INTEGRAL EQUATIONS:

A SURVEY OF PAST AND CURRENT DEVELOPMENTS

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MATTHEW HAANTUMBULA KALUWA

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PROFESSOR E. F. BARTHOLOMEUSZ
DISSERTATION SUPERVISOR AND
INTERNAL EXAMINER

EXTERNAL EXAMINER

D E C L A R A T I O N

I hereby declare that this dissertation is my own work and that it has not been previously submitted for degree purposes here or at any other University.

C O N T E N T S

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CHAPTER ONE

GENERAL INTRODUCTION

1.1 INTRODUCTION

A functional equation in which the unknown appears under an integral sign is called an INTEGRAL EQUATION. Integral equations occur in the mathematical contexts of a number of branches of science. In particular, they come up frequently in boundary value problems of potential theory, fluid dynamics, Mechanics, Engineering, Elasticity, and other applied sciences.

1.2 SOME PROBLEMS WHICH GIVE RISE TO INTEGRAL EQUATIONS

Amongst the many problems which reduce to integral equations, the following may be listed and formulated.

1). DIRICHLET'S PROBLEM (REF: [19], pp137 - 152).

Find the function which is harmonic in the interior of a region; and which takes stipulated values on the boundary of the region.

2). NEUMANN'S PROBLEM (REF: [19], p 153)

Find a function, harmonic in a region, when the values of the normal derivative are known on the boundary

The essential device in the conversion of each boundary value problem to the solution of equivalent integral equations is the use of appropriate Green's functions, illustrated below in a simple one-dimensional example which contains the essential features of the method as applied to the problems of Dirichlet and Neumann in 2 and 3 - dimensions.

A LOADED ELASTIC STRING

Consider a weightless elastic string (figure 1.2.1), stretched between two horizontal points O and A. Suppose that a weight W is hung from the elastic string and that in equilibrium the position of the weight is at a distance ξ from O and at a depth η below OA. If W is small, compared to the initial tension T in the string, it can be assumed that the tension of the string remains T during further stretching.

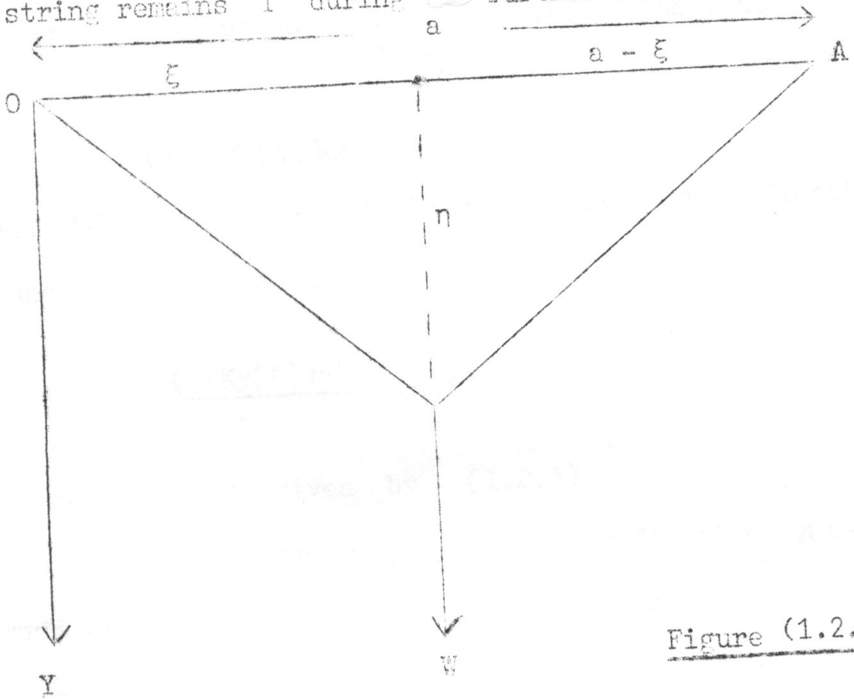


Figure (1.2.1)

For vertical equilibrium

$$T(\eta/\xi) + T\{\eta/(a-\xi)\} - W = 0 \qquad (1.2.1)$$

giving

$$\eta = \frac{W(a-\xi)\xi}{(Ta)} \qquad (1.2.2)$$

Combining this with the geometrical relation

$$y = \frac{x\eta}{\xi}, \quad 0 \leq x \leq \xi$$

$$= \frac{(a-x)\eta}{a-\xi}, \quad \xi \leq x \leq a \qquad (1.2.3)$$

The equation of the deflected string can be expressed in the form

$$y = \frac{W}{T} G(x,\xi) \qquad (1.2.4)$$

where $G(x,\xi) = \frac{x}{a}(a-\xi), \quad 0 \leq x \leq \xi$ (1.2.5)

$$\frac{\xi}{a}(a-x), \quad \xi \leq x \leq a$$

This function $G(x,\xi)$ called a Green's function, could now be used to set up an integral equation for the deflection $y(x)$ in the more complex and interesting case of a uniform heavy string resting on an elastic foundation which subjects it to an upthrust density proportional to the displacement at each point.

Let w be the linear density of the string and K the elastic contribution of the foundation. The load W 'at ξ' is given by

$$(w - Ky(\xi))d\xi \tag{1.2.6}$$

and its contribution $\delta y(x)$ to the overall deflection function $y(x)$ equals

$$\frac{(w-Ky(\xi))d\xi}{T} \, G(x,\xi) \tag{1.2.7}$$

where $G(x,\xi)$ is given by (1.2.5)

Assuming linearity, the overall deflection $y(x)$ can be written down by superposition as follows:

$$y(x) = \int_{\xi=0}^{a} \frac{w-Ky(\xi)}{T} \, G(x,\xi)d\xi \, . \tag{1.2.8}$$

Thus $y(x)$ is presented as a solution of the integral equation

$$y(x) + \frac{K}{T}\int_0^a G(x,\xi) \, y(\xi)d\xi = \frac{w}{T}\int_0^a G(x,\xi)d\xi \tag{1.2.9}$$

called a Fredholm integral equation of type 2.

This result, derived using heuristic arguments can be readily justified by general Green's Function Theory (Ref., [5], sec.3, p.277).

The numerical solution of this problem (a loaded elastic string) is discussed in chapter six.

1.3 A BRIEF HISTORICAL REVIEW OF INTEGRAL EQUATIONS

The first conscious direct use and solution of an integral equation go back to Abel. He published two papers, one in 1823 and the second in 1826 [Ref: [1], p.2] in which he considered a generalization of the so-called isochronous pendulum problem which reduces to

$$t(h) = \frac{1}{(2g)^{\frac{1}{2}}} \int_0^h \frac{s'(y)dy}{(h-y)^{\frac{1}{2}}} \tag{1.3.1}$$

where $t(h)$ denotes the time fall to the origin 0, from some point A, which is at a height h above 0, and $\mathcal{S} = s(y)$ is the intrinsic equation of the curve measured from 0. Abel solved the equation (1.3.1) to give

$$s(y) = \frac{(2g)^{\frac{1}{2}}}{\pi} \int_0^y \frac{t(x)}{(y-x)^{\frac{1}{2}}} dx \tag{1.3.2}$$

Abel also showed that the solution of the integral equation

$$f(x) = \int_a^x \frac{\phi(y)dy}{(x-y)^{\alpha}} \quad , \quad 0 < \alpha < 1 \tag{1.3.3}$$

is given by

$$\phi(y) = \frac{\sin\pi\alpha}{\pi} \cdot \frac{d}{dy} \left\{ \int_a^y \frac{f(x)dx}{(y-x)^{1-\alpha}} \right\} \tag{1.3.4}$$

Abel's achievement in solving (1.3.3) is quite remarkable.

After Abel, was Liouville, who, in 1837, considered the initial value problem

$$\phi''(x) + [e^2 - \sigma(x)]\,\phi(x) = 0, \tag{1.3.5}$$

with $\phi(a) = 1$, $\phi'(a) = 0$, and showed that it could be transformed into equations of the form

$$\phi(s) + \lambda \int_a^s K(s,t)\,\phi(t)\,dt = f(s) \tag{1.3.6}$$

which are now called Volterra integral equations of the second kind. Liouville solved his integral equation by the Method of successive approximation.

Vito Volterra (1860 - 1940) was the first to recognise the importance of the theory and to consider it systematically. He first met with an integral equation in 1884 when dealing with the problem of distribution of electric charges on a segment of the surface of a sphere. However, it was not until 1896 that he seriously studied them. To avoid some of the difficulties of existence and uniqueness of solutions, Volterra investigated the solution of both the first and second kind, given respectively by

$$\int_a^s K(s,t)\,\phi(t)\,dt = f(s) \qquad (1.3.7)$$

and

$$\phi(s) - \lambda\int_a^s K(s,t)\,\phi(t)\,dt = f(s);$$

in which the Kernel K satisfies the condition

$$K(s,t) \equiv 0, \quad \text{if} \quad t > s. \qquad (1.3.8)$$

This corresponds to the simple case of a system of algebraic Linear equations where the elements of the determinant above the main diagonal are all zero.

The integral equation (1.3.7) can be recast as an integral equation (1.3.6) so that it can easily be solved by Picard's process of successive approximation.

The next significant development is due to IVAR FREDHOLM(1866-19 who published two papers [Ref: [11], pp. 39-46; [12], pp.365-390] with particular reference to linear integral equations of the type

$$\int_D K(s,t)\,\phi(t)\,dt = f(s) \qquad (1.3.9)$$

and

(1.3.10)

where it was assumed that :

$K(s,t)$ was continuous on D^2;

λ is constant parameter;

the data function $f(s)$ was continuous on D;

$\phi(s)$ the unknown function was also continuous;

D being the fixed base domain.

Equations of this kind are now called <u>Fredholm integral equations</u> of the first and second type, respectively. He showed that any harmonic function ϕ that satisfies the Dirichlet problem also satisfies an equation of the form (1.3.10).

These papers by Fredholm placed the subject of integral equations for the first time on a firm mathematical footing and established once and for all useful criteria in regard to:

(a) existence of solutions ;

(b) uniqueness of solutions;

(c) construction of solutions.

In his subsequent researches on such equations, Fredholm was also led to the related equation, namely

$$\psi(s) + \mu \int_D \overline{K(t,s)}\, \psi(t)\, dt = g(s) \tag{1.3.11}$$

called the transposed equation. The associated homogeneous equations (1.3.10) and (1.3.11) are

$$\phi(s) + \int_D K(s,t)\, \phi(t)\, dt = 0 \tag{1.3.12}$$

$$\psi(s) + \mu \int_D \overline{K(t,s)}\, \psi(t)\, dt = 0 \tag{1.3.13}$$

Fredholm, following an observation of Volterra several years earlier, showed that an equation of the type (1.3.10) could, by suitable discretization, be regarded as a limiting form of a system of linear equations, and tentatively carried out this limiting process and subsequently justified the passage to the limit directly.

The theory and methods of solutions of the Fredholm equations are discussed in Chapters 2, 3 and 5.

Followed by the epoch-making discovery of Fredholm, Hilbert wrote a series of six papers in the period from 1904 to 1910. He considered integral equations with real, symmetric kernels $(K(s,t) = K(t,s),\ K$ real) (see Chapter 4), and the theory of orthogonal transformations. He showed that if λ is not a characteristic value of the real, symmetric kernel, then the solution of

$$f(s) + \lambda \int_a^b K(s,t)\,\phi(t)\,dt = \phi(s)$$

is given by

$$\phi(s) = f(s) + \lambda \sum_{k=1}^{\infty} \frac{(\phi_k, f)}{\lambda_k - \lambda}\,\phi_k(s) \qquad (1.3.14)$$

the series converging absolutely and uniformly for all $s \in [a,b]$, when K is continuous. In the second half of his life, Hilbert applied integral equations to both kinetic theory of gases and to the theory of radiation.

At the sametime Schmidt, Reisz, Banach & Frechet investigated the very same problem in more general terms. They introduced new concepts like the normed complete function spaces of Banach and Hilbert; linear and compact operators in these spaces. These led directly to the development of Functional Analysis in mathematics in its present form.

So far only non-singular integral equations have been considered. However, singular integral equations of the form

$$a(s)\phi(s) + \frac{b(s)}{\pi} \int_D \frac{\phi(t)}{t-s}\,dt + \lambda \int_D K(s,t)\phi(t)\,dt = f(s) \qquad (1.3.15)$$

$$s \in D$$

are also of particular importance, even though they are not discussed in this thesis. The foundations of the theory of these equations were laid down almost simultaneously but independently, by Hilbert and Poincaré in the first decade of this century. However, it was not until 1921 that Noether publihed her theorems for singular equations. A year later Carleman gave his solution to (1.3.16) in the particular case when b is a constant, $K \equiv 0$, and $D \equiv [-1,1]$. For the next two or three decades, progress in the theory was due entirely to Russian Mathematicians and mention must be made of the contribution of Muskhelishvili, whose interest in these equations was aroused by his work in the theory of elasticity.

CHAPTER TWO

THE THEORY OF FREDHOLM EQUATIONS AND FREDHOLM'S THEOREMS

2.1 INTRODUCTION

In a vast majority of cases solutions of integral equations are executed by recourse to iterative computational procedures. Such procedures demand a certainity of:

(a) the existence of a solution;

(b) the uniqueness of a solution; and

(c) the convergence of the iterative procedure used.

Important significance, therefore, attaches to an analysis of the integral equation prior to its solution. This analysis can in every case be conducted using general theorems established by Fredholm for the particular class of linear integral equation

$$\phi(s)-\lambda \int_D K(s,t)\phi(t)dt \;=\; f(s) \qquad (2.1.1)$$

called Fredholm integral equations of the second kind, for Kernels K, data functions f and solution functions ϕ in prescribed function spaces.

A fairly complete statement of the possibilities in regard to the solution of (2.1.1) in a prescribed function space for wide categories of kernels K and data function f is embodied in what is known as the Fredholm Alternatives theorem. This particular theorem is a complete analogue of a corresponding theorem relating to the solution of a system of Linear Algebraic equations in a given finite dimensional vector space.

The existence of such an analogue for integral equations is not altogether surprising, because, there is a special kind of of kernel K_n, of the form $\sum_1^n \alpha_i(s)\beta_i(t)$, $a\le s,t\le b$ (or $(s,t)\in D^2$ where $D\equiv[a,b]$) called degenerate for which the integral equation (2.1.1) can be easily shown to be exactly equivalent, in system of equations of the type

$$x_i - \lambda \sum_{j=1}^{n} a_{ij} x_j = f_i , \; i = 1,2,\ldots,n. \qquad (2.1.2)$$

Moreover, non-degenerate kernels K in important special spaces (e.g. $K \varepsilon L_2[D^2]$ or $K \varepsilon C [D^2]$, etc) can be uniformly approximated by sequences of degenerate kernels in the same space. It can be expected, therefore, that the solution alternatives that carry over from algebraic systems to integral equations with degenerate kernels, can be extended by a suitable limiting procedure to the case of integral equations with certain fairly wide classes of non-degenerate kernels.

The limiting procedures referred to, call for careful execution, and appeal to well known standard theorems in Analysis. In all cases the proof of the Fredholm alternatives is conducted in two main stages. In stage (1), an approximating sequence (K_n) of degenerate kernels that converges uniformly to K in the given function space is introduced, and a corresponding sequence of approximating integral equations is set up. In stage (2), standard justifiable limiting procedures are used to extend the algebraic equation solution alternatives for the sequence of degenerate approximating integral equations to the original limiting equation

For L_2 - kernels (i.e. kernels $K:D^2 \to C$) with $||K|| = \iint_{D^2} |K(s,t)|^2 ds dt$, where integrals exist, at least, in the Lebesgue sense, stage (1) is implemented on the following lines. Approximating degenerate kernels K_n are available such that $||K - K_n||$ can be made $<$ any $\varepsilon(>0)$, $\forall$ $s,t \varepsilon D^2$ with a choice of n sufficiently large, by a well known property of Hilbert Spaces. K is partitioned by the relation

$$K = K_n + (K - K_n) \qquad (2.1.3)$$

Introducing the operator notation

then integral equation (2.1.1):

$$\phi(s) - \lambda \int_D K(s,t)\, \phi(t)\, dt = f(s)$$

has the equivalent operator form

$$\phi - \lambda K \circ \phi = f.$$

With (2.1.3), this may be re-written as

$$\phi - \lambda[K_n + (K - K_n)]\circ\phi = f,$$

which is rearranged as

$$\phi - \lambda(K - K_n) \circ \phi = (f + \lambda K_n \circ \phi) = g. \qquad (2.1.4)$$

Use may now be made of the fact that for $|\lambda| < \frac{1}{||K-K_n||}$ equations (2.1.4) has unique solutions of the form

$$\phi = g + \lambda(R \circ g); \text{ where } (g = f + \lambda K_n \circ \phi), \qquad (2.1.5)$$

where R is a 'resolvent kernel' available explicitly. Rearrangement of (2.1.5) leads to an equivalent integral equation for ϕ with degenerate kernels. This implements stage (1), and prepares the problem for stage (2) in which the alternatives for degenerate kernels are extended by admissible limit procedures.

This is basically the method used by Mikhlin (Ref: [19], Sec. 8, pp34 - 46; Sec. 10, pp59-66), Tricomi (Ref: [32], Sec. 2.3, pp55-64, Sec. 2.4, pp64-66), and others to establish the Fredholm theorems for L_2-kernels and $f, \phi \in L_2[D]$, where $L_2[D]$ is the Hilbert Space of square Lebesgue integrable functions f and g,

ie. $\int_D |f(s)|^2 ds = ||f||^2 \equiv$ finite, and $(f,g) = \int_D f(t)\, \overline{g(t)}\, dt$

with integrals, at least, in the Lebesgue sense.

In this chapter it is proposed to treat the very special case (though not unduly restricted) of continuous real valued kernels K and continuous f, ϕ with

$$(f, g) = \int_D f(t)\, g(t)\, dt.$$

This choice is dictated by the fact that,

(a) the proofs can easily be extended to complex valued continuous kernels if so desired;

(b) the proof does not depend on previous results such as the existence of solutions for sufficiently "small" kernels K, and gains in elegence from this;

(c) the proof uses concepts and theorems closely allied to the theory of compact operators in which Fredholm integral operators play a special role.

2.2 THE SOLUTION ALTERNATIVES FOR LINEAR SYSTEMS OF ALGEBRAIC EQUATIONS.

Let $A = (a_{ij})_{n \times n}$, $a_{ij} \in C$ (or R),

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, \quad b_i \in C \text{ (or } R),$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad x_i \in C \text{ (or } R),$$

$\lambda \in C$ (or R); $I = (\delta_{ij})_{n \times n}$.

Then the system of linear equations

$$(I - \lambda A)x = b \quad \text{or } x_i - \lambda \sum_{j=1}^{n} a_{ij} x_j = b_i, \quad (i = 1,2,\ldots,n) \tag{2.2.1}$$

admits the following alternatives.

(A) The det$(I - \lambda A) \neq 0$. In this case $(I - \lambda A)^{-1}$ exists and (2.2.1) has a unique solution, (given explicitly by Cramer's rule) $x = (I - \lambda A)^{-1} b$, for every choice $b \in C^n$; in particular, the trivial solution $x = 0$ for the choice $b = 0$.

(B) The $\det(I - \lambda A) = 0$. In this case $(I - \lambda A)^{-1}$ does not exist and

$$(I - \lambda A)x = 0 \tag{2.2.2}$$

admits a maximal finite number r of linearly independent non-zero solutions $x_1, x_2, \ldots, x_r$ called eigen vectors (which may, if necessary, be presented as an orthormal set), *with associated values λ.*

(C) In the case (A), the transposed system

$$(I - \bar{\lambda}A^*)y = C, A^* = (a^*_{ij} = \overline{a_{ji}}) \tag{2.2.3}$$

yields a unique solution for any choice $C \in C^n$ and the transposed homogeneous system

$$(I - \bar{\lambda}A^*)y = 0 \tag{2.2.4}$$

is trivial (i.e. admits the unique solution $y = 0$).

(D) In the case (B), the homogeneous transposed equation (2.2.4) also has r linearly independent solutions $y_1, y_2, \ldots, y_r$ and equation (2.2.1) admits a solution <u>iff</u> $(b, y_i) = 0$, i.e. b must be orthogonal to every <u>non-trivial solution</u> (eigen function) $y_1, y_2, \ldots, y_r$ of equation (2.2..4) for eigen value $\bar{\lambda}$ (the values of $\lambda_i \in C$ for which equation (2.2.1) admit non-trivial solutions) If equation (2.2.1) is consistent for eigen value λ, with eigen vectors $(x_1, x_2, \ldots, x_r)$ and $x = x_0$ is a solution, then a general solution is $x = x_0 + \lambda_1 x_1 + \lambda_2 x_2 +, \ldots\ldots + \lambda_r x_r$ where λ_i are free constants, which may be fixed if necessary by the added orthogonality condition $(x, x_i) = 0, i = 1, 2, \ldots, r$.

This standard result in Linear algebra will be assumed. Proofs of this may be found in SMIRNOV, Vol. III, (Ref: [29] Sec. 2, pp 32 - 62).

2.3 THE SOLUTION ALTERNATIVES FOR FREDHOLM'S INTEGRAL EQUATION (2.1.1): $\phi - \lambda K o \phi = f$ OF TYPE (2).

The integral equation (2.1.1);

$$\phi(s) - \lambda \int_D K(s,t)\, \phi(t)dt = f(s)$$

where $D \equiv \{s,t/s,t \in [a,b]\}$; $f,\phi \in C[D]$ ($\equiv$ the space of continuous real valued functions on D^2); and with fixed $\lambda \in C$;

$(A)_1$ has a unique solution $\phi \in [D]$ for each choice $f \in C[D]$, in particular, the trivial solution $\phi = 0$ for the choice $f = 0$. In this case the parameter λ is called regular.

$(B)_1$ The associated homogeneous equation

$$\phi(s) - \lambda \int_D K(s,t)\, \phi(t)\, dt = 0, \text{ for } \lambda = \lambda_0 \qquad (2.3.1)$$

possesses a finite positive number r of linearly independent solutions $\phi_1, \phi_2, ----, \phi_r$ (which may be assumed orthonormalized by Gram-Schmidt procedure), λ_0 is called an eigen value of multiplicity r of K with associated eigen functions $\phi_1, \phi_2, ---, \phi_r$.

$(C)_1$ In the first case (A), the transposed integral equation $\Psi(s) - \bar{\lambda} \int_D K(t,s)\, \Psi(t)dt = g(s)$ associated with (2.1.1), also possesses a unique solution for every $g \in C[D]$.

In the second case $(B)_1$, the transposed homogeneous equation

$$\Psi(s) - \lambda_0 \int_D K(t,s)\, \Psi(t)dt = 0 \qquad (2.3.3)$$

also has exactly r linearly independent solutions $\Psi_1, \Psi_2, ----, \Psi_r$ (which also may be assumed orthonormal).

$(D)_1$ If in case $(B)_1$ where equation (2.1.1) has for $\lambda = \lambda_0$ eigen functions $\phi_1, \phi_2, -----, \phi_r$ and (2.3.2) has for λ_0 associated eigen functions $\Psi_1, \Psi_2, ----, \Psi_r$, the inhomogeneous equation (2.1.1) is consistent and therefore, admits solutions iff the data function f satisfies the orthogonality condition $(f, \Psi_i) = \int_D f(s)\, \Psi_i(s)ds = 0$, $(i = 1,2,----, r)$. In this case if ϕ_o is a solution of (2.1.1) the general solution of (2.1.1) is

$$\phi_o + C_1\,\phi_1 + C_2\,\phi_2 + ---- + C_r\,\phi_r\,.$$

This can be made unique by imposition of the r orthogonality conditions $(\phi, \phi_i) = 0$.

PRELIMINARY SPECIAL RESULTS

For the special case of continuous real valued kernel K and continuous real valued functions f and ϕ, with $(f,g) = \int_D f(t)\, g(t)dt$. The proof given briefly below employs the following standard results in real analysis.

DEFINITION 1.

$F = \{f : D \to R\}$ is said to be a uniformly bounded functi[on] set iff, there exists a fixed constant $M > 0$, such that

$$|f(s)| < M. \text{ for all } f \in F, \text{ for all } s \in D \qquad (2.3.4)$$

DEFINITION 2.

The functión set F is said to be equicontinuous on D iff, given any ε, there exists $\delta(\varepsilon)$ such that for every function $f \in F$, and all $s \in D$, $|f(s_2) - f(s_1)| < \varepsilon$, for $|s_2 - s_1| < \delta(\varepsilon)$ (2.3.5)

ARZELA'S THEOREM (The Principle of Accumulation for f)

Let F be (i) a uniformly bounded function set on D

(ii) equicontinuous on D

Then, $\exists$ a sequence $f_1, f_2, \text{-----} = (f_n)$ in F which converges uniformly to a function f on D. (2.3.6)

A STANDARD APPLICATION

Let (a) (K_n) be a kernel sequence in $C[D^2]$ that converges uniformly to K on D^2

(b) h be any member of the space H of piece wise continuous functions $D \to R$ with uniformly bounded norm.

i.e. $||h||^2 = \int_D |h(t)|^2\, dt <$ fixed M.

Then the function sequence

$$\mathcal{G} = \{g_n : D \to R / g_n(s) = \int_D K_n(s,t)\, h(t)dt, \quad h \in H,$$

is both uniformly bounded and equicontinuous, and by (2.3.6) contain a sub-sequence (g_n) that tends uniformly to a continuous function $g \in C[D]$. (2.3.7)

The uniform boundedness and equicontinuity of these function sequences can be established by confirming the definitions using the Schwarz inequality for integrals.

NOTE.

In the course of the proof given below, the result (2.3.7) will be frequently applied. The sub-sequence $(g_{n1}, g_{n2}, ----)$ in this result will be denoted by (g_n) with the understanding that all sequences will be modified to retain only $n = n_1, n_2, ---$. The proof being recast accordingly will be described by the single term re-indexing.

WEIRSTRASS THEOREM

$K \in C[D^2] \Rightarrow \exists$ a sequence of degenerate kernels (K_n) in $C[D^2]$ which converges uniformly to K on D^2 (2.3.8).

PROOF.

STAGE (1). The Algebraic Equivalent for degenerate kernels

Given
$$\phi - \lambda K_n \circ \phi = f, \quad \text{where } K_n = \sum_{1}^{n} \alpha_i(s)\beta_i(t) \qquad (2.3.9)$$

and, α_i and β_i are linearly independent, $f, \phi \in C[D]$, $K_n \in C[D^2]$,

$$\phi(s) - \lambda \int_D \left(\sum_{1}^{n} \alpha_i(s)\, \beta_i(t)\, \phi(t)\, dt\right) = \phi(s) - \lambda \sum_{1}^{n}\left(\int_D \beta_i(t)\phi(t)dt\right)\alpha_i($$

$$= f(s)$$

giving
$$\phi(s) - \lambda \sum_{j=1}^{n} x_j \alpha_j(s) = f(s) \qquad (2.3.10)$$

where $x_j = (\phi, \beta_j)$, $j = 1,2,---,n$.

Taking inner products of (2.3.10) in succession with β_i,

$$x_i - \lambda \sum_{j=1}^{n} \alpha_{ij}\, x_j = f_i, \qquad (2.3.11)$$

where $f_i = (f, \beta_i)$, $\alpha_{ij} = (\beta_i, \alpha_j)$

(2.3.10) => (2.3.11).

Again given (2.3.11), with

$$\phi(s) = f(s) + \lambda \sum_{1}^{n} x_j \, \alpha_j(s),$$

it can easily be shown that

$$\sum_{1}^{n} \alpha_i(s) \, [\phi(s) - \lambda K o \phi - f] = 0$$

$$\Rightarrow \quad \phi(s) - \lambda K o \, \phi(s) = f(s), \text{ by linear independence of } \{ \alpha_i \},$$

so that (2.3.11) => (2.3.10)

$\therefore$ (2.3.10) <=> (2.3.11).

The Fredholm's theorems $(A)_1, (B)_1 (C)_1$ and $(D)_1$, now follow immediately for K_n, from their algebraic counterparts A, B, C and D as can easily be verified.

<u>STAGE 2.</u> <u>The Approximating degenerate Kernel Sequence K_n for any $K \in C[D^2]$ and the approximate integral equation sequence for (2.1.1)</u>

<u>Case 1</u> λ is NOT an eigen value (or λ is regular) for an infinite subsequence of (K_n) which is selected as our new approximating sequence (K_n) for $K(s,t)$. (2.3.9) delivers a unique solution $\rho_n(s)$, for every choice $f \in C[D]$ ($\rho_n = 0$, $f = 0$). Two cases arise here.

<u>Case 1.1.</u> Given $K(s,t) \in C[D^2]$, $\exists$ a uniformly approximating kernel sequence (K_n) in $C[D^2]$

(i) that uniformly converges to K on D^2

(ii) so that for all $f \in C[D]$, equation (2.3.9) has a <u>unique</u> solution $\rho_n(s)$ in $C[D]$ (c.f. 2.3.7).

$\| \quad \|^2 \leq M$ (fixed & same $\forall n, \forall f$).

STEP 1

For each $f \in C[D]$, $\rho_n \in H$, $||h||^2 < M$

$\Rightarrow \{ \lambda \int_D K_n(s,t)\, \rho_n(t)dt \;/\; n \in N\}$ is uniformly bounded and equicontinuous on D.

$\Rightarrow \exists$ a subsequence $g_n(s) = \lambda \int_D K_n(s,t)\rho_n(t)dt$ which converges uniformly to $g(s)$.

STEP 2

Re-indexing $g_n(s)$ and representing the corresponding kernel sub-sequence by (K_n), then $\exists$ a sequence of kernels (K_n) so that for each $f \in C[D]$, $g_n(s) = \lambda \int_D K_n(s,t)\rho_n(t)dt$ converges uniformly to $g(s)$ in $C[D]$, by (2.3.7), where

$\rho_n(s) = f(s) - \lambda \int_D K_n(s,t)\, \rho_n(t)dt$ converges uniformly to

$$f(s) - y(s) = \phi(s) \tag{2.3.12}$$

By uniform convergence of $(K_n(s,t).\, \rho_n(t))$ to $K(s,t)\,.\,\rho(t)$ in D and taking limits of (2.3.12), as $n \to \infty$

$$\phi(s) = f(s) - \lambda \int_D K(s,t)\, \phi(t)dt \text{ ---------- } (2.1.1)$$

CONCLUSION (1)

For each $f \in C[D]$, $\exists$ a solution $\phi \in C[D]$.

CONCLUSION (2)

For $f = 0$, $\rho_n = 0$, each K_n and $\rho_n(s) \to \phi(s)$ $\therefore \phi = 0$

CONCLUSION (3)

For any $f \in C[D]$, $f \neq 0$, let $\rho_n(s) \to \phi_1(s)$ for one solution sub-sequence ρ_n and $\rho_n(s) \to \phi_2(s)$ for another.

$\Rightarrow \phi_1(s) + \lambda \int_D K(s,t)\, \phi_1(t)dt = f(s)$

$\phi_2(s) + \lambda \int_D K(s,t)\, \phi_2(t)dt = f(s)$

$\therefore (\phi_1 - \phi_2)(s) + \lambda \int_D K(s,t)(\phi_1 - \phi_2)t = 0 \Rightarrow \phi_1 = \phi_2$

$\therefore$ <u>In case 1.1</u>, for regular vaues of λ, $\exists$ a unique solution for every choice <u>$f \in C[D]$</u>, in particular $\phi = 0$ for $f = 0$.

<u>Case 1.2</u> $\exists$ at least one $f \in C[D]$ so that $||\rho_n|| \to \infty$ as $n \to \infty$ where $\phi = \rho_n$ satisfies (2.3.9) uniquely.

Take $\delta_n = \dfrac{\rho_n}{||\rho_n||}$, $n \in N$ and dividing throughout by $||\rho_n||$ in (2.3.12), it follows that, $\exists$ a unique δ_n such that

$$\delta_n(s) - \frac{f(s)}{||\rho_n||} = \lambda \int_D K_n(s,t)\, \delta_n(t)\, dt \tag{2.3.13}$$

with $||\delta_n||^2 = 1, \forall_n$, i.e. $\delta_n \in H$($\equiv$ space of piecewise continuous functions).

Since $g_n(s) = \lambda \int_D K_n(s,t)\delta_n(t)dt$ is uniformly bounded and equicontinuous on D, by (2.3.7)

$\exists$ a subsequence g_n that tends uniformly to $g(s) \in C[D]$, by (2.3.12). Taking limits of (2.3.13), as $n \to \infty$

$\exists\, g(s) \in C[D]$ with $||g|| = 1$ such that $g(s) = \lambda\int_D K(s,t)g(t)dt$.

Thus <u>in case 1.2</u>, the homogeneous equation (2.3.1), admits at least one non-trivial solution $\phi(s) = g(s)$.

<u>Case 2</u>

λ is an eigen value for all but a finite number of (K_n). Discarding this finite set and re-indexing (K_n). Then every $\phi(s) - \lambda\int_D K_n(s,t)\,\phi(t)dt = 0$ admits at least one normalized solution, $\phi = \delta_n$, since $||\delta_n||^2 = 1$

$$\delta_n(s) = \lambda \int_D K_n(s,t)\, \delta_n(t)\, dt \tag{2.3.14}$$

is uniformly bounded and equicontinuous on D, by (2.3.7).

As before, $\exists$ a sub-sequence (δ_n) converging uniformly to $C[D]$.

Also $||\phi||^2 = \lim_{n\to\infty} ||\delta_n||^2 = 1$

Taking limits of (2.3.14), as $n \to \infty$

$$\phi(s) = \lambda \int_D K(s,t)\phi(t)dt \quad \text{where} \quad ||\phi|| = 1.$$

Hence (2.3.1) admits normalized solutions.

FURTHER RESULTS

Suppose $\phi(s) = \lambda \int_D K(s,t)\phi(t)dt$ has exactly r linearly independent (orthonormalized) solutions $\phi_1, \phi_2, --- \phi_r$.

Let $(K_n) \equiv$ a degenerate kernel sequence converging uniformly to K.

STEP 1

With each $n \in N$, associate the set of r functions

$$\delta_{ni}(s) = \phi_i(s) - \lambda\int_D K_n(s,t)\phi_i(t)dt, (i=1,2,---, r) \tag{2.3.15}$$

and take

$$K_n^1(s,t) = K_n(s,t) + \frac{1}{\lambda}\sum_1^r \delta_{ni}(s)\phi_i(t) \tag{2.3.16}$$

Now,

$$\delta_{ni}(s) = \phi_i(s) - \lambda \int_D K_n(s,t)\phi_i(t)dt$$

$$\to \phi_i(s) - \lambda \int_D K(s,t)\, \phi_i(t)dt$$

$$= \phi_i(s) - \phi_i(s) = 0.$$

$\therefore$ $\delta_{ni}(s)$ converges uniformly to 0 on D, $\therefore$ $K_n^1(s,t)$ converges uniformly to $K(s,t)$

CONCLUSION

$K_n^1(s,t)$ given by (2.3.16) is a degenerate kernel sequence that converges uniformly to $K(s,t)$ on D^2

STEP 2

$$\lambda \int_D K_n^1(s,t)\phi_i(t)dt = \lambda \int_D K_n(s,t)\phi_i(t)dt$$

$$+ \lambda \times \frac{1}{\lambda} \sum_1^r \delta_{ni}(s) \int_D \phi_i^2(t)dt$$

$$= \phi_i(s) .$$

$\therefore$ Each $K_n^1(s,t)$ defined by (2.3.16) has eigen value λ and eigen functions $\phi_1, \phi_2, ---, \phi_r$, same as $K(s,t)$.

STEP 3

Suppose there is an infinite sub-sequence of K_n^1 with an additional linearly independent eigen function $(\phi_{n,r+1})$.

Re-indexing (K_n^1) and asserting that the new sequence (K_n^1) admits eigen functions $\phi_1, \phi_2, ----, \phi_r, \phi_{n,r+1}$ associated with eigen value λ. Now (K_n^1) converges uniformly to K on D^2, $||\phi_{n,r+1}||^2 = 1$ so that $\phi_{n,r+1} \in H$,

$$\phi_{n,r+1}(s) = \lambda \int_D K_n^1(s,t)\phi_{n,r+1}(t)dt \qquad (2,3.17)$$

which is a uniformly bounded and equicontinuous function set.

$\therefore$ $\exists$ a sub-sequence of $(\phi_{n,r+1})$ that defines a function $\phi_{r+1}(s)$ in C[D] to which it converges uniformly on D as before

Taking limits of (2.3.17), as $n \to \infty$

$$\phi_{r+1}(s) = \lambda \int_D K(s,t)\phi_{r+1}(t)dt$$

This gives an eigen function ϕ_{r+1} of K, where $||\phi_{r+1}|| = 1$ and $(\phi_{r+1}, \phi_i) = 0$, and $(r+1)$ linearly independent eigen functions $\phi_1, \phi_2, ----, \phi_r, \phi_{r+1}$, for K associated with λ which contradicts the initial premise.

CONCLUSION

$\exists$ an infinite sub-sequence of (K_n^1) with eigen functions besides $\phi_1, \phi_2, \ldots, \phi_r$.

STEP 4

For $K_n^1(t,s)$ which converges uniformly to $K(t,s)$, any new $K_n^1(t,s)$ has exactly r eigen functions $\Psi_{n1}, \Psi_{n2}, \ldots, \Psi_{nr}$.

$\Psi_{ni}(s) = \lambda\int_D K_n^1(t,s)\,\Psi_{ni}(t)dt$ is uniformly bounded and equicontinuous on D, and as $n \to \infty$ the same argument as that used earlier gives a function ψ_i so that

$$\psi_i(s) = \lambda\int_D K(t,s)\,\Psi_i(t)dt, \quad \text{for each } i.$$

STEP 5

If $K(t,s)$ has $(r+1)$ orthonormal eigen functions for λ, then step 4 $\Rightarrow$ $K(s,t)$ has $(r+1)$ eigen functions for λ which contradicts the initial premise.

Thus $K(t,s)$ has exactly r eigen functions $\Psi_1, \Psi_2, \ldots, \Psi_r$ with λ.

Now, suppose (2.1.1)

$$\phi(s) - \lambda\int_D K(s,t)\,\phi(t)dt = f(s)$$

has a solution, λ being an eigen value with multiplicity r and eigen functions $\phi_1, \phi_2, \ldots, \phi_r$.

Then, by Step 5, $\exists\ \Psi_1, \Psi_2, \ldots, \Psi_r$ so that

$$\Psi_i(s) = \lambda\int_D K(t,s)\Psi_i(t)dt, \quad i = 1,2,\ldots,r$$

$$\Rightarrow \quad (f,\Psi_i) = (\phi,\Psi_i) - \lambda\int_D ds\left(\int_D K(s,t)\phi(t)dt\right) \times \Psi_i(s)$$

which reduces to

$$(f,\Psi_i) = (\phi,\Psi_i) - \int_D \phi(t)\,\Psi_i(t)dt = (\phi,\Psi_i) - (\phi,\Psi_i) = 0$$

By Step 3,

$K_n^1(t,s)$ converges uniformly to $K(t,s)$ in D^2

$K_n^1(t,s)$ has exactly r orthonormal eigen functions

$\{\Psi_{1,n}, \Psi_{2,n}, \text{-----}, \Psi_{r,n}\}$, with λ, and

$\Psi_{i,n}(s) = \lambda \int_D K_n^1(t,s)\Psi_{i,n}(t)dt, \; ||\Psi_{i,n}|| = 1$

is uniformly bounded and equicontinuous sequence in $C[D]$

which defines exactly r (orthonormal) functions

$\Psi_1, \Psi_2, \text{-----}, \Psi_r$ in $C[D]$.

Now, for each $i = 1,2,\text{-----}, r$

$\lim_{n\to\infty} (f,\Psi_{i,n}) \; (= \varepsilon_{in}) = (f,\Psi_i) = 0.$

Introduce

$$f_n(s) = f(s) - \sum_{i=1}^{r} \varepsilon_{in} \Psi_{in}(s) \text{ which } \to f(s), \text{ as } n\to\infty$$

Then for each $n \notin i$,

$$(f_n, \Psi_{in}) = 0$$

$\therefore$ By degenerate Fredholm theory,

$f_n(s) = \phi(s) - \lambda \int_D K_n^1(s,t)\phi(t)dt$

has a unique solution $\phi(s) = \rho_n(s)$ given the orthogonality condition $(\rho_n,\phi_i) = 0.$

Thus we have the situation of case 1, in which case

$\rho_n(s) \to \rho(s)$ where $(\rho,\phi_i) = 0$, and

$f(s) = \lim_{n\to\infty} f_n(s) = \phi(s) - \lambda\int_D K(s,t)\,\phi(t)dt,$

where $||\rho_n|| \to \infty$, as $n\to\infty$ as in case (1.2).

(2.1.1) admits a normalized solution $\rho(s)$, where

$(\rho, \phi_i) = 0, \; i = 1,2,\text{-----}, r.$

This would demand the existence of an $(r+1)^{th}$ eigen function ρ in addition to $\phi_1, \phi_2, \text{-----}, \phi_r$ which

CONCLUSION.

$$(f,\Psi_i) = 0, \; i = 1,2,\text{-----}, r,$$

=> (2.1.1) has a unique solution $\phi = \rho$ which is orthogonal to $\phi_1, \phi_2, \text{-----}, \phi_r$.

This completes the Fredholm theorems for all $K(s,t) \in C[D^2]$.

2.4 ILLUSTRATIVE EXAMPLE

Solve the integral equation

$$\phi(s) - \lambda \int_0^\pi [\cos^2 s \cos 2t + \cos 3 s \cos^3 t]\phi(t)dt = 0 \qquad (2.4.1)$$

Solution

The equation (2.4.1) becomes

$$\phi(s) = \lambda \cos^2 s \int_0^\pi \cos 2t \; \phi(t)dt + \lambda \cos 3s \int_0^\pi \cos^3 t \; \phi(t)dt$$

$$= C \lambda \cos^2 s + C_2 \lambda \cos 3s \qquad (2.4.2)$$

where, $C_1 = \int_0^\pi \phi(t) \cos 2t \; dt$ (2.4.3)

$$C_2 = \int_0^\pi \phi(t) \cos^3 t \; dt$$

Substituting (2.4.2) into (2.4.3), and integrating,

$$(1 - \frac{\lambda\pi}{4})C_1 = 0 \qquad (2.4.4)$$

$$(1 - \frac{\lambda\pi}{8})C_2 = 0$$

The eigen values are $\lambda_1 = \frac{4}{\pi}$, $\lambda_2 = \frac{8}{\pi}$, which is finite by theorem $(A)_1$.

For $\lambda_1 = \frac{4}{\pi}$ the eigen function becomes $\phi_1(s) = \cos^2 s$;

and for $\lambda_2 = \frac{8}{\pi}$, the eigen function becomes $\phi_2(s) = \cos 3s$.

Thus by $(B)_1$, the eigen values $\lambda_1 = \frac{4}{\pi}$, $\lambda_2 = \frac{8}{\pi}$ have corresponding linearly independent eigen functions

$\phi_1(s) = \cos^2 s$; $\phi_2(s) = \cos 3s$.

The transposed equation of (2.4.1) is

$$\Psi(s) + \mu \int_0^{\pi} [\cos^2 t \cos 2s + \cos 3t \cos^3 s]\, \Psi(t)dt = 0 \qquad (2.4.5)$$

and solving this equation as before, the eigen values becomes $\mu_1 = \frac{4}{\pi}$, $\mu_2 = \frac{8}{\pi}$ and their linearly independent eigen functions $\Psi_1(s) = \cos 2s$; $\Psi_2(s) = \cos^3 s$ which confirms $(C)_1$.

For a particular $f(s) = \cos 3s$, equation (2.4.1) becomes

$$\phi(s) - \lambda \int_0^{\pi} [\cos^2 s \cos 2t + \cos 3s \cos^3 t]\phi(t)dt = \cos 3s \qquad (2.4.6)$$

and $\lambda_1 = \mu_1 = \frac{4}{\pi}$. Using $(D)_1$

$$(f(s), \Psi_1(s)) = \int_0^{\pi} \cos 3s \cos 2s \, ds = 0,$$

ie. $f(s)$ is orthogonal to $\Psi_1(s)$ and the equation (2.3.6) is consistent and $\therefore$ admits solutions.

CHAPTER THREE

NEUMANN SERIES, RESOLVENT KERNEL AND FREDHOLM DETERMINANT FORM OF THE RESOLVENT KERNEL

3.1 NEUMANN SERIES SOLUTION FOR FREDHOLM EQUATION (2.1.1) AND L_2 - KERNELS.

The Fredholm integral equation (2.1.1):

$$\phi(s) = f(s) + \lambda\int_a^b K(s,t)\,\phi(t)dt$$

may be solved by the method of successive approximations, in the following steps

$$\phi(s) = f(s) + \lambda\int_a^b K(s,t)\,\phi(t)dt$$

$$= f(s) + \lambda\int_a^b K(s,t)\,[f(t) + \lambda\int_a^b K(t,x)\phi(x)dx]\,dt$$

$$= f(s) + \lambda\int_a^b K(s,t)\,f(t)dt + \lambda^2\int_a^b K_2(s,t)\phi(t)\,dt$$

where $K_2(s,t) = \int_{x=a}^{b} K(s,x)K(x,t)dx.$

Proceeding formally, assuming the validity of change of order of integration at each step, we finally arrive at a solution in the form of the infinite series (called a Neumann Series) which may be expressed in the form

$$\phi(s) = f(s) + \lambda\int_a^b R(s,t;\lambda)\,f(t)\,dt$$

where

$$R(s,t;\lambda) = \sum_1^\infty \lambda^{(n-1)}\,K_n(s,t) \tag{3.1.1}$$

and the sequence of iterated Kernels (K_n) is defined by the recursive relation; $K_1(s,t) = K(s,t)$

$$K_2(s,t) = \int_a^b K(s,x)K(x,t)dx$$

$$K_n(s,t) = \int_a^b K(s,x)\, K_{n-1}(x,t)dx \tag{3.1.2}$$

$$= (K \circ K_{n-1})(s,t) = (K_{n-1} \circ K)(s,t)$$

$R(s,t;\lambda)$ is called the resolvent kernel of the equation (2.1.1).

The validity of this solution depends on the uniform convergence of (3.1.1). Using (3.1.2) in conjunction with the Schwarz inequality and Fubini's theorem with the assumption that $K \in L_2[D^2]$ and $f \in L_2[D]$. It is possible to show (Ref: [31], Sec. 2.1, pp49-51) that the series '$R(s,t;\lambda)f(t)$' has majorant

$$A(s)\ B(t)\ |\lambda| \sum_{n=0}^{\infty} (|\lambda|N)^n \tag{3.1.3}$$

where $A(s) = [\int_a^b K^2(s,t)dt]^{\frac{1}{2}}$, $B(t) = [\int_a^b K^2(s,t)ds]^{\frac{1}{2}}$

and $||K||^2 = \int_a^b \int_a^b K^2(s,t)ds\ dt \leq N^2$,

resulting in the theorem that follows.

<u>THEOREM</u> (3.1.4)

To each quadratically integrable kernel $K(s,t)$, there corresponds a resolvent kernel $R(s,t;\lambda)$ which is an analytic function of λ, regular at least inside the circle $|\lambda| < ||K||^{-1}$ and represented there by the power series (3.1.1). Let the maximal domain of existence of the resolvent kernel in the complex λ-plane be H. Then, if $f(x)$ belongs to the class L_2, the unique quadratically integrable solution of Fredholm's equation (2.1.1), valid in H, is given by the formula

$$\phi(s) = f(s) - \lambda \int_a^b R(s,t;\lambda)\, f(t)\, dt \tag{3.1.5}$$

(Ref: [31], Sec. 2.1, p 52).

The resolvent $R(s,t;\lambda)$ can be represented as a Neumann series only in the disc $|\lambda| < \frac{1}{||K||}$

3.2 THE PARTICULAR CASE OF THE VOLTERRA KERNEL

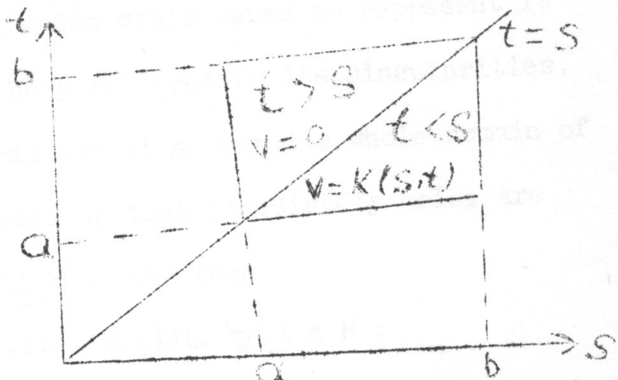
The Volterra Kernel $v(s,t)$ may be regarded as a kernel of type (3.1) with the added stipulation that

$$V(s,t) = 0 \ (t > s)$$

$$= K(s,t), \ (t < s);$$

$$(s,t) \in D^2 \equiv \begin{matrix} a < s < b \\ a < t < b \end{matrix}$$

where $K \in L_2[D^2]$.



In this case the Neumann series constructed on the same lines as in (3.1) is found to have majorant

$$A(s), B(t) \ |\lambda| \sum_{n=0}^{\infty} \frac{(N|\lambda|)^n}{\sqrt{n!}} \qquad (3.2.1)$$

where $A(s)$, $B(t)$, and N are defined as in (3.1.3) which converges uniformly on the whole complex λ-plane, since the power series $\sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}$ has an infinite radius of convergence.

Therefore, this series can be used to represent $R(s,t;\lambda)$, on the whole λ-plane.

This is the essential distinction of Volterra kernels as against Fredholm kernels.

3.3 THE FREDHOLM DETERMINANT FORM FOR $R(s,t;\lambda)$

The Neumann Series (3.1.1) for the equation (2.1.1), represents the resolvent $R(s,t;\lambda)$ as a regular function of λ, each $s,t \in D$ in the restricted domain $|\lambda| < \frac{1}{||K||}$. This function may theoretically be continued analytically into the whole λ-plane.

It would clearly be advantageous to have an explicit form for the function $R(s,t;\lambda)$ which would serve to represent it on the whole λ-plane, and help to identify its singularities. If $R(s,t;\lambda)$ represents this function over its whole domain of regularity H, then the questions that immediately arise are

(i) Will the solution still have the form

$$\phi(s) = f(s) + \lambda\int_H R(s,t;\lambda)\, f(t)dt, \quad \forall\, \lambda \in H;$$

(ii) Will $R(s,t;\lambda)$ satisfy the fundamental relation

$$R(s,t;\lambda)+K(s,t) = \lambda\int K(s,x)R(x,t;\lambda)dx \qquad (3.3.1)$$

$$= \lambda\int R(s,x;\lambda)K(x,t)dx$$

or $\quad R + K = \lambda K o H.$

The answers to both (i) and (ii) are in the affirmative. The affirmation of (ii) follows at once by Analytic Continuation and the affirmation of (i) results (c.f. Theorem (3.1.4).

A related question (iii) is: Can the singularities of the complete analytic function $R(s,t;\lambda)$ be identified and classified, and what solution can be expected when λ = one of these singularities.

One representation of the complete analytic function $R(s,t;\lambda)$ was given by Fredholm (Ref: [31], Sec.2.3, pp57-58). Using a discretization of the integral equation into a sequence of systems of linear algebraic equations and passing to the limit as the discretization was made infinitesmal.

An essential tool in the justification of the limiting procedure was Hadamard's theorem (Ref: [31], Appendix II, P223) for the bounds of a class of infinite determinants.

As might be expected the limiting form of Cramer's determinant solution for algebraic system takes the form of a ratio of two determinants in the Fredholm presentation, in which $R(s,t;\lambda)$ has the form

$$R(s,t;\lambda) = \frac{D(s,t;\lambda)}{D(\lambda)} \tag{3.3.2}$$

provided that $D(\lambda) \neq 0$. Here the limiting forms $D(s,t;\lambda)$ are power series in λ which have the forms

$$D(s,t;\lambda) = K(s,t;\lambda) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} B_n(s,t)\lambda^n \tag{3.3.3}$$

$$D(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} C_n \lambda^n \tag{3.3.4}$$

where the respective coefficients are given by the formulae

$$B_n(s,t) = \underbrace{\int_a^b \cdots \int_a^b}_{n\text{ - times}} \begin{vmatrix} K(s,t) & K(s,t_1) & \cdots & K(s,t_n) \\ K(t_1,t) & K(t_1,t_1) & \cdots & K(t_1,t_n) \\ K(t_2,t) & K(t_2,t_1) & \cdots & K(t_2,t_n) \\ \cdots & \cdots & \cdots & \cdots \\ K(t_n,t) & K(t_n,t_1) & \cdots & K(t_n,t_n) \end{vmatrix} dt_1 \cdots dt_n \tag{3.3.5}$$

and $B_o(s,t) = K(s,t)$

$$C_n = \underbrace{\int_a^b \cdots\cdots \int_a^b}_{n\text{ - times}} \begin{vmatrix} K(t_1,t_1) & K(t_1,t_2) & \cdots & K(t_1,t_n) \\ K(t_2,t_1) & K(t_2,t_2) & \cdots & K(t_2,t_n) \\ K(t_3,t_1) & K(t_3,t_1) & \cdots & K(t_3,t_n) \\ \cdots & \cdots & \cdots & \cdots \\ K(t_n,t_1) & K(t_n,t_2) & \cdots & K(t_n,y_n) \end{vmatrix} dt_1 \cdots dt_n \qquad (3.3.6)$$

The function $D(s,t;\lambda)$ is called the <u>Fredholm Minor</u> and $D(\lambda)$ the <u>Fredholm determinant</u>.

When the kernel $K(s,t)$ is bounded or $\int_a^b\int_a^b K^2(s,t)ds\,dt < \infty$, each of the series (3.3.3) and 3.3.4) converges for all values of λ. The resolvent kernel (3.1.1) is an analytic function of λ, except for those values of λ which are zeros of the function $D(\lambda)$. The latter are the singularities of the resolvent kernel $R(s,t;\lambda)$, and are isolated singularities of pole type.

Only in very rare cases is it possible to compute the coefficients $B_n(s,t)$ and C_n of the series (3.3.3) and (3.3.4) from the formulae (3.3.5) and (3.3.6). However, these formulae yield the recursion relations;

$$B_n(s,t) = C_n K(s,t) - n\int_a^b K(s,x)B_{n-1}(x,t)dx \qquad (3.3.7)$$

$$C_n = \int_a^b B_{n-1}(x,x)dx \qquad (3.3.8)$$

(Ref: [18], Sec. 13, pp 73 -76).

Knowing that the coefficients $C_o = 1$, $B_o(s,t) = K(s,t)$, formulae (3.3.7) and (3.3.8) can be used to find C_1, $B_1(s,t)$, C_2, $B_2(s,t)$, C_3, in successive steps.

While this recursive procedure may alleviate the computational difficulties in this formulation, the main computational problem remains in the general case.

CHAPTER FOUR

HILBERT - SCHMIDT THEORY FOR FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND WITH SYMMETRIC KERNELS.

4.1 INTRODUCTION

A vast number of problems in Physics and Engineering may be formulated as differential equations subject to the boundary conditions of Dirichlet or Neumann or mixed type.

In most such cases, the linear differential operator D of the problem is selfadjoint. In such cases a Green's function $K(s,t)$, necessarily symmetric in s and t, may be used to recast the problem in the form of an integral equation where the boundary conditions are built into the equation (c.f., Chap.1, problem on a loaded string). It is evident, therefore, that Fredholm equations with symmetric kernels call for special attention.

The theory of such integral equations has been developed in detail by HILBERT and SCHMIDT, and provides the most effective means of ensuring the existence and uniqueness of solutions of boundary value problems (of Sturm Louville type) in their differential form, and of constructing such solutions by numerical procedures.

4.2 DEFINITIONS

A complex valued kernel is called symmetric if it coincides with its own conjugate. i.e. $\overline{K(s,t)} = K(t,s)$, $(s,t) \varepsilon\ D^2$, $D \equiv [a, b]$. For real kernel K, $K(t,s) = K(s,t)$.

An integral equation with a symmetric kernel is called a symmetric integral equation.

If a Kernel is symmetric, then, as is easily verified, all

$$K_2(s,t) = \int_D K(s,x)K(x,t)dx = \int_D \overline{K(t,x)}\ \overline{K(x,s)}\ dx$$

$$= \overline{K_2(t,s)}$$

$$K_3(s,t) = \int_D \overline{K(t,x)}\ \overline{K_2(x,s)}\ dx = \overline{K_3(t,s)}\ .$$

and so on.

Examples

The Kernels $s + t$, $\ln|s-t|$, $i(s-t)$ are symmetric. The kernel $i(s+t)$ is not symmetric, since in this case $\overline{K(t,s)} = -\overline{K(s,t)}$.

The fundamental property of such kernels is the subject of the theorem stated and proved for continuous real symmetric complex Valued kernels of a more general class [REF: [19], Sec. 11 - 13, pp 67 - 81].

(4.3.1)

4.3 THEOREM (Existence of Eigen Values)

Let K be a real valued symmetric continuous kernel on $(s,t)\epsilon D^2$, where $D \equiv \{a \leq s,t \leq b\}$. Then an eigen value λ_1 and associated normalized eigen function ϕ_1 exist

i.e. $\phi_1(s) = \lambda_1 \int_D K(s,t)\phi_1(t)dt, \quad s \in D$

($\frac{1}{\lambda_1} = K_1$ shall be called the corresponding characteristic values of K as in Anderssen (Ref: [1], p. 11)).

This characteristic value K_1 is the maximum value of the integral form $J(\phi,\phi) = \iint_{D^2} K(s,t)\ \phi(s)\ \phi(t)dsdt$

in the sample space $\Sigma_1 = \{\phi \in C[D] \ / \ ||\phi|| = 1\}$, and is actually attained at the 'point' $\phi_1 \in \Sigma_1$ where

$$K_1\phi_1(s) = \int_D K(s,t)\ \phi_1(t)dt.$$

REMARK

This particular theorem is an analogue of the following well known result (assumed here) in algebraic quadratic form R^n namely:

If $\phi(x,x) = \sum_{i,j=1}^{n} C_{ij}x_i x_j$; $(C_{ji} = C_{ij})$ is

a quadratic form on R^n, then ϕ attains a maximum value $K_1^{(n)}$ at a point

$x^{(1)} = (x_1^{(1)}, x_2^{(1)}, \text{-----}, x_n^{(1)})$ in the sample space

$(\underline{x} \in R^n \,\big|\, ||x|| = 1)$, where

$$K_1^{(n)}\, \underline{x}^{(1)} = \sum_{i,j=1}^{n} C_{ij}\, x_j^{(1)} \tag{4.3.2}$$

PROOF

STEP 1 The case of real symmetric degenerate Kernels

A symmetric degenerate kernel $K_n(s,t) = \sum_{i=1}^{n} \alpha_i(s)\beta_i(t) = \sum_{i=1}^{n} \alpha_i(t)\beta_i(s)$ can always be recast in the form

$$K(s,t) = \sum_{i,j=1}^{n} C_{ij}\, w_i(s)\, w_j(t) \quad , \quad (C_{ij} = C_{ji})$$

and $\Omega = \{w_i \mid i = 1,2,\text{---}, N\}$ is an orthonormal finite function set in Σ_1 (Ref: [5] ,

In this form it is easy to see that the quadratic Form

$$J_n(\phi,\phi) = \iint_{D^2} K_n(s,t)\phi(s)\phi(t)\, ds\, dt, \ \phi \in \Sigma_1$$

is exactly equivalent to an algebraic quadratic form

$$Q_n(x,x) = \sum_{i,j=1}^{n} C_{ij}\, x_i\, x_j \ , \ \underline{x} = (x_i) \in R^n$$

the (1-1) correspondence between $\underline{x}$ and ϕ being given by

STEP 2 Extension to any real continuous symmetric kernel $K(s,t)$

Any such kernel K can be approximated uniformly on D^2 by a sequence (K_n) of symmetric degenerate kernels in the same space.

$$\text{Let } J_n(\phi,\phi) = \iint_{D^2} K_n(s,t)\, \phi(s)\, \phi(t)\, ds\, dt,$$

$$J(\phi,\phi) = \iint_{D^2} K(s,t)\, \phi(s)\, \phi(t)\, ds\, dt$$

By step 1,

$$\operatorname*{Max}_{\phi\varepsilon\Sigma_1} J_n(\phi,\phi) = K_1^{(n)} \text{ is attained at } \phi_1^{(n)} \varepsilon \Sigma_1$$

$$\text{where } K_1(n)\, \phi_1^{(n)} = \int_D K_n(s,t)\, \phi_1^{(n)}(t)\, dt.$$

Furthermore by Schwarz inequality

$$|J(\phi,\phi)|^2 \leq \iint_{D^2} |K(s,t)|^2\, dsdt \times |\int_D \phi(s)\, \phi(t)|^2\, dsdt$$

$$\leq \iint_{D^2} |K(s,t)|^2\, dsdt = B, \quad \phi\, \varepsilon\, \Sigma_1.$$

and $\{J(\phi,\phi) | \phi\, \varepsilon\, \Sigma_1\}$ is $\therefore$ a bounded set in R,

so that $\sup_{\phi\varepsilon\Sigma_1} J(\phi,\phi) = K_1$ exists.

We now proceed to show that the supremum is actually attained by $J(\phi,\phi)$ at some point $\phi_1\, \varepsilon\, \Sigma_1$. In fact,

$K_1 = \lim_{n\to\infty} K_1^{(n)}$ and $(\phi_1^{(n)})$ $(\phi\, \varepsilon\, \Sigma_1)$ tends uniformly to the required function ϕ_1 as is established below.

STEP 3.1

By Schwarz's inequality, any $\varepsilon > o$, $(s,t) \varepsilon D^2$,

and for each $n \varepsilon N$, $\phi \varepsilon \Sigma_1$

$$|J_n(\phi,\phi) - J(\phi,\phi)|^2 = |\iint_{D^2} [K_n(s,t) - K(s,t)]\phi(s)\,\phi(t)dsdt|^2$$

$$\leq (b-a)^2\,\varepsilon^2$$

$\therefore$ for each $\phi \varepsilon \Sigma_1$, any $\varepsilon > o$,

$$|J_n - J| \leq \varepsilon(b-a), \text{ for all } n > n_0$$

$=> \quad |\sup J_n - \sup J| \leq \varepsilon(b-a)$, for all $n > n_0$

$=> |K_1^{(n)} - K_1| \leq \varepsilon(b-a)$, since $\max J_n = K_1^{(n)}$; and $\max J = K_1$

$\therefore \quad K_1^{(n)} \to K_1$ as $n \to \infty$ (4.3.3)

Corollary

The set of real max $\{K_1^{(n)}\}$ is bounded, $\forall n \varepsilon N$.

STEP 3.2

Consider the set $\{\phi_1^{(n)} | n \varepsilon N\}$ which is known to have the property

$$K_1^{(n)}\,\phi_1^{(n)}(s) = \int_D K_n(s,t)\,\phi_1^{(n)}(t)\,dt \tag{4.3.4}$$

for each $n \varepsilon N$, $\phi_1^{(n)} \varepsilon \Sigma_1$.

Now $(K_n) \to$ uniformly to K in D^2.

$K_1^{(n)}\,\phi_1^{(n)}$ and $\therefore$ $(\phi_1^{(n)})$ is a uniformly bounded equicontinuous function set as was shown in Chapter 2.

By Arzela's theorem, there exists a sequence denoted by $(\phi_1^{(n)})$ after reindexing so that $(\phi_1^{(n)}) \to$ uniformly to a specific function ϕ_1 in $C[D]$. After reindexing we, therefore, have

$$(K_n) \to \text{uniformly to } K \text{ on } D^2$$

$$(\phi_1^{(n)}) \to \text{uniformly to } \phi_1 \text{ on } D$$

$$K_1^{(n)} \to K_1$$

By uniform convergence, it follows that,

$$J(\phi,\phi) = \lim_{n\to\infty} J_n(\phi_1^{(n)}, \phi_1^{(n)}) = \lim_{n\to\infty} K_1^{(n)} = K_1$$

<u>Also</u> $(\phi_1, \phi_1) = \lim_{n\to\infty} (\phi_1^{(n)}, \phi_1^{(n)}) = 1, \therefore \phi_1 \in \Sigma_1$

<u>Also</u> $K\phi_1 - \int_D K(s,t)\phi_1(t)dt = \lim [K_1^{(n)} \phi_1^{(n)} - \int_D K_n(s,t)\phi_1^{(n)}(t) dt] = 0$

Hence the theorem.

4.4 THE SEQUENCE OF MAXIMIZATIONS/MINIMIZATION PROBLEMS

Let (i) Σ_1 represent the function space $\{\phi \in C[D] / ||\phi|| = 1\}$

(ii) $J(\phi,\phi)$ assume positive values in Σ_1

Then the first maximization problem for the leading positive eigen values λ_1 or first positive characteristic value K_1 is the problem stated in the previous theorem which asserted that

$K(\phi,\phi)$ attained its first positive max $K_1 (= \frac{1}{\lambda_1})$

at $\phi_1 = \Sigma_1$, where

Similarly if $J(\phi,\phi)$ happens to take negative values in Σ_1, we have a corresponding first minimization problem for $J(\phi,\phi)$ which is equivalent to the maximization problem for $-J(\phi,\phi)$, whose solution by the theorem (4.3.1) delivers the first negative characteristic value $K_{-1}(= \frac{1}{\lambda_{-1}})$ and its associated eigen function ϕ_{-1}.

The sample space Σ_1 is now subjected to the additional constraint $(\phi,\phi_1) = 0$ to give a second sample space

$$\Sigma_2 = \{ \phi \in C[D] \; / \; ||\phi|| = 1, (\phi,\phi_1) = 0\} .$$

If $J(\phi,\phi)$ takes positive values (and / or negative values) in Σ_2, we formulate the second maximization/minimization problem namely; maximize $J(\phi,\phi)$ in Σ_2, where J attains positive values. This maximization problem is now shown to be equivalent to the following associated first problem.

Maximize $J_1(\phi,\phi) = \iint\limits_{D^2} K_1(s,t)\phi(s)\phi(t)\, dsdt$ in Σ_1

where $$K_1(s,t) = K(s,t) - \frac{\phi_1(s)\phi_1(t)}{\lambda_1} .$$

By theorem (4.3.1),

$J_1(\phi,\phi)$ attains a positive maximum $K_2(= \frac{1}{\lambda_2})$

at a point $\phi_2 \in \Sigma_1$ where

$$K_2 \phi_2 (s) = \int_D K_1(s,t)\phi_2(t)\, dt \qquad (4.4.2)$$

Note also the following:

(i) $(\phi_2,\phi_1) = 0$. This follows directly from (4.4.2) by using $K_1 = K - \frac{\phi_1(s)\,\phi_1(t)}{\lambda_1}$, so that the solution

$\phi_2 \in \Sigma_2 \subset \Sigma_1$

(ii) for $\phi \in \Sigma_2$, $J(\phi,\phi) = J_1(\phi,\phi)$ also is easily confirmed and, therefore, the maximum of J in Σ_2 (which is K_2) is attained at the point $\phi_2 \in \Sigma_2$, and furthermore

(iii) $K_2\,\phi_2(s) = \int_D K_1(s,t)\phi_2(t)dt = \int_D K(s,t)\phi_2(t)dt.$

This yields the second positive characteristic value $(K_2 = \frac{1}{\lambda_2})$ and second eigen function ϕ_2 for K so that $K_1 \geq K_2$.

Similarly, if J takes negative values we have the second **minimum** characteristic value K_{-2}, with associated eigen function ϕ_{-2}.

We next proceed to the sample space

$\Sigma_3 = \{\phi \in C[D] / ||\phi|| = 1, (\phi,\phi_1) = 0, (\phi_1,\phi_2) = 0\}$. If J takes positive values we maximize J in Σ_3 by introducing the equivalent maximization problem for J_2 with

$K_2 = K - \frac{\phi_1(s)\,\phi_1(t)}{\lambda_1} - \frac{\phi_2(s)\,\phi_2(t)}{\lambda_2}$ in Σ_1 and getting a solution (K_3,ϕ_3) in Σ_3, as before.

Step by step we generate a sequence of; positive eigen values $0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3$ ------, with orthonormal eigen functions ϕ_1, ϕ_2, ---- ; and negative eigen values $0 \geq \lambda_{-1} \geq \lambda_{-2}$----, with orthonormal eigen functions $\phi_{-1}, \phi_{-2}, \phi_{-3}$, ---- . We rearrange the solution system in each of magnitude to get a simple system of eigen values λ_1, λ_2,----, where $|\lambda_1| \leq |\lambda_2| \leq$----- and eigen functions ϕ_1, ϕ_2, -----

$$\Lambda : \lambda_1, \lambda_2, \lambda_3, \text{------}, \quad 0 \leq |\lambda_1| < |\lambda_2| < \text{-----}$$

$$\Phi : \phi_1, \phi_2, \phi_3, \text{------}$$

4.5 PROPERTIES OF THE SETS (Λ , Φ)

P(1) The eigen function set Φ is orthonormal on D

Proof

By construction, the eigen function sets $\Phi_+ = \{\phi_1, \phi_2, \text{-----}\}$, and $\Phi_- = \{\phi_{-1}, \phi_{-2}, \text{----}\}$ are separately orthonormal. It remains to show that $\phi_n \in \Phi_+$ is orthogonal to $\Phi_{-m} \in \Phi_-$.

$$K_n \phi_n(s) = \int_D K(s,t)\phi_n(t)dt \Rightarrow K_n(\phi_{-m}, \phi_n) = \int_D \phi_{-m}(s)ds(\int_D K(s,t)\phi_n(t)dt$$

$$\therefore \quad K_n(\phi_{-m}, \phi_n) = \int \phi_n(t)dt \int (K(s,t)\phi_{-m}(s)ds) = K_{-m}(\phi_{-m}, \phi_n)$$

$$\Rightarrow (\phi_{-m}, \phi_n) = 0, \quad \text{since} \quad K_n \neq K_{-m} \quad .$$

$\therefore$ Φ is a single orthonormal function set on D.

P(2) The series $\sum_1^\infty \frac{\phi_i^2(s)}{\lambda_i^2} = T(s)$ and $\sum_1^\infty \frac{1}{\lambda_i^2}$ both converge. $T(s)$ being a continuous bounded function on D.

Proof.

This is an immediate consequence with Bessel's inequality for orthonormal vector sets : viz (if $\varepsilon = \{e_i / i \in I\}$ is any orthonormal set of vectors in an inner product vector space X and $x \in X$, $C_i = (x, e_i)$. Then $\sum_{i \in I} |C_i|^2 \leq ||x||^2$).

In this case the component C_i of $K(s,t)$ for any fixed $s \in D$ is $\int_D K(s,t)\, \phi_i(t) = \frac{\phi_i(s)}{\lambda_i}$.

$\therefore$ Bessel's inequality $\Rightarrow \sum_1^\infty \frac{\phi_i^2(s)}{\lambda_i^2} \leq \int_D K^2(s,t)dt = T(s)$

which is continuous and bounded on D.

$\therefore \forall\ s \in D,\ \sum_1^\infty \frac{\phi_i^2(s)}{\lambda_i^2} < M \Rightarrow \sum_1^\infty \frac{\phi_i^2(s)}{\lambda_i^2}$ converges uniformly to $\Lambda(s) \leq T(s)$ on D.

By Lebesgue's convergence theorem a series of convergent positive functions can be integrated term by term.

$$\therefore \sum_1^\infty \frac{1}{\lambda_i^2} \int_D |\phi_i(s)|^2 ds = \int_D \Lambda(s)ds$$

$$\leq \iint_D |K(s,t)|^2 dsdt = \int_D T(s)ds = B$$

$\therefore \sum_1^\infty \frac{1}{\lambda_i^2}$ converges.

Corollary.

If Λ is an infinite set $\lambda_n \to \infty$ as $n \to \infty$

P(3) The set (Λ, Φ) represents the totality of all eigen values with eigen functions ϕ_n of K.

Proof Let $\lambda = \sigma$, with $\phi = X$ represent a new eigen value and associated eigen function

i.e. $\frac{1}{\sigma} X(s) = \int_D K(s,t)\, X(t)\, dt.$

The case when $\sigma \neq$ any λ_i

For each $i \in I$ as before $\frac{1}{\sigma}(X, \phi_i) = \frac{1}{\lambda_i}(X, \phi_i)$

and $\therefore$ $(X, \phi_i) = 0$, any $i \in I$.

The case when σ = some λ_i, with associated eigen functions $\phi_1, ----, \phi_r$ (λ_i has multiplicity r or necessarily finite) and X is linear independent of $\phi_1, ----\phi_r$.

In this case we can orthonormalize the set

$X, \phi_1, ----, \phi_r$ so that $(X, \phi_i) = 0$

Thus in all cases $(X, \phi_i) = 0$, $\forall\, i \in I$, any new eigen function X is orthogonal to every $\phi_i \in \Phi$ and $||X|| = 1$.

i.e. $X \in$ each of the sample space Σ_1, Σ_2, ---- .

$\therefore$ $J(X, X) \leq K_n$, every n.

i.e. $\iint_D K(s,t)\, X(s)\, X(t) ds\, dt \leq K_n$, every n.

i.e. $\int_D X(t)\, dt\, (\frac{1}{\sigma} X(t)) \leq K_n \Rightarrow \frac{1}{\sigma} ||X||^2 \leq$ every K_n.

$||X||^2 \leq \frac{\sigma}{\lambda}$, every $n \to 0$, by the corollary in P(2) since

$K(s,t)$ does not admit eigen functions besides those in the set Φ .

DEFINITION

Λ = totality of eigen values of K is called the spectrum of K.

Φ = the orthonormal set of eigen functions generated by K.

P(4) $K(s,t)$ is degenerate <=> Λ is finite.

Proof

(i) If K is degenerate => Λ is finite, follows at once by reduction to an equivalent algebraic system.

(ii) Suppose $\Lambda = \lambda_1 \lambda_2, \text{-----}, \quad \lambda_n$,

then after a sequence of n maximum/minimum problems for $J(\phi,\phi)$,

$$J(\phi,\phi) = 0, \quad \forall \phi \in \Sigma_n \ .$$

$$\Rightarrow \iint_D K(s,t)\phi(s)\phi(t)dsdt = 0, \quad \forall \phi \in \Sigma_n$$

$$\Rightarrow \iint_D K_n(s,t)\phi(s)\phi(t)ds\, d(t) = 0, \forall \phi \in \Sigma_1$$

$$\Rightarrow K_n(s,t) = 0 \Rightarrow K(s,t) - \sum_1^n \frac{\phi_i(s)\phi_i(t)}{\lambda_i} = 0$$

=> K is degenerate.

P(5) All eigen values $\lambda_i(\varepsilon\ \Lambda)$ are real.

Proof $\phi(s) = \lambda \int_D K(s,t)\,\phi(t)\,dt$

$$\int_D \phi(s)\,\overline{\phi(s)}\,ds = \lambda \iint_D \overline{\phi}(s) K(s,t)\,\phi(t)\,ds\,dt$$

Also $\int_D \overline{\phi(s)}\,\phi(s)\,ds = \bar{\lambda} \iint_D \phi(s)\,\overline{K(s,t)}\,\bar{\phi}(t)\,ds\,dt$

$$= \bar{\lambda} \iint_D \bar{\phi}(t)\,K(s,t)\,\phi(s)\,ds\,dt$$

Hence $\lambda = \bar{\lambda}$.

Thus λ is real.

*P(6) THE EXPANSION THEOREM

Let (i) $\Lambda = (\lambda_i)$ and $\Phi = (\phi_i)$ be the complete sequences of eigen values and orthonormal eigen functions of a real symmetric kernel K on D^2.

(ii) h be any piecewise function $D \to R$.

Then the integral K - transform

$$g(s) = \int_D K(s,t)\,h(t)\,dt$$

can be developed in a (Fourier) series $\sum_1^{\infty} g_i\,\phi_i(s)$, $g_i = (g,\phi_i)$, in Φ which converges absolutely and uniformly to $g(s)$ on D.

Proof

Step 1 $g_i = (g,\phi_i) = \int_D \phi_i(s)\,ds\,(\int_D K(s,t)\,h(t)\,dt)$

$$= \int_D h(t)dt \int_D K(s,t)\,\phi_i(s)ds$$

$$= \int_D \frac{h(t)\,\phi_i(t)}{\lambda_i}\,dt = \frac{h_i}{\lambda_i},\quad h_i = (h,\phi_i)$$

$$\therefore \quad g_i = \frac{h_i}{\lambda_i} \tag{4.5.1}$$

Corollary.

By Bessel's inequality for Fourier components

$$\sum_1^\infty h^2{}_i < \infty \tag{4.5.2}$$

Step 2.

Consider the series $\sum_1^\infty g_i\,\phi_i(s) = \sum_1^\infty \frac{h_i}{\lambda_i}\,\phi_i(s)$

By the Cauchy inequality, $\left| \sum_m^n \frac{h_i}{\lambda_i}\phi_i \right|^2 \leq \sum_m^n h^2{}_i \times \sum_m^n \frac{\phi^2{}_i(s)}{\lambda_i{}^2}$ (4.5.3)

Now by P(2), $\sum_1^\infty \frac{\phi_i{}^2(s)}{\lambda_i{}^2}$ converges uniformly to

$$T(s) \in C[D], \quad s \in D.$$

By (4.5.2), $\sum_m^n h^2{}_i < \text{any } \varepsilon^2$, for all $n > m > n_o(\varepsilon)$

and $\sum_m^n \frac{\phi_i{}^2(s)}{\lambda_i{}^2} < T(s) < M^2, \forall s \in D.$

(4.5.3) $\Rightarrow \quad \therefore \left| \sum_m^n \frac{h_i}{\lambda_i}\phi_i(s) \right| < M\,\varepsilon, \quad \forall\; n > m > n_o(\varepsilon), \forall S \in D,$

any $\varepsilon > o$.

By the General principle of convergence

$\sum_{1}^{\infty} g_i \phi_i(s)$ converges uniformly to a sum

$$\gamma(s),\ \gamma \in C(D),$$

i.e. $$\sum_{1}^{\infty} g_i \phi_i(s) = \gamma(s) \qquad (4.5.4)$$

Step 3

Let (i) $$K_n = K(s,t) - \sum_{1}^{n} \frac{\phi_i(s)\phi_i(t)}{\lambda_i}$$

(ii) $$\gamma_n(s) = \sum_{1}^{n} \frac{h_i}{\lambda_i} \phi_i(s)$$ (= n^{th} partial sum of $\sum_{1}^{\infty} g_i \phi_i\ s = \gamma(s)$).

then, $$g(s) = \int_D K_n(s,t)\, h(t)\, dt + \sum_{1}^{n} \frac{h_i \phi_i(s)}{\lambda_i}$$

thus $$g(s) - \gamma_n(s) = \int_D K_n(s,t)\, h(t)\, dt$$

and for any choice of a function $w \in C(D)$.

$$\int_D w(s)\,(g(s) - \gamma_n(s))ds = \iint_{D^2} K_n(s,t)\, w(s)\, h(t)\, ds\, dt$$

$$= J_n(w,h) \qquad (4.5.5)$$

By the previous section

$$J_n(\phi,\phi) \leq \frac{1}{\lambda_n} ||\phi||^2,\ \forall\ \phi \in C[D] .$$

Also $$wh = \frac{(w + h)^2 - w^2 - h^2}{2}$$

$$\therefore \quad J_n(w,h) = \tfrac{1}{2}[J_n(w + h,\ w + h) - J_n(w,w) - J_n(h,h)].$$

$$\leq \frac{||w+h||^2 + ||w||^2 + ||h||^2}{}$$

Further more for all choices w and h with norms bounded by M; and by Schwarz inequality.

$$||w+h||^2 + 2(w,h) + ||h||^2 \leq 4M^2$$

$$\therefore \quad J_n(w,h) \leq \frac{3M^2}{\lambda_n}, \text{ for all } w,h \text{ with norms bounded by } M.$$

Using this estimate in (4.5.5)

$$\int_D w(s)\,(r(s) - \gamma_n(s))ds \leq \frac{3M^2}{\lambda_n}, \text{ where } \lambda_n \to \infty \text{ as } n \to \infty.$$

$$\Rightarrow \quad \lim_{n\to\infty} \int_D w(s)(r(s) - \gamma_n(s))ds = 0 \qquad (4.5.6)$$

for any choice w,h, with norm $< M$.

By uniform convergence of $\gamma_n(s)$ to $\gamma(s)$ on D,

$$\int_D w(s)(r(s) - \gamma(s))ds = 0,$$

any choice w, so that $||w|| <$ some M.

$\therefore$ choosing $w(s) = r(s) - \gamma(s)$. $\gamma(s) = -\sum_1^\infty c_i \phi_2(s) = r(s)$ the convergence being absolute and uniform as established elsewhere. Hence the theorem.

F(7). Eigen Function Expansions for the Sequence of iterated Kernels $K_2, K_3, \ldots\ldots$

$$K_n(s,t) = \int_D K(s,\sigma)K_{n-1}(\sigma,t)d\sigma$$

$$K_2(s,t) = \int_D K(s,\sigma)\,K(\sigma,t)d\sigma = K \text{ - transform of } K(\sigma,t)$$

for some fixed t.

$$p(6) \quad \Rightarrow \quad \sum_{1}^{\infty} \frac{k_i(t)}{\lambda_i} \phi_i(s).$$

$$k_i(t) = (K(\sigma,t)\, \phi_i(\sigma)) = \int_D K(\sigma,t)\, \phi_i(\sigma) d\sigma = \frac{1}{\lambda_i} \phi_i(t)$$

$$\therefore K_2(s,t) = \sum_{1}^{\infty} \frac{\phi_i(s)\phi_i(t)}{\lambda^2_i}, \quad \text{for each } t.$$

The convergence being absolutely and uniformly in both s and t in D^2.

Now,

$$\sum_{m}^{n} \left| \frac{\phi_i(s)}{\lambda_i} \frac{\phi_i(t)}{\lambda_i} \right| \leq \sum_{m}^{n} \left| \frac{\phi_i^2(s)}{\lambda^2_i} \right| \times \sum_{m}^{n} \left| \frac{\phi_i^2(t)}{\lambda^2_i} \right|$$

Dini's theorem states that if $\sum_{1}^{\infty} u_i(s)$ is a convergent series of positive functions that tends point wise to $u(s)$ in D, then the convergence is necessarily uniform.

$\therefore$ $\sum_{1}^{\infty} \frac{\phi_i^2(s)}{\lambda_i^2}$, $\sum_{1}^{\infty} \frac{\phi_i^2(t)}{\lambda^2_i}$ converge uniformly to $A(s)$, $A(t)$ on D.

$$\therefore \sum_{m}^{n} \left| \frac{\phi_i(s)\, \phi_i(t)}{\lambda^2_i} \right| < \varepsilon^2, \; n > m > n_0, \quad \forall \; s,t \in D^2 .$$

$\therefore$ $\sum_{1}^{\infty} \frac{\phi_i(s)\, \phi_i(t)}{\lambda^2_i}$ converge absolutely and uniformly on D^2

$\therefore$ $K_2(s,t) = \sum_{1}^{\infty} \frac{\phi_i(s)\, \phi_i(t)}{\lambda^2_i}$ converges absolutely and uniformly on D^2.

Corollary

$\sum_1^\infty \frac{\phi_i(s)}{\lambda_i^2} = K_2(s,s)$ converges absolutely and uniformly on D^2 (c.f. P(2)).

$$K_3(s,t) = \int_D K(s,\sigma) K_2(\sigma, t) d\sigma = \sum_1^\infty \frac{K_i(t)}{\lambda_i} \phi_i(s)$$

$$K_i(t) = \int_D K_2(\sigma,t)\,\phi_i(\sigma) d\sigma = \frac{\phi_i(t)}{\lambda_i^2}$$

$$\therefore \quad K_3(s,t) = \sum_1^\infty \frac{\phi_i(s)\,\phi_i(t)}{\lambda_i^3}$$ which as before

converges absolutely and uniformly on D^2. Continuing in this way,

$$K_n(s,t) = \int_D K(s,\sigma) K_{n-1}(\sigma,t) d\sigma = \sum_1^\infty \frac{K_i(t)\phi_i(s)}{\lambda_i^n}, \quad n \geq 2.$$

$$K_i(t) = \int_D K_{n-1}(\sigma,t)\,\phi_i(\sigma) d\sigma = \frac{\phi_i(t)}{\lambda_i^{n-1}}$$

$$\therefore \quad K_n(s,t) = \sum_1^\infty \frac{\phi_i(s)\,\phi_i(t)}{\lambda_i^n}$$ which converges

absolutely and uniformly to D^2, $n \geq 2$.

P(8). An explicit expression for Resolvent Kernel $R(s,t;\lambda)$.

For $|\lambda| <$ every $|\lambda_i|$, $R(s,t;\lambda) = \sum_1^\infty K_n(s,t)\lambda^{n-1}$

which is a Neumann series.

Now,

$$R(s,t;\lambda) = K(s,t) + \lambda K_2 + \lambda^2 K_3 + \lambda^3 K_4 + \text{------}$$

$$= K + \lambda \sum_1^\infty \frac{\phi_i(s)\phi_i(t)}{\lambda_i^2} + \lambda^2 \sum_1^\infty \frac{\phi_i(s)\phi_i(t)}{\lambda_i^3} + \text{------}$$

$$= K + \sum_1^\infty \phi_i(s)\phi_i(t)\left(\frac{\lambda}{\lambda_i^2} + \frac{\lambda^2}{\lambda_i^3} + \frac{\lambda^3}{\lambda_i^4} + \text{---}\right), |\lambda| < |\lambda_i|$$

$$\therefore \quad R(s,t,\lambda) = K(s,t) + \lambda \sum_{i}^{\infty} \frac{\phi_i(s)\,\phi_i(t)}{\lambda_i(\lambda_i - \lambda)} \; ; \; |\lambda|<|\lambda_i|$$

Since $\left(\frac{\lambda}{\lambda^2_i} + \frac{\lambda^2}{\lambda^3_i} + \frac{\lambda^3}{\lambda^4_i} + \cdots\cdots\right)$ reduces to a geometric series whose sum is $\frac{\lambda}{\lambda_i(\lambda_i - \lambda)}$.

This expansion continues analytically on the whole λ-plane as a meromophic function of λ with simple poles at $\lambda = \lambda_i$, residues at $\lambda = \lambda_i = \frac{\phi_i(s)\,\phi_i(t)}{\lambda_i}$ which can be used to find ϕ_i .

P(9) Explicit solution for Fredholm equation with symmetric Kernels K, in terms of its spectrum Λ and total orthonormal eigen function set Φ .

Let $\psi(s) - \lambda \int_D K(s,t)\psi(t)dt = f(s)$, $f \in C[D]$,

Then $\psi(s) - f(s) = \lambda \int_D K(s,t)\,\psi(t)dt$

$= \lambda \times$ K-transform of ψ

$$\therefore \quad \psi(s) - f(s) = \lambda \sum_{i}^{\infty} \frac{C_i\phi_i(s)}{\lambda_i}, \quad C_i = (\psi,\phi_i)$$

by p(6) — (4.5.6)

$((4.5.6), \phi_i) \Rightarrow C_i - (f,\phi_i) = \frac{\lambda C_i}{\lambda_i}; \; (\phi_i,\phi_j) = 0, \; i \neq j$

$= \quad i = j$

$$\Rightarrow C_i\left(1 - \frac{\lambda}{\lambda_i}\right) = f_i\left(=(f,\phi_i)\right)$$

$$\Rightarrow C_i = \frac{\lambda_i f_i}{\lambda_i - \lambda}$$

$$(4.5.6) \Rightarrow \psi(s) - f(s) = \lambda \sum_{1}^{\infty} \frac{\lambda_i f_i}{\lambda_i(\lambda_i - \lambda)} \phi_i(s)$$

$$\therefore \psi(s) = f(s) + \lambda \sum_{1}^{\infty} \frac{f_i'}{\lambda_i - \lambda} \phi_i(s).$$

It is noted that

(i) $\lim_{n\to\infty} \int_D (K(s,t) - \sum_{1}^{n} \frac{\phi_i(s)\phi_i(t)}{\lambda_i})dsdt = 0 \not\Rightarrow K(s,t) = \sum_{1}^{\infty} \frac{\phi_i(s)\phi_i(t)}{\lambda_i}$

(ii) if K is positive definite, i.e. $J(\phi,\phi) > 0, \forall \phi \in \Sigma_1$,

or if $\sum_{1}^{\infty} \frac{\phi_i(s)\phi_i(t)}{\lambda_i}$ converges uniformly then the spectrum Λ is positive and the representation $\sum_{1}^{\infty} \frac{\phi_i(s)\phi_i(t)}{\lambda_i^{\,n}} = K_n(s,t)$

applies for $n = 1$ as well.

The results above for real symmetric Kernels can be extended to general symmetric kernels $K \in C[D^2]$. (Ref.: [19], Sec. 12, PP73 - 81; Sec 17, PP99 - 103, Secs. 18 & 19, PP. 103 - 105).

(4.6) AN ILLUSTRATIVE EXAMPLE

Take the case of the real, continuous, symmetric Kernel on $D^2 = 0 \le s,t \le 1$, defined by

$$K(s,t) = \begin{cases} s(t-1), & 0 \le s \le t \\ t(s-1), & t \le s \le 1 \end{cases} \qquad (4.6.1)$$

which is the Green's function for the differential operator

$$L = \frac{d^2}{ds^2} \text{ on } D = [0, 1]$$

under homogeneous boundary conditions $\phi(1) = 0 = \phi(o)$.

The complete set (Λ, Φ) of eigen values and eigen functions of K can be computed from the eigen value problem for L, Viz., complete solutions of ϕ, and λ where

$$\phi''(s) + \lambda\, \phi(s) = 0, \quad s \in D \equiv (o \leq s \leq 1) \tag{4.6.2}$$

$$\phi(0) = 0 = \phi(1) \tag{4.6.3}$$

The boundary conditions (4.6.3) demand solutions ϕ of (4.6.2) of periodic type

$$\phi(s) = C_1 \operatorname{Cos} \sqrt{\lambda}\, s + C_2 \operatorname{Sin} \sqrt{\lambda}\, s.$$

The boundary conditions yield the eigen value set

$$\Lambda = \{ \lambda_n = n^2\pi^2, \; n = 1, 2, ---- \} \tag{4.6.4}$$

and the associated normalized eigen function set

$$\Phi = \{ \phi_n(s) = \sqrt{2} \operatorname{Sin} n\pi s, \; n = 1, 2, ----- \}. \tag{4.6.5}$$

The theorems listed above can now be confirmed and illustrated.

1. the first eigen value π^2 is the maximum of

$$J(\phi,\phi) = \int_0^1\int_0^1 K(s,t)\phi(s)\phi(t)\,ds\,dt, \; \phi \in \Sigma_1 = \{\phi \in C[D]/||\phi||^2 = 1\}$$

$$= \int_0^1 \phi(s)ds \left[\int_0^1 K(s,t)\phi(t)dt\right]$$

$$= \int_0^1 \phi(s)ds \left[\int_0^s t(s-1)\phi(t)dt + \int_s^1 s(t-1)\phi(t)dt\right]$$

which is found after some computation to reduce to

$$J(\phi,\phi) = \phi^2(1) - 2\int_0^1 \phi(s)\, \psi(s)\, ds,$$

where $\psi(s) = \int_0^s t\, \phi(t)\, dt$;

The explicit expression for the resolvent kernel $R(s,t;\lambda)$ is given by

$$R(s,t;\lambda) = K(s,t) + \lambda \sum_{1}^{\infty} \frac{2 \operatorname{Sin} n\Pi s \operatorname{Sin} n\Pi t}{n^2\Pi^2(n^2\Pi^2-\lambda)},$$

where $K(s,t)$ is defined as in (4.6.1), for $|\lambda|<|n^2\Pi^2|$. This expansion continues analytically on the whole λ-plane as a meromophic function of λ with simple poles at $\lambda = n^2\Pi^2$, $n = 1, 2, ----$.

The explicit solution for

$$\phi(s) - \lambda \int_0^1 K(s,t)\, \phi(t)dt = f(s), \quad f(= s) \in C[D]$$

is
$$\phi(s) = s + \lambda \sum_{1}^{\infty} \frac{\int_0^1 \operatorname{Sin} n\Pi s\, ds}{n^2 \Pi^2 - \lambda} \operatorname{Sin} n\Pi s$$

and
$$\int_0^1 s \operatorname{Sin} n\Pi s\, ds = \int_0^1 s\, d\left(-\frac{\cos n\Pi s}{n\Pi}\right) = \frac{(-1)^{n+1}}{n\Pi}$$

$$\therefore \quad \phi(s) = s + \frac{\lambda}{\Pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n^2\Pi^2-\lambda)} \operatorname{Sin} n\Pi s.$$

6. The orthogonal condition is met, since

$$\int_0^1 \phi_1(s)\phi_2(s)ds = \int_0^1 \operatorname{Sin}\Pi s \operatorname{Sin} 2\Pi s\, ds = 0$$

7. The particular case of $K(s,t)$ which converges to

$\sum_{1}^{\infty} \frac{2\phi_i(s)\, \phi_i(t)}{i^2\Pi^2}$ in the mean.

$\sum_{1}^{\infty} \frac{\operatorname{Sin} \Pi ns \operatorname{Sin}\Pi nt}{n^2\, \Pi^2}$ is uniform convergent on $[0, 1]^2$.

CHAPTER FIVE

A REVIEW OF NUMERICAL METHODS FOR FREDHOLM EQUATIONS OF THE SECOND KIND.

5.1 INTRODUCTION.

This chapter presents a survey of Numerical Methods for linear Fredholm integral equations of the second kind i.e.

$$\phi(s) = f(s) + \int_D K(s,t)\,\phi(t)dt, \qquad (5.1.1$$

where, $D \equiv [a,b]$, $a \le s,t \le b$, the data function $f \in C[D]$ and the Kernel $K \in C[D^2]$ are given and the interval $[a,b]$ may be finite or infinite,

The sections that follow will discuss a variety of methods for Fredholm equations of the type (5.1.1). Much of the emphas will be on practical aspects of the methods, and proofs of theoretical results will generally be omitted.

5.2 THE NYSTROM METHOD

Suppose a and b are finite, $f \in C[D]$ and $k \in C[D^2]$ then the simplest way to deal with the integral equation (5.1.1) is to approximate the integral by one of the standard quadrature rules (e.g. the simpson or Gauss rules). The quadrature rule is written as

$$\int g(t)dt = \sum_{i=1}^{n} w_{ni}\, g(S_{ni}), \qquad (5.$$

where $\{S_{ni}\}_{i=1}^{n}$ and $\{W_{ni}\}_{i=1}^{n}$ are the points and weights for the particular n-point quadrature, rule on the interval $[a,b]$.

If ϕ_n denotes the approximate solution of (5.1.1) obtained by replacing the integral in (5.1.1) by the quadrature rule (5.2.1), then ϕ_n satisfies

$$\phi_n(s) = f(s) + \sum_{i=1}^{n} w_{ni} K(s, s_{ni}) \phi_n(s_{ni}). \tag{5.2.2}$$

A set of n linear equations for the n unknowns $\phi_n(s_{n1})$, ----, $\phi_n(s_{nn})$ may be obtained by setting $s = s_{nj}$, $j = 1$, ----, n, to give

$$\sum_{i=1}^{n} [\delta_{ij} - w_{ni} K(s_{nj}, s_{ni})] \phi_n(s_{ni}) = f(s_{ni}), \qquad j = 1, ---, n. \tag{5.2}$$

where

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

This set of linear equations, together with (5.2.2) constitute the Nystrom method. The only assumption required in convergence theory of the Nystrom method, beyond the continuity of f and K, is that the integration rule (5.2.1) should converge to the exact integral as $n \to \infty$ for all continuous functions g. The analysis carried out in the space $C[D]$, then shows that ϕ_n converges to ϕ as $n \to \infty$, or in other words, $\phi_n(s)$ converges uniformly to $\phi(s)$.

The Nystrom method is easily programmed for the Computer. The program uses either Simpson's rule or the Gauss rule, and, if appropriate, solves the resulting system of Linear equations by means of an iterative Method.

In brief, the iterative method is as follows: First, choose a suitable integer n of moderate size, for which the equations (5.2.3) can be solved directly. For $m > n$, write

$$\phi_m(s) = f(s) + \int_a^b K_m(s,t)\,\phi_m(t)\,dt.$$

as

$$\begin{aligned}\phi_m &= f + K_m\,\phi_m \\ &= f + (K_m - K_n)\,\phi_m + K_n\,\phi_m\,, \\ &= f + (K_m - K_n)(f + K_m\phi_m) + K_n\,\phi_m\,,\end{aligned}$$

and this equation is solved iteratively, i.e. , given $\phi_m^{(o)}$, one constructs $\phi_m^{(1)}$, $\phi_m^{(2)}$, ---, by

$$(I - K_n)\,\phi_m^{(k+1)} = f + (K_m - K_n)f + (K_m - K_n)\,K_m\,\phi_m^{(k)}\,.$$

Convergence of the iterative scheme is guaranteed, if n is sufficiently large.

5.3 PRODUCT INTEGRATION METHOD.

Suppose the kernel takes the form

$$K(s,t) = h(s,t)\,m(s,t) \qquad (5.3.1)$$

where $m(s,t)$ is a smooth function of t for each t in D, whereas $h(s,t)$ is singular. Then the product - integration method may be very appropriate particularly so if the singular factor $h(s,t)$ can be chosen to be a function of a standard type, such as, $\ln|s - t|$, or $|s - t|^\alpha$, $\alpha > -1$.

A product-integration rule is obtained by approximating just the smooth part of the integrand by an interpolating function and again integrating exactly.

More precisely, given $n > o$, select a set of n quadrature points S_{n1}, -----, S_{nn} in $[a,b]$, and a corresponding n - dimensional space of interpolating functions, which will be assumed has a basis $\{L_{n1}, -----L_{nn}\}$ satisfying

$$L_{ni}(S_{nj}) = \delta_{ij} . \tag{5.3.2}$$

Given a function $g \in C[D]$, denote by $p_n g$ the unique function in the n-dimensional space that coincides with g at S_{ni},------- S_{nn}. It is easily seen that

$$p_n g(t) = \sum_{i=1}^{n} L_{ni}(t) g(S_{ni}) ; \tag{5.3.3}$$

for it then follows immediately from (5.3.2) that

$$p_n g(t_{nj}) = g(t_{nj}) , \quad j = 1,----, n.$$

Since we are here seeking an appropriate solution for

$$\phi(s) = f(s) + \int_a^b h(s,t)m(s,t)\phi(t)dt, \quad a \leq s \leq b,$$

we approximate just the smooth factor $M(s,t)\phi(t)$ by the interpolatory approximation (5.3.3). Thus the approximate equation is

$$\phi_n(s) = f(s) + \int_a^b h(s,t) \sum_{i=1}^{n} L_{ni}(t) M(S,S_{ni})\phi_n(S_{ni})dt \tag{5.3.4}$$

$$= f(s) + \sum_{i=1}^{n} W_{ni}(S) M(S,S_{ni})\phi_n(S_{ni}),$$

where

$$W_{ni}(S) = \int_a^b h(s,t) L_{ni}(t)dt \tag{5.3.5}$$

Equation (5.3.4) is analogous to the Nystrom equation (5.2.2). In just the same way we obtain from (5.3.4) a set of linear equations for the unknowns $\phi_n(S_{ni})$,----,$\phi_n(S_{nn})$,

$$\sum_{i=1}^{n} [\delta_{ij} - W_{ni}(S_{nj}) M(S_{nj}, S_{ni})]\phi_n(S_{ni}) = f(S_{nj}), \tag{5.3.}$$

and then use (5.3.4) as a natural interpolation formula for values of $\phi_n(s)$ between the quadrature point.

Convergence of this method is ensured provided f and M are continuous, and h satisfies

$$\sup_{s \in D} \int_a^b |h(s,t)|\,dt < \infty \qquad (5.3.7)$$

together with

$$\lim_{s\to T} \int_a^b |h(s,t) - h(T,t)|\,dt = 0, \quad a \le T \le b, \qquad (5.3.8)$$

then ϕ_n exists for all n sufficiently large, and converges uniformly to ϕ.

(5.4) COLLOCATION METHOD AND VARIANTS

The collocation method is an expansion method: For each n one selects a basis set U_{n1}, ----, U_{nn} of real-valued functions defined on D, and approximates ϕ by a linear combination

$$\phi_n = \sum_{i=1}^{n} a_{ni} U_{ni} . \qquad (5.4.1)$$

The coefficients a_{ni} are then fixed by collocating at n selected points S_{n1}, ----, S_{nn} in the interval, that is by requiring

$$\phi_n(S_{nj}) = f(S_{nj}) + \int_a^b K(S_{nj},t)\,\phi_n(t)\,dt, \quad j = 1, ----, n. \qquad (5$$

Immediately, a set of n linear equations for the coefficients a_{n1}, ----, a_{nn} is obtained,

$$\sum_{i=1}^{n} \{U_{ni}(S_{nj}) - \int_a^b K(S_{nj}',t)\, U_{ni}(t)dt\} a_{ni} = f(s_{ni}), \qquad (5.4.3)$$

$$j = 1,2,---, n$$

A variant of the above, suggested by the form of the exact equation $\phi = f + K\phi$, is the approximation of ϕ by

$$\phi_n = f + \sum_{i=1}^{n} b_{ni}\, U_{ni}, \qquad (5.4.4)$$

instead of (5.4.1). The collocation procedure then leads to a set of linear equations with the same left-hand side matrix as (5.4.3), but with the right hand side replaced by

$$\int_a^b K(S_{nj},t)\, f(t)\, dt.$$

Another variant is obtained by substituting the collocation solution (5.4.1) into the right-hand side of (5.1.1), to obtain a new approximation ϕ_n, which we may refer to as the iterated, collocation method, and which is given by

$$\phi_n^1(s) = f(s) + \int_a^b K(s,t)\phi_n(t)\, dt \qquad (5.4.5)$$

$$= f(s) + \sum_{i=1}^{n} a_{ni} \int_a^b K(s,t)\, U_{ni}(t)dt \qquad (5.4.6)$$

where the coefficients a_{ni} are given by (5.4.3).

A condition that is certainly essential if the collocation method is to have any reasonable hope of success, is that the nxn matrix $\{U_{nj}(S_{nj})\}$ be non-singular.

The convergence, for the piecewise-polynomial, of the collocation approximation ϕ_n to the exact solution ϕ can usually be established without difficulty. It just requires that the nodes $\{S_{nj}\}$ be sensibly chosen, so that for every $g \in C[D]$, the unique linear combination of U_{n1},----, U_{nn} that coincides with g at S_{n1},---, S_{nn} should converge uniformly to g as $n \to \infty$.

On the other hand, for the case of polynomials (and trignometric functions) it is impossible to choose the nodes $\{S_{nj}\}$ so that the unique polynomial of degree $< n$ that coincides with g at the nodes $S_{n1}, ---, S_{nn}$ Converges uniformly to g for all continuous functions g. However, it is well known that some choices of nodes have very much better interpolation properties than others, and that among the best are the Chebyshev nodes and the Clenshaw-Curtis nodes, because convergence is assured provided K satisfies (5.3.7) and (5.3.8) [REF: [1], Sec. 4, pp 62-66].

BATEMAN METHOD

Suppose K is a smooth Kernel, if we wish to use the collocation method, choose the basis function $U_{n1}, ---, U_{nn}$ as

$$U_{ni}(s) = K(s, S_{ni}). \quad i = 1, ---, n \tag{5.5.1}$$

where $\{S_{ni}\}$ is a suitable set of points in $[a,b]$ satisfying

$$a \le S_{n1} < ------ < S_{nn} \le b \tag{5.5.2}$$

Assume that tese points are chosen so that the set $\{U_{n1}, ---, U_{nn}\}$ so obtained is linearly independent.

The Bateman Method is obtained by using the basis functions (5.5.1) in the variant (5.4.4) of the collocation method, so that we have as our approximate solution

$$\phi_n(s) = f(s) + \sum_{i=1}^{n} b_{ni} K(S, S_{ni}) . \tag{5.5.3}$$

Choosing the points already used in the definition (5.5.1), the equations that determine the coefficients in (5.5.3) are

$$\sum_{i=1}^{n} [K(S_{nj},S_{ni}) - \int_a^b K(S_{nj},t)K(t,S_{ni})dt] b_{ni} = \int_a^b K(S_{nj},t)f(t)dt \qquad (5.5.4)$$

It should be noted that the Bateman method is little discussed by numerical Analysts, perhaps because no satisfactory convergence theory exists, so that one has no guidance in choosing the points $\{S_{nj}\}$, and that the matrix on the left hand side of (5.5.4) will usually be not very well conditioned. One point in its favour is that the choice of the points $\{S_{nj}\}$ is not tied to any particular quadrature rule.

5.6 GALERKIN METHOD AND VARIANTS

The Galerkin Method, like the collocatiom method, is an expansion method, in which the exact solution ϕ is approximated by

$$\phi_n = \sum_{i=1}^{n} a_{ni} U_{ni} \quad \text{where for each } n \ \{U_{ni}/i=1,---,n\}, \qquad (5.6.$$

in the set of basis function, as before.

The coefficients in the Galerkin method are determined by

$$(U_{nj},\phi_n) = (U_{nj},f) + (U_{nj},K\phi_n), \; j=1, ---, n, \qquad (5.6.2$$

where $(g,h) = \int_a^b \overline{g(t)}\, h(t)\, w(t)dt \equiv$ inner product in $L_2[D]$ space and w is a non-negative weight function which equals to 1 unless otherwise stated, giving the linear equations

$$\sum_{i=1}^{n} [(U_{nj},U_{ni}) - (U_{nj},K U_{ni})]a_{ni} = (U_{nj},f), \quad i = 1,----, n \qquad (5.6.3$$

Often these integrals will be evaluated numerically. If K is smooth, then there is at least one situation in which the calculation of the approximate integrals can be made relatively efficient: if the interval is $[-1,1]$ and the basis functions are Chebyshev polynomials, and if $w(t)$ is chosen to be $(1 - t^2)^{\frac{1}{2}}$ then the fast Fourier transform technique may be used.

The Galerkin method converges, iff $f \in L_2[D]$ and K is a compact operator in $L_2[D^2]$, and its error approaches asymptoticall the least possible error for an approximation of the form (5.6.1).

A variant is obtained if ϕ_n denotes the Galerkin approximation, then the iterated Galerkin approximation is

$$\phi_n^1 = f + K\phi_n = f + \sum_{i=1}^{n} a_{ni} K U_{ni} \qquad (5.6.4)$$

It may be shown quite generally (see, [1] p. 69) that ϕ_n^1 converges to ϕ faster than ϕ_n does : specifically, that

$$||\phi_n^1 - \phi|| \leq \beta \, ||\phi_n - \phi|| \qquad (5.6.5)$$

where $\beta_n \to 0$ as $n \to \infty$

Another variant of the Galerkin method known as the Petrov-Galerkin method is obtained by replacing (5.6.3) by

$$\sum_{i=1}^{n} [(V_{nj}, U_{ni}) - (V_{nj}, K U_{ni})] a_{ni} = (V_{nj}, f), \qquad j = 1, 2, ---, n. \tag{5.6.6}$$

where $V_{n1}, ----, V_{nn}$ is a second set of linearly independent functions in $L_2[D]$.

CHAPTER SIX

A NUMERICAL SOLUTION TO THE PROBLEM

- A LOADED ELASTIC STRING

6.1 THE PROBLEM

The methematical formulation of this problem is dealt in Chapter One, pages 2 and 3. The present Chapter examines and implements the numerical solution of the integral equation (1.2.9):

$$y(x) + \frac{k}{T}\int_0^a G(x,\xi)y(\xi)d\xi = \frac{w}{T}\int_0^a G(x,\xi)d\xi$$

where,

$G(x,\xi)$ is the Green's function defined in (1.2.5);

$y(x)$ the deflection defined in (1.2.8) is the solution function;

w is the linear density of the string;

k is the resistant elastic constant of the foundation; and

T is the initial tension in the string.

$G(x\ \xi)$ is symmetric, continuous, and absolutely integrable square.

In fact,

$$\int_0^a\int_0^a |G(x,\xi)|^2\,dxd\xi = \frac{a^4}{180} + \frac{a^4}{180} = \frac{a^4}{90}\ . \tag{6.1.1}$$

$$\therefore\quad \int_0^a\int_0^a |G(x,\xi)|^2 dxd\xi < \infty\ .$$

Introducing a non-dimensional length $y = x/a$ and writing $\lambda = k/v$, with $[0,a] = [0,1]$, the integral equation (1.2.9) takes the form:

$$Y(x) + \lambda\int_0^1 G(x,s)Y(s)ds = \int_0^1 G(x,s)ds \tag{6.1.2}$$

There are several established numerical methods used to find ...lutions of integral equations [c.f. chapter 5]. The methods ...st favoured in practice fall into two classes: those based on ...tegration formulae - quadrature methods (e.g. the Nystrom and ...e product integration methods) and those which are expansion ...thods in particular the Rayleigh-Ritz, Galerkin and collocation ...thods. In the later class of methods, an approximate eigen ...nction is obtained as a linear combination of chosen functions. ...e former class of methods, though, not always the best, perhaps, ...e the simplest to implement in a straight forward way, and the ...herent and truncation errors are minimal. For these reasons, we ...all use Simpson's rule, a more accurate quadrature rule, to solve ...uation (6.2.1). For a detailed discussion of these methods c.f., ...], pp 167-341 and [1], pp 51-71.

Using Simpson's rule, with the notation

$$...x) = Y_{xi} \; ; \; Y(s) = Y_{sj}$$

...e integral equation (6.2.1) reduces to a system of linear ...quations of the form:

$$Y_{xi} + \frac{\lambda h}{3} [G(x_i, s_o) Y_{so} + G(x_i, s_{20}) Y_{s20}$$

$$+ 4 \sum_{j=0}^{a} G(x_i, s_{2j+1}) Y s_{2j+1} + 2 \sum_{j=1}^{a} G(x_i, s_{2j}) Y s_{2j}]$$

$$= \tfrac{1}{2}\left(\frac{x_i}{20} - \frac{x_i^2}{20} \right), \; i = 0, 1, \text{-----}, 20. \qquad (6.2.2).$$

The algebraic unknowns comprise the values Y_{xi} of the unknown y at 21 equispaced stations from 0 to 1.

$$\text{Let}\quad Y = \begin{pmatrix} y_{xo} \\ y_{xi} \\ - \\ - \\ - \\ y_{x20} \end{pmatrix} .$$

The integral operator

$$\int_0^1 G(x,s)Y(s)ds$$

is represented as a linear combination of the set Y, using Simpson's rule for a spacing $h = \frac{1}{20}$.

The error in the computation of each integral may be estimated to be around 2.17×10^{-9}, since the error of the Simpson's rule can in this case be computed from

$$\left| -\left(\frac{h/2}{180}\right)^4 (b-a)\, G^4(\xi,h) \right| .$$

The values $G(x_i, s_j)$; $i,j=0,1,$ ----,20 are accurate to 16 decimal places. Thus in the resulting algebraic system

$$AY = B$$

the elements of the coefficient matrix A and the data vector B are computed to within an accuracy of $O(10^{-16})$.

The actual computation of a solution vector is undertaken for e particular case $\lambda=1$, using a minimac, micro computer with a b-routine for the inversion of the coefficient matrix A(for details the computation o.f computer print outs in Appendix 1).

An estimate of the error involved in the computation of A^{-1}, this program, may be attained by the method of residuals which es the error estimate (E) given by

$$E = Max||AA^{-1} - I||,$$

ich in the present case could be assessed as $E = 7 \times 10^{-17}$.

Taking the larger error propagation of $O(10^{-9})$, the relative esulting error of 21 operations may be estimated to be around .x2.17x10^{-9}, which would amount to $O(10^{-8})$.

Thus to the accuracy such that the maximum error is of $O(10^{-8})$, he solutions of the discretized problem is presented below in ppendix 3.

APPENDIX 1

COMPUTER PROGRAM FOR THE SOLUTION OF THE INTEGRAL EQUATION OF THE SECOND KIND BY SIMPSON'S RULE.

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