_\delta(\mathrm{a})$ $(\mathrm{a} \in \mathrm{A}, \| \mathrm{a} \| = 1)$ for some $0 < \delta < \frac{1}{3}$.

Let $0 < \delta < 1$ and assume that ϕ is a linear functional on A such that $\phi(\mathrm{a}) \in \mathrm{sp}_\delta(\mathrm{a})$ for each $\mathrm{a} \in \mathrm{A}$, with $\| \mathrm{a} \| = 1$. Since $\phi(1) \in \mathrm{sp}_\delta(1)$, it follows that $| 1 - \phi(1) | < \delta$. Moreover by Lemma 4.2.1, ϕ is continuous with $\| \phi \| \leq 1 + \delta$. We now pick $\mathrm{a} \in \mathrm{A}$ with $\| \mathrm{a} \| = 1$ and define an entire function $\mathrm{f} : \mathbb{C} \to \mathbb{C}$ by

$$\mathrm{f}(\mathrm{z}) = \phi(1)^{-1}\ \big(\exp(\mathrm{za})\big)\ (\mathrm{z} \in \mathbb{C}).$$

Since $0 < \delta < 1$ and $| 1 - \phi(1) | < \delta$, then $\phi(1) \neq 0$. It follows that

$$| \mathrm{f}(\mathrm{z}) | \leq | \phi(1)^{-1} | \, \| \exp(\mathrm{za}) \| \leq \frac{1+\delta}{1-\delta} \mathrm{e}^{|\mathrm{z}|}.$$

We now estimate

$$\frac{\phi(\mathrm{a}^2)}{\phi(1)} - \left(\frac{\phi(\mathrm{a})}{\phi(1)}\right)^2 = -\sum_{\mathrm{n}=1}^{\infty} \frac{1}{(\alpha_\mathrm{n})^2}$$

where α_n are the zeros of f arranged in such a way that $| \alpha_1 | \leq | \alpha_2 | \leq ....$

It follows that $\phi\big(\exp(\alpha_\mathrm{n}\ \mathrm{a})\big) = \phi(1)\ \mathrm{f}(\alpha_\mathrm{n}) = 0$, so that $0 \in \mathrm{sp}_{\delta \|\exp(\alpha_\mathrm{n}\mathrm{a})\|}\ \big(\exp(\alpha_\mathrm{n}\mathrm{a})\big)$. Hence $\frac{1}{\varepsilon} < \| \exp(\alpha_\mathrm{n}\mathrm{a}) \| \, \| \big(\exp(-\alpha_\mathrm{n}\mathrm{a})\big) \| \leq \mathrm{e}^{2|\alpha_\mathrm{n}|}$ and therefore $\frac{1}{2}\log(\frac{1}{\delta}) \leq | \alpha_\mathrm{n} |$. This gives the first estimate for $| \alpha_\mathrm{n} |$ just as in main theorem of [13], identified as Theorem 4.1.1 in this dissertation. Then the second estimate $\frac{\log(2)}{3}\mathrm{n} \leq | \alpha_\mathrm{n} |$ obtained from Jensen's formular (Theorem 2.4.9) also works with $\delta < \frac{1}{3}$. It follows that

$$\left| \sum_{\mathrm{n}=1}^{\infty} \frac{1}{(\alpha_\mathrm{n})^2} \right| \leq \frac{4}{\log(1/\delta)}\left(1 + \frac{18}{(\log\ 2)^2}\right)$$

is in line with the estimate in [13], which is just 4.1.1 in our dissertation. Equivalently,

$$\left| \sum_{\mathrm{n}=1}^{\infty} \frac{1}{(\alpha_\mathrm{n})^2} \right| = | \phi(\mathrm{a}^2) - \phi(1)^{-1}\ \phi(\mathrm{a})^2 | \leq$$

$$\frac{4|\,\phi(1)\,|}{\log(1/\delta)}\Big(1+\frac{18}{(\log\,2)^2}\Big)\;\leq\;\frac{4(1+\delta)}{\log(1/\delta)}\Big(1+\frac{18}{(\log\,2)^2}\Big)$$

and therefore

$$\begin{aligned}|\,\phi(a^2)\;-\;\phi(a)^2\,|\;&\leq\;|\,\phi(a^2)\;-\;\phi(1)^{-1}\,\phi(a)^2\,|\;+\;|\,\phi(a)^2\,|\,|\,\phi(1)^{-1}\;-1\,|\\ &\leq\;\frac{4(1+\delta)}{\log(1/\delta)}\Big(1+\frac{18}{(\log\,2)^2}\Big)\;+\;\frac{1}{|\,\phi(1)\,|}\,|\,\phi(a)^2\,|\,|\,1\;-\;\phi(1)\;\,|\\ &\leq\;\frac{4(1+\delta)}{\log(1/\delta)}\Big(1+\frac{18}{(\log\,2)^2}\Big)\;+\;\frac{1}{|\,\phi(1)\,|}\delta(1+\delta)^2\\ &\leq\;\frac{4(1+\delta)}{\log(1/\delta)}\Big(1+\frac{18}{(\log\,2)^2}\Big)\;+\;\frac{\delta(1+\delta)^2}{1-\delta}\;=\;\Gamma(\delta).\end{aligned}$$

Finally, since $\lim_{t\to 0}\Gamma(t)\;=\;0$ and $1-\delta\;<\;|\,\phi(1)\,|\;\leq\;\|\,\phi\,\|$, the first assertion in our theorem holds. The task is now to prove the second assertion in the theorem. We first point out that the functionals involved in that assertion are automatically continuous. We now assume towards a contradiction that the second assertion in the theorem fails. Then there are $\tau, v > 0$, a sequence (ϕ_n) of linear functionals on A, and a sequence (b_n) in A such that $\mathrm{mult}(\phi_n)\;<\;\frac{1}{n}$, $\;\|\,\phi_n\,\|\;>\;v$ and $\phi_n(b_n)\;\notin \mathrm{sp}_\tau(b_n)$ for each $n\in\mathbb{N}$. (ϕ_n) is continuous for each $n\;\in\mathbb{N}$ and the sequence (ϕ_n) is bounded. We then consider the continuous linear functional $\mathbf{\Phi}$ on the Banach algebra $A^{\mathcal{U}}$ given by $\langle\mathbf{\Phi},\mathbf{a}\rangle\;=\;\lim_{\mathcal{U}}\,\langle\phi_n,a_n\rangle\;\big(\mathbf{a}=(a_n)\in A^{\mathcal{U}}\big)$. It follows that $\mathrm{mult}(\mathbf{\Phi})=\lim_{\mathcal{U}}\mathrm{mult}(\phi_n)\;=\;0$ and therefore $\mathbf{\Phi}$ is multiplicative. Since $\|\,\phi_n\,\|\;>\;v$, it follows that $\|\,\mathbf{\Phi}\,\|\;=\;\lim_{\mathcal{U}}\;\|\,\phi_n\,\|\;\geq\;v$. We now consider the element $\mathbf{b}=(b_n)\in\;A^{\mathcal{U}}$ and $z\;=\;\lim_{\mathcal{U}}\,\langle\phi_n,b_n\rangle\;=\;\mathbf{\Phi}(\mathbf{b})$. Since $\mathbf{\Phi}$ is a non-zero multiplicative linear functional on $A^{\mathcal{U}}$, it follows that $z\in\;\mathrm{sp}(\mathbf{b})$. Since $\phi_n(b_n)\;\notin\mathrm{sp}_\tau(b_n)$, then it follows that $\|\,(b_n-\,\langle\phi_n,b_n\rangle 1)^{-1}\,\|\;\leq\;\tau^{-1}$ for each $n\in\mathbb{N}$ and this implies that $(b_n-\,\langle\phi_n,b_n\rangle 1)\;\in\;A^{\mathcal{U}}$ is invertible with inverse $(b_n-\,\langle\phi_n,b_n\rangle 1)^{-1}$. Since $\mathbf{b}-z\mathbf{1}=\lim_{\mathcal{U}}\,(b_n-\,\langle\phi_n,b_n\rangle 1)$, it follows that $\mathbf{b}-z\mathbf{1}$ is invertible, which contradicts the already proven fact that $z\in\mathrm{sp}(\mathbf{b})$.

4.2.3 Corollary ([4], Corollary 4.3)

Let A be a unital Banach algebra. Then for each $\varepsilon\;>0$ there is a $\delta>0$ such that if ϕ is a linear functional on A with $\mathrm{dist}\big(\phi(a),\mathrm{sp}(a)\big)\;<\;\delta\;\big(a\;\in A,\|\,a\,\|=1\;\big)$, then $\mathrm{mult}(\phi)\;<\;\varepsilon$.

Proof

Let $\delta>0$ and let ϕ be a linear functional on A with $\mathrm{dist}\big(\phi(a),\mathrm{sp}(a)\big)\;<\;\delta$ for each

$a \in A$ with $\| a \| = 1$. Let $\lambda \in sp(a)$. Since $| \phi(a) - \lambda | < \delta$, it follows that $| \phi(a) | < | \lambda | + \delta$. Thus we have $| \phi(a) | < \| a \| + \delta$, implying that

$$\phi(a) \in sp_{\delta}(a)$$

by Lemma 2.5.9 . The conclusion follows from Theorem 4.2.2.

We are mindful of the following facts in order to conclude this way. The $\delta > 0$ required in this corollary should not exceed 1. Since unital map ϕ is such that $\phi(a) \in sp_{\delta}(a)$, then we have $jmult(\phi) \to 0$ which further implies that $mult(\phi) < \varepsilon$ and $\| \phi \| > v$ for interval $0 < v < 1$.

4.2.4 Definition ([4], p. 252)

Let A be a unital Banach algebra. Then A is properly infinite if it contains elements a_1, a_2, b_1, b_2 such that

$$a_i b_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases} \quad (i, j = 1, 2).$$

4.2.5 Proposition ([4], Proposition 4.7)

Let A be a properly infinite unital Banach algebra. Then A has no non- zero multiplicative linear functionals and A is Almost Multiplicative Near Multiplicative (AMNM).

Proof

Let ϕ be a continuous linear functional on A such that $mult(\phi) < \delta$ with $\delta < 1$. Then $| \phi(1) || 1 - \phi(1) | = | \phi(1) - \phi(1)^2 | < \delta$. Therefore, either $| \phi(1) | < \sqrt{\delta}$ or $| 1 - \phi(1) | < \sqrt{\delta}$. In the first case, we have $| \phi(a) | | 1 - \sqrt{\delta} | < | \phi(a) | | 1 - | \phi(1) || \leq | \phi(a) | | 1 - \phi(1) | = | \phi(a) - \phi(a)\phi(1) | = | \phi(a1) - \phi(a)\phi(1) | \leq \delta \| a \|$ for each $a \in A$ and so

$$\| \phi \| \leq \frac{\delta}{| 1 - \sqrt{\delta} |}.$$

We now claim that the case $| 1 - \phi(1) | < \sqrt{\delta}$ does not occur for δ small enough. Indeed let $a_1, a_2, b_1, b_2 \in A$ satisfy Definition 4.2.4. Then

(i) $| \phi(1) - \phi(a_i)\phi(b_i) | < \delta \| a_i \| \| b_i \| \quad (i = 1, 2)$

and

(ii) $| \phi(a_i)\phi(b_j) | < \delta \| a_i \| \| b_j \| \quad (i, j = 1, 2, \ i \neq j)$

Then it follows that

$| 1 - \phi(a_1)\phi(b_1)\phi(a_2)\phi(b_2) | \geq 1 - | \phi(a_1)\phi(b_2) || \phi(a_2)\phi(b_1) |$

$\geq 1 - \delta^2 \| a_1 \|\| b_2 \|\| a_2 \|\| b_1 \|$, which goes to 1 as $\delta \to 0$.

On the other hand,

$$| 1 - \phi(a_1)\phi(b_1)\phi(a_2)\phi(b_2) | \leq | 1 - \phi(a_1)\phi(b_1) | + | \phi(a_1)\phi(b_1) || 1 - \phi(a_2)\phi(b_2) |$$

$$\leq \sqrt{\delta} + | \phi(a_1)\phi(b_1) | + | \phi(a_1)\phi(b_1) | \left(\sqrt{\delta} + | \phi(a_2)\phi(b_2) | \right)$$

$$\leq \left(\sqrt{\delta} + \delta \| a_1 \|\| b_1 \| \right) + (1 + \delta)^2 \| a_1 \|\| b_1 \| \left(\sqrt{\delta} + \delta \| a_2 \|\| b_2 \| \right),$$

which goes to 0 as $\delta \to 0$.

4.2.6 Corollary ([4], Corollary 4.8)

Let A be a unital AMNM algebra. Then for each $\varepsilon > 0$, there is a $\delta > 0$, such that if ϕ is a linear functional on A with $\phi(a) \in \mathrm{sp}_\delta(a) \left(a \in A, \| a \| = 1\right)$, then $\| \phi - \psi \| < \varepsilon$ for some non-zero multiplicative linear functional ψ on A.

Proof

Applying Theorem 4.2.2 , we gather the following facts:

(i) $\mathrm{jmult}(\phi) \to 0$ implying that $\mathrm{mult}(\phi) < \rho$ given $\rho > 0$.

(ii) $\| \phi \| \leq K$ for some $K > 0$. Hence $\phi \in \mathcal{B}(A, \mathbb{C})$.

These two facts imply Definition 2.11.13 on Almost Multiplicative Near Multiplicative (AMNM)

Chapter 5

Approximately spectrum-preserving maps and the fourth solution

The aim of this chapter is to identify the approximate Jordan multiplicative linear maps in terms of the ε-pseudospectrum's approximate classical Gleason-Kahane-Żelazko's Theorem for commutative Banach algebras developed in Chapter 4.

5.1 Spectrally dominated functions

5.1.1 Theorem ([4], Theorem 5.1)

Let X and Y be Banach spaces, let A and B be unital standard operator algebras on X and Y respectively, and let $\Phi : \mathrm{A} \to \mathrm{B}$ be a linear map. Then Φ has the form $\Phi(\mathrm{T}) = \mathrm{STS}^{-1}$ $(\mathrm{T} \in \mathrm{A})$ for some isomorphism $\mathrm{S} : \mathrm{X} \to \mathrm{Y}$ or $\Phi(\mathrm{T}) = \mathrm{RT}^{*}\mathrm{R}^{-1}$ $(\mathrm{T} \in \mathrm{A})$ for some isomorphism $\mathrm{R} : \mathrm{X}^{*} \to \mathrm{Y}$ in either the following cases:

(1) the map Φ is bijective and $\mathrm{sp}(\Phi(\mathrm{T})) \subset \mathrm{sp}(\mathrm{T})$ for each $(\mathrm{T} \in \mathrm{A})$ or

(2) the map Φ is surjective and $\mathrm{sp}(\Phi(\mathrm{T})) = \mathrm{sp}(\mathrm{T})$ for each $(\mathrm{T} \in \mathrm{A})$.

Proof

This proof is given according to [4]

Assume that (1) holds. Then $\Phi(\mathrm{T})$ is invertible in $\mathcal{B}(\mathrm{Y})$ whenever $\mathrm{T} \in \mathrm{A}$

is invertible in $\mathcal{B}(\mathrm{X})$ (and therefore whenever T is invertible in A). This means that we can have isomorphisms $\mathrm{S}_1 : \mathrm{Y} \to \mathrm{X}$ and $\mathrm{S}_2 : \mathrm{X} \to \mathrm{Y}$ such that $\Phi(\mathrm{T}) = \mathrm{S}_2\mathrm{TS}_1$ $(\mathrm{T} \in \mathrm{A})$ or isomorphisms $\mathrm{R}_1 : \mathrm{Y} \to \mathrm{X}^{*}$ and $\mathrm{R}_2 : \mathrm{X}^{*} \to \mathrm{Y}$ such that $\Phi(\mathrm{T}) = \mathrm{R}_2\mathrm{T}^{*}\mathrm{R}_1$ $(\mathrm{T} \in \mathrm{A})$. We are thus reduced to proving that

$$\Phi(\mathrm{I}) = \mathrm{I}\ .$$

It follows that

$$\mathrm{sp}(\Phi(\mathrm{I})) \subset \mathrm{sp}(\mathrm{I}) = \{1\}$$

and so the operator

$$\mathrm{Q} = \Phi(\mathrm{I}) - \mathrm{I}$$

is quasi nilpotent. Let $\mathrm{V} \in \mathrm{B}$ and set

$$\mathrm{U} = \Phi^{-1}(\mathrm{V} - \mathrm{Q}) \tag{5.1}$$

Then it follows that

$$\mathrm{I} + \mathrm{V} = \Phi(\mathrm{U} + \mathrm{I})$$

so that

$$1 + \mathrm{sp}(\mathrm{V}) = \mathrm{sp}(\mathrm{I} + \mathrm{V}) = \mathrm{sp}\Big(\Phi(\mathrm{I} + \mathrm{U})\Big) \subset \mathrm{sp}(\mathrm{I} + \mathrm{U}) = 1 + \mathrm{sp}(\mathrm{U})$$

and therefore

$$\mathrm{sp}(\mathrm{V}) \subset \mathrm{sp}(\mathrm{U}) \tag{5.2}$$

According to the seminal representation of Φ, it follows that it is continuous. By (5.2), we have that $\mathrm{r(V)} \leq \mathrm{r(U)}$. By (5.1), we have that $\| \mathrm{U} \| \leq \| \Phi^{-1} \| \| \mathrm{V} - \mathrm{Q} \|$. It follows that

$$\mathrm{r(V)} \leq \mathrm{r(U)} \leq \| \mathrm{U} \| \leq \| \Phi^{-1} \| \| \mathrm{V} - \mathrm{Q} \|$$

Hence it follows that Q is in the radical of B, by Theorem 2.8.11(iv), which is zero and therefore $\Phi(\mathrm{I}) = \mathrm{I}$ as required.

We now assume that (2) holds. We only need to show that Φ is injective. Suppose $\Phi(\mathrm{T}) = 0$ for some $\mathrm{T} \in \mathrm{A}$. For each $\mathrm{U} \in \mathrm{A}$ we have

$$\mathrm{sp}(\mathrm{U}) = \mathrm{sp}(\Phi(\mathrm{U} - \mathrm{T})) = \mathrm{sp}(\mathrm{U} - \mathrm{T})$$

and so $\mathrm{r(U)} \leq \| \mathrm{U} - \mathrm{T} \|$. Hence the operator T lies in the radical of A by Theorem 2.8.11(iii), which is zero.

5.1.2 Definition ([4], p. 240)

A key notion to studying continuity of a linear map $\Phi : \mathrm{X} \to \mathrm{Y}$, where X and Y are Banach spaces, is that of the separating space $\mathcal{S}(\Phi)$ of Φ which is defined as follows : $\mathcal{S}(\Phi) = \{ \mathrm{y} \in \mathrm{Y} :$ there exists $(\mathrm{x_n}) \to 0$ in X and $(\Phi(\mathrm{x_n})) \to \mathrm{y}$ in Y $\}$.

Below are some useful properties of the separating space as given by [4].

(i) The space $\mathcal{S}(\Phi)$ is a closed subspace of Y.

(ii) $\mathcal{S}(\Phi)$ measures the closability of Φ.

(iii) The linear map Φ is closed if and only if $\mathcal{S}(\Phi) = \{0\}$.

Specifically property (iii) has been included in the proof of Corollary 5.1.4. We also wish to remind our readers that the concept of the separating space is not new to this dissertation. The first encounter is in the proof of Proposition 2.11.4. The second encounter is in the proof of Theorem 3.1.4 .

5.1.3 Proposition ([4], Proposition 5.2)

Let X be a Banach space, let A be a Banach algebra, and let $\Phi : X \to A$ be a linear map. Then the following assertions are equivalent.

(i) Φ is spectrally dominated by an upper semicontinuous function.

(ii) $r\big(\Phi(x)\big) \le \operatorname{dist}\big(\Phi(x), \mathcal{S}(\Phi)\big)$ for each $x \in X$.

(iii) There exist $M > 0$ such that $r\big(\Phi(x)\big) \le M \| x \|$ for each $x \in X$.

Proof

Suppose that Φ is spectrally dominated by an upper semicontinuous function $\psi : X \to [0, \infty)$. Let (x_n) be a sequence in X with $\lim x_n = 0$ and $\lim \Phi(x_n) = a$ for some $a \in A$ and let $x \in X$. For every $n \in \mathbb{N}$, let p_n be the complex polynomial with coefficients in A given by

$$p_n(z) = z\Phi(x_n) + \Big(\Phi(x) - \Phi(x_n)\Big) \quad (z \in \mathbb{C}).$$

Then

$$r(p_n(z)) \le \| z\Phi(x_n) + \big(\Phi(x) - \Phi(x_n)\big) \| \le | z | \| \Phi(x_n) \| + \| \Phi(x) - \Phi(x_n) \|$$

and

$$r(p_n(z)) = r\Big(\Phi(zx_n) + \Big(\Phi(x) - \Phi(x_n)\Big)\Big) \le \psi\Big(zx_n + \big(x - x_n\big)\Big)$$

for each $z \in \mathbb{C}$. We arrive at

$$r\big(\Phi(x)\big)^2 = r\big(p_n(1)\big)^2 \le \sup_{|z| = R} r\big(p_n(z)\big) \sup_{|z| = R^{-1}} r\big(p_n(z)\big)$$

$$\le \sup_{|z| = R} \psi\Big(zx_n + \big(x - x_n\big)\Big) \sup_{|z| = R^{-1}} \Big(| z | \| \Phi(x_n) \| + \| \Phi(x) - \Phi(x_n) \| \Big)$$

$$\le \sup_{|z| = R} \psi\Big(zx_n + \big(x - x_n\big)\Big) \Big(R^{-1} \| \Phi(x_n) \| + \| \Phi(x) - \Phi(x_n) \| \Big)$$

for all $n \in \mathbb{N}$ and $R > 0$. Letting first $n \to \infty$ and taking into account the upper semicontinuity of ψ we obtain

$$r\big(\Phi(x)\big)^2 \le \psi(x)\Big(R^{-1} \| a \| + \| \Phi(x) - a \| \Big)$$

for each $R > 0$. By letting $R \to \infty$ we then arrive at

$$r\big(\Phi(x)\big)^2 \le \psi(x) \| \Phi(x) - a \| .$$

Since the preceding property inequality holds for each $a \in \mathcal{S}(\Phi)$, we have

$$r\big(\Phi(x)\big) \le \Big(\psi(x)\Big)^{\frac{1}{2}} \operatorname{dist}\Big(\Phi(x) , \mathcal{S}(\Phi)\Big)^{\frac{1}{2}} \quad (x \in X) \tag{5.3}$$

We now observe that the map

$$x \rightarrow \ \mathrm{dist}\Big(\Phi(x)\ ,\ \mathcal{S}(\Phi)\Big)^{\frac{1}{2}}$$

is continuous on X. Indeed,

$$\mathrm{dist}\Big(\Phi(x)\ ,\ \mathcal{S}(\Phi)\Big)^{\frac{1}{2}} \ = \ \|\ \big(Q\Phi\big)(x)\ \|^{\frac{1}{2}}$$

where Q is the quotient map from A onto $A/\mathcal{S}(\Phi)$ and the norm on the right side is taken on $A/\mathcal{S}(\Phi)$. Consequently (5.3) shows that Φ is spectrally dominated by the upper semicontinuous function

$$x \rightarrow \ \Big(\psi(x)\Big)^{\frac{1}{2}}\ \mathrm{dist}\Big(\Phi(x)\ ,\ \mathcal{S}(\Phi)\Big)^{\frac{1}{2}} \quad (x \in X).$$

By iterating this process we then arrive at

$$r\big(\Phi(x)\big) \ \leq \ \Big(\psi(x)\Big)^{\frac{1}{2^n}}\ \mathrm{dist}\Big(\Phi(x)\ ,\ \mathcal{S}(\Phi)\Big)^{\frac{1}{2}\ +\ \frac{1}{2^2}\ +\ \cdots\ +\ \frac{1}{2^n}} \qquad (x \in X)$$

for each $n \in \mathbb{N}$ and, letting $n \ \rightarrow\ \infty$ we obtain

$$r\big(\Phi(x)\big) \ \leq \ \mathrm{dist}\Big(\Phi(x)\ ,\ \mathcal{S}(\Phi)\Big) \quad (x \in X)$$

which gives the second assertion in the proposition. We now assume (ii) holds. Of course , this property can be written in the form

$$r\big(\Phi(x)\big) \ \leq \ \|\ \big(Q\Phi\big)(x)\ \| \ \leq \ \|\ Q\Phi\ \|\|\ x\ \| \quad (x \ \in X),$$

where Q stands for the quotient map from A onto $A/\mathcal{S}(\Phi)$. This gives the third assertion. Finally, it is obvious that (iii) implies (i).

5.1.4 Corollary ([4], Corollary 5.4)

Let X be a Banach space, let A be a Banach algebra, and let $\Phi : X \rightarrow A$ be a linear map. If Φ is surjective and spectrally dominated by an upper semicontinuous function, then $\mathcal{S}(\Phi) \subset \mathrm{rad}(A)$. Accordingly, if in addition A is semisimple, then Φ is continuous.

Proof

Let $a \in \mathcal{S}(\Phi)$. By Corollary 5.1.3(ii) , then for each $b \in A$, we have that

$$r(b) \ \leq \ \|\ b \ - \ a\ \| \ .$$

By Theorem 2.8.11, we have that $a \in \mathrm{rad}(A)$. It follows that $\mathcal{S}(\Phi) \subset \mathrm{rad}(A)$. Since A is semisimple , then Φ is a closed map by Definition 5.1.2 (iii) . This implies that Φ is continuous.

5.1.5 Corollary ([4], Corollary 5.5)

Let A be a Banach algebra, let B be a semisimple Banach algebra, and let $\Phi : A \to B$ be a surjective linear map. Assume that there is $\varepsilon > 0$ such that

$\mathrm{sp}\big(\Phi(a)\big) \subset \mathrm{sp}_\varepsilon(a)$ $(a \in A, \ \| a \| = 1)$, then Φ is continuous.

Proof

Since $\mathrm{sp}\big(\Phi(a)\big) \subset \mathrm{sp}_\varepsilon(a)$ $(a \in A, \ \| a \| = 1)$, then

$$\mathrm{r}\big(\Phi(a)\big) \leq \| a \| + \varepsilon = 1 + \varepsilon \leq \Big(1 + \varepsilon\Big) \| a \|$$

By Proposition 5.1.3 , Φ is spectrally dominated by a semicontinuous function

$$\Big(1 + \varepsilon\Big) \| a \| .$$

Accordingly, by Corollary 5.1.4 , $\mathcal{S}(\Phi) \subset \mathrm{rad}(B)$. Since B is semisimple, then Φ is continuous as required.

5.2 Superreflexive Banach algebras and Jordan multiplicative linear maps

The main results of this chapter rely heavily on superreflexive Banach algebras, which are Banach spaces in which every finitely representable subset of it is reflexive.

5.2.1 Lemma ([4], Lemma 5.6)

Let X and Y be Banach spaces and let $\Phi : \mathcal{B}(X) \to \mathcal{B}(Y)$ be a surjective linear map such that $\mathrm{sp}\big(\Phi(T)\big) \subset \mathrm{sp}_\delta(T)$ $(T \in \mathcal{B}(X), \ \| T \| = 1)$ for some $\delta > 0$. Then Φ is continuous, $\kappa(\Phi) \leq 1 + \delta$, and $\mathrm{sp}_\varepsilon\big(\Phi(T)\big) \subset \mathrm{sp}_{\delta(\|T\| + \frac{\varepsilon}{k}) + \frac{\varepsilon}{k}}(T)$ $\big(T \in \mathcal{B}(X)\big)$ for all $\varepsilon > 0$ and $0 < k < \kappa(\Phi)$.

Proof

By Corollary 5.1.5, Φ is continuous. We now claim that

$$\mathrm{sp}\big(\Phi(T)\big) \subset \mathrm{sp}_{\delta\|T\| + \varrho}(T) \ \ \big(T \in \mathcal{B}(X), \ \varrho > 0\big).$$

Indeed, if $T \neq 0$, then we have $\mathrm{sp}\big(\Phi(T)\big) =$

$$\| T \| \, \mathrm{sp}\Big(\Phi\Big(\frac{T}{\| T \|}\Big)\Big) \subset \| T \| \, \mathrm{sp}_\delta\Big(\frac{T}{\| T \|}\Big) = \mathrm{sp}_{\delta\|T\|}(T) \subset \mathrm{sp}_{\delta\|T\|+\varrho}(T) \qquad (5.4)$$

On the other hand, if $T = 0$, then it is obvious. We now consider $T \in \mathcal{B}(X)$, $\varepsilon > 0$ and $0 < k < \kappa(\Phi)$. Then we choose

$$k < \tau < \kappa(\Phi) \quad \text{and} \quad 0 < \varrho < \frac{\varepsilon}{k} - \frac{\varepsilon}{\tau} + \frac{\delta\varepsilon}{k} - \frac{\delta\varepsilon}{\tau} .$$

Let $z \in \mathrm{sp}_\varepsilon\big(\Phi(T)\big)$. Then $z \in \mathrm{sp}\big(\Phi(T) + S\big)$ for some $S \in \mathcal{B}(Y)$ with $\| S \| < \varepsilon$ by Lemma 2.5.12 . Let $S = \Phi(R)$ for some $R \in \mathcal{B}(X)$ with $\| R \| < \frac{\varepsilon}{\tau}$. Using (5.4) , we have

$$z \in \mathrm{sp}\big(\Phi(T) + S\big) = \mathrm{sp}\big(\Phi(T + R)\big) \subset \mathrm{sp}_{\delta\|T+R\|+\varrho}(T + R)$$

Then

$$z \in \mathrm{sp}_{\delta\left(\|T\|+\frac{\varepsilon}{\tau}\right)+\varrho}(T + R) \subset \mathrm{sp}_{\delta\left(\|T\|+\frac{\varepsilon}{\tau}\right)+\varrho+\frac{\varepsilon}{\tau}}(T)$$

by Lemma 2.5.12 . Then finally

$$z \in \mathrm{sp}_{\delta\left(\|T\|+\frac{\varepsilon}{k}\right)+\frac{\varepsilon}{k}}(T)$$

by Lemma 2.5.6 so that

$$\mathrm{sp}_\varepsilon\big(\Phi(T)\big) \subset \mathrm{sp}_{\delta\left(\|T\|+\frac{\varepsilon}{k}\right)+\frac{\varepsilon}{k}}(T) .$$

Finally, by taking $T = 0$, we get

$$\varepsilon\mathbb{D} = \mathrm{sp}_\varepsilon\big(\Phi(0)\big) \subset \mathrm{sp}_{\frac{\delta\varepsilon}{k}+\frac{\varepsilon}{k}}(0) = \frac{(1+\delta)\varepsilon}{k}\mathbb{D}$$

for all $\varepsilon > 0$ and $0 < k < \kappa(\Phi)$. This clearly implies that $\kappa(\Phi) \leq 1 + \delta$.

5.2.2 Lemma ([4], Lemma 5.7)

Let X and Y be Banach spaces and let $\Phi : \mathcal{B}(X) \to \mathcal{B}(Y)$ be a continuous linear map such that $\mathrm{sp}(T) \subset \mathrm{sp}_\delta\big(\Phi(T)\big)$ $\Big(T \in \mathcal{B}(X),\ \| T \| = 1\Big)$ for some $\delta > 0$. Then $1 - \delta \leq \| \Phi \|$ and $\mathrm{sp}_\varepsilon(T) \subset \mathrm{sp}_{\delta(\|T\|+\varepsilon)+\nu\varepsilon}\big(\Phi(T)\big)$ $\big(T \in \mathcal{B}(X)\big)$ for all $\varepsilon > 0$ and $\| \Phi \| < \nu$.

Proof

By assumption, $\mathrm{sp}(T) \subset \mathrm{sp}_\delta\big(\Phi(T)\big)$. It follows that

$$\mathrm{sp}(T) \subset \mathrm{sp}_\delta\big(\Phi(T)\big) \subset \mathrm{sp}_{\delta(\|T\|+\varepsilon)+\varrho}\big(\Phi(T)\big) \quad \big(T \in \mathcal{B}(X),\ \varrho > 0\big) .$$

We now consider $T \in \mathcal{B}(X)$, $\varepsilon > 0$ and $\| \Phi \| < \nu$. Then we pick

$$0 < \varrho < (\nu - \| \Phi \|)\varepsilon.$$

Let $z \in \mathrm{sp}_\varepsilon(T)$. Then $z \in \mathrm{sp}\big(T + S\big)$ for some $S \in \mathcal{B}(X)$ with $\| S \| < \varepsilon$.

$$z \in \mathrm{sp}\big(T + S\big) \subset \mathrm{sp}_{\delta\|T+S\|+\varrho}\big(\Phi(T+S)\big)$$

$$\subset \mathrm{sp}_{\delta\left(\|T\|+\varepsilon\right)+\varrho}\big(\Phi(T)+\Phi(S)\big) \subset \mathrm{sp}_{\delta\left(\|T\|+\varepsilon\right)+\varrho+\|\Phi\|\varepsilon}\big(\Phi(T)\big) \subset \mathrm{sp}_{\delta\left(\|T\|+\varepsilon\right)+\nu\varepsilon}\big(\Phi(T)\big),$$

which establishes the inclusion in the lemma. Finally, by taking $T = 0$ in the inclusion we get

$$\varepsilon\mathbb{D} = \mathrm{sp}_\varepsilon(0) \subset \mathrm{sp}_{\delta\varepsilon+\nu\varepsilon}\big(\Phi(0)\big) = (\delta+\nu)\varepsilon\mathbb{D}$$

for all $\varepsilon > 0$ and $\| \Phi \| < \nu$. This clearly implies that $1 - \delta \leq \| \Phi \|$ for some $\delta > 0$.

5.2.3 Theorem ([4], Theorem 5.8)

Let X and Y be superreflexive Banach spaces. Then the following assertions hold.

(1) For each $\mathrm{K}, \varepsilon > 0$ there is a $\delta > 0$ such that if $\Phi : \mathcal{B}(\mathrm{X}) \to \mathcal{B}(\mathrm{Y})$ is a bijective linear map with $\mathrm{sp}\big(\Phi(\mathrm{T})\big) \subset \mathrm{sp}_\delta(\mathrm{T})$ $(\mathrm{T} \in \mathcal{B}(\mathrm{X}),\ \| \mathrm{T} \| = 1)$ and $\| \Phi \|, \| \Phi^{-1} \| < \mathrm{K}$, then $\min\big\{\mathrm{mult}(\Phi), \mathrm{amult}(\Phi)\big\} < \varepsilon$.

(2) For each $\mathrm{K}, \varepsilon > 0$ there is a $\delta > 0$ such that if $\Phi : \mathcal{B}(\mathrm{X}) \to \mathcal{B}(\mathrm{Y})$ is a bijective linear map with $\mathrm{jmult}(\Phi) < \delta$ and $\| \Phi \|, \| \Phi^{-1} \| < \mathrm{K}$, then $\mathrm{sp}\big(\Phi(\mathrm{T})\big) \subset \mathrm{sp}_\varepsilon(\mathrm{T})$ $(\mathrm{T} \in \mathcal{B}(\mathrm{X}), \| \mathrm{T} \| = 1)$.

Proof

Suppose the first assertion in the theorem fails to be true. Then there exist $\mathrm{K}, \tau > 0$ and a sequence (Φ_n) of bijective linear maps from $\mathcal{B}(\mathrm{X})$ onto $\mathcal{B}(\mathrm{Y})$ with the properties that

$$\mathrm{sp}\big(\Phi_\mathrm{n}(\mathrm{T})\big) \subset \mathrm{sp}_{\frac{1}{\mathrm{n}}}(\mathrm{T}) \ \ (\mathrm{T} \in \mathcal{B}(\mathrm{X}), \| \mathrm{T} \| = 1), \tag{5.5}$$

$$\| \Phi_\mathrm{n} \|, \| {\Phi_\mathrm{n}}^{-1} \| < \mathrm{K}, \tag{5.6}$$

$$\min\big\{\mathrm{mult}(\Phi_\mathrm{n}), \mathrm{amult}(\Phi_\mathrm{n})\big\} \geq \tau \tag{5.7}$$

for each $\mathrm{n} \in \mathbb{N}$. On account of (5.6), the map

$$\mathbf{\Phi} = (\Phi_\mathrm{n}) : \mathcal{B}(\mathrm{X})^{\mathcal{U}} \subset \mathcal{B}(\mathrm{X}^{\mathcal{U}}) \longrightarrow \mathcal{B}(\mathrm{Y})^{\mathcal{U}} \subset \mathcal{B}(\mathrm{Y}^{\mathcal{U}})$$

is a continuous bijective linear map $\big($ with inverse given by $\mathbf{\Phi^{-1}} = ({\Phi_\mathrm{n}}^{-1})$ $\big)$. Our next concern consists in proving that $\mathbf{\Phi}$ shrinks the spectrum. Let $\mathbf{T} = (\mathrm{T}_\mathrm{n}) \in \mathcal{B}(\mathrm{X})^{\mathcal{U}}$. We claim that

$$\mathrm{sp}_{\frac{\varepsilon}{\mathrm{K}+1}}\big(\mathbf{\Phi}(\mathbf{T})\big) \subset \mathrm{sp}_\varepsilon(\mathbf{T}) \ (\varepsilon > 0).$$

Indeed, let $0 < \varepsilon$ and pick $\varepsilon < \varrho < (1 + \frac{1}{\mathrm{K}})\varepsilon$. Since $\kappa(\Phi_\mathrm{n}) = \| {\Phi_\mathrm{n}}^{-1} \|^{-1} > \mathrm{K}^{-1}$, then we have

$$\mathrm{sp}_{\frac{\varrho}{\mathrm{K}+1}}\big(\Phi_\mathrm{n}(\mathrm{T}_\mathrm{n})\big) \subset \mathrm{sp}_{\frac{1}{\mathrm{n}}\left(\|\mathrm{T}_\mathrm{n}\| + \frac{\varrho\mathrm{K}}{(\mathrm{K}+1)}\right) + \frac{\varrho\mathrm{K}}{(\mathrm{K}+1)}}(\mathrm{T}_\mathrm{n}) \ \ (\mathrm{n} \in \mathbb{N})$$

by Lemma 5.2.1 . Since

$$\lim_{\mathcal{U}} \frac{1}{\mathrm{n}} \| \mathrm{T}_\mathrm{n} \| + \lim_{\mathcal{U}} \frac{1}{\mathrm{n}}\Big(\frac{\varrho\mathrm{K}}{(\mathrm{K}+1)}\Big) + \lim_{\mathcal{U}}\Big(\frac{\varrho\mathrm{K}}{(\mathrm{K}+1)}\Big) = \frac{\varrho\mathrm{K}}{(\mathrm{K}+1)} < \varepsilon ,$$

It follows that

$$\mathrm{sp}_{\frac{\varepsilon}{\mathrm{K}+1}}\big(\mathbf{\Phi}(\mathbf{T})\big) \subset \mathrm{sp}_\varepsilon(\mathbf{T}) \ (\varepsilon > 0)$$

as we claimed. Furthermore, by Lemma 2.5.7 , it follows that

$$\mathrm{sp}\Big(\mathbf{\Phi}(\mathbf{T})\Big) \;=\; \bigcap\nolimits_{\varepsilon>0} \mathrm{sp}_{\frac{\varepsilon}{\mathrm{K}+1}}\Big(\mathbf{\Phi}(\mathbf{T})\Big) \;\subset\; \bigcap\nolimits_{\varepsilon>0} \mathrm{sp}_{\varepsilon}(\mathbf{T}) \;=\; \mathrm{sp}(\mathbf{T}) \;.$$

By Lemma 2.2.14, we know that $\mathcal{B}(\mathrm{X})^{\mathcal{U}}$ and $\mathcal{B}(\mathrm{Y})^{\mathcal{U}}$ are standard operator algebras on $\mathrm{X}^{\mathcal{U}}$ and $\mathrm{Y}^{\mathcal{U}}$ respectively. Then by Theorem 5.1.1(1) , we see that $\mathbf{\Phi}$ is either a homomorphism or an anti-homomorphism. Then from Lemma 2.11.10, we see that

$$\lim_{\mathcal{U}} \min\Big\{\mathrm{mult}(\Phi_{\mathrm{n}}) \;,\; \mathrm{amult}(\Phi_{\mathrm{n}})\Big\} \;=\; \min\Big\{\lim_{\mathcal{U}} \mathrm{mult}(\Phi_{\mathrm{n}}) \;,\; \lim_{\mathcal{U}} \mathrm{amult}(\Phi_{\mathrm{n}})\Big\}$$

$$=\; \min\Big\{\mathrm{mult}(\mathbf{\Phi}) \;,\; \mathrm{amult}(\mathbf{\Phi})\Big\} \;=\; 0,$$

which contradicts (5.7) . Now we are going to prove the second assertion in the theorem. To obtain a contradiction, suppose the assertion is false. Then there exist $\tau > 0$, a sequence (Φ_{n}) of bijective linear maps from $\mathcal{B}(\mathrm{X})^{\mathcal{U}}$ onto $\mathcal{B}(\mathrm{Y})^{\mathcal{U}}$, and a sequence $(\mathrm{T}_{\mathrm{n}})$ in $\mathcal{B}(\mathrm{X})^{\mathcal{U}}$ such that

$$\mathrm{jmult}(\Phi_{\mathrm{n}}) \;<\; \frac{1}{\mathrm{n}} \tag{5.8}$$

$$\|\, \Phi_{\mathrm{n}} \,\|, \|\, {\Phi_{\mathrm{n}}}^{-1} \,\| \;\;<\; \mathrm{K} \tag{5.9}$$

$$\|\, \mathrm{T}_{\mathrm{n}} \,\| \;\;=\; 1 \tag{5.10}$$

and

$$\mathrm{sp}\Big(\Phi_{\mathrm{n}}(\mathrm{T}_{\mathrm{n}})\Big) \;\not\subset\; \mathrm{sp}_{\tau}(\mathrm{T}_{\mathrm{n}}) \tag{5.11}$$

for each $\mathrm{n} \in \mathbb{N}$. Since $\mathcal{B}(\mathrm{Y})^{\mathcal{U}}$ is ultraprime, using Theorem 2.11.11 , we replace (5.8) by

$$\lim_{\mathcal{U}} \min\Big\{\mathrm{mult}(\Phi_{\mathrm{n}}), \mathrm{amult}(\Phi_{\mathrm{n}})\Big\} \;=\; 0.$$

We now consider the bijective continuous linear map

$$\mathbf{\Phi} \;=\; (\Phi_{\mathrm{n}}) : \mathcal{B}(\mathrm{X})^{\mathcal{U}} \;\subset\; \mathcal{B}(\mathrm{X}^{\mathcal{U}}) \to \mathcal{B}(\mathrm{Y})^{\mathcal{U}} \;\subset\; \mathcal{B}(\mathrm{Y}^{\mathcal{U}}) \;.$$

Then

$$\min\Big\{\mathrm{mult}(\mathbf{\Phi}), \mathrm{amult}(\mathbf{\Phi})\Big\} \;=\; \lim_{\mathcal{U}} \min\Big\{\mathrm{mult}(\Phi_{\mathrm{n}}), \mathrm{amult}(\Phi_{\mathrm{n}})\Big\} \;=\; 0$$

so that $\mathbf{\Phi}$ is either a homomorphism or an anti-homomorphism. Our next objective is to show that $\mathbf{\Phi}$ preserves the spectrum. We first suppose that $\mathbf{\Phi}$ is an isomorphism. Then there exist an invertible operator $\mathbf{S} \in \mathcal{B}\Big(\mathrm{X}^{\mathcal{U}}, \mathrm{Y}^{\mathcal{U}}\Big)$ such that

$$\mathbf{\Phi}(\mathbf{T}) \;=\; \mathbf{S}\mathbf{T}\mathbf{S}^{-1} \;\; \Big(\mathbf{T} \in \mathcal{B}(\mathrm{X})^{\mathcal{U}}\Big).$$

Given $\mathbf{T} \in \mathcal{B}(\mathrm{X})^{\mathcal{U}}$ and $\mathrm{z} \in \mathbb{C}$, we have $\mathbf{\Phi}(\mathbf{T}) - \mathrm{z}\mathbf{I} = \mathbf{S}(\mathbf{T} - \mathrm{z}\mathbf{I})\mathbf{S}^{-1}$. Thus

$\mathbf{\Phi}(\mathbf{T}) - \mathrm{z}\mathbf{I}$ is an invertible operator if and only if so is $\mathbf{T} - \mathrm{z}\mathbf{I}$. Hence

$\mathrm{sp}\big(\mathbf{\Phi}(\mathbf{T})\big) = \mathrm{sp}(\mathbf{T})$ as required. We now suppose that $\mathbf{\Phi}$ is an anti-isomorphism. Then the map

$$\mathbf{\Psi} : \mathcal{B}(\mathrm{X})^{\mathcal{U}} \to \Big\{\mathbf{S}^* \;:\; \mathbf{S} \in \mathcal{B}(\mathrm{X})^{\mathcal{U}}\Big\}, \qquad \Psi(\mathbf{T}) = \mathbf{\Phi}(\mathbf{T})^* \;\big(\mathbf{T} \in \mathcal{B}(\mathrm{X})^{\mathcal{U}}\big)$$

is an isomorphism. $\Big\{\mathbf{S}^* \;:\; \mathbf{S} \in \mathcal{B}(\mathrm{X})^{\mathcal{U}}\Big\}$ is a standard operator algebra on $\big(\mathrm{Y}^{\mathcal{U}}\big)^*$. There is an invertible operator $\mathbf{R} \in \mathcal{B}\Big(\mathrm{X}^{\mathcal{U}}, \big(\mathrm{Y}^{\mathcal{U}}\big)^*\Big)$ such that

$$\Psi(\mathbf{T}) = \mathbf{R}\mathbf{T}\mathbf{R}^{-1} \;\big(\mathbf{T} \in \mathcal{B}(\mathrm{X})^{\mathcal{U}}\big).$$

Consequently, we have

$$\mathbf{\Phi}(\mathbf{T}) = \mathbf{S}\mathbf{T}^*\mathbf{S}^{-1} \;\big(\mathbf{T} \in \mathcal{B}(\mathrm{X})^{\mathcal{U}}\big)$$

where $\mathbf{S} = (\mathbf{R}^*)^{-1} \in \mathcal{B}\Big(\big(\mathrm{X}^{\mathcal{U}}\big)^*, \mathrm{Y}^{\mathcal{U}}\Big)$. Given $\mathbf{T} \in \mathcal{B}(\mathrm{X})^{\mathcal{U}}$ and $\mathrm{z} \in \mathbb{C}$, we have $\mathbf{\Phi}(\mathbf{T}) - \mathrm{z}\mathbf{I} = \mathbf{S}(\mathbf{T} - \mathrm{z}\mathbf{I})^*\mathbf{S}^{-1}$. Thus $\mathbf{\Phi}(\mathbf{T}) - \mathrm{z}\mathbf{I}$ is an invertible operator if and only if so is $(\mathbf{T} - \mathrm{z}\mathbf{I})^*$ if and only if so is $\mathbf{T} - \mathrm{z}\mathbf{I}$. This implies that $\mathrm{sp}\big(\mathbf{\Phi}(\mathbf{T})\big) = \mathrm{sp}(\mathbf{T})$. Finally, let $\mathbf{T} = (\mathrm{T_n}) \in \mathcal{B}(\mathrm{X})^{\mathcal{U}}$. Since $\mathrm{sp}\big(\mathbf{\Phi}(\mathbf{T})\big) = \mathrm{sp}(\mathbf{T})$, it follows that $\mathrm{sp}\big(\Phi_{\mathrm{n}}(\mathrm{T_n})\big) \subset \mathrm{sp}_{\tau}(\mathrm{T_n})$ for some $\mathrm{n} \in \mathbb{N}$. This is a contradiction.

5.2.4 Lemma ([4], Lemma 5.9)

Let X and Y be Banach spaces and let $\Phi : \mathcal{B}(\mathrm{X}) \to \mathcal{B}(\mathrm{Y})$ be a continuous linear map. If $\| \Phi - \Psi \| < \varepsilon$ for some $\varepsilon > 0$ and some epimorphism or anti-epimorphism $\Psi : \mathcal{B}(\mathrm{X}) \to \mathcal{B}(\mathrm{Y})$, then $\mathrm{sp}\big(\Phi(\mathrm{T})\big) \subset \mathrm{sp}_{\varepsilon(\varepsilon+\|\Phi\|)}(\mathrm{T})$ $(\mathrm{T} \in \mathcal{B}(\mathrm{X}), \| \mathrm{T} \| = 1)$.

Proof

Let $\mathrm{T} \in \mathcal{B}(\mathrm{X})$ with $\| \mathrm{T} \| = 1$. Then $\| (\Phi - \Psi)(\mathrm{T}) \| < \varepsilon$. Lemma 2.5.12 yields

$$\mathrm{sp}\big(\Phi(\mathrm{T})\big) = \mathrm{sp}\Big(\Psi(\mathrm{T}) + (\Phi - \Psi)(\mathrm{T})\Big) \subset \mathrm{sp}_{\varepsilon}(\Psi(\mathrm{T})).$$

If $\mathrm{z} \in \mathrm{sp}_{\varepsilon}(\Psi(\mathrm{T}))$, then

$$\varepsilon^{-1} < \| \big(\mathrm{zI_y} - \Psi(\mathrm{T})\big)^{-1} \| = \| \Big(\Psi\big(\mathrm{zI_x} - \mathrm{T}\big)\Big)^{-1} \|$$

$$= \| \Psi\Big(\big(\mathrm{zI_x} - \mathrm{T}\big)^{-1}\Big) \| \leq \| \Psi \| \| \big(\mathrm{zI_x} - \mathrm{T}\big)^{-1} \|$$

and therefore $\mathrm{z} \in \mathrm{sp}_{\varepsilon\|\Psi\|}(\mathrm{T})$ by Definition 2.5.3 . On the other hand, since $\| \Psi \| < \varepsilon + \| \Phi \|$ with respect to $\| \Phi - \Psi \| < \varepsilon$, then we have

$$\mathrm{sp}(\Phi(\mathrm{T})) \subset \mathrm{sp}_{\varepsilon}(\Psi(\mathrm{T})) \subset \mathrm{sp}_{\varepsilon\|\Psi\|}(\mathrm{T}) \subset \mathrm{sp}_{\varepsilon(\varepsilon + \|\Phi\|)}(\mathrm{T}),$$

as required.

5.2.5 Theorem ([4], Theorem 5.10)

Let H be a separable Hilbert space. Then the following assertions hold.

(i) For each $\mathrm{K}, \varepsilon > 0$ there is a $\delta > 0$ such that $\Phi : \mathcal{B}(\mathrm{H}) \to \mathcal{B}(\mathrm{H})$ is a bijective linear map with

$$\mathrm{sp}(\Phi(\mathrm{T})) \subset \mathrm{sp}_{\delta}(\mathrm{T}) \ \Big(\mathrm{T} \in \mathcal{B}(\mathrm{H}), \ \| \mathrm{T} \| = 1\Big)$$

and $\| \Phi \|, \ \| \Phi^{-1} \| < \mathrm{K}$, then

$$\| \Phi - \Psi \| < \varepsilon$$

for some automorphism or anti-automorphism $\Phi : \mathcal{B}(\mathrm{H}) \to \mathcal{B}(\mathrm{H})$.

(ii) For each $\mathrm{K}, \varepsilon > 0$ there is a $\delta > 0$ such that $\Phi : \mathcal{B}(\mathrm{H}) \to \mathcal{B}(\mathrm{H})$ is a continuous linear map with $\| \Phi \| < \mathrm{K}$ and

$$\| \Phi - \Psi \| < \delta$$

for some automorphism or anti-automorphism $\Psi : \mathcal{B}(\mathrm{H}) \to \mathcal{B}(\mathrm{H})$, then

$$\mathrm{sp}(\Phi(\mathrm{T})) \subset \mathrm{sp}_{\varepsilon}(\mathrm{T}) \ \Big(\mathrm{T} \in \mathcal{B}(\mathrm{H}), \ \| \mathrm{T} \| = 1\Big).$$

Proof

Let $\varepsilon > 0$ and set $\varepsilon' = \min\{\varepsilon, 1\}$. By Corollary 2.11.17 , there is a $\varrho > 0$ such that if $\Phi : \mathcal{B}(\mathrm{H}) \to \mathcal{B}(\mathrm{H})$ is a linear map with $\mathrm{jmult}(\Phi) < \varrho$, $\mathrm{K}^{-1} < \kappa(\Phi)$ and $\| \Phi \| < \mathrm{K}$, then $\| \Phi - \Psi \| < \varepsilon'$ for some epimorphism or anti-epimorphism $\Psi : \mathcal{B}(\mathrm{H}) \to \mathcal{B}(\mathrm{H})$. We now apply Theorem 5.2.3(1) to get $\delta > 0$ such that if $\Phi : \mathcal{B}(\mathrm{X}) \to \mathcal{B}(\mathrm{Y})$ is a bijective linear map with

$$\mathrm{sp}(\Phi(\mathrm{T})) \subset \mathrm{sp}_{\delta}(\mathrm{T}) \ \Big(\mathrm{T} \in \mathcal{B}(\mathrm{H}), \ \| \mathrm{T} \| = 1\Big)$$

and $\| \Phi \|, \ \| \Phi^{-1} \| < \mathrm{K}$, then $\min\{\mathrm{mult}(\Phi), \mathrm{amult}(\Phi)\} < \varrho$. Then $\mathrm{jmult}(\Phi) < \varrho$ and $\mathrm{K}^{-1} < \kappa(\Phi)$ implies that $\| \Phi - \Psi \| < \varepsilon'$ for some epimorphism or anti-epimorphism $\Psi : \mathcal{B}(\mathrm{H}) \to \mathcal{B}(\mathrm{H})$. Finally, since Φ is invertible and

$$\| \Phi - \Psi \| < 1,$$

it follows that Ψ is invertible by Lemma 5.2.4 . Therefore Ψ is either an automorphism or anti-automorphism. The task is now to prove the second assertion. Let $\varepsilon > 0$. Then we take $\delta > 0$ such that $\delta(\delta + \mathrm{K}) < \varepsilon$. Suppose $\Phi : \mathcal{B}(\mathrm{H}) \to \mathcal{B}(\mathrm{H})$ is a continuous linear map with $\| \Phi \| < \mathrm{K}$ and $\| \Phi - \Psi \| < \delta$ for some automorphism or anti-automorphism $\Psi : \mathcal{B}(\mathrm{H}) \to \mathcal{B}(\mathrm{H})$. On account of Lemma 5.2.4, we have

$$\mathrm{sp}(\Phi(\mathrm{T})) \subset \mathrm{sp}_{\varepsilon}(\mathrm{T}) \ \Big(\mathrm{T} \in \mathcal{B}(\mathrm{H}), \ \| \mathrm{T} \| = 1\Big).$$

5.2.6 Theorem ([4], Theorem 5.11)

Let X and Y be superreflexive spaces. Then for each $k, K, \varepsilon > 0$ there is a $\delta > 0$ such that if $\Phi : \mathcal{B}(X) \to \mathcal{B}(Y)$ is a surjective linear map with

$$\mathrm{dist}_H\Big(\mathrm{sp}\big(\Phi(T)\big), \mathrm{sp}(T)\Big) < \delta \quad (T \in \mathcal{B}(X), \ \| T \| = 1),$$

$\kappa(\Phi) > k$, and $\| \Phi \| < K$, then

$$\min\big\{\mathrm{mult}(\Phi), \mathrm{amult}(\Phi)\big\} < \varepsilon.$$

Proof

The conditions set by this theorem (Theorem 5.2.6) are naturally satisfied by Theorem 5.2.3(1). However, we supply the details of the proof in order to expose its dependance on several other results other than just Theorem 5.2.3. It should also be noted that Theorem 5.2.7 depends on this result. The proof goes as follows:

On the contrary, suppose that there exist $K, \tau > 0$ and a sequence (Φ_n) of surjective linear maps from $\mathcal{B}(X)$ onto $\mathcal{B}(Y)$ such that

$$\sup_{\|T\|=1} \mathrm{dist}_H\Big(\mathrm{sp}\big(\Phi_n(T)\big), \mathrm{sp}(T)\Big) \to 0 \tag{5.12}$$

$$\kappa(\Phi_n) > k, \qquad \| \Phi_n \| < K \quad (n \in \mathbb{N}) \tag{5.13}$$

and

$$\min\big\{\mathrm{mult}(\Phi_n), \mathrm{amult}(\Phi_n)\big\} \geq \tau \quad (n \in \mathbb{N}) \tag{5.14}$$

Let (ε_n) be a sequence of positive numbers with

$$\lim\ (\varepsilon_n) = 0$$

and

$$\sup_{\|T\|=1} \mathrm{dist}_H\Big(\mathrm{sp}\big(\Phi_n(T)\big), \mathrm{sp}(T)\Big) < \varepsilon_n \quad (n \in \mathbb{N}).$$

On account of (5.12), the definition of the Hausdorff distance and Lemma 2.5.10, we have

$$\mathrm{sp}\big(\Phi_n(T)\big) \subset \mathrm{sp}(T) + \varepsilon_n\mathbb{D} \subset \mathrm{sp}_{\varepsilon_n}(T) \quad (T \in \mathcal{B}(X), \ \| T \| = 1) \tag{5.15}$$

and

$$\mathrm{sp}(T) \subset \mathrm{sp}\big(\Phi_n(T)\big) + \varepsilon_n\mathbb{D} \subset \mathrm{sp}_{\varepsilon_n}\big(\Phi_n(T)\big) \quad (T \in \mathcal{B}(X), \ \| T \| = 1) \tag{5.16}$$

for each $n \in \mathbb{N}$. On account of (5.13), we can consider the continuous linear operator

$$\mathbf{\Phi} = (\Phi_n) : \ \mathcal{B}(X)^{\mathcal{U}} \subset \mathcal{B}(X^{\mathcal{U}}) \to \mathcal{B}(Y)^{\mathcal{U}} \subset \mathcal{B}(Y^{\mathcal{U}}).$$

From (5.13) and Lemma 2.3.8 , it follows that $\kappa(\mathbf{\Phi}) = \lim_{\mathcal{U}} \kappa(\Phi_n) \geq k$ so that $\mathbf{\Phi}$ is a continuous surjective linear map from $\mathcal{B}(X)^{\mathcal{U}}$ onto $\mathcal{B}(Y)^{\mathcal{U}}$. We claim that $\mathbf{\Phi}$ preserves the spectrum. Let $\mathbf{T} \in \mathcal{B}(X)^{\mathcal{U}}$. On account of (5.15) and the property that $\kappa(\Phi_n) > k$ for each $n \in \mathbb{N}$, we arrive at

$$\mathrm{sp}_{\frac{\varepsilon}{K+1}}\big(\mathbf{\Phi}(\mathbf{T})\big) \subset \mathrm{sp}_{\varepsilon}(\mathbf{T}) \ (\varepsilon > 0)$$

just in the same manner as in Theorem 5.2.3(1). Here the sequence $(\frac{1}{n})$ is being replaced by the sequence (ε_n). We thus have

$$\mathrm{sp}\big(\mathbf{\Phi}(\mathbf{T})\big) = \bigcap_{\varepsilon>0} \mathrm{sp}_{\frac{\varepsilon}{K+1}}\big(\mathbf{\Phi}(\mathbf{T})\big) \subset \bigcap_{\varepsilon>0} \mathrm{sp}_{\varepsilon}(\mathbf{T}) = \mathrm{sp}(\mathbf{T}).$$

We now prove the converse inclusion in much the same way, the only difference being in the application of Lemma 5.2.2 instead of Lemma 5.2.1. We begin by proving that

$$\mathrm{sp}_{\frac{\varepsilon}{K+1}}(\mathbf{T}) \subset \mathrm{sp}_{\varepsilon}\big(\mathbf{\Phi}(\mathbf{T})\big) \ (\varepsilon > 0)$$

Let $0 < \varepsilon$ and pick $\varepsilon < \varrho < (1+\frac{1}{K})\varepsilon$. On account of (5.16) and the property that $\| \Phi_n \| < K$ for each $n \in \mathbb{N}$, Lemma 5.2.2 now gives

$$\mathrm{sp}_{\frac{\varrho}{K+1}}(T_n) \subset \mathrm{sp}_{\varepsilon_n\left(\|T_n\|+\frac{\varrho K}{(K+1)}\right)+\frac{\varrho K}{(K+1)}}\big(\Phi_n(T_n)\big) \quad (n \in \mathbb{N}).$$

Since

$$\lim_{\mathcal{U}}\Big(\varepsilon_n \| T_n \|\Big) + \lim_{\mathcal{U}}\Big(\varepsilon_n \frac{\varrho K}{(K+1)}\Big) + \lim_{\mathcal{U}}\Big(\frac{\varrho K}{(K+1)}\Big) = \frac{\varrho K}{(K+1)} < \varepsilon,$$

we now have

$$\mathrm{sp}(\mathbf{T}) = \bigcap_{\varepsilon>0} \mathrm{sp}_{\frac{\varepsilon}{K+1}}(\mathbf{T}) \subset \bigcap_{\varepsilon>0} \mathrm{sp}_{\varepsilon}\big(\mathbf{\Phi}(\mathbf{T})\big) = \mathrm{sp}\big(\mathbf{\Phi}(\mathbf{T})\big),$$

as desired. Finally, since $\mathbf{\Phi}$ is a surjective linear map preserving the spectrum and, by Lemma 2.2.14, $\mathcal{B}(X)^{\mathcal{U}}$ and $\mathcal{B}(Y)^{\mathcal{U}}$ are standard operator algebras on $X^{\mathcal{U}}$ and $Y^{\mathcal{U}}$, respectively. From Theorem 5.1.1(2), we see that $\mathbf{\Phi}$ is either a homomorphism or an anti-homomorphism. Then by Lemma 2.11.10,

$$\lim_{\mathcal{U}} \min\Big\{\mathrm{mult}(\Phi_n)\ ,\ \mathrm{amult}(\Phi_n)\Big\} = \min\Big\{\mathrm{mult}(\mathbf{\Phi})\ ,\ \mathrm{amult}(\mathbf{\Phi})\Big\} = 0,$$

which is a contradiction.

5.2.7 Theorem ([4], Theorem 5.12)

Let H be a separable Hilbert space. Then for each $k, K, \varepsilon > 0$ there is a $\delta > 0$ such that if $\Phi : \mathcal{B}(H) \to B(H)$ is a surjective linear map with

$$\mathrm{dist}_H\Big(\mathrm{sp}\big(\Phi(T)\big), \mathrm{sp}(T)\Big) < \delta \quad (T \in \mathcal{B}(X),\ \| T \| = 1),$$

$\kappa(\Phi) \;>\; \mathrm{k}, \quad \text{and} \quad \|\,\Phi\,\| \;<\; \mathrm{K}, \quad \text{then}$

$$\|\,\Phi \;-\; \Psi\,\| \;<\; \varepsilon$$

for some automorphism or anti-automorphism $\Phi : \mathcal{B}(\mathrm{H}) \to \mathrm{B}(\mathrm{H})$.

Proof

The proof of the first part of Theorem 5.2.5 carries over to this result; the only difference being in the application of Theorem 5.2.6 instead of Theorem 5.2.3(1).

Chapter 6

Conclusion

This dissertation was an attempt to answer the so-called Kaplansky's problem (item 3.1.1) which was given as follows:

Let A and B be complex Banach algebras. Suppose $\Phi : A \to B$ is a unital invertibility preserving linear map from A into B such that $\mathrm{sp}\big(\Phi(a)\big) \subset \mathrm{sp}(a)$ $(a \in A)$. Is it true that Φ is a Jordan homomorphism?

In this dissertation, we have shown through Examples 3.1.5, 3.1.6 and 3.1.7 that this question may be negative when the map is not surjective or when the Banach algebras are not semisimple. Because of these concerns, the so-called Kaplansky's problem has been answered via the Bernard Aupetit's open problem (item 3.1.8) which was given as follows:

Let A and B be semisimple Banach algebras and let $\Phi : A \to B$ be a surjective linear map such that $\mathrm{sp}(\Phi(a)) = \mathrm{sp}(a)$ $(a \in A)$. Is it true that Φ is a Jordan isomorphism?

The so-called Kaplansky's problem has four answers in this dissertation, all of them via the Bernard Aupetit's open problem.

The first result (result 3.2.7) shows that a surjective spectrum-preserving map between Von Neumann algebras is a Jordan homomorphism.

The second result (result 3.3.12) shows that a surjective spectra isometry between finite dimensional semisimple Banach algebras is a Jordan homomorphism.

The third result (result 3.4.14) shows that a bijective spectrum-preserving map $\Phi : A \to B$ such that A= $\mathcal{B}(X)$ and B= $\mathcal{B}(Y)$ being algebras of all bounded maps defined on Banach spaces X and Y respectively is a Jordan homomorphism.

The fourth result (Theorems 5.2.3, 5.2.6 and 5.2.7) shows that a bijective almost multiplicative or anti-multiplicative map $\Phi : A \to B$ such that it almost preserves the spectra, where A= $\mathcal{B}(X)$ and B= $\mathcal{B}(Y)$, being algebras of all bounded maps defined on superreflexive Banach spaces X and Y respectively, is a Jordan homomorphism.

None of them answers the so-called Kaplansky's problem completely, but they are all motivating results with a view of accelerating the search for an eventual solution. However result four is stronger in implications than the rest.

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