CONVEXITIES IN T_0-QUASI-METRIC SPACES

By

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ABSTRACT

In this thesis, we recall three types of convexities in metric spaces, namely; Menger convexity, Takahashi convexity and M-convexity. We then generalise these convexities to the framework of T_0-quasi-metric spaces. Since the concept of convexities heavily relies on the concept of betweenness, a fundamental concept in the study of axiomatic geometry, we begin by generalising the concept of betweenness to T_0-quasi-metric spaces. We show that Takahashi convexity implies Menger convexity in T_0-quasi-metric spaces. Lastly, we generalise the concept of M-convexity to T_0-quasi-metric setting and present some best approximations in these spaces.

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DEDICATION

I dedicate this work to my parents Godfrey Nyirongo and Monica Lukhele and, my siblings Joseph, Gloria, Emmanuel and Edith, for the support and words of courage they have rendered. To my grandfather Mr Francis Nyirongo and his wife Mrs Aginess Ngulube Nyirongo for their invaluable encouragement, financial and moral support.

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INDEX OF NOTATION

Below is a list of symbols that will be frequently used and a brief indication of their meaning.

(X,d)	a metric space
(X,q)	quasi-metric space
$B_d(x,r)$	Open ball of radius r centred at x
$C_d(x,r)$	Closed ball of radius r centred at x
$S_d(x,\delta)$	Sphere of radius δ centred at x
$cl(A)$	closure of set A
$\mathbb{N}$	the set of natural numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{Q}$	the set of rational numbers
inf	Infimum(greatest lower bound)
sup	supremum(least upper bound)
$\max(\vee)$	maximum
$\min(\wedge)$	minimum
$\mathscr{P}_0(X)$	the set of all nonempty subsets of X
$C_b(X)$	the space of continuous bounded functions
$[xyz]_d$	y is between x and z in a metric space
$B^d(x,y)$	the set of points between x and y
$\mathcal{F}(X)$	the space of real valued function
$\mathcal{CB}_0(X)$	sub-collection of bounded convex elements of $\mathscr{P}_0(X)$.

CHAPTER 1

INTRODUCTION

1.1. Background

The concept of convex metric spaces which was first introduced by Menger [21] in 1928 has received considerable attention by many different scholars [28], [29], [20], [11] and so on. Menger used the concept of betweenness to develop the theory of convexity in metric spaces as follows: Let (X, d) be a metric space and $x, y, z \in X$. Then z is said to be between x and y if and only if $x \neq z \neq y$, implies $d(x, z) + d(z, y) = d(x, y)$. A metric space (X, d) is said to be convex [21] provided it contains for each two points at least one point between them. Thereafter, Menger [21] pioneered the Fundamental Theorem of convexity, which states that: if (X, d) is a complete and convex metric space, then any two points of X can be joined by a metric segment. This theorem is very important in the study of the geometry of metric spaces. Henceforth, many other definitions of convexity in metric spaces have been put forward. For example, in 1970, Takahashi [28], defined convexity in the following way: Let (X, d) be a metric space. A mapping $W : X \times X \times [0, 1] \longrightarrow X$ is said to be a Takahashi convex structure (TCS) on X if for all $x, y \in X, \lambda \in [0, 1]$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y)$$

for all $u \in X$. The metric space (X, d) together with a Takahashi convex structure (TCS) W is called a convex metric space. Also, in 1988 Khalil [11] defined convexity in a metric space in the following way: Let (X, d) be a metric space and $x, y \in X$ with $d(x, y) = \lambda$. Then (X, d) is said to be M-convex if

$$C_d(x, r) \cap C_d(y, \lambda - r) = \{z_r\},$$

where $r \in [0, \lambda]$ and $C_d(x, r) = \{y \in X : d(x, y) \leq r\}$.
If we remove the symmetry property from a metric, we have a quasi-metric defined as : Let X be a nonempty set and consider a function $q : X \times X \longrightarrow [0, \infty)$. Then, q is called a quasi-metric on X if

(i) $q(x, x) = 0$ whenever $x \in X$. (ii) $q(x, z) \leq q(x, y) + q(y, z)$ whenever $x, y, z \in X$. Further, we shall say that q is a T_0-quasi-metric provided that q also satisfies the following T_0-condition: For each $x, y \in X$, $q(x, y) = 0 = q(y, x) \implies x = y$. The pair (X, q) is called a T_0-quasi-metric space.
Recently, Künzi [19] in 2016 defined and investigated a convexity structure in the sense of Takahashi in the more general T_0-quasi-metric spaces. He proved that various important

results about convexity structure in metric spaces can be generalised to quasi-metric setting. He also showed later that convexity structure occurs naturally in asymmetric normed real vector spaces and in q-hyperconvex T_0-quasi-metric spaces. Some studies on quasi-metric spaces have been based on generalising the well known results in metric spaces to the quasi-metric setting. For example, in [14], [19] and [24] we find the generalisations of classical results in hyperconvex metric spaces to the quasi-metric setting. Results in these articles confirm the surprising fact that many classical results about hyperconvexity do not make essential use of the symmetry of the metric and, therefore, this is our motivation of generalising other convexities to the framework of quasi-metric spaces.
In this thesis, we recall three types of convexities in metric spaces, namely; Menger convexity, Takahashi convexity and M-convexity. We then generalise these convexities to the framework of T_0-quasi-metric spaces. Since the concept of convexities heavily relies on the concept of betweenness, it is natural to start with the concept of betweenness in T_0-quasi-metric spaces. We observe that, in T_0-quasi-metric spaces, there are at least five types of betweenness, namely; q-betweenness, q^{-1}-betweenness, q^s-betweenness, q^+-betweenness and q, q^{-1}-betweenness. We show that q, q^{-1}-betweenness implies q^+-betweenness in T_0-quasi-metric spaces (see Proposition 4.1.6). Also, we show that q-betweenness does not necessarily imply q^{-1}-betweenness (see Example 4.1.3). Thereafter, we generalise the concept of Menger convexity, from metric settings to the framework of T_0-quasi-metric space. Furthermore, we recall Takahashi convexity structure in T_0-quasi-metric spaces. Then, we show that Takahashi convexity implies Menger convexity in T_0-quasi-metric spaces (see Proposition 4.3.27). Lastly, we generalise the concept of M-convexity to T_0-quasi-metric settings and present some best approximations in these spaces. Also, we observe that, in T_0-quasi-metric spaces if (X, q) is M-convex, then (X, q^{-1}) is M-convex too. However, M-convexity of (X, q) does not necessarily imply M-convexity of (X, q^s) (see Example 4.4.5).

1.2. Organisation of the dissertation

This dissertation is organized as described below.

Chapter 1. This chapter presents a background on convexities in metric spaces as investigated by different scholars and the organisation of the dissertation.

Chapter 2. In this chapter, we recall some of the important definitions to be used throughout the dissertation. In the first section we give definitions and some examples of metric spaces, thereafter, we present a summary of notions related to normed spaces. The second section briefly discusses the concept of quasi-metric spaces. Thereafter, we discuss the concept of topologies and completeness in relation to quasi-metric spaces. Section three briefly discusses the concept of asymmetric normed spaces.

Chapter 3. In this chapter, we recall convexities in metric spaces, namely; Menger convexity, Takahashi convexity and M-convexity. Since these convexities rely on the concept of betweenness, a fundamental concept to axiomatic geometry, the first section gives the concept of betweenness in metric spaces, which was introduced by Blumenthal [4]. Thereafter, we recall the properties of betweenness. In the second section, we recall Menger convexity which was introduced by Karl Menger [21] and also recall the Menger Theorem of convexity. In the third section, we discuss a notion of convexity for metric spaces which was introduced in [28] by W. Takahashi. Then, we recall some geometric and topological properties which result when a uniqueness assertion is added to Takahashi's requirements. Thereafter, we will end this section by recalling the connection between Takahashi and Menger convexity in metric spaces. In the third section of this chapter, we start by recalling the definition of strong convex metric spaces and M-convex metric spaces. Then we show that a metric space (X, d) is strongly convex if and only if it is M-convex (see Lemma 3.4.8). Thereafter, we will show that if (X, d) is strictly convex, then it is M-convex. We will end this section by showing that a metric space (X, d) is M-convex if and only if any two points x and y of X can be joined by a unique curve of length $d(x, y)$. In the fifth section, we recall best approximations in M-convex metric spaces.

Chapter 4. In this chapter, we start our own investigations. In the first section, we introduce the concept of betweenness and midpoint in T_0-quasi-metric spaces which was introduced by Blumenthal [4]. We show that q-betweenness does not necessarily imply q^{-1}-betweenness (see Example 4.1.3). Thereafter, we will show that q, q^{-1}-betweenness implies q^{+}-betweenness in T_0-quasi-metric spaces. We will end this section by generalising a well known result [4, Theorem 12.1] for the relation of betweenness in metric spaces, to the setting of T_0-quasi-metric spaces. In the second section, we generalise the concept of Menger convexity [21], from metric settings to the framework of T_0-quasi-metric spaces. In the third section, we recall the convexity structure in the sense of Takahashi in T_0-quasi-metric spaces. We will end this section by showing the relationship between Takahashi and Menger convexity in T_0-quasi-metric spaces. In the fourth section, we generalise the concept of M-convexity from the metric setting to the framework of T_0-quasi-metric spaces. In the fifth section, we generalise the concept of best approximations in M-convex metric spaces.

Chapter 5. In this chapter, we discuss the findings and results of our work, and present some open problems to be studied in future.

Chapter 6 This chapter is the conclusion of this thesis.

CHAPTER 2

PRELIMINARIES

In this chapter, we recall some basic concepts to be used throughout the dissertation. For more details, we refer the reader to [9], [19], [15], [24], [6]. In some cases we provide the proofs as a motivation for generalisations to come, and for the sake of the reader.

2.1. Metric spaces

In this section, we recall the definition of a metric space and give some examples.

Definition 2.1.1. Let X be a set and $d : X \times X \longrightarrow [0, \infty)$ be a function mapping $X \times X$ into the set of nonnegative real numbers. Then d is called a pseudometric on X if

(i) $d(x, x) = 0$ for all $x \in X$

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$

The pair (X, d) is called a pseudometric space. If further, for $x \neq y$ we have that

$$d(x, y) > 0,$$

then d is a metric on X and the pair (X, d) is called a metric space.

Example 2.1.2. Let $X = l_\infty$ be a space whose elements consist of all bounded sequences $(x_n)_{n=1}^{\infty}$ of real numbers, with the distance between $x = (x_n)_{n=1}^{\infty}$ and $y = (y_n)_{n=1}^{\infty}$ taken as

$$d_\infty(x, y) = \sup_{1 \leq i < \infty} |x_i - y_i|$$

for all $x, y \in X$, then (X, d_∞) is a metric space.

Example 2.1.3. Let $\mathcal{F}(X)$ be a space of real valued functions $f : X \longrightarrow \mathbb{R}$, together with a special point $x_0 \in X$. Then x_0 induces a pseudometric on the space $\mathcal{F}(X)$ where $d : \mathcal{F}(X) \times \mathcal{F}(X) \longrightarrow [0, \infty)$ is given by

$$d(f, g) = |f(x_0) - g(x_0)|$$

for all $f, g \in \mathcal{F}(X)$.

Definition 2.1.4. Let (X, d) be a metric space. The open ball of radius $\varepsilon > 0$ centred at $x \in X$ is the set

$$B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}.$$

Similarly, a closed ball of radius $\varepsilon > 0$ centered at $x \in X$ is the set

$$C_d(x, \varepsilon) = \{y \in X : d(x, y) \leq \varepsilon\}.$$

We note that the collection of all open balls forms a base for a topology $\tau(d)$, and it is called the topology induced by the metric d on X.

Definition 2.1.5. Let (X, d) be a metric space and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. We say that (x_n) converges to a point x if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$d(x_n, x) < \varepsilon.$$

In this case we say that x is a limit of the sequence (x_n) in X and we write $\lim_{n\to\infty} x_n = x$.

Definition 2.1.6. Let (X, d) be a metric space.

(i) A sequence $(x_n)_{n\in\mathbb{N}}$ in X is Cauchy if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$,

$$d(x_n, x_m) \leq \varepsilon.$$

(ii) A metric space (X, d) is complete if every Cauchy sequence is convergent in X.

Definition 2.1.7. Let (X, d) be a metric space. Given $A \subseteq X$ and $x \in X$, the distance from a point x to a set A is defined by

$$dist(x, A) = \inf\{d(x, a) : a \in A\}.$$

Definition 2.1.8. Let (X, d) be a metric space. Given $A, B \subseteq X$, we define a Hausdorff distance between sets A and B by

$$d(A, B) = \max\big\{\sup_{a\in A}\inf_{b\in B} d(a, b), \sup_{b\in B}\inf_{a\in A} d(a, b)\big\}.$$

Definition 2.1.9. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $x_0 \in X$. A function $f : X \longrightarrow Y$ is said to be continuous at a point $x_0 \in X$ if given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$d_Y(f(x), f(x_0)) < \varepsilon$$

whenever $d_X(x, x_0) < \delta$ for all $x \in X$.

The function $f : X \longrightarrow Y$ is said to be continuous on X if it is continuous at each point of X.

Definition 2.1.10. Let (X, d_X) and (Y, d_Y) be metric spaces. A mapping $f : X \longrightarrow Y$ is said to be an isometry or isometric map provided that

$$d_Y(f(x), f(y)) = d_X(x, y)$$

whenever $x, y \in X$.

Two metric spaces (X, d_X) and (Y, d_Y) will be called isometric provided that there exists a bijective isometry $f : X \longrightarrow Y$ between them.

We next recall the concept of a norm defined on a vector space X over a field $\mathbb{F}$.

Definition 2.1.11. Let X be a vector space over a field $\mathbb{F}$. A norm on X is a map $||.|| : X \longrightarrow [0, \infty)$ that satisfies the following properties, for all $x, y, z \in X$ and $\alpha \in \mathbb{F}$;

(i) $||x|| \geq 0$.

(ii) $||x|| = 0$ if and only if $x = 0$.

(iii) $||\alpha x|| = |\alpha| ||x||$.

(iv) $||x + y|| \leq ||x|| + ||y||$.

A normed vector space is a pair $(X, ||.||)$, where X is a vector space and $||.||$ a norm on X.

We give some examples of a normed vector space over a field $\mathbb{F}$.

Example 2.1.12. Let $X = l^p$ be a space whose elements are sequences $x = (x_i)_{i=1}^{\infty}$ of complex numbers such that $\sum_{i=1}^{\infty} |x_i|^p < \infty$. For each $x \in X$ define

$$||x||_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}} \quad \text{for } 1 \leq p < \infty.$$

Then $(X, ||.||_p)$ is a normed vector space.

Example 2.1.13. Let $X = \mathcal{C}[a, b]$ be the set of all continuous real valued functions on a closed interval $[a, b]$. For $x \in X$, define

$$||x||_\infty = \sup_{a \leq t \leq b} |x(t)| \,.$$

Then $(X, ||x||_\infty)$ is a normed vector space.

We note that each norm on X induces a metric d by setting $d(x, y) = ||x - y||$ whenever $x, y \in X$.

2.2. Quasi-metric spaces

In this section, we recall the definition of quasi-pseudometric spaces and give some of their properties.

Definition 2.2.1. Let X be a nonempty set and consider a function $q : X \times X \longrightarrow [0, \infty)$. Then, q is called a quasi-pseudometric on X if

(i) $q(x, x) = 0$ whenever $x \in X$.

(ii) $q(x, z) \leq q(x, y) + q(y, z)$ whenever $x, y, z \in X$.

In addition, we shall say that q is a T_0-quasi-metric provided that q also satisfies the following T_0-condition: For each $x, y \in X$,

$$q(x, y) = 0 = q(y, x) \implies x = y.$$

The pair (X, q) is called a T_0-quasi-metric space.

Remark 2.2.2. If q is a quasi-metric on a set X, then $q^{-1} : X \times X \longrightarrow [0, \infty)$ defined by $q^{-1}(x, y) = q(y, x)$ for every $x, y \in X$, often called the conjugate or (dual) quasi-metric of q, is also a quasi-metric on X. The quasi-metric on a set X such that $q = q^{-1}$ is a metric. Note that if (X, q) is a T_0-quasi-metric space, then $q^s = \max\{q, q^{-1}\} = q \vee q^{-1}$ and $q^+ = q + q^{-1}$ are called the associated metrics of q on X.

Example 2.2.3. Let $X = [0, \infty)$ and define $q(x, y) = \max\{x - y, 0\}$ for $x, y \in X$. Then (X, q) is a T_0-quasi-metric space.

Proof. Let $x, y, z \in X$. Then we see that $q(x, x) = \max\{x - x, 0\} = 0$. Also, we note that

$$q(x, y) = \max\{x-y, 0\} = \max\{x-z+z-y, 0\} \leq \max\{x-z, 0\}+\max\{z-y, 0\} = q(x, z)+q(z, y).$$

Now, we observe that $q^s(x, y) = |x - y|$ whenever $x, y \in X$. If $q(x, y) = 0 = q(y, x)$, then it implies that $q^s(x, y) = 0$. Since q^s is a metric , therefore, we have that $x = y$ and so (X, q) is a T_0-quasi-metric space. □

The following describes some concepts related to asymmetric topologies of a quasi-metric space [6].

The topology $\tau(q)$ of a quasi-metric space (X, q) can be defined starting from the family $\nu_q(x)$ of neighbourhoods of an arbitrary point $x \in X$: For any $V \subseteq X$, we have that $V \in \nu_q(x)$ if and only if there exists $\delta > 0$ such that $B_q(x, \delta) = \{y \in X : q(x, y) < \delta\} \subseteq V$ if and only if there exist $\varepsilon > 0$ such that $C_q(x, \varepsilon) = \{y \in X : q(x, y) \leq \varepsilon\} \subseteq V$. A set $A \subseteq X$ is $\tau(q)$-open if and only if for every $x \in A$, there exists $\delta = \delta_x > 0$ such that $B_q(x, \delta) \subset A$. We shall say that A is a q-neighbourhood of x or that the set A is q-open.

Taking into consideration lack of symmetry, a quasi metric q generates three different topologies (see [6]), that we recall next.

Definition 2.2.4. Let (X, q) be a quasi-metric space, and $x \in X$ and $\delta > 0$. Then:

(i) the topology $\tau(q)$ is generated by the quasi-metric q, where the open and closed balls are described as follows: $B_q(x, \delta) \subseteq X$, where $B_q(x, \delta) = \{y \in X : q(x, y) < \delta\}$, and $C_q(x, \delta) \subseteq X$, where $C_q(x, \delta) = \{y \in X : q(x, y) \leq \delta\}$.

(ii) the topology $\tau(q^{-1})$ is generated by the quasi-metric q^{-1}, where the open and closed balls are described as follows: $B_{q^{-1}}(x, \delta) \subseteq X$, where $B_{q^{-1}}(x, \delta) = \{y \in X : q(y, x) < \delta\}$, and $C_{q^{-1}}(x, \delta) \subseteq X$, where $C_{q^{-1}}(x, \delta) = \{y \in X : q(y, x) \leq \delta\}$.

(iii) the topology $\tau(q^{-1})$ is generated by the quasi-metric q^{-1}, where the open and closed balls are described as follows: $B_{q^s}(x, \delta) \subseteq X$, where $B_{q^s}(x, \delta) = \{y \in X : q^s(x, y) < \delta\}$, and $C_{q^s}(x, \delta) \subseteq X$, where $C_{q^s}(x, \delta) = \{y \in X : q^s(x, y) \leq \delta\}$.

The following Propositions give some properties of asymmetric topologies:

Proposition 2.2.5. *Let q and ρ be quasi-metrics on X inducing the asymmetric topologies $\tau(q)$, $\tau(q^{-1})$ and $\tau(\rho)$, $\tau(\rho^{-1})$ respectively. Then, $\tau(q^s)$ is finer than $\tau(\rho^s)$ if and only if for all $x \in X$ and $\varepsilon > 0$, there is a $\delta > 0$ such that $B_{q^s}(x, \varepsilon) \subseteq B_{\rho^s}(x, \delta)$.*

Proof. Suppose that $\tau(\rho^s) \subseteq \tau(q^s)$. Let $x \in X$ and $\varepsilon > 0$ such that $B_{\rho^s}(x, \varepsilon)$ is open in $\tau(\rho^s)$, so it's open in $\tau(q^s)$. Since the q^s-open balls form a basis for $\tau(q^s)$, then for all $x \in X$, there is a $\delta > 0$ such that $x \in B_{q^s}(x, \delta) \subseteq B_{\rho^s}(x, \varepsilon)$ by the definition of a basis.

Conversely, suppose that for all $x \in X$ and $\varepsilon > 0$, there is a $\delta > 0$ such that $B_{q^s}(x, \delta) \subseteq B_{\rho^s}(x, \varepsilon)$. We need to show that $\tau(\rho^s) \subseteq \tau(q^s)$. Let U be open in $\tau(\rho^s)$, we must show that it is open in $\tau(q^s)$. Let $x \in U$. Since the ρ^s-open balls forms a basis for $\tau(\rho^s)$, there is an $\varepsilon > 0$ such that $x \in B_{\rho^s}(x, \varepsilon) \subseteq U$. By assumption, there is a $\delta > 0$ such that $x \in B_{q^s}(x, \delta) \subseteq B_{\rho^s}(x, \varepsilon)$. Thus, $x \in B_{q^s}(x, \delta) \subseteq U$. Since $x \in U$ was arbitrary, U is open in $\tau(q^s)$. Therefore, $\tau(\rho^s) \subseteq \tau(q^s)$. □

Proposition 2.2.6. *([6, Proposition 1.5]) Let (X, q) be a quasi-metric space, then whenever $x \in X$ and $\epsilon > 0$,*

(i) any ball $B_q(x, \epsilon)$ is $\tau(q)$-open, $B_{q^{-1}}(x, \epsilon)$ is $\tau(q^{-1})$-open and $C_q(x, \epsilon)$ is $\tau(q^{-1})$-closed. The ball $C_q(x, \epsilon)$ need not be $\tau(q)$-closed. Also, the following inclusions hold

$$B_{q^s}(x, \epsilon) \subseteq B_q(x, \epsilon) \quad \text{and} \quad B_{q^s}(x, \epsilon) \subseteq B_{q^{-1}}(x, \epsilon),$$

with the similar inclusions for the closed balls.

(ii) The topology $\tau(q^s)$ is finer than the topologies $\tau(q)$ and $\tau(q^{-1})$, indeed $\tau(q^s) = \tau(q) \vee \tau(q^{-1})$. This means that :

- *any $\tau(q)$-open(closed) set is $\tau(q^s)$-open(closed), similar results hold for the topology $\tau(q^{-1})$.*
- *the identity mappings from $(X, \tau(q^s))$ to $(X, \tau(q))$ and to $(X, \tau(q^{-1}))$ are continuous,*
- *a sequence $(x_n)_{n \in \mathbb{N}}$ in X is $\tau(q^s)$-convergent if and only if it is $\tau(q)$-convergent and $\tau(q^{-1})$-convergent.*

(iii) the topologies $\tau(q)$ and $\tau(q^{-1})$ are T_0, but not necessarily T_1 (and so nor T_2, in contrast to the case of metric spaces). The topology $\tau(q)$ is T_1 if and only if $q(x, y) > 0$ whenever $x \neq y$. In this case, $\tau(q^{-1})$ is also T_1 and, as a bitopological space, X is pairwise Hausdorff.

Proof. (i) Let $y \in X$ be such that $q(x, y) < \delta$ and $\varepsilon = \delta - q(x, y) > 0$. If $z \in X$ is such that $q(y, z) < \varepsilon$ then $q(x, z) \leq q(x, y) + q(y, z) < q(x, y) + \varepsilon = \delta$, this shows that $B_q(x, \varepsilon) \subseteq B_q(x, \delta)$. Hence $B_q(x, \epsilon)$ is $\tau(q)$-open. Similarly, let $y \in X$ with $q(x, y) > \epsilon$ and $\delta = q(x, y) - \epsilon > 0$. If $z \in X$ is such that $q(z, y) = q^{-1}(y, z) < \delta$ then $q(x, y) \leq q(x, z) + q(z, y) < q(x, z) + \delta$, so that $B_{q^{-1}}(x, \delta) \subseteq B_{q^{-1}}(x, \epsilon)$. Hence $B_{q^{-1}}(x, \delta)$ is $\tau(q^{-1})$-open

Next, we prove that $C_q(x, \epsilon)$ is $\tau(q^{-1})$-closed. Let $y \in X$ be such that $y \notin X \backslash C_q(x, \epsilon)$ and setting $\delta = q(x, y) - \epsilon > 0$. Then $B_{q^{-1}}(x, \delta) \cap C_q(x, \epsilon) = \emptyset$, or equivalently $B_{q^{-1}}(x, \delta) = X \backslash C_q(x, \epsilon)$. Indeed, if $z \in B_{q^{-1}}(x, \delta) \cap C_q(x, \epsilon)$, with $q(z, y) = q^{-1}(y, z) < \delta$, then we have that $q(x, y) \leq q(x, z) + q(z, y) < \epsilon + q(z, y) < \epsilon + \delta = q(x, y)$, a contradiction. Therefore, $X \backslash C_q(x, \epsilon)$ is $\tau(q^{-1})$-open and so $C_q(x, \epsilon)$ is $\tau(q^{-1})$-closed.

Since $q(x,y) \leq q^s(x;y)$ and $q^{-1}(x,y) \leq q^s(x,y)$ for all $x, y \in X$, then the given inclusions hold.

(ii) Suppose that $q^{-1}(x,y) \leq q^s(x,y)$. From $q^s(x,y) = \max\{q(x,y), q^{-1}(x,y)\}$, we have that $A \in \tau(q^s)$ is equivalent to the fact that for every $x \in A$ there exists a quasi-metric q and $\delta > 0$ such that $B_{q^s}(x,\delta) \subseteq A$. Since $q(x,y) < \delta$ implies that $q^s(x,y) \leq q(x,y) < \delta$, we have $B_q(x,\delta) \subseteq B_{q^s}(x,\delta) \subseteq A$, so that $A \in \tau(q)$. Hence, $B_{q^s}(x,\delta) \subseteq B_q(x,\delta)$ and so by Proposition 2.2.5 we have $\tau(q) \subseteq \tau(q^s)$. Similarly, if $q(x,y) \leq q^s(x,y)$, then $A \in \tau(q^s)$ with $x \in A$ implies that there exist a quasi-metric q and $\delta > 0$ such that $B_{q^s}(x,\delta) \subseteq A$. As $q(y,x) < \delta$ implies $q(x,y) \leq q(y,x) < \delta$, it follows that $B_{q^{-1}}(x,\delta) \subseteq B_{q^s}(x,\delta) \subset A$, so that $A \in \tau(q^{-1})$. Hence, $B_{q^s}(x,\delta) \subseteq B_{q^{-1}}(x,\delta)$ and so by Proposition 2.2.5 we have that $\tau(q^{-1}) \subseteq \tau(q^s)$. Therefore, we conclude that the topology $\tau(q^s)$ is finer than the topologies $\tau(q)$ and $\tau(q^{-1})$.

(iii) If x and y are distinct points in the quasi-metric space (X,q) then $\max\{q(x,y), q(y,x)\} > 0$. If $q(x,y) > 0$, then $y \notin B_q(x,\varepsilon)$ where $\varepsilon = q(x,y)$. Similarly, if $q(y,x) > 0$, then $x \notin B_q(y,\delta)$, where $\delta = q(y,x)$. Consequently, $\tau(q)$ is T_0 and $\tau(q^{-1})$ as well.

Next, suppose that $q(x,y) > 0$ for every $x \neq y$. Then $y \notin B_q(x, q(x,y))$. Since $q(y,x) > 0$ too, $x \notin B_q(y, q(y,x))$, showing that the topology $\tau(q)$ is T_1. Similarly, $\tau(q^{-1})$ is T_1.

Conversely suppose that $\tau(q)$ is T_1 and let $x, y \in X$, $x \neq y$. Then, there exists a quasi-metric q and $\delta > 0$ such that $x \notin B_q(y,\delta)$, which implies that $q(x,y) \geq \delta$. Also $B_{q^{-1}}(x,\delta) \cap B_q(x,\delta) = \emptyset$ where $\delta > 0$ is given by $2\delta = q(x,y) > 0$. Indeed if $z \in B_{q^{-1}}(x,\delta) \cap B_q(x,\delta)$, then

$$q(x,y) \leq q(x,z) + q(z,y) < \delta + \delta = q(x,y)$$

a contradiction, which shows that $(X, \tau(q), \tau(q^{-1}))$ is a pairwise Hausdorff.

□

We present the following standard definitions.

Definition 2.2.7. If (X,q) is a quasi-metric space, then the pair $(B_q(x,r), B_{q^{-1}}(x,s))$ where $x \in X$ and $r, s \in [0,\infty)$ is called a double ball. In general, $(B_q(x_i,r_i), (B_{q^{-1}}(x_i,s_i))_{i \in I}$, with $x_i \in X$ and $r_i, s_i \in [0,\infty)$, is called a family of double balls.

Definition 2.2.8. Let (X,q) be a quasi-metric space. For any subset A of X, we call $cl_{\tau(q)}A \cap cl_{\tau(q^{-1})}A$ the double closure of A. Moreover, if $A = cl_{\tau(q)}A \cap cl_{\tau(q^{-1})}A$, then we say A is a doubly closed subset of X.

Definition 2.2.9. A subset A of a quasi-metric space (X,q) is said to be bounded if there exists a real number $M > 0$ such that $q(x,y) \leq M$ whenever $x, y \in A$. Equivalently, a subset A of (X,q) is bounded if there is an $x \in X$ and $r, s > 0$ such that $A \subseteq C_q(x,r) \cap C_{q^{-1}}(x,s)$.

Definition 2.2.10. Given a subset A of a quasi-metric space (X,d), the diameter of A, denoted by $diam(A)$ is defined as $diam(A) = \sup\{q(x,y) : x, y \in A\}$. We say that set A is bounded if $diam(A) < \infty$.

Definition 2.2.11. Let (X, q) be a quasi-metric space. Given $A, B \in \mathscr{P}_0(X)$ and $x \in X$, a mapping $q_H : \mathscr{P}_0(X) \times \mathscr{P}_0(X) \longrightarrow [0, \infty)$ defined by

$$q_H(A, B) = \max\left\{\sup_{a \in A} dist(a, B),\ \sup_{b \in B} dist(b, A)\right\}.$$

is said to be the extended Hausdorff (-Bourbaki) quasi-pseudometric on $\mathscr{P}_0(X)$. It is known that q_H is an extended T_0-quasi-metric when restricted to the set of all nonempty subsets A of X which satisfy $A = cl_{\tau(q)}A \cap cl_{\tau(q^{-1})}A$.

The following describe some properties of maps between two quasi-pseudometric spaces;

Definition 2.2.12. Let (X, q_X) and (Y, q_Y) be quasi-metric spaces. A map $f : (X, q_X) \longrightarrow (Y, q_Y)$ is said to be an isometry provided that $q_Y(f(x), f(y)) = q_X(x, y)$ whenever $x, y \in X$, that is, f is distance preserving.

Two quasi metric spaces (X, q_X) and (Y, q_Y) will be called isometric provided that there exists a bijective isometry $f : (X, q_X) \longrightarrow (Y, q_Y)$ between them.

We next recall some basic concepts related to the convergence of sequences in quasi-metric spaces;

Definition 2.2.13. Let (X, q) be a quasi-metric space.

(i) A sequence (x_n) converges to x with respect to $\tau(q)$, called q-convergence or left-convergence and denoted by $x_n \xrightarrow{q} x$, if and only if

$$q(x, x_n) \longrightarrow 0.$$

(ii) A sequence (x_n) converges to x with respect to $\tau(q^{-1})$, called q^{-1}-convergence or right-convergence and denoted by $x_n \xrightarrow{q^{-1}} x$, if and only if

$$q(x_n, x) \longrightarrow 0.$$

(iii) A sequence (x_n) q^s-converges to x if it is both left and right q-convergent to x. That is

$$x_n \xrightarrow{q^s} x \iff x_n \xrightarrow{q} x \text{ and } x_n \xrightarrow{q^{-1}} x.$$

Definition 2.2.14. A sequence (x_n) in a quasi-metric (X, q) is called

(i) left q-Cauchy if for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ and $x \in X$ such that

$$\forall n \geq n_\varepsilon \quad q(x, x_n) < \varepsilon.$$

(ii) right q-Cauchy if for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ and $x \in X$ such that

$$\forall n_\varepsilon, \quad q(x_n, x) < \varepsilon.$$

(iii) left K-Cauchy if for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$\forall n, k : n_\varepsilon \geq k \geq n, \quad q(x_k, x_n) < \varepsilon.$$

(iv) right K-Cauchy if for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$\forall n, k : n_\varepsilon \geq k \geq n, \quad q(x_n, x_k) < \varepsilon.$$

(v) q^s-Cauchy if for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\forall n, k \geq n_\varepsilon \quad q(x_n, x_k) < \varepsilon.$$

Remark 2.2.15. (i) q^s-Cauchy $\Longrightarrow$ left K-Cauchy $\Longrightarrow$ left q-Cauchy. The same implications holds for the corresponding right notations. However, none of the above implications are reversible.

(ii) A sequence (x_n) in a T_0-quasi-metric space is left Cauchy with respect to q if and only if it is right Cauchy with respect to q^{-1}.

(iii) A q-convergent sequence is left q-Cauchy and a q^{-1}-convergent is a right q-Cauchy.

(iv) A sequence is q^s-Cauchy if and only if it is both left and right K-Cauchy.

The following results concerning sequences in quasi-metric spaces are true.

Proposition 2.2.16. *Let (x_n) be a sequence in a T_0-quasi-metric space (X, q):*

(i) If (x_n) is $\tau(q)$-convergent to x and $\tau(q^{-1})$-convergent to y, then $q(x, y) = 0$.

(ii) If (x_n) is $\tau(q)$-convergent to x and $q(y, x) = 0$, then (x_n) is $\tau(q^{-1})$-convergent to y.

(iii) If (x_n) is left K-Cauchy and has a subsequence which is $\tau(q)$-convergent to x, then (x_n) is $\tau(q)$-convergent to x.

(iv) If (x_n) is left K-Cauchy and has a subsequence which is $\tau(q^{-1})$-convergent to x, then (x_n) is $\tau(q^{-1})$-convergent to x.

Proof. (i) Suppose that (x_n) is $\tau(q)$-convergent to x and $\tau(q^{-1})$-convergent to y , then $x_n \to x$ if and only if $q(x, x_n) \longrightarrow 0$ as $n \to \infty$ and $x_n \to y$ if and only if $q(x_n, y) \longrightarrow 0$ as $n \to \infty$. Now, using triangle inequality and letting $n \to \infty$, we have that

$$q(x, y) \leq q(x, x_n) + q(x_n, y) \longrightarrow 0$$

and so we obtain that $q(x, y) \leq 0$ which implies that $q(x, y) = 0$.

(ii) Suppose that (x_n) is $\tau(q)$-convergent to x and $q(y, x) = 0$, then $x_n \to x$ if and only if $q(x, x_n) \longrightarrow 0$ as $n \to \infty$ and $q(y, x) = 0$. Then using the triangle inequality and letting $n \longrightarrow \infty$, we have

$$q(y, x_n) \leq q(y, x) + q(x, x_n) = q(x, x_n) \longrightarrow 0$$

and so $q(y, x_n) \longrightarrow 0$ as $n \to \infty$. Hence, (x_n) is $\tau(q^{-1})$-convergent to y.

(iii) Suppose that (x_n) is left K-Cauchy and (x_{n_k}) is a subsequence of (x_n) such that $\lim_{k \longrightarrow \infty} q(x, x_{n_k}) = 0$. For $\varepsilon > 0$ choose n_0 such that $n_0 \leq m \leq n$ implies $q(x_m, x_n) < \varepsilon$, and let $k_0 \in \mathbb{N}$ be such that $n_{k_0} \geq n_0$ and $q(x, x_{n_k}) < \varepsilon$ for all $k \geq k_0$. Then, for $n \geq n_{k_0}$, $q(x, x_n) \leq q(x, x_{n_{k_0}}) + q(x_{n_{k_0}}, x_n) < 2\varepsilon$.

(iv) Suppose that (x_n) is left K-Cauchy such that there exists a subsequence (x_{n_k}) which is $\tau(q^{-1})$-convergent to some $x \in X$. For $\varepsilon > 0$ let $k_0 \in \mathbb{N}$ be such that for all $k \geq k_0$, $q(x_{n_k}, x) < \varepsilon$, and let $n_0 \in \mathbb{N}$ be such that for all $m, n \in \mathbb{N}$, $n_0 \leq m < n$ implies $q(x_m, x_n) < \varepsilon$. For $n \geq n_{k_0}$ let $k > k_0$ be such that $n_k \leq n$, $k \in \mathbb{N}$. Then $q(x_n, x) \leq q(x_n, x_{n_k}) + q(x_{n_k}, x) < 2\varepsilon$.

□

Definition 2.2.17. Let (X, q) be a quasi-metric space. We say that (X, q) is

(i) left K-complete provided that any left K-Cauchy sequence in X is q-convergent.

(ii) right K-complete provided that any right K-Cauchy sequence in X is q^{-1}-convergent.

Definition 2.2.18. Let (X, q) be a T_0-quasi-metric space. Then (X, q) is called bicomplete provided that the associated metric space (X, q^s) is complete.

2.3. Asymmetric normed spaces

In this section, we recall the definition of an asymmetric norm on a real vector space X and give some examples.

Definition 2.3.1. ([6, p.10]) Let X be a real vector space and $\|.| : X \longrightarrow [0, \infty)$ be a mapping of X into the set $[0, \infty)$. Then $\|.|$ is called an asymmetric semi-norm on X if for all $x, y \in X$ and $\alpha \in [0, \infty)$ we have that

(i) $\|\alpha x| = \alpha \|x|$

(ii) $\|x + y| \leq \|x| + \|y|$

If in addition, we have

(iii) $\|x| = \|-x| = 0$ if and only if $x = 0$,

then $\|.|$ is called an asymmetric norm on X, and the pair $(X, \|.|)$ is called an asymmetric normed space.

We note that each asymmetric norm on X induces a T_0-quasi-metric q by setting $q(x, y) = \|x - y|$ whenever $x, y \in X$.

Remark 2.3.2. ([6, Remark 1]) If $\|.|$ is an asymmetric norm on a real vector space X, then the function $|.\| : X \longrightarrow [0, \infty)$ defined by $|x\| = \|-x|$, whenever $x \in X$ is also an asymmetric norm on X called the conjugate to $\|.|$. We note that the symmetrisation of the asymmetric norm $\|.|$ is the function $||.|| : X \longrightarrow [0, \infty)$ given by $||x|| = \max\{\|x|, |x\|\}$ whenever $x \in X$ and is called a norm on X.

We now look at some examples of an asymmetric norm on a real vector spaces:

Example 2.3.3. ([6, Example 1.2]) If $X = \mathbb{R}$ is a real vector space, consider the asymmetric norm defined for all $x \in \mathbb{R}$ by $||x| = x^+$ where $x^+ = \max\{x, 0\}$ is the positive part of x. Then $|x|| = x^- = \max\{-x, 0\}$ and $||x|| = \max\{x^+, x^-\} = |x|$.
Example 2.3.4. ([19, Example 3]) Let (X, q) be a T_0-quasi-metric space, $C_b(X)$ be the real vector space of continuous bounded real valued functions on X and $\|f| = \sup_{x\in X}(\max\{f(x) - 0, 0\})$ whenever $f \in C_b(X)$. Then $\|f|$ is an asymmetric norm on $C_b(X)$.

Proof. We show that $||f|$ is an asymmetric norm on $C_b(X)$. Let $f, g \in C_b(X)$ and $\alpha \in \mathbb{R}$.

We show that the three conditions of asymmetric norm are satisfied

(i)
$$\begin{aligned} ||\alpha f| &= \sup_{x\in X}[\max\{\alpha f(x) - 0, 0\}] \\ &= \sup_{x\in X}[\alpha \max\{f(x) - 0, 0\}] \\ &= \alpha \sup_{x\in X}[\max\{f(x) - 0, 0\}] \\ &= \alpha ||f|. \end{aligned}$$

(ii)
$$\begin{aligned} ||f + g| &= \sup_{x\in X}[\max\{f(x) + g(x) - 0, 0\}] \\ &\leq \sup_{x\in X}[\max\{f(x) - 0, 0\} + \max\{g(x) - 0, 0\}] \\ &\leq \sup_{x\in X}(\max\{f(x) - 0, 0\}) + \sup_{x\in X}(\max\{g(x) - 0, 0\}) \\ &= ||f| + ||g|. \end{aligned}$$

(iii) We first observe that $||f|| = \max\{||f|, || - f|\}$ is a norm on $C_b(X)$. Now, if $||f| = 0$ and $|| - f| = 0$, and (X, q) is a T_0-quasi-metric space, then $||f|| = \max\{||f|, || - f|\} = \max\{0, 0\} = 0$. Since $||f||$ is a norm on $C_b(X)$, then we have that $f = 0$.
Conversely, if $f = 0$, then we have that $||f| = \sup_{x\in X}[\max\{0 - 0, 0\}] = \sup_{x\in X}(0) = 0$ and $|| - f| = \sup_{x\in X}[\max\{-0 - 0, 0\}] = \sup_{x\in X}(0) = 0$. Hence, we have that $||f| = 0 = || - f|$ if and only if $f = 0$.

$\square$

CHAPTER 3

CONVEXITIES IN METRIC SPACES

In 1970, Takahashi [28] introduced a convex structure on a metric space which is called Takahashi convex structure and is a generalization of the convex structure in the ordinary sense. Earlier to it, in 1928, Menger [21] proposed another concept of convexity in a metric space, which is known as Menger convexity and is a generalization of convexity in the usual sense as well. Later on Khalil [11], defined strong convex metric spaces which is also the generalization of convexity in ordinary sense.Therefore, we note that all the three convexities are the generalization of the ordinary convexity.
In this chapter, we recall convexities in metric spaces, namely; Menger convexity, Takahashi convexity, strong and M-convexity (see [21], [28], [11]). We will see that all these convexities are the generalization of convexity structure in an ordinary sense and the converse is not true. Since these convexities rely on the concept of betweenness, a fundamental concept to the study of axiomatic geometry, we start by recalling betweenness in metric spaces. Thereafter, we recall convexities in metric spaces and some best approximations for M-convex metric spaces.

3.1. Betweenness in metric spaces

In this section, we recall the concept of betweenness in metric spaces. This notion was introduced by Blumenthal [4].
Definition 3.1.1. ([4, Definition 12.1]) Let (X, d) be a metric space. A point $z \in X$ is said to be between x and y if and only if $x \neq z \neq y$,

$$d(x, y) = d(x, z) + d(z, y).$$

We shall symbolize this relationship by $[xzy]_d$ to mean z is between x and y in a metric space X. We must also note that $[xzy]_d$ implies that x, y and z are pairwise distinct, since $d(x, y) > 0$.
Definition 3.1.2. ([23, Definition 2.1.0]) Let (X, d) be a metric space and $x, y, z \in X$. For any points $x, y, z \in X$, the set

$$B^d(x, y) = \{z \in X : d(x, z) + d(z, y) = d(x, y)\}$$

is called the metric segment of x and y.

Definition 3.1.3. ([4, Definition 13.3]) Let (X, d) be a metric space and $x, y, z \in X$. Then z is said to be a midpoint of x and y if and only if $x \neq y \neq z$,

$$d(x, z) = d(z, y) = \frac{d(x, y)}{2}.$$

Theorem 3.1.4. ([4, Theorem 12.1]) In a metric space (X, d), the relation of betweenness has the following properties:

(i) $[xzy]_d$ implies $[yzx]_d$ (symmetry of the outer points).

(ii) If $[xzy]_d$ then neither $[xyz]_d$ nor $[zxy]_d$ holds (special inner points).

(iii) $[xzy]_d$ and $[xyw]_d$ are equivalent to $[xzw]_d$ and $[zyw]_d$.

(iv) If $x, y \in X$, the set $\bar{B}^d(x, y) = \{x\} \cup \{y\} \cup B^d(x, y)$ is closed, where $B^d(x, y)$ is the set of all points between x and y.

Proof. (i) Let x and y be distinct points of X. If $[xzy]_d$ then $d(x, z) + d(z, y) = d(x, y)$. Since X is a metric space, then the symmetry condition gives $d(y, z) + d(z, x) = d(y, x)$ which implies that $[yzx]_d$.

(ii) Since $[xzy]_d$ implies that $x \neq z \neq y$, $d(x, z) + d(z, y) = d(x, y)$ and $[xyz]_d$ implies that $x \neq y \neq z$, $d(x, y) + d(y, z) = d(x, z)$. From the two equalities we have $2d(z, y) = 0$ which implies that $z = y$, which contradicts that $y \neq z$. Hence, $[xyz]_d$ cannot hold. Similarly, since $[xzy]_d$ implies that $x \neq z \neq y$, $d(x, z) + d(z, y) = d(x, y)$ and $[zxy]_d$ implies that $z \neq x \neq y$, $d(z, x) + d(x, y) = d(z, y)$. From these two equalities we obtain $2d(x, z) = 0$, which implies that $x = z$, contradicting that $x \neq z$. Therefore, $[zxy]_d$ cannot hold too.

(iii) Suppose $[xzy]_d$ and $[xyw]_d$ holds, then we have $x \neq y \neq z$, $d(x, z) + d(z, y) = d(x, y)$ and $x \neq y \neq w$, $d(x, y) + d(y, w) = d(x, w)$ for all $x, y, z, w \in X$. Now, from $d(x, z) + d(z, y) = d(x, y)$ and $d(x, y) + d(y, w) = d(x, w)$ we get

$$d(x, w) = d(x, z) + d(z, y) + d(y, w). \tag{3.1}$$

Applying the triangle inequality to Equation 3.1, we have

$$\begin{aligned} d(x, w) &= d(x, z) + d(z, y) + d(y, w) \\ &\geq d(x, z) + d(z, w) \\ &\geq d(x, w). \end{aligned}$$

This implies that $d(x, z) + d(z, w) = d(x, w)$. Since each two points are pairwise distinct, we have that $[xzw]_d$. Now, substituting $d(x, z) + d(z, w) = d(x, w)$ in Equation 3.1, we obtain $d(z, w) = d(z, y) + d(y, w)$ and also, since each two points are pairwise distinct, we have that $[zyw]_d$. The converse follows directly from the above argument.

(iv) We show that if z is a limit point (or accumulation point) of $\bar{B}^d(x,y)$, with $x \neq y \neq z$, then $z \in \bar{B}^d(x,y)$. Since z is a limit point, there exists a sequence $\{z_n\}$ that converges to z, where $z_n \in \bar{B}^d(x,y)$. Hence, we have

$$d(x,z_n) + d(z_n,y) = d(x,y)$$

for all $n \in \mathbb{N}$. From continuity of a metric d, we have

$$d(x,z) + d(z,y) = d(x,y)$$

and so we have that $z \in \bar{B}^d(x,y)$. Therefore, $\bar{B}^d(x,y)$ is a closed set.

□

3.2. Menger convexity in metric spaces

In this section, we recall the concept of Menger convexity which was introduced by Karl Menger [21] in 1928. He used the idea of betweenness to define convexity in metric spaces. We start by recalling the definition of convexity in the usual sense and give some examples of in this case:

Definition 3.2.1. Let X be a linear space and $\lambda \in [0,1]$. A subset A of X is said to be convex if for all $x,y \in A$, we have

$$x\lambda + (1-\lambda)y \in A.$$

A point of the form $x\lambda + (1-\lambda)y$, $\lambda \in [0,1]$ is called a convex combination of x and y.

Example 3.2.2. ([23]) Consider the interval $[a,b] \in \mathbb{R}$. We show that this interval is a convex set. Let $u,v \in [a,b]$ be two arbitrary elements. We need to prove that $tu+(1-t)v \in [a,b]$ for all $t \in [0,1]$. Since $u,v \in [a,b]$, then $u,v \leq b$. As $t \in [0,1]$, it follows that $tu + (1-t)v \leq b$. Using a similar argument $tu + (1-t)v \leq a$. As u and v were arbitrarily chosen, then $tu + (1-t)v \in [a,b]$ for all $u,v \in [a,b]$ and $t \in [0,1]$.

We now recall Menger convexity in metric spaces and later show that convexity in ordinary sense implies Menger convexity but the converse is not true.

Definition 3.2.3. ([21, Definition 2.1.1]) A metric space (X,d) is said to be Menger convex if for every $x,y \in X$ and for each $t \in [0,1]$, there exists $z \in X$ satisfying the following two conditions

(i) $d(x,z) = td(x,y)$

(ii) $d(z,y) = (1-t)d(x,y)$.

Remark 3.2.4. ([8, Remark 2.8]) A metric space (X,d) is said to be Menger convex if for every $x,y \in X$, there exists a point $z \in X$ such that

$$d(x,z) + d(z,y) = d(x,y).$$

We notice that we have just added conditions (i) and (ii) in Definition 3.2.4 to obtain the result in above remark and so we observe that Menger convexity implies betweenness in metric spaces but the converse is not true.

The following example shows that Menger convexity does not imply convexity in the usual sense:

Example 3.2.5. ([16, p.112]) Let $X = \mathbb{Q}$ be the set of rational numbers in $[0, 1]$ and d be a metric defined by $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is Menger convex. To see this, let $x, y \in X$, $x \neq y$, set $z = \frac{x+y}{2} \in X$, then we have

$$d(x,z) = \left| x - \frac{x+y}{2} \right| = \left| \frac{2x-x-y}{2} \right| = \frac{1}{2} |x-y| = \frac{1}{2} d(x,y)$$

and

$$d(y,z) = \left| y - \frac{x+y}{2} \right| = \left| \frac{2y-x-y}{2} \right| = \frac{1}{2} |y-x| = \frac{1}{2} d(x,y).$$

That is, $d(x, y) = d(x, z) + d(z, y)$ for all $x, y, z \in X$.
On the other hand, the set of all rational numbers in $[0, 1]$ is not convex. Since for every two rational numbers there is an irrational number between them. Hence it is not possible to join two points without leaving the set.

The following Proposition gives an equivalent definition of Menger convex metric space. This result will be extended to T_0-quasi-metric setting with minor modification.

Proposition 3.2.6. *([23, Lemma 2.1.2]) Let (X, d) be a metric space. Then (X, d) is Menger convex if and only if for every $x, y \in X$ with $x \neq y$ we have*

$$C_d(x, r) \cap C_d(y, r - \lambda) \neq \emptyset,$$

where $r = d(x, y)$ and every $\lambda \in [0, r]$.

Proof. Suppose that for every $x, y \in X$ with $x \neq y$ we have

$$C_d(x, \lambda) \cap C_d(y, r - \lambda) \neq \emptyset,$$

where $r = d(x, y)$ and every $\lambda \in [0, r]$. Let $x, y \in X$, $0 \leq t \leq 1$. Then $0 \leq tr \leq r$. Let $r_1 = tr$ and $r_2 = r - tr$ .i.e $r_1 + r_2 = r = d(x, y)$. Then there exists $z \in C_d(x, r_1) \cap C_d(y, r_2)$ such that $d(x, z) \leq r_1$ and $d(z, y) \leq r_2$.
Now, using the triangle inequality we have,

$$d(x, y) \leq d(x, z) + d(z, y) \leq r_1 + r_2 = r = d(x, y)$$

and so $d(x, y) = d(x, z) + d(z, y)$, thus we have that $d(x, z) = r_1 = tr = td(x, y)$ and $d(z, y) = r_2 = r - tr = (1 - t)d(x, y)$. Hence, (X, d) is Menger convex metric space.
Conversely, suppose that (X, d) is Menger convex. Let x and y be distinct points of X such that $d(x, y) = r$ and $\lambda \in [0, r]$. We want to show that

$$C_d(x, \lambda) \cap C_d(y, r - \lambda) \neq \emptyset.$$

To do this, let $t = \frac{\lambda}{r}$, so that $0 \leq t \leq 1$, and by Menger convexity of (X, d), there exists $z \in X$ such that

$$d(x, z) = td(x, y) = tr = \lambda \quad \text{and} \quad d(z, y) = (1-t)d(x, y) = (1-t)r = r - \lambda.$$

This implies that $z \in C_d(x, \lambda)$ and $z \in C_d(y, r-\lambda)$ and so $z \in C_d(x, \lambda) \cap C_d(y, r-\lambda)$. Hence,

$$C_d(x, \lambda) \cap C_d(y, r-\lambda) \neq \emptyset.$$

□

Remark 3.2.7. A subset A of a convex metric space (X, d) is said to be Menger convex if for every $x, y \in A$, any point between x and y also lies in A.

We now give the fundamental theorem of metric convexity which was introduced by Menger (1928) in [21]. We start by stating, without proof the Caristi's Theorem, and later use it to prove Lemma 3.2.11. The proof of Caristi's Theorem can be found in [8].

Definition 3.2.8. Consider a function $f : X \longrightarrow X$ and a point $x_0 \in X$. The function f is said to be upper (resp. lower) semi-continuous at the point x_0 if

$$f(x_0) \geq \lim_{x \to x_0} \sup f(x) \qquad \left(\text{resp. } f(x_0) \leq \lim_{x \to x_0} \inf f(x)\right).$$

Definition 3.2.9. ([8, Definition 1.2.1]) Let (X, d) be a metric space. A self-mapping $f : X \longrightarrow X$ is said to be a Caristi's mapping if there exists a lower semi-continuous function $\varphi : X \longrightarrow [0, +\infty)$ such that

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)),$$

for all $x \in X$.

The following theorem will be used to prove Lemma 3.2.11 and Lemma 3.2.12 and later these Lemmas will be used to prove Theorem 3.2.14.

Theorem 3.2.10. (Caristi's Theorem) Let (X, d) be a complete metric space and let $\varphi : X \longrightarrow [0, +\infty)$ be a lower semi-continuous and bounded function. Suppose that $f : X \longrightarrow X$ is an arbitrary self-mapping which satisfies:

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)), x \in X.$$

Then f has a fixed point.

Lemma 3.2.11. *([8, Lemma 2.1]) Let (X, d) be a complete metric space with $x, y \in X, x \neq y$, and suppose $0 < \lambda < d(x, y)$. Let $S = S(x, y, \lambda) = \{z \in B^d(x, y) : d(x, z) \leq \lambda\} \cup \{x\}$. Then there exists a point $z_\lambda \in X$ such that*

(i) $z_\lambda \in S(x, y, \lambda)$.

(ii) $u \in B^d(x, y)$ *and* $[xz_\lambda u]_d$ *implies* $d(x, u) > \lambda$.

Proof. (i) For each $z \in S$ with $d(x,z) < \lambda$ there exists y_z such that $[xzy_\lambda]_d$. In this case we define a mapping $G : S \longrightarrow S$ by $G(z) = y_\lambda$ if $d(x,z) < \lambda$ and $G(z) = z$ if $d(x,z) \geq \lambda$. Also define $\varphi : S \longrightarrow (0,\infty)$ by $\varphi(z) = \lambda - d(x,z)$. Since the metric d is continuous and λ is a constant, then φ is continuous on S. For $z \in S$ and from $d(x,G(z)) = d(x,z) + d(z,G(z))$, we have

$$\begin{aligned} d(z,G(z)) &= d(x,G(z)) - d(x,z) \\ &= \lambda - \lambda + d(x,G(z)) - d(x,z) \\ &= \lambda - d(x,z) - (\lambda - d(x,G(z))) = \varphi(z) - \varphi(G(z)). \end{aligned}$$

.

Since S is a closed subset of a complete metric space X, then by Caristi's theorem, we have $G(z) = z$ for some $z \in S$. This implies that $d(x,z) = \lambda$ and so we can choose $z_\lambda = z \in S$

(ii) Let $u \in B^d(x,y)$ and that there exists $z' \in S$ with $d(x,z') \leq \lambda$ such that $[xz'u]_d$. Then it follows that $d(x,u) > \lambda$. Hence the result.

□

Lemma 3.2.12. *([8, Lemma 2.2]) Let (X,d) be a complete Menger convex metric space with $x, y \in X, x \neq y$, and suppose $0 < \lambda < d(x,y)$. Then there exists $z' \in S$ such that $[xz'y]_d$ and $d(x,z') = \lambda$.*

Proof. By Lemma 3.2.11, there exist $z_\lambda \in S$ such that

(i) $z_\lambda \in S(x,y,\lambda)$.

(ii) $u \in B^d(x,y)$ and $[xz_\lambda u]_d$ implies $d(x,u) > \lambda$.

Let $\lambda' = d(x,y) - \lambda$ and again applying Lemma 3.2.11 we obtain $y_\lambda \in X$ such that

(i)$'$ $y_{\lambda'} \in S(y, z_\lambda, \lambda')$.

(ii)$'$ $u \in B^d(y, z_\lambda)$ and $[yy_{\lambda'}u]_d$ implies $d(y,u) > \lambda'$.

Case 1. Suppose $z_\lambda = y_{\lambda'}$. Since $z_\lambda = y_{\lambda'} \in S(y, z_\lambda, \lambda')$, we have $d(y, z_\lambda) = d(y, y_{\lambda'}) \leq \lambda'$ and also $z_\lambda \in S(x,y,\lambda)$, gives $d(x,z_\lambda) \leq \lambda$. Now, using the triangle inequality

$$\begin{aligned} d(x,y) &\leqslant d(x,z_\lambda) + d(z_\lambda, y) \\ &\leqslant \lambda + \lambda' = d(x,y). \end{aligned}$$

Thus, $d(x,y) = d(x,z_\lambda) + d(z_\lambda, y)$. Since $d(x,y) = \lambda + \lambda' = d(x,z_\lambda) + d(z_\lambda, y)$, then it follows that $d(x,z_\lambda) = \lambda$.

Case 2. Suppose $z_\lambda \neq y_{\lambda'}$. In this case since X is convex in the sense of Menger, there exists $w \in X$ such that $[z_\lambda w y_{\lambda'}]_d$. By assumption the relations $[xz_\lambda y]_d$, $[z_\lambda y_{\lambda'} y]_d$ and $[z_\lambda w y_{\lambda'}]_d$ hold. It follows from transitivity of betweenness that $[xwy]_d$, $[xz_\lambda w]_d$, $[ywz_\lambda]_d$ and $[yy_{\lambda'}]_d$

also hold. Now, $[xwy]_d$ and $[xz_\lambda w]_d$ imply $d(x,w) > \lambda$ by (ii) while $[ywz_\lambda]_d$ and $[yy_{\lambda'}w]_d$ imply $d(y,w) > \lambda'$ by $(ii)'$. Therefore, $d(x,y) = d(x,w) + d(w,y) > \lambda + \lambda' = d(x,y)$. This is a contradiction and so using the same argument as in case 1, we have that $d(x,w) = \lambda$. □

Next, we state without proof Banach Extension Theorem, which will be used to prove Theorem 3.2.14.

Theorem 3.2.13. (Banach Extension Theorem) Let f be a bounded linear functional on a subspace Z of a normed space X. Then there exists a bounded linear functional $\hat{f}$ on X which is an extension of f to X and has the same norm $||\hat{f}||_X = ||f||_Z$ where $||\hat{f}||_X = \sup_{x \in X, ||x||=1} |\hat{f}(x)|$ and $||\hat{f}||_Z = \sup_{x \in Z, ||x||=1} |\hat{f}(x)|$.

We end this section by looking at the fundamental theorem of convexity which was pioneered by Menger[21]. This theorem is very important in the study of the geometry of metric spaces.

Theorem 3.2.14. If (X,d) is a complete and Menger convex metric space, then any two points $x, y \in X$ can be joined by a metric segment.i.e there exists an isometry $\varphi : [0, d(x,y)] \longrightarrow X$ with $\varphi(0) = x$ and $\varphi(d(x,y)) = y$.

Proof. Let $x_0, x_1 \in X, x_0 \neq x_1$. By Lemma 3.2.12, there exists $x_{1/2} \in X$ such that $d(x_0, x_{1/2}) = d(x_{1/2}, x_1) = \frac{1}{2} d(x_0, x_1)$ i.e $x_{1/2}$ is a midpoint of the pair (x_0, x_1). Let $\lambda = d(x_0, x_1)$ and define the mapping F by taking

$$F(0) = x_0,\ F(\lambda/2) = x_{1/2},\ F(\lambda) = x_1.$$

Again by Lemma 3.2.12, there exists points $x_{1/4}$ and $x_{3/4}$ which is a pair of $(x_0, x_{1/2})$ and $(x_{1/2}, x_1)$ respectively. Define

$$F(\lambda/4) = x_{1/4},\ F(3\lambda/4) = x_{3/4}.$$

We show that F is an isometry on the set $A = \{0, \lambda/4, \lambda/2, 3\lambda/4, \lambda\}$. That is,

$$d(F(0),F(\lambda/4)) = d(x_0,x_{1/4}) = \frac{1}{4}d(x_0,x_1) = \frac{\lambda}{4} = \left|0-\frac{\lambda}{4}\right| = d(0,\lambda/4).$$
$$d(F(0),F(\lambda/2)) = d(x_0,x_{1/2}) = \frac{1}{2}d(x_0,x_1) = \frac{\lambda}{2} = \left|0-\frac{\lambda}{2}\right| = d(0,\lambda/2).$$
$$d(F(0),F(3\lambda/4)) = d(x_0,x_{3/4}) = \frac{3}{4}d(x_0,x_1) = \frac{3\lambda}{4} = \left|0-\frac{3\lambda}{4}\right| = d(0,3\lambda/4).$$
$$d(F(0),F(\lambda)) = d(x_0,x_1) = \lambda = |0-\lambda| = d(0,\lambda).$$
$$d(F(\lambda/4),F(\lambda/2)) = d(x_{1/4},x_{1/2}) = d(x_0,x_{1/4}) = \frac{1}{4}d(x_0,x_1) = \frac{\lambda}{4} = \left|\frac{\lambda}{4}-\frac{\lambda}{2}\right| = d(\lambda/4,\lambda/2).$$
$$d(F(\lambda/4),F(3\lambda/4)) = d(x_{1/4},x_{3/4}) = 2d(x_0,x_{1/4}) = \frac{1}{2}d(x_0,x_1) = \frac{\lambda}{2} = \left|\frac{\lambda}{4}-\frac{3\lambda}{4}\right| = d(\lambda/4,3\lambda/4).$$
$$d(F(\lambda/4),F(\lambda)) = d(x_{1/4},x_1) = \frac{3}{4}d(x_0,x_1) = \frac{3\lambda}{4} = \left|\frac{\lambda}{4}-\lambda\right| = d(\lambda/4,\lambda).$$
$$d(F(\lambda/2),F(3\lambda/4)) = d(x_{1/2},x_{3/4}) = \frac{1}{4}d(x_0,x_1) = \frac{\lambda}{4} = \left|\frac{\lambda}{2}-\frac{3\lambda}{4}\right| = d(\lambda/2,3\lambda/4)$$
$$d(F(\lambda/2),F(\lambda)) = d(x_{1/2},x_1) = d(x_0,x_{1/2}) = \frac{1}{2}d(x_0,x_1) = \frac{\lambda}{2} = \left|\frac{\lambda}{2}-\lambda\right| = d(\lambda/2,\lambda).$$
$$d(F(3\lambda/4),F(\lambda)) = d(x_{3/4},x_1) = d(x_{1/2},x_{3/4}) = \frac{1}{4}d(x_0,x_1) = \frac{\lambda}{4} = \left|\frac{3\lambda}{4}-\lambda\right| = d(3\lambda/4,\lambda).$$

Hence, the mapping F is an isometry on the set $A = \{0,\lambda/4,\lambda/2,3\lambda/4,\lambda\}$. By induction we obtain the points $\{x_{p/2^n}\}, 1 \leq p \leq 2^n - 1$ for all $n \in \mathbb{N}$ in X such that the mapping $F : p\lambda/2^n \longrightarrow x_{p/2^n}$ is an isometry. Since $\{p\lambda/2^n\}$ is a dense subset of $[0,\lambda]$ with F an isometry defined on this set, and since X is complete, by Banach extension theorem, we can extend F to the entire interval $[0,\lambda]$ by the function φ, and thus obtaining a metric segment in X joining x_0 and x_1. □

3.3. Takahashi convexity in metric spaces

In this section, we recall the concept of Takahashi convexity in metric spaces. This concept was introduced by Wataru Takahashi [28] in 1970 and later it was extensively studied by Machado ([20]) and Talman([29]). We will end this section by showing that Takahashi convexity implies Menger convexity.

Definition 3.3.1. ([29, Definition 1.1]) Let (X,d) be a metric space. A mapping $W : X \times X \times [0,1] \longrightarrow X$ is said to be a Takahashi convex structure (TCS) on X if for all $x,y \in X$ and $\lambda \in [0,1]$,

$$d(u,W(x,y,\lambda)) \leq \lambda d(u,x) + (1-\lambda)d(u,y)$$

for all $u \in X$. The metric space (X,d) together with a convex structure is called a convex metric space.

Note 3.3.2. A Banach space and each of its convex subsets are also convex metric spaces but the converse is not true.
Definition 3.3.3. ([28]) A subset K of a convex metric space X is said to be convex if $W(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda \in [0, 1]$.

The following example shows that Takahashi convexity does not imply convexity in the usual sense:
Example 3.3.4. ([29, p.112]) Let X be the set of rational numbers in $[0, 1]$ and d be a metric defined by $d(x, y) = |x - y|$ for all $x, y \in X$, and a mapping $W : X \times X \times [0, 1] \longrightarrow X$ defined by $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$. Then we have

$$\begin{aligned} d(u, W(x, y, \lambda)) &= |u - W(x, y, \lambda)| \\ &\leq \lambda|u - x| + (1 - \lambda)|u - y| \\ &= \lambda d(u, x) + (1 - \lambda)d(u, y), \end{aligned}$$

for all $u \in X$. Hence W is a Takahashi convex structure on X, and so (X, d) is convex in the sense of Takahashi.
However, the set of all rational numbers in $[0, 1]$ is not convex. Since for every two rational numbers there is an irrational number between them. Hence it is not possible to join two points without leaving the set.
Definition 3.3.5. ([20]) A TCS W on a metric space (X, d) is said to have Property (I) provided that

$$d(W(x, y, \lambda_1), W(x, y, \lambda_2)) = |\lambda_1 - \lambda_2| d(x, y)$$

for all $x, y, z \in X$ and $\lambda_1, \lambda_2 \in [0, 1]$.

We now give two examples of convex metric spaces in the sense of Takahashi (see [28]) :
Example 3.3.6. ([28, Example 2]) Let (X, d) be a metric space with the following properties;

(a) For all $x, y \in X$,

$$d(x - y, 0) = d(x, y)$$

(b) For all $x, y \in X$ and $\lambda \in [0, 1]$

$$d(\lambda x + (1 - \lambda)y, 0) \leq \lambda d(x, 0) + (1 - \lambda)d(y, 0).$$

Then the metric space (X, d) is convex.

Proof. Let $x, y \in X$ and for every $\lambda \in [0, 1]$, define a mapping $W : X \times X \times [0, 1] \longrightarrow X$ by $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$. We show that (X, d) is convex in the sense of Takahashi. Thus for all $u \in X$ and property (a), we have

$$\begin{aligned} d(u, W(x, y, \lambda)) &= d(\lambda x + (1 - \lambda)y, u) \\ &= d(\lambda x + (1 - \lambda)y - u, 0) \end{aligned}$$

Now, we can write $u = \lambda u + (1-\lambda)u$ and using property (b) we have,

$$\begin{aligned} d(u, W(x,y,\lambda)) &= d(\lambda x + (1-\lambda)y - u, 0) \\ &= d(\lambda(x-u) + (1-\lambda)(y-u), 0) \\ &\leq \lambda d(x-u,0) + (1-\lambda)d(y-u,0) \\ &= \lambda d(u,x) + (1-\lambda)d(u,y). \end{aligned}$$

Hence, $W(x,y,\lambda)$ is a convex structure on X and so (X,d) is a convex metric space. □

Example 3.3.7. ([28, Example 1]) Let I be the unit interval $[0,1]$ and X be the family of closed intervals $[a_i, b_i]$ such that $0 \leq a_i \leq b_i \leq 1$. For $I_i = [a_i, b_i]$, $I_j = [a_j, b_j]$ and $\lambda \in [0,1]$, we define a mapping W by $W(a_i, b_j, \lambda)$=$[\lambda a_i + (1-\lambda)a_j, \lambda b_i + (1-\lambda)b_j]$ and define a metric d in X by a Hausdorff distance i.e

$$d(I_i, I_j) = \sup_{a\in I}\{|\inf_{b\in I_i}\{|a-b|\} - \inf_{c\in I_j}\{|a-c|\}|\}.$$

Then the metric space (X,d) is convex.

Proof. We prove that (X,d) is a convex metric space by showing that $d(I_k, W(I_i, I_j, \lambda)) \leq \lambda d(I_k, I_i) + (1-\lambda)d(I_k, I_j)$. By the Hausdorff distance

$$\begin{aligned} &\lambda d(I_k, I_i) + (1-\lambda)d(I_k, I_j) \\ &= \lambda\left(\sup_{a\in I}\{|\inf_{b\in I_j}\{|a-b|\} - \inf_{c\in I_i}\{|a-c|\}|\}\right) + (1-\lambda)\left(\sup_{a\in I}\{|\inf_{b\in I_k}\{|a-b|\} - \inf_{c\in I_j}\{|a-c|\}|\}\right) \\ &\geq \sup_{a\in I}\left\{|\inf_{b\in I_k}\{\lambda|a-b|\} - \inf_{c\in I_i}\{\lambda|a-c|\} + \inf_{b\in I_k}\{(1-\lambda)|a-b|\} - \inf_{c\in I_j}\{(1-\lambda)|a-c|\}|\right\} \\ &= \sup_{a\in I}\left\{|\inf_{b\in I_k}\{\lambda|a-b|\} + \inf_{b\in I_k}\{(1-\lambda)|a-b|\} - \left(\inf_{c\in I_j}\{\lambda|a-c|\} + \inf_{c\in I_i}\{(1-\lambda)|a-c|\}\right)|\right\} \\ &\geq \sup_{a\in I}\left\{|\inf_{b\in I_k}\{|a-b|\} - \left(\inf_{c\in I_j}\{\lambda|a-c|\} + \inf_{c\in I_i}\{(1-\lambda)|a-c|\}\right)|\right\} \end{aligned}$$

Now, we have that

$$\begin{aligned} \inf_{c\in I_i}\{\lambda|a-c|\}| + \inf_{c\in I_j}\{(1-\lambda)|a-c|\} &= \inf_{c\in[\lambda a_i, \lambda b_i]}\{|a-c|\} + \inf_{c\in[(1-\lambda)a_j,(1-\lambda)b_j]}\{|a-c|\} \\ &\geq \inf_{c\in[\lambda a_i+(1-\lambda)a_j, \lambda b_i+(1-\lambda)b_j]}\{|a-c|\} \\ &= \inf_{c\in W(I_i,I_j,\lambda)}\{|a-c|\}. \end{aligned}$$

Thus, we have that

$$\begin{aligned} \lambda d(I_k, I_i) + (1-\lambda)d(I_k, I_j) &\geq \sup_{a\in I}\{|\inf_{b\in I_k}\{|a-b|\} - \inf_{c\in W(I_i,I_j,\lambda)}\{|a-c|\}|\} \\ &= d(I_k, W(I_i, I_j, \lambda)). \end{aligned}$$

Hence, $d(I_k, W(I_i, I_j, \lambda)) \leq \lambda d(I_k, I_i) + (1-\lambda)d(I_k, I_j)$ and so (X,d) is convex. □

We have the following properties of convex metric spaces;

Proposition 3.3.8. *([28, Proposition 1]) Let $\{K_\alpha\}_{\alpha\in A}$ be a family of convex subsets of convex metric space (X,d), then $\bigcap_{\alpha\in A} K_\alpha$ is also a convex subset of (X,d).*

Proof. Let $\{K_\alpha\}_{\alpha\in A}$ be a family of convex subsets of the metric space X and let $\mathcal{K}=\bigcap_{\alpha\in A} K_\alpha$. For any $x,y\in\mathcal{K}$, then by the definition of indexed family of sets, $x,y\in K_\alpha$ for all $\alpha\in A$. Since each of these sets are convex , then for all $\alpha\in A$ and $\lambda\in[0,1]$, $W(x,y,\lambda)\in K_\alpha$. Hence, we have that $W(x,y,\lambda)\in\mathcal{K}$. Since x and y are arbitrary in $\mathcal{K}$ and so $\bigcap_{\alpha\in A} K_\alpha$ is convex subset of (X,d). □

Proposition 3.3.9. *([28, Proposition 2]) The open balls $B_d(x,\varepsilon)$ and closed balls $C_d(x,\varepsilon)$ are convex subsets of the convex metric space (X,d).*

Proof. Let $y,z\in B_d(x,\varepsilon)$ and fix $\alpha\in[0,1]$, then $W(y,z,\alpha)\in X$. We need to show that $W(y,z,\alpha)\in B_d(x,\varepsilon)$. Now, since X is a convex metric space, we have

$$d(x,W(y,z,\alpha))\leq \alpha d(x,y)+(1-\alpha)d(x,z)<\alpha\varepsilon+(1-\alpha)\varepsilon=\varepsilon.$$

Therefore, $W(y,z,\alpha)\in B_d(x,\varepsilon)$ and so $B_d(x,\varepsilon)$ is convex.
Similarly, we can show that $C_d(x,\varepsilon)$ is a convex subset of (X,d). □

Proposition 3.3.10. *([28, Proposition 3]) Suppose that (X,d) is a convex metric space and $\lambda\in[0,1]$ then*

$$d(x,y)=d(x,W(x,y,\lambda))+d(W(x,y,\lambda),y)$$

for all $x,y\in X$.

Proof. Since X is a convex metric space, and using the triangle inequality, we obtain

$$\begin{aligned}
d(x,y) &\leq d(x,W(x,y,\lambda))+d(W(x,y,\lambda),y)\\
&\leq \lambda d(x,x)+(1-\lambda)d(x,y)+\lambda d(x,y)+(1-\lambda)d(y,y)\\
&=(1-\lambda)d(x,y)+\lambda d(x,y)\\
&=d(x,y).
\end{aligned}$$

Therefore, we have that $d(x,y)=d(x,W(x,y,\lambda))+d(W(x,y,\lambda),y)$ for all $x,y\in X$ and $\lambda\in[0,1]$. □

Proposition 3.3.11. *([29, Proposition 1.2]) Let W be a TCS on a metric space (X,d). If $x,y\in X$ and $\lambda\in[0,1]$, then*

(i) $W(x,y,1)=x$ and $W(x,y,0)=y$.

(ii) $W(x,x,\lambda)=x$.

Proof. (i) Since W is a TCS on X, then for any $x,y\in X$ and $\lambda\in[0,1]$, we obtain that $d(x,W(x,y,1))\leq 1.d(x,x)+(1-1)d(x,y)=0+0.d(x,y)=0$ and so we have that $d(x,W(x,y,1))\leq 0$ which implies $d(x,W(x,y,1))=0$ and therefore, $W(x,y,1)=x$ for all $x,y\in X$. A similar argument shows that $W(x,y,0)=y$ for all $x,y\in X$.

(ii) Since W is a TCS on a metric space X, then for every $x, y \in X$ and $\lambda \in [0,1]$, we have that $d(x, W(x, x, \lambda)) \leq \lambda d(x, x) + (1-\lambda)d(x, x) = \lambda.0 + (1-\lambda).0 = 0$ and so we have $d(x, W(x, x, \lambda)) \leq 0$ which implies that $d(x, W(x, x, \lambda)) = 0$ and so $W(x, x, \lambda) = x$ whenever $x \in X$.

□

Lemma 3.3.12. *([28, Proposition 1.2]) Let (X, d) be a Takahashi convex metric space. For any x and y in X and $\lambda \in [0,1]$, we have*

$$d(x, W(x, y, \lambda)) = (1-\lambda)d(x, y)$$

and

$$d(W(x, y, \lambda), y) = \lambda d(x, y)$$

Proof. Let $x, y \in X$ and $\lambda \in [0,1]$, we want to show that $d(x, W(x, y, \lambda)) = (1-\lambda)d(x, y)$. Since W is a TCS on X , we have that

$$d(x, W(x, y, \lambda)) \leq \lambda d(x, x) + (1-\lambda)d(x, y) = (1-\lambda)d(x, y). \tag{3.2}$$

Next, using the triangle inequality and W being a TCS on X, we have

$$\begin{aligned} d(x, y) &\leq d(x, W(x, y, \lambda)) + d(W(x, y, \lambda), y) \\ &\leq d(x, W(x, y, \lambda)) + (\lambda d(x, y) + (1-\lambda)d(y, y)) \\ &= \lambda d(x, y) + d(x, W(x, y, \lambda)), \end{aligned}$$

which implies that

$$(1-\lambda)d(x, y) \leq d(x, W(x, y, \lambda)) \tag{3.3}$$

Hence, by combining (3.2) and (3.3), we have $d(x, W(x, y, \lambda) = (1-\lambda)d(x, y)$.
Similarly, we have

$$d(W(x, y, \lambda), y) \leq \lambda d(x, y) + (1-\lambda)d(y, y) = \lambda d(x, y). \tag{3.4}$$

Also, using the triangle inequality and W being a TCS on X, we obtain

$$\begin{aligned} d(x, y) &\leq d(x, W(x, y, \lambda)) + d(W(x, y, \lambda), y) \\ &\leq d(W(x, y, \lambda), y) + (\lambda d(x, x) + (1-\lambda)d(x, y)) \\ &= (1-\lambda)d(x, y) + d(W(x, y, \lambda), y), \end{aligned}$$

which implies that

$$\lambda d(x, y) \leq d(x, W(x, y, \lambda)) \tag{3.5}$$

Hence, combining (3.4) and (3.5) we obtain $d(W(x, y, \lambda), y) = \lambda d(x, y)$. □

Definition 3.3.13. ([28, p.145]) Let (X, d) be a metric space. We say that (X, d) is strictly convex provided that for each $x, y \in X$ and $\lambda \in [0, 1]$, there exists a unique $W(x, y, \lambda) \in X$ such that

(i) $d(x, W(x, y, \lambda)) = (1 - \lambda)d(x, y)$

(ii) $d(W(x, y, \lambda), y) = \lambda d(x, y)$.

Definition 3.3.14. ([29, Definition 1.3]) Let W be a TCS on a metric space (X, d). We say that W is a strict TCS if for any $w \in X$ there exists $(x, y, \lambda) \in X \times X \times [0, 1]$ for which

$$d(z, w) \leq \lambda d(z, x) + (1 - \lambda)d(z, y),$$

for every $z \in X$, then $w = W(x, y, \lambda)$.

Lemma 3.3.15. *([29, Lemma 1.4]) Let W be a strict TCS on a metric space (X, d). Then for every $x, y \in X$ and $\alpha, \beta \in [0, 1]$, we have*

$$W(W(x, y, \beta), y, \alpha) = W(x, y, \alpha\beta).$$

Proof. Let $z \in X$. Then we have

$$\begin{aligned} d(z, W(W(x, y, \beta), y, \alpha)) &\leq \alpha d(z, W(x, y, \beta)) + (1 - \alpha)d(z, y) \\ &\leq \alpha\left(\beta d(z, x) + (1 - \beta)d(z, y)\right) + (1 - \alpha)d(z, y) \\ &= \alpha\beta d(z, x) + (1 - \alpha\beta)d(z, y). \end{aligned}$$

Since W is strict TCS on X, then we have $W(W(x, y, \beta), y, \alpha) = W(x, y, \alpha\beta)$. □

Definition 3.3.16. ([29]) Let (X, d) be a metric space with TCS W. We say a TCS W has

(i) property (S) provided that

$$d(W(x, y, \lambda), W(x', y', \lambda)) \leq \lambda d(x, x') + (1 - \lambda)d(y, y')$$

whenever $x, y, x', y' \in X$ and $\lambda \in [0, 1]$.

(ii) condition (C) provided that $W(x, y, \lambda) = W(y, x, 1 - \lambda)$ whenever $x, y \in X$ and $\lambda \in [0, 1]$.

(iii) Property (J) if, $W(W(x, y, \beta), y, \alpha) = W(x, y, \alpha\beta)$. whenever $x, y \in X$ and $\alpha, \beta \in [0, 1]$.

We now investigate the continuity property of a strict TCS W.

Theorem 3.3.17. ([29, Theorem 1.5]) If W is a strict TCS on a metric space (X, d), then for every pair $x, y \in X$ with $x \neq y$ and $\lambda \in [0, 1]$ the function $h : [0, 1] \longrightarrow X$ defined by $h(\lambda) = W(x, y, \lambda)$ is an embedding of $[0, 1]$ into X.

Proof. Let $\lambda_1, \lambda_2 \in [0, 1]$, and assume, without loss of generality, that $\lambda_1 < \lambda_2$. Then by

Lemma 3.3.12 and Lemma 3.3.15, we obtain that

$$\begin{aligned}
d(W(x,y,\lambda_1),W(x,y,\lambda_2)) &= d(W(x,y,\lambda_2\frac{\lambda_1}{\lambda_2}),W(x,y,\lambda_2))\\
&= d(W(W(x,y,\lambda_2),y,\frac{\lambda_1}{\lambda_2}),W(x,y,\lambda_2)) \quad \text{by Lemma 3.3.15}\\
&= (1-\frac{\lambda_1}{\lambda_2})d(W(x,y,\lambda_2),y) \quad \text{by Lemma 3.3.12}\\
&= (1-\frac{\lambda_1}{\lambda_2})\lambda_2 d(x,y) \quad \text{by Lemma 3.3.12}\\
&= (\lambda_2-\lambda_1)d(x,y).
\end{aligned}$$

Hence the proof. □

It does not appear that even a unique TCS W is necessarily continuous as a function from $X \times X \times [0,1]$ to X. However, we have the following:

Theorem 3.3.18. ([29, Theorem 1.7]) Let W be a TCS on a metric space (X,d). Then W is continuous at a point (x,x,λ) of $X \times X \times [0,1]$.

Proof. Let (x_n,y_n,λ_n) be a sequence in $X \times X \times [0,1]$ which converges to (x,x,λ). In view of Proposition 3.3.11, it suffices to show that $(W(x_n,y_n,\lambda_n))$ converges to x. Let $\varepsilon > 0$, since the sequences (x_n) and (y_n) both converge to x, there is an $N \in \mathbb{N}$ such that $d(x,x_n) \leq \varepsilon$ and $d(x,y_n) \leq \varepsilon$ for all $n \geq N$. Now,

$$\begin{aligned}
d(x,W(x_n,y_n,\lambda_n)) &\leq \lambda_n d(x,x_n) + (1-\lambda_n)d(x,y_n)\\
&= \lambda_n\varepsilon + (1-\lambda_n)\varepsilon\\
&= \varepsilon, \quad \text{for all} \quad n \geq N.
\end{aligned}$$

Hence, $W(x_n,y_n,\lambda_n)$ converges to $x = W(x,x,\lambda)$. □

The following Theorem shows that if X is compact, then the sequence $(W(x_{n_k},y_{n_k},\lambda_{n_k}))$ converges to (x,y,λ) with $x \neq y$.

Theorem 3.3.19. ([29, Theorem 1.8]) Let W be a strict TCS on a compact metric space (X,d). Then W is continuous as a function from $X \times X \times [0,1]$ to X.

Proof. Let (x_n,y_n,λ_n) be a sequence in $X \times X \times [0,1]$ which converges to (x,y,λ), and let w be a limit point of the sequence $(W(x_n,y_n,\lambda_n))$. Choose a subsequence $(W(x_{n_k},y_{n_k},\lambda_{n_k}))$ which converges to w. Then for any $z \in X$, we have

$$d(z,W(x_{n_k},y_{n_k},\lambda_{n_k})) \leq \lambda_{n_k}d(z,x_{n_k}) + (1-\lambda_{n_k})d(z,y_{n_k})$$

for all $n \in \mathbb{N}$. By continuity of a metric d, we have

$$d(z,w) \leq \lambda d(z,x) + (1-\lambda)d(z,y).$$

By Definition 3.3.14, we have that $w = W(x,y,\lambda)$. It follows that $W(x,y,\lambda)$ is the only limit point of the sequence $(W(x_n,y_n,\lambda_n))$. Since X is compact, $(W(x_n,y_n,\lambda_n)$ must converge to $W(x,y,\lambda)$. □

In the next proposition we show that Takahashi convexity implies Menger convexity. However, the converse is not yet known.

Proposition 3.3.20. *([23, Proposition 2.1.8]) Let (X, d) be a metric space. If (X, d) is Takahashi convex, then it is Menger convex.*

Proof. Suppose that (X, d) is Takahashi convex metric space, let $x, y \in X$ and let $\lambda \in [0, 1]$. Let $z = W(x, y, 1-\lambda)$. By Lemma 3.3.12, we have that

$$\begin{aligned} d(x, z) &= d(x, W(x, y, 1-\lambda)) \\ &= (1 - (1-\lambda))d(x, y) \\ &= \lambda d(x, y). \end{aligned}$$

And

$$\begin{aligned} d(z, y) &= d(W(x, y, 1-\lambda), y) \\ &= (1-\lambda)d(x, y). \end{aligned}$$

Hence, Takahashi convexity implies Menger convexity. □

3.4. M-convexity in metric spaces

In this section, we first recall the concept of strong convexity in metric spaces. Thereafter, we recall M-convexity in metric spaces and study the convexity of balls in relation to proximinality of convex sets in M- convex metric spaces [11]. We begin by recalling the definition of a strongly convex metric space as defined by Borsuk [23] in 1959:

Definition 3.4.1. ([23, Definition 2.2.1]) A metric space (X, d) is said to be strongly convex if for every $x, y \in X$, and for every $t \in [0, 1]$, there exists a unique $z \in X$ such that

(i) $d(x, z) = td(x, y)$

(ii) $d(z, y) = (1-t)d(x, y)$.

Remark 3.4.2. A metric space (X, d) is said to be strongly convex if and only for every $x, y \in X$, there exists a unique point $z \in X$ such that

$$d(x, z) + d(z, y) = d(x, y).$$

This can easily be seen when we add conditions (i) and (ii) in Definition 3.4.1.

We note that every $t \in [0, 1]$ determines a unique value of a point z in the segment $B^d(x, y)$. The following example shows that strong convexity does not imply convexity in the usual sense:

Example 3.4.3. ([16, p.112]) Let X be the set of rational numbers in $[0, 1]$ and d be a metric defined by $d(x, y) = |x - y|$ for all $x, y \in X$. Let $x, y \in X$, $x \neq y$, then we can find a unique point $z = \frac{x+y}{2} \in X$, such that,

$$d(x, y) = d(x, z) + d(z, y)$$

for all $x, y, z \in X$. Thus, (X, d) is a strong convex metric space
However, the set of all rational numbers in $[0, 1]$ is not convex. Since for every two rational numbers there is an irrational number between them. Hence it is not possible to join two points without leaving the set.

Remark 3.4.4. We notice that every strongly convex metric space is Menger convex, but the converse is not always true as can be seen from the example below.

Example 3.4.5. ([23, Example 1.6]) Let $X = \mathbb{R}^2$ be a Euclidean space and define a metric $d : X \times X \longrightarrow [0, \infty)$ by

$$d(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2\}$$

where $x = (x_1, y_1)$, $y = (x_2, y_2)$. Then (X, d) is Menger convex but not strongly convex.

Proof. Let $x, y \in X, x \neq y$, set $z = \left(\frac{x_1 + y_1}{2}, \frac{x_2 + y_2}{2}\right)$ then

$$\begin{aligned} d(x, z) &= \max\left\{\left|x_1 - \frac{x_1 + y_1}{2}\right|, \left|x_2 - \frac{x_2 + y_2}{2}\right|\right\} = \max\left\{\left|\frac{x_1 - y_1}{2}\right|, \left|\frac{x_2 - y_2}{2}\right|\right\} \\ &= \frac{1}{2}\max\left\{|x_1 - y_1|, |x_2 - y_2|\right\} = \frac{1}{2}d(x, y). \end{aligned}$$

A similar argument shows that $d(z, y) = \frac{1}{2}d(x, y)$ for all $x, y \in X$. Hence,

$$d(x, z) + d(z, y) = \frac{1}{2}d(x, y) + \frac{1}{2}d(x, y) = d(x, y).$$

Therefore, (X, d) is Menger convex metric space. To see that it is not strongly convex. Consider the points $x = (0, 0)$, $y = (1, 0)$ and fix $t = \frac{1}{2} \in [0, 1]$. We notice that

$$d(x, y) = \max\{|1 - 0|, |0 - 0|\} = \max\{1, 0\} = 1.$$

Since (X, d) is Menger convex, we obtain the following equations;

$$d(x, z) = \frac{1}{2} \quad \text{and} \quad d(z, y) = \frac{1}{2} \tag{3.6}$$

where $z = (z_1, z_2)$. Now, we observe that two distinct points $z = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $z = \left(\frac{1}{2}, 0\right)$ satisfies the two equations in 3.6. Therefore, there is no unique value of z that solves the two equations and so (X, d) is not a strong convex metric space. □

Definition 3.4.6. ([11, Definition 1.1]) A metric space (X, d) is called M-convex if for every two distinct points x and y in X with $d(x, y) = \lambda$, and for all $r \in [0, \lambda]$, there exists a unique $z_r \in X$ such that

$$C_d(x, r) \cap C_d(y, \lambda - r) = \{z_r\}.$$

Remark 3.4.7. If $z_r = \frac{1}{2}d(x, y)$, then z_r is the midpoint of x and y.

The following Proposition shows that strong convexity and M-convexity are equivalent in a convex metric space.

Proposition 3.4.8. *([23, Lemma 2.2.2]) Let (X, d) be a convex metric space. Then (X, d) is strongly convex if and only if it is M-convex.*

Proof. We first show that if (X, d) is M-convex then it is a strongly convex metric space. Suppose that X is M-convex. Let $x, y \in X$ be such that $d(x, y) = r$ and $0 \leq t \leq 1$. Then $0 \leq tr \leq r$. Let $r_1 = tr$ and $r_2 = r - tr$ then we have that $r_1 + r_2 = r = d(x, y)$. Since X is M-convex,

$$C_d(x, r_1) \cap C_d(y, r_2) = \{z_t\},$$

this implies that $d(x, z_t) \leq r_1$ and $d(z_t, y) \leq r_2$. Considering, the triangle inequality, we have that

$$d(x, y) \leq d(x, z_t) + d(z_t, y) \leq r_1 + r_2 = r = d(x, y)$$

and so $d(x, y) = d(x, z_t) + d(z_t, y)$. Hence, we obtain

$$d(x, z_t) = r_1 = tr = td(x, y) \quad \text{and} \quad d(z_t, y) = r_2 = r - tr = (1 - t)r = (1 - t)d(x, y).$$

Next, we show that $z_t \in X$ is unique. Suppose that there also exist $z_t^{'} \in X$ such that $d(x, z_t^{'}) = td(x, y)$ and $d(z_t^{'}, y) = (1 - t)d(x, y)$. Since X is M-convex, then

$$z_t^{'} \in C_d(x, r_1) \cap C_d(y, r_2) = \{z_t\}$$

and so $z_t = z_t^{'}$. Hence (X, d) is a strong convex metric space.
Conversely, suppose X is strongly convex. Let $x, y \in X$ with $d(x, y) = \lambda$ and $r \in [0, \lambda]$. We show that

$$C_d(x, r) \cap C_d(y, \lambda - r) = \{z_r\}.$$

Let $t = \dfrac{r}{\lambda}$ so that $0 \leq t \leq 1$. Then by the strong convexity of X, there exists a unique $z_r \in X$ such that $d(x, z_r) = td(x, y) = r$ and $d(z_r, y) = (1 - t)d(x, y) = \lambda - r$. That is,

$$z_r \in C_d(x, r) \cap C_d(y, \lambda - r).$$

Suppose that there also exist $z_r' \in X$ such that $z_r' \in C_d(x, r) \cap C_d(y, \lambda - r)$. Then we have $d(x, z_r') \leq r$ and $d(y, z_r') \leq \lambda - r$. Now,

$$\begin{aligned} d(x, y) &\leq d(x, z_r') + d(z_r', y) \\ &\leq r + \lambda - r \\ &= \lambda = d(x, y) \end{aligned}$$

which implies that $d(x, y) = d(x, z_r') + d(z_r', y)$. Hence, we obtain

$$d(x, z_r') = r \quad \text{and} \quad d(y, z_r') = \lambda - r.$$

Since

$$d(x, z_r') = r = d(x, z_r) \quad \text{and} \quad d(y, z_r') = \lambda - r = d(y, z_r),$$

it follows that $z_r' = z_r$ and so we have

$$C_d(x, r_1) \cap C_d(y, r_2) = \{z_r\}$$

and so (X, d) is M-convex metric space. □

Definition 3.4.9. ([11, p.585]) Let (X, d) be a metric space and $A \subseteq X$. Then A is said to be convex if for every $x, y \in A$, then

$$C_d(x, (1-t)\lambda) \cap C_d(y, t\lambda) \subseteq A$$

for all $t \in [0, 1]$ where $d(x, y) = \lambda$.

Definition 3.4.10. ([23, Definition 2.1.3]) An M-convex metric space (X, d) is said to be strictly convex if for all $x, y \in C_d(z, r)$ with $\lambda = d(x, y)$, we have

$$C_d(x, (1-t)\lambda) \cap C_d(y, t\lambda) \subseteq B_d(z, r).$$

for all $0 < t < 1$ and all $z \in X$ and $r > 0$.

We note that for normed linear spaces, it was proved in [12], that strict convexity characterises M-convexity. We recall the following:

Theorem 3.4.11. ([11, Theorem 2.4]) Let (X, d) be a strictly convex metric space, then (X, d) is M-convex.

Proof. Let $x, y \in X$ and $\lambda = d(x, y)$. Since (X, d) is strictly convex then we have that

$$E(t) = C_d(x, (1-t)\lambda) \cap C_d(y, t\lambda) \neq \emptyset$$

for all $t \in (0, 1)$. Suppose that $z_1, z_2 \in E(t)$, then we have that $d(x, z_1) \leqslant (1-t)\lambda, \quad d(x, z_2) \leqslant (1-t)\lambda$ and $d(y, z_1) \leqslant t\lambda, \quad d(y, z_2) \leqslant t\lambda$. Now consider

$$d(x, y) \leqslant d(x, z_1) + d(z_1, y) \leqslant (1-t)\lambda + t\lambda = \lambda = d(x, y),$$

and so $d(x, y) = d(x, z_1) + d(z_1, y)$. That is z_1 is between x and y and so we have that $d(x, z_1) = (1-t)\lambda$ and $d(y, z_1) = t\lambda$.

Similarly, we obtain $d(x, y) = d(x, z_2) + d(z_2, y)$. That is z_2 is between x and y and so we have that $d(x, z_2) = (1-t)\lambda$ and $d(y, z_2) = t\lambda$. Let $d(z_1, z_2) = \alpha$ and since z_1 and z_2 between x and y, then $\alpha < \lambda$ and X being strictly convex, then we have that

$$C_d(z_1, (1-s)\alpha) \cap C_d(z_2, s\alpha) \subseteq C_d(x, (1-t)\lambda) \cap C_d(y, t\lambda) \subseteq B_d(z, r)$$

for all $s \in (0, 1)$. This implies that

$$C_d(z_1, (1-s)\alpha) \cap C_d(z_2, s\alpha) \subseteq B_d(z, r).$$

Since z_1 and z_2 between x and y then we have that $d(x, y) = d(x, z_1) + d(z_1, z_2) + d(z_2, y)$ and so

$$d(z_1, z_2) = d(x, y) - d(x, z_1) - d(z_2, y). \tag{3.7}$$

But we know that $d(x, y) = d(x, z_2) + d(z_2, y)$, thus substituting this in 3.7, we obtain

$$d(z_1, z_2) = d(x, z_2) - d(x, z_1) = \lambda - t\lambda - (\lambda - t\lambda) = 0.$$

Since d is a metric, we get $z_1 = z_2$. Hence (X, d) is an M-convex metric space. □

The converse of Theorem 3.4.11 need not be true as is clear from the following example.
Example 3.4.12. ([11, p.16-18]) Let $X = C_d(x, \rho)$ be closed ball of $S_{2,r}$ of radius ρ with $\frac{\pi r}{4} < \rho < \frac{\pi r}{2}$, where $S_{2,r}$ is a spherical space whose elements are ordered 3-tuples $x = (x_1, x_2, x_3)$ of reals with $x_1^2 + x_2^2 + x_3^2 = r^2$, and distance d is defined for each pair of elements x and y as

$$d(x, y) = r \cos^{-1}\left(\frac{x_1y_1 + x_2y_2 + x_3y_3}{r^2}\right).$$

To see that (X, d) is M-convex. Let $x, y \in X$ such that $d(x, y) = \dfrac{\pi r}{4} = \lambda$, then we need to find a point $z \in X$ such that

$$C_d(x, \mu) \cap C_d(y, \lambda - \mu) = \{z\}$$

where $\mu \in \left[0, \dfrac{\pi r}{4}\right]$. Now, let

$$z \in C_d\left(x, \frac{\pi r}{8}\right) \cap C_d\left(y, \frac{\pi r}{8}\right).$$

Then we have that $d(x, z) = d(y, z) = \frac{1}{2}d(x, y) = \dfrac{\pi r}{8}$. Suppose that there exists another $z' \in C_d\left(x, \dfrac{\pi r}{8}\right) \cap C_d\left(y, \dfrac{\pi r}{8}\right)$, then $d(x, z') = d(y, z') = \frac{1}{2}d(x, y) = \dfrac{\pi r}{8}$. Thus we have that $d(x, z) = \dfrac{\pi r}{8} = d(x, z')$ and so $z = z'$. Therefore,

$$C_d\left(x, \frac{\pi r}{8}\right) \cap C_d\left(y, \frac{\pi r}{4} - \frac{\pi r}{8}\right) = \{z\}.$$

Hence, (X, d) is M-convex. To see that (X, d) is not strictly convex. We just have to check that the conditions for strict convexity are not satisfied. That is, for any $x, y \in C_d\left(z, \dfrac{\pi r}{2}\right)$ we have that $d(x, w) \leqslant \dfrac{\pi r}{2}, \quad d(y, w) \leqslant \dfrac{\pi r}{2} \implies d(z, w) < \dfrac{\pi r}{2}$. However, letting x and y be the points of X such that $d(x, z) = d(x, z) = \frac{\pi r}{2} \implies x, y \in C_d\left(z, \dfrac{\pi r}{2}\right)$, then all the points $w \in X$ between x and y have the property that $d(x, w) = \frac{\pi r}{2} = d(y, w) \implies d(z, w) = \frac{\pi r}{2}$, contradicting strictly convexity.

We recall the characterization of M-convexity by Khalil(1988) in [11], using line segment.
Definition 3.4.13. ([11, p.580]) Let (X, d) be a metric space. For $x, y \in X$, a curve joining x and y in X is an image under a one-to-one continuous map γ of a closed interval $I = [a, b]$ into X such that $\gamma(a) = x$ and $\gamma(b) = y$.
Definition 3.4.14. ([11, p.580]) Let (X, d) be a metric space, $I = [a, b]$ a closed interval and $\gamma : I \longrightarrow X$ a curve. We define the length of a curve denoted by $l(\gamma)$ as

$$l(\gamma) = \sup_n \sum_{i=1}^{n} d(\gamma(a_{i-1}), \gamma(a_i))$$

where the supremum is taken over all $n \in \mathbb{N}$ and $\{a = a_0, a_1 \ldots, a_n = b\}$ is a partition of $[a, b]$.

Let (X, d) be an M-convex metric space and $x, y \in X$. For each n, let us define a set $E(n) \subseteq X$ as follows:

$$E(0) = \{a(0, 0) = x,\ a(0, 1) = y\}$$

$$E(1) = \{a(1,0) = x, a(1,1), a(1,2) = y\}$$

where $a(1,1)$ is the midpoint of x and y. Assume that $E(n)$ has been defined such that $E(n) = \{a(n,0) = x, a(n,1), a(n,2), \ldots, a(n,2^n) = y\}$, where $a(n,k) = mid(a(n,k-1), a(n,k+1)), 0 < k < 2^n$. Then we define $E(n+1)$ as follows:

$$E(n+1) = \{a(n+1,0) = x, a(n+1,1), a(n+1,2), \ldots, a(n+1,2^{n+1}) = y\},$$

where

$$a(n,k) = \begin{cases} mid(a(n, \frac{(k-1)}{2}), a(n, \frac{(k+1)}{2})) & \text{if k is odd} \\ a(n, \frac{k}{2}) & \text{if k is even} \end{cases}$$

for $0 < k < 2^n$. For each n we have,

$$d(x,y) = \sum_{0}^{2^n-1} d(a(n,k), a(n,k+1)).$$

Further, for all positive integers $0 < t < 2^n$, we have

$$d(a(n,t), a(n,s)) = \sum_{k=s}^{2^n-1} d(a(n,k), a(n,k+1)).$$

Set $A(x,y) = \cup_{n=0}^{\infty} E(n)$ and $G[x,y] = cl_{\tau(d)} A(x,y)$. Then $G[x,y]$ is called a line segment joining x and y.

Definition 3.4.15. ([11, p.582]) Let (X,d) be M-convex and for every $x, y \in X$ such that $d(x,y) = \lambda$ and each $t \in [0,1]$, define

$$L(x,y) = \bigcup_{0 \le t \le 1} (C_d(x,(1-t)\lambda) \cap C_d(y,t\lambda)) = \bigcup_{0 \le r \le \lambda} (C_d(x,\lambda - r) \cap C_d(y,r)).$$

It follows from the construction of $G[x,y]$ and Definition 3.3.7 that $L[x,y] = G[x,y]$.

Theorem 3.4.16. ([11, Theorem 1.1]) Let (X,d) be an M-convex metric space and $x, y \in X$. Then $G[x,y]$ is a curve of minimum length joining x to y.

Proof. Let $d(x,y) = 1$, and $\mathbb{Q}$ be the rational numbers in $[0,1]$. Define the map: $\gamma : Q \longrightarrow A[x,y]$ such that $\gamma(r) = a(n,k) \in E(n)$, when $r = \frac{k}{2^k}$ and $0 < k < 2^n$. We show that γ is an isometry from Q onto $A(x,y)$. Now

$$\begin{aligned} d(\gamma(r), \gamma(s)) = d(a(n,r), a(n,s)) &= \sum_{k=s}^{2^n-1} d(a(n,k), a(n,k+1)) \\ &= \sum_{k=0}^{2^n-1} d(a(n,k), a(n,k+1)) = d(r,s). \end{aligned}$$

This implies that $d(\gamma(r), \gamma(s)) = d(r,s)$ and so γ is an isometry. Let $t \in [0,1]$, and t_n be a decreasing sequence in Q such that $t_n \longrightarrow t$. Since t_n is a Cauchy sequence in Q and γ is an isometry on Q, it follows that $\gamma(t_n)$ is a Cauchy sequence in $G[x,y]$. Further, $d(\gamma(t_n), x)$ is a decreasing sequence. Thus $d(\gamma(t_n), x) \longrightarrow t$. Since (X, d) is M-convex, there exists $z \in X$ such that

$$C_d(x, t) \cap C_d(y, 1 - t) = \{z\}.$$

We claim that $\gamma(t_n)$ converges to z. Since $d(\gamma(t_n), x)$ is a decreases to t then $d(\gamma(t_n), x) > t$, and using the fact that (X, d) is M-convex, then we can find a unique z_n such that

$$C_d(x, t) \cap C_d(\gamma(t_n, r_n - t) = \{z_n\},$$

where $r_n = d(\gamma(t_n), x)$. Furthermore, $d(y, \gamma(t_n)) + d(\gamma(t_n), x) = d(x, y)$, and $d(\gamma(t_n), z_n) + d(z_n, x) = d(\gamma(t_n), x)$. It follows that $d(y, z_n) + d(z_n, x) = d(x, y)$. But $d(z_n, x) = t$, since (X, d) is M-convex then we have that $z_n = z$. Since $d(\gamma(t_n), z_n) \longrightarrow 0, \quad \gamma(t_n) \longrightarrow z$.
We now define $\gamma(t) = z$. This establishes the extension of γ from Q to $[0, 1]$. But γ is an isometry on Q. Hence γ is continuous (and an isometry by construction) on $[0, 1]$. This completes the proof of the theorem. □

We give the characterisation of an M-convex metric space.
Theorem 3.4.17. ([11, Theorem 1.2]) Let (X, d) be a metric space. Then (X, d) is M-convex if and only if any two points x and y in X can be joined by a unique curve of length $d(x, y)$.

Proof. Let (X, d) be M-convex, and $x, y \in X$, with $d(x, y) = \lambda$. By Theorem 3.4.16, there is a curve of length λ joining x to y, namely $G[x, y]$. Let γ be another curve of length λ joining x to y. Since γ is connected and (X, d) is M-convex, it follows that $z_t = xt + (1 - t)y \in \gamma$ for all $t \in [0, 1]$. Hence $G[x, y) \subset \gamma$. Now, since $l(\gamma) = l(G[x, y]) = \lambda$. Hence $\gamma = G(x, y)$.
Conversely, let (X, d) be such that any two points of X are joined by a unique curve of minimum length. If $x, y \in X, d(x, y) = \lambda$, let γ be the unique curve of length λ joining x to y. Let $E(t) = C_d(x, (1-t)\lambda) \cap C_d(y, t\lambda), 0 \leq t \leq 1$. Since γ is connected, $E_1(t) = \gamma \cap S(x, (l - t)\lambda)) \neq \emptyset, E_2(t) = \gamma \cap S(y, t\lambda) \neq \emptyset$. We claim that $E(t) \neq \emptyset$. For if $E(t) = \emptyset$ then there exists $z_1 \in E_1$ and $z_2 \in E_2$ such that $d(z_1, z_2) > \varepsilon > 0$. The set $\{\gamma^{-1}(x), \gamma^{-1}(z_1), \gamma^{-1}(z_2), \gamma^{-1}(y)\}$ is a partition of the domain of γ. Hence

$$l(\gamma) \geq d(x, z_1) + d(z_1, z_2) + d(z_2, y) > d(x, y) + \varepsilon.$$

Hence $E(t) \neq \emptyset$. Now, we claim $E(t) \subseteq \gamma$. Let $z \in E(t)$. By the hypothesis, there exist unique curves γ_1^t and γ_2^t joining x to z and z to y, respectively, such that $l(\gamma_1^t) = d(x, z)$ and $l(\gamma_1^t) = d(z, y)$. We may assume by using a standard scaling down method that the domain of γ_1^t is $[0, (1 - t)\lambda]$, and the domain of γ_2^t is $[(1 - t)\lambda, \lambda]$. Consider the function $\gamma_0^t : [0, \lambda] \longrightarrow X$, defined by

$$\gamma_0^t(s) = \begin{cases} \gamma_1^t(s) & \text{on } [0, (1 - t)\lambda] \\ \gamma_2^t(s) & \text{on } [(1 - t)\lambda, \lambda] \end{cases}$$

Then $\gamma_0^t(0) = x, \gamma_0^t((1-t)\lambda) = z, \gamma_0^t(\lambda) = y$. Also γ_0^t is continuous, and $l(\gamma_0^t) = l(\gamma_1^t) + l(\gamma_2^t) = d(x, y) = l(\gamma)$. Consequently, $\gamma_0^t = \gamma$, for all t. Hence $E(t) \subset \gamma$. Finally, $E(t)$ consists of one point for each $t \in [0,1]$. For otherwise, one can easily see that $l(\gamma) > d(x,y)$. Thus (X,d) is M-convex. $\square$

3.5. Best Approximation in Metric Spaces

In this section, we recall the concept of best approximation in a convex metric space (X,d) and give some properties of the metric d in terms of proximinality and Chebychevity of some subset G of X.

Definition 3.5.1. Let G be a closed subset of a metric space (X,d). For a given $x \in X \setminus G$, a best approximation or nearest point to x from G is an element $y \in G$ such that

$$d(x,y) = dist(x,G) = \inf_{z \in G} d(x,z).$$

The set of all best approximations to x from G is denoted by $P_G(x)$. That is

$$P_G(x) = \{y \in G : dist(x,G) = d(x,y)\}.$$

Definition 3.5.2. Let G be a closed subset of a metric space (X,d). Then;

(i) G is called Proximinal if each $x \in X$ has a best approximation in G i.e $P_G(x) \neq \emptyset$ for each $x \in X$.

(ii) G is called Chebyshev if each $x \in X$ has a unique best approximation in G. i.e the set $P_G(x)$ consists of a singleton point.

(iii) A set valued function $p : X \longrightarrow \mathscr{P}(X)$, mapping each $x \in X$ to the set $P_G(x)$ is called the nearest point map or a metric projection.

We now give some properties of the metric d in terms of proximinality and Chebyshevity of some set in G.

Theorem 3.5.3. ([11, Theorem 2.1]) Let (X,d) be a Menger convex metric space. Then the following are equivalent.

(i) (X,d) is M-convex.

(ii) $C_d(z,r)$ is Chebyshev for all $z \in X, r > 0$.

(iii) $P_A(x) \cap P_A(y) = \emptyset$ for $x \neq y$ and all closed balls A in X.

Proof. $(i) \implies (ii)$ Let $C_d(z,r)$ be any closed ball. Let $x \in X \setminus C_d(z,r)$ and $d(x,z) = s = r + \lambda$. Then $dist(x, C_d(z,r)) = \lambda$. Since X is M-convex, then $C_d(x,\lambda) \cap C_d(z,r) = \{y\}$ for some $y \in X$. This implies that $P_{C_d(z,r)}(x) = \{y\}$ and so $C_d(z,r)$ is Chebyshev.
$(ii) \implies (iii)$ Suppose that $P_A(x) \cap P_A(y) \neq \emptyset$ for some $A = C_d(z,r)$ and some $x, y \in A$, then $\omega \in P_A(x) \cap P_A(y)$,

$$d(\omega,x) = d(\omega,y) = dist(\omega,A).$$

Then we have that x and y are best approximating elements of $A = C_d(z, r)$ and therefore, contradicting (ii). Hence, $P_A(x) \cap P_A(y) = \emptyset$ for some $x, y \in X$ and $x \neq y$.
$(iii) \implies (i)$ Let $x, y \in X$ and $d(x, y) = \lambda$. By convexity of (X, d), there exists a $t \in [0, 1]$ such that $C_d(x, (1-t)\lambda) \cap C_d(y, t\lambda) = E(t) \neq \emptyset$
If $z_1, z_2 \in E(t)$ and $z_1 \neq z_2$, then

$$x \in P_A(z_1) \cap P_A(z_2) \neq \emptyset$$

where $A = C_d(y, t\lambda)$. However this contradicts (ii). Hence (X, d) is M-convex. □

Definition 3.5.4 ([11])**.** Let (X, d) be a metric space. For $x \in X$ and $r > 0$, the set $S_d(x, r) = \{y \in X : d(x, y) = r\}$ is called a sphere centred at x with radius $r > 0$.
Theorem 3.5.5. ([11, Theorem 2.2]) Let (X, d) be an M-convex metric space. The following statements are equivalent:

(i) closed balls in X are convex.

(ii) If A is a closed convex subset in X and $x \notin A$, then $P_A(x)$ is convex.

Proof. $(i) \implies (ii)$ Suppose that $C_d(x, \delta)$ is convex. Let A be a closed convex subset of X and $x \notin A$. We show that the set $P_A(x)$ is convex. If $P_A(x) = \emptyset$ then it is trivially convex. Suppose that $P_A(x)$ is Chebyshev, that is $P_A(x) = \{z\}$, then $P_A(x)$ is convex, since every singleton set is convex. Suppose that $P_A(x) \neq \emptyset$, and let $z_1, z_2 \in P_A(x)$ such that $d(z_1, z_2) = \lambda$. Since $z_1, z_2 \in P_A(x)$, we have that $d(x, z_1) = d(x, z_2) = dist(x, A)$. So that if $dist(x, A) = r$, then we have that $d(x, z_1) = d(x, z_2) = r$ then $z_1, z_2 \in C_d(x, r)$. Since we have assumed that closed balls are convex, they contain a curve joining the points z_1 and z_2. That is, $L[z_1, z_2] \subseteq C_d(x, r)$. Also since A is convex, we get $L[z_1, z_2] \subseteq A$. Consequently, $L[z_1, z_2] \subseteq S_d(x, r)$ and so $l(L[z_1, z_2]) = dist(x, A) = r$. Hence $L[z_1, z_2] \subseteq P_A(x)$. Therefore, $P_A(x)$ is convex.
$(ii) \implies (i)$ Let $C_d(z, r)$ be a closed ball in (X, d) and $x, y \in C_d(z, r)$. Suppose that $C_d(z, r)$ is not convex, that is, $C_d(z, (1-t)\lambda) \cap C_d(z, t\lambda) \not\subset C_d(z, r)$ where $\lambda = d(x, y)$. By Theorem 3.4.17 , x and y can be joined by a unique curve of length $\lambda = d(x, y)$. If γ is this curve, then by Theorem 3.4.17 γ is a convex closed set in (X, d). By connectedness of γ, there exists at least two points z_1 and z_2 such that $\{z_1, z_2\} \subseteq \gamma \cap S_d(x, r)$. Then $z_1, z_2 \in P_\gamma(z)$. However, $L[z_1, z_2] \subsetneq P_\gamma(z)$. This contradicts (ii), and so $C_d(z, r)$ is a convex. □

Theorem 3.5.6. ([11, Theorem 2.3]) Let (X, d) be an M-convex metric space in which every proximinal convex set is Chebyshev. Then $C_d(x, r)$ is convex for all $z \in X$ and $r > 0$.

Proof. Let $x, y \in C_d(x, r)$ with $\lambda = d(x, y)$. If possible, let $\{C_d(x, \lambda - r) \cap C_d(y, r)\} \notin C_d(z, r)$ for some $r \in [0, \lambda]$. By Theorem 3.4.17, there exists distinct points $z_1, z_2 \in S_d(x, r)$ such that $L[z_1, z_2] = \gamma$ is not contained in $C_d(z, r)$. Since γ is compact (being the continuous image of $[0, d(z_1, z_2)]$), then γ is proximinal. But $z_1, z_2 \in P_\gamma(z)$ contradicting Chebyshevity of γ, since γ is convex. Hence $C_d(z, r)$ is convex. □

Theorem 3.5.7. ([11, Theorem 2.5]) Let (X, d) be a strictly convex metric space. Then every proximinal convex set in (X, d) is Chebyshev.

Proof. Let $G \subseteq X$ be proximinal and convex. Let $z \in X \setminus G$ such that $P_G(z)$ contains more than one element. Let $dist(z, G) = r$, and consider $\{z_1, z_2\} \subseteq P_G(z)$. Since $\{z_1, z_2\} \subseteq P_G(z)$, then we have $d(z, z_1) = d(z, z_2) = dist(z, G) = r$ and so $\{z_1, z_2\} \subseteq C_d(z, r)$. Let $d(z_1, z_2) = \lambda$. Since $\{z_1, z_2\} \subseteq C_d(z, r)$, it follows from the strict convexity of (X, d) that

$$w(t) = C_d(z_1, (1-t)\lambda) \cap C_d(z_2, t\lambda) \in B_d(z, r).$$

The convexity of G implies that $w(t) \in G$. Since every strictly convex space is M-convex, we have that $w(t)$ is a singleton set. Hence, $d(z_1, z_2) = \lambda = 0$ and so $z_1 = z_2$. Hence G is Chebyshev. $\square$

CHAPTER 4

CONVEXITIES IN T_0-QUASI-METRIC SPACES

In this chapter, we begin investigating convexities in T_0-quasi-metric spaces, namely; Menger convexity, Takahashi convexity, strong and M-convexity. In metric spaces we saw that, these convexities rely on the concept of betweenness, a fundamental concept in axiomatic geometry. Therefore, we start by presenting betweenness in the more general setting of T_0-quasi-metric spaces. Thereafter, we present convexities in T_0-quasi-metric spaces and some best approximations for M-convex T_0-quasi-metric spaces.

4.1. Betweenness in T_0-quasi-metric spaces

In this section, we discuss the concept of betweenness and midpoints in T_0-quasi-metric spaces. This concept was first introduced by Blumenthal ([4]). We show that q-betweenness does not necessarily imply q^{-1}-betweenness.

Definition 4.1.1. (Compare with Definition 3.1.1) Let (X, q) be a T_0-quasi-metric space and $x, y, z \in X$. Then a point z is said to be;

(i) q-between x and y if and only if $x \neq z \neq y$,

$$q(x, z) + q(z, y) = q(x, y).$$

(ii) q^{-1}-between x and y if and only if $x \neq z \neq y$,

$$q(y, z) + q(z, x) = q(y, x).$$

(iii) q^s-between x and y if and only if $x \neq z \neq y$,

$$q^s(x, z) + q^s(z, y) = q^s(x, y).$$

(iv) q, q^{-1}-between x and y if and only if $x \neq z \neq y$,

$$q(x, z) + q(z, y) = q(x, y) \quad \text{and} \quad q(y, z) + q(z, x) = q(y, x).$$

(v) q^+-between x and y if and only if $x \neq z \neq y$,

$$q^+(x, z) + q^+(z, y) = q^+(x, y),$$

where $q^+ = q + q^{-1}$.

We shall symbolize these relations in Definition 4.1.1 by $[xzy]_q$, $[xzy]_{q^{-1}}$, $[xzy]_{q^s}$, $[xzy]_q^{q^{-1}}$ and $[xzy]_{q^+}$ respectively. Also, since q is a T_0-quasi-metric, then $[xzy]_q$ implies x, y, z are pairwise distinct, since $q(x, y) > 0$.

Remark 4.1.2. The following example shows that q-betweenness does not necessarily imply q^{-1}-betweenness.

Example 4.1.3. Let $X = \{1, 2, 3, 4\}$ be a four point set, and let q be a T_0-quasi-metric defined by the distance matrix

$$M = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix}$$

that is, $q(i, j) = q_{i,j}$ whenever $i, j \in X$. We show that 3 is q-between 2 and 4 but 3 is not q^{-1}-between 2 and 4. We notice that $q(2, 4) = 2$, $q(2, 3) = 1$, $q(3, 4) = 1$. Hence

$$q(2, 3) + q(3, 4) = q(2, 4).$$

This implies that 3 is q-between 2 and 4.

On the other hand, $q(4, 2) = 1$, $q(4, 3) = 1$ and $q(3, 2) = 1$. Hence

$$q(4, 3) + q(3, 2) \neq q(4, 2).$$

Therefore, 3 is not q^{-1}-between 2 and 4.

The following Example shows that q^s-betweenness does not necessarily implies q^{-1}-betweenness.

Example 4.1.4. Consider again the four point set $X = \{1, 2, 3, 4\}$, and let q be a T_0-quasi-metric defined by the distance matrix

$$M = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix}$$

that is, $q(i, j) = q_{i,j}$ whenever $i, j \in X$. Then, one sees that $q^s = q \vee q^{-1}$ is defined by the matrix

$$M^s = \begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix}$$

We show that 3 is q^s-between 2 and 4. To see this we notice that, $q^s(2, 4) = 2$, $q^s(2, 3) = 1$ and $q^s(3, 4) = 1$. Hence,

$$q^s(2, 3) + q^s(3, 4) = q^s(2, 4).$$

Therefore, 3 is q^s-between 2 and 4.

Remark 4.1.5. We notice in Example 4.1.4 above that 3 is q^s-between 2 and 4 but 3 is not q^{-1}-between 2 and 4. In general, we note that q^s-between x and y does not necessarily imply q^{-1}-between x and y.

Proposition 4.1.6. *Let (X, q) be a T_0-quasi-metric space and $x, y, z \in X$. If z is q, q^{-1}-between x and y, then z is q^+-between x and y.*

Proof. Suppose that z is q, q^{-1}-between x and y, then $q(x,z)+q(z,y) = q(x,y)$ and $q(y,z)+q(z,x) = q(y,x)$. Adding the two equations $q(x,z) + q(z,y) = q(x,y)$ and $q(y,z) + q(z,x) = q(y,x)$ gives

$$q(x,z) + q(z,x) + q(z,y) + q(y,z) = q(x,y) + q(y,x).$$

Therefore,

$$q^+(x,z) + q^+(z,y) = q^+(x,y)$$

and so z is q^+-between x and y. □

We now introduce the concept of midpoints in T_0-quasi-metric space.

Definition 4.1.7. (Compare with Definition 3.1.3) Let (X, q) be a T_0-quasi-metric space and $x, y, z \in X$. Then z is said to be a

(i) q-midpoint of x and y if $x \neq z \neq y$,

$$q(x,z) = q(z,y) = \frac{1}{2}q(x,y).$$

(ii) q^{-1}-midpoint of x and y if $x \neq z \neq y$,

$$q(y,z) = q(z,x) = \frac{1}{2}q(y,x).$$

(iii) q^s-midpoint of x and y on (X, q^s) if and only if $x \neq z \neq y$,

$$q^s(x,z) = q^s(z,y) = \frac{1}{2}q^s(x,y).$$

(iv) q, q^{-1}-midpoint of x and y if and only if $x \neq y \neq z$,

$$q(x,z) = q(z,y) = \frac{q(x,y)}{2} \quad \text{and} \quad q(y,z) = q(z,x) = \frac{q(y,x)}{2}.$$

Let us note that the second condition in Definition 4.1.7 is just the first condition formulated for the dual T_0-quasi-metric. Evidently Definition 4.1.7 is analogous to Definition 3.1.3.

Remark 4.1.8. We notice that the following example shows that a point z can be q-midpoint of x and y but not necessarily q^{-1}-midpoint of x and y.

Example 4.1.9. Let $X = \{1, 2, 3, 4\}$ be a four point set, and let q be a T_0-quasi-metric defined by the distance matrix

$$M = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix}$$

that is, $q(i,j) = q_{i,j}$ whenever $i, j \in X$. We show that 2 is q-midpoint of 1 and 3 but 2 is not q^{-1}-midpoint of 1 and 3. We notice that $q(1,3) = 2$, $q(1,2) = 1$, $q(2,3) = 1$. Hence

$$q(1,2) = q(2,3) = \frac{1}{2}q(1,3).$$

This implies that 2 is q-midpoint of 1 and 3.
However, $q(3,1) = 1$, $q(3,2) = 1$ and $q(2,1) = 1$. Hence

$$q(3,2) = q(2,1) \neq \frac{1}{2}q(3,1).$$

Therefore, 2 is not q^{-1}-midpoint of 1 and 3.

Example 4.1.10. Consider again the four point set $X = \{1,2,3,4\}$, and let q be a T_0-quasi-metric defined by the distance matrix

$$M = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix}$$

that is, $q(i,j) = q_{i,j}$ whenever $i, j \in X$. Then, as before q^s is given by the matrix

$$M^s = \begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix}$$

We show that 2 is a q^s-midpoint of 1 and 3. To see this we notice that, $q^s(1,3) = 2$, $q^s(1,2) = 1$ and $q^s(2,3) = 1$. Hence,

$$q^s(1,2) = q^s(2,3) = \frac{1}{2}q^s(1,3).$$

Therefore, 2 is a q^s-midpoint of 1 and 3 on (X, q^s).

Remark 4.1.11. We notice in the Example 4.1.16 above that 2 is a q^s-midpoint of 1 and 3 but 2 is not a q^{-1}-midpoint 1 and 3. In general, we note that q^s-midpoint of x and y does not necessarily imply q^{-1}-midpoint of x and y.

We now generalize a well known result introduced by Blumenthal [4] in metric spaces to the setting of T_0-quasi-metric spaces.

Theorem 4.1.12. (Compare with Theorem 3.1.4)

Let (X, q) be a T_0-quasi-metric space. Then the relation of q, q^{-1}-betweenness has the following properties:

(i) If $[xzy]_q^{q^{-1}}$ then $[yzx]_q^{q^{-1}}$ (symmetry of the outer points).

(ii) If $[xzy]_q^{q^{-1}}$ then neither $[xyz]_q^{q^{-1}}$ nor $[zxy]_q^{q^{-1}}$ holds (special inner points).

(iii) $[xzy]_q^{q^{-1}}$ and $[xys]_q^{q^{-1}}$ are equivalent to $[xzs]_q^{q^{-1}}$ and $[zys]_q^{q^{-1}}$.

(iv) If $x, y \in X$, the set $\bar{B}_q^{q^{-1}}(x,y) = \{x\} \cup \{y\} \cup B_q^{q-1}(x,y)$, where $B_q^{q-1}(x,y)$ is the set of all points q, q^1-between x and y is $\tau(q^s)$-closed.

Proof. (i) If $[xzy]_q^{q^{-1}}$ then $x \neq z \neq y$, $q(x,y) = q(x,z) + q(z,y)$ and $q(y,x) = q(y,z) + q(z,x)$ and so by definition, we have that $[yzx]_q^{q^{-1}}$.

(ii) Since $[xzy]_q^{q^{-1}}$ implies $x \neq z \neq y$,

$$q(x,z) + q(z,y) = q(x,y) \tag{4.1}$$

and

$$q(y,z) + q(z,x) = q(y,x) \tag{4.2}$$

Also, $[xyz]_q^{q^{-1}}$ implies $x \neq y \neq z$,

$$q(x,y) + q(y,z) = q(x,z) \tag{4.3}$$

and

$$q(z,y) + q(y,x) = q(z,x) \tag{4.4}$$

Adding Equation 4.1 and Equation 4.3 we obtain

$$q(z,y) + q(y,z) = 0 \tag{4.5}$$

Adding Equation 4.2 and Equation 4.4 gives

$$q(y,z) + q(z,y) = 0 \tag{4.6}$$

Furthermore, adding Equation 4.5 and Equation 4.6 we have

$$2(q(z,y) + q(y,z)) = 0.$$

This implies that $q^+(z,y) = 0$. Since q^+ is a metric, $z = y$. This contradicts that $z \neq y$. Hence, $[xyz]_q^{q^{-1}}$ does not hold. Similarly, $[zxy]_q^{q^{-1}}$ cannot hold too.

(iii) Suppose that $[xzy]_q^{q^{-1}}$ holds, then we have that $x \neq y \neq z$,

$$q(x,z) + q(z,y) = q(x,y) \tag{4.7}$$

and

$$q(y,z) + q(z,x) = q(y,x) \tag{4.8}$$

Also, suppose that $[xys]_q^{q^{-1}}$ holds, then we have that $x \neq y \neq s$,

$$q(x,y) + q(y,s) = q(x,s) \tag{4.9}$$

and

$$q(s,y) + q(y,x) = q(s,x) \tag{4.10}$$

Adding Equation 4.7 and Equation 4.9 obtain the equation

$$q(x,z)+q(z,y)+q(y,s)=q(x,s) \tag{4.11}$$

Also, adding Equation 4.8 and Equation 4.10 we have

$$q(y,z)+q(z,x)+q(s,y)=q(s,x) \tag{4.12}$$

Now, applying the triangle inequality to Equation 4.11, we have

$$\begin{aligned} q(x,s) &= q(x,z)+q(z,y)+q(y,s) \\ &\geq q(x,z)+q(z,s) \\ &\geq q(x,s) \end{aligned}$$

which implies that $q(x,s)=q(x,z)+q(z,s)$. Now, substituting this result in Equation 4.11, we obtain $q(z,s)=q(z,y)+q(y,s)$. Similarly, applying the triangle inequality to Equation 4.12, we obtain $q(s,x)=q(s,z)+q(z,x)$ and also substituting this results in Equation 4.12 we have $q(s,z)=q(s,y)+q(y,z)$. Since each two points are pairwise distinct and $q(x,s)=q(x,z)+q(z,s)$ and $q(s,x)=q(s,z)+q(z,x)$, then we have that $[xzs]_q^{q^{-1}}$. Also, since each two points are pairwise distinct and also $q(z,s)=q(z,y)+q(y,s)$ and $q(s,z)=q(s,y)+q(y,z)$, then we obtain $[zys]_q^{q^{-1}}$. The converse follows directly from above argument.

(iv) We show that if z is a $\tau(q^s)$-accumulation element of $\bar{B}_q^{q^{-1}}(x,y)$, with $x\neq z\neq y$, then $z\in \bar{B}_q^{q^{-1}}(x,y)$. Since z is a $\tau(q^s)$-accumulation point, there exist a sequence $\{z_n\}$ that is $\tau(q)$-convergent and $\tau(q^{-1})$-convergent to z, where $z_n\in \bar{B}_q^{q^{-1}}(x,y)$. Therefore,

$$q(x,z_n)+q(z_n,y)=q(x,y)$$

and

$$q(y,z_n)+q(z_n,x)=d(y,x)$$

for all $n\in\mathbb{N}$. From the continuity of the T_0-quasi-metrics q and q^{-1}, we obtain

$$q(x,z)+q(z,y)=q(x,y)$$

and

$$q(y,z)+q(z,x)=q(y,x).$$

Therefore, $z\in \bar{B}_q^{q^{-1}}(x,y)$ and so $\bar{B}_q^{q^{-1}}(x,y)$ is $\tau(q^s)$-closed.

. □

4.2. Menger convexity in T_0-quasi-metric spaces

In this section, we generalise Menger convexity, which was introduced by Karl Menger [21] in 1928, to the framework of T_0-quasi-metric spaces.

Definition 4.2.1. (Compare with Definition 3.2.3) A T_0-quasi-metric space (X, q) is said to be Menger convex if for every distinct points x and y of X and $t \in [0, 1]$, there exists a point $z \in X$ such that

(i) $q(x, z) = tq(x, y)$ and $q(z, y) = (1 - t)q(x, y)$.

(ii) $q(y, z) = (1 - t)q(y, x)$ and $q(z, x) = tq(y, x)$.

We notice that condition (ii) of Definition 4.2.1 is formulated for the dual T_0-quasi-metric. We observe that if q has symmetry, then we have that $q(x, z) = q(z, x) = tq(x, y) = tq(y, x)$ which implies $q(x, z) = tq(x, y)$. Similarly, $q(z, y) = (1 - t)q(x, y) = q(y, z) = (1 - t)q(y, x)$ which implies $q(z, y) = (1-t)q(x, y)$. Hence, obtaining the conditions (i) and (ii) of Definition 3.2.3.

Remark 4.2.2. (Compare with Remark 3.2.4) We notice that a T_0-quasi-metric space (X, q) is said to be Menger convex if for any two distinct points x and y of X, there exists a point $z \in X$ with $x \neq z \neq y$ such that

$$q(x, z) + q(z, y) = q(x, y) \quad \text{and} \quad q(y, z) + q(z, x) = q(y, x).$$

We notice that we have just added conditions (i) and (ii) respectively in Definition 4.2.1 to obtain the result in above remark.

Example 4.2.3. (Compare with Example 3.2.5) Let $X = \mathbb{Q}$ be the set of rational numbers and q be a T_0-quasi-metric defined by $q(x, y) = \max\{x - y, 0\}$ for all $x, y \in X$. Then (X, q) is a Menger convex T_0-quasi-metric space.

Proof. Let $x, y \in X$, $x \neq y$, set $z = \frac{x+y}{2} \in X$, then we have

$$\begin{aligned} q(x, z) &= \max\left\{x - z, 0\right\} = \max\left\{x - \frac{x+y}{2}, 0\right\} \\ &= \max\left\{\frac{2x - x - y}{2}, 0\right\} = \frac{1}{2}\left(\max\left\{x - y, 0\right\}\right) = \frac{1}{2}q(x, y) \end{aligned}$$

and

$$\begin{aligned} q(z, y) &= \max\left\{z - y, 0\right\} = \max\left\{\frac{x+y}{2} - y, 0\right\} \\ &= \max\left\{\frac{x - y}{2}, 0\right\} = \frac{1}{2}\left(\max\left\{x - y, 0\right\}\right) = \frac{1}{2}q(x, y) \end{aligned}$$

Thus, $q(x, z) + q(z, y) = \frac{1}{2}q(x, y) + \frac{1}{2}q(x, y) = q(x, y)$ for all $x, y, z \in X$.
In a similar manner,

$$\begin{aligned} q(y, z) &= \max\left\{y - z, 0\right\} = \max\left\{y - \frac{x+y}{2}, 0\right\} \\ &= \max\left\{\frac{2y - y - x}{2}, 0\right\} = \frac{1}{2}\left(\max\left\{y - x, 0\right\}\right) = \frac{1}{2}q(y, x) \end{aligned}$$

and

$$q(z,x) = \max\left\{z-x, 0\right\} = \max\left\{\frac{x+y}{2} - x, 0\right\}$$
$$= \max\left\{\frac{y-x}{2}, 0\right\} = \frac{1}{2}\left(\max\left\{y-x, 0\right\}\right) = \frac{1}{2}q(y,x)$$

Thus, $q(y,z) + q(z,x) = \frac{1}{2}q(y,x) + \frac{1}{2}q(y,x) = q(y,x)$ for all $x, y, z \in X$. Therefore, by Remark 4.2.2 (X, q) is a Menger convex T_0-quasi-metric space. $\square$

The following Proposition is adapted from Proposition 3.2.6, but here we use the property of double balls.

Proposition 4.2.4. *Let (X, q) be a T_0-quasi-metric space. Then (X, q) is Menger convex if and only if for every $x, y \in X$ with $x \neq y$ we have*

$$C_q(x,r) \cap C_{q^{-1}}(x,s) \cap C_q(y, \lambda_1 - r) \cap C_{q^{-1}}(y, \lambda_2 - s) \neq \emptyset,$$

where $q(x,y) = \lambda_1, \quad q(y,x) = \lambda_2$ and $0 \leq r \leq \lambda_1$ and $0 \leq s \leq \lambda_2$.

Proof. Suppose that for every $x, y \in X$ with $x \neq y$ we have

$$C_q(x,r) \cap C_{q^{-1}}(x,s) \cap C_q(y, \lambda_1 - r) \cap C_{q^{-1}}(y, \lambda_2 - s) \neq \emptyset,$$

where $q(x,y) = \lambda_1, \quad q(y,x) = \lambda_2$ and $0 \leq r \leq \lambda_1$ and $0 \leq s \leq \lambda_2$. Let $x, y \in X, t \in [0,1]$ with $\lambda_1 = q(x,y)$ and $\lambda_2 = q(y,x)$. Then $0 \leq t\lambda_1 \leq \lambda_1$ and $0 \leq t\lambda_2 \leq \lambda_2$. Let $r_1 = t\lambda_1$ and $r_2 = \lambda_1 - t\lambda_1 = (1-t)\lambda_1$. Also, let $s_1 = t\lambda_2$ and $s_2 = \lambda_2 - t\lambda_2 = (1-t)\lambda_2$. There exists

$$z \in C_q(x, r_1) \cap C_{q^{-1}}(x, s_1) \cap C_q(y, r_2) \cap C_{q^{-1}}(y, s_2),$$

so that $q(x,z) \leq r_1, \quad q(z,x) \leq s_1$ and $q(y,z) \leq s_2, \quad q(z,y) \leq r_2$.
Consider

$$\begin{aligned} q(x,y) &\leq q(x,z) + q(z,y) \\ &\leq r_1 + r_2 \\ &= \lambda_1 = q(x,y). \end{aligned}$$

Then $q(x,y) = q(x,z) + q(z,y)$. Also,

$$\begin{aligned} q(y,x) &\leq q(y,z) + q(z,x) \\ &\leq s_1 + s_2 \\ &= \lambda_2 = q(y,x). \end{aligned}$$

Then $q(y,x) = q(y,z) + q(z,x)$. Therefore, we have

(i) $q(x,z) = r_1 = tq(x,y)$ and $q(z,y) = r_2 = (1-t)q(x,y)$

(ii) $q(y,z) = s_1 = (1-t)q(y,x)$ and $q(z,x) = s_2 = tq(x,y)$.

Hence (X,q) is Menger convex.

Conversely, suppose (X,q) is Menger convex. Let $x,y \in X$ be such that $q(x,y) = \lambda_1$ and $q(y,x) = \lambda_2$. Let $r \in [0,\lambda_1]$ and $s \in [0,\lambda_2]$. We want to show that

$$C_q(x,r) \cap C_{q^{-1}}(x,s) \cap C_q(y,\lambda_1 - r) \cap C_{q^{-1}}(y,\lambda_2 - s) \neq \emptyset.$$

Let $t = \frac{r}{\lambda_1}$, then $t \in [0,1]$, and by the Menger convexity of (X,q) there exists $z \in X$ such that

$$q(x,z) = tq(x,y) = t\lambda_1 = r$$

and

$$q(z,y) = (1-t)q(x,y) = (1-t)\lambda_1 = \lambda_1 - r.$$

Also, by letting $t = \frac{s}{\lambda_2}$, we have $t \in [0,1]$, and by Menger convexity of (X,q), there exists $z \in X$ such that

$$q(y,z) = (1-t)q(x,y) = (1-t)\lambda_1 = \lambda_2 - s$$

and

$$q(z,x) = tq(x,y) = t\lambda_2 = s.$$

This implies that

$$z \in C_q(x,r) \cap C_{q^{-1}}(x,s) \cap C_q(y,\lambda_1 - r) \cap C_{q^{-1}}(y,\lambda_2 - s).$$

Hence

$$C_q(x,r) \cap C_{q^{-1}}(x,s) \cap C_q(y,\lambda_1 - r) \cap C_{q^{-1}}(y,\lambda_2 - s) \neq \emptyset.$$

□

4.3. Takahashi convexity in T_0-quasi-metric spaces

We now recall the definition of Takahashi convexity in T_0-quasi-metric spaces. This was extensively studied by Künzi [19]. We will later show the relationship between Takahashi convexity and Menger convexity in T_0-quasi-metric spaces.

Definition 4.3.1. (Compare with Definition 3.3.1) Let (X,q) be a T_0-quasi-metric space. A mapping $W : X \times X \times [0,1] \longrightarrow X$ is said to be a Takahashi convex structure (TCS) on X if for all $x,y \in X$ and $\lambda \in [0,1]$, the following two conditions are satisfied:

(i) $q(u, W(x,y,\lambda)) \leq \lambda q(u,x) + (1-\lambda)q(u,y)$ and

(ii) $q(W(x,y,\lambda),u) \leq \lambda q(x,u) + (1-\lambda)q(y,u)$ whenever $u \in X$.

The T_0-quasi-metric space (X,q) together with a convex structure W is called a convex T_0-quasi-metric space.

We note that condition (ii) is formulated for the dual T_0-quasi-metric and so by definition if W is TCS for (X,q), then it is also a TCS for (X,q^{-1}).

Definition 4.3.2 (Compare with Definition 3.3.3). Let (X, q) be a T_0-quasi-metric space with TCS W on X. A subset K of X is said to be convex provided that $W(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda \in [0, 1]$.

The following proposition shows that the intersection of convex subsets in T_0-quasi-metric spaces is convex:

Proposition 4.3.3. *(Compare with Proposition 3.3.8) Let (X, q) be a T_0-quasi-metric space with a TCS on X. Let $\{K_\alpha\}_{\alpha \in A}$ be a family of convex subsets of X, then $\bigcap_{\alpha \in A} K_\alpha$ is convex.*

Proof. Follows in the same way as Proposition 3.3.8. □

The following proposition shows that the closed and open backwards and forward balls in T_0-quasi-metric space with TCS are convex:

Proposition 4.3.4. *(Compare with Proposition 3.3.9) Let (X, q) be a T_0-quasi-metric space with W as TCS on X. Then for any $x \in X$ and $\delta > 0$ the open balls $B_q(x, \delta)$ and $B_{q^{-1}}(x, \delta)$, and the closed balls $C_q(x, \delta)$ and $C_{q^{-1}}(x, \delta)$ in X are convex subsets of X.*

Proof. Follows in the same way as Proposition 3.3.9. □

Proposition 4.3.5. *([19, Remark 2]) Let (X, q) be a T_0-quasi-metric space with W as TCS on X, then W is a TCS for the metric $q^+ = q + q^{-1}$ on X.*

Proof. Let $x, y \in X$ and $\lambda \in [0, 1]$. Since W is a TCS on (X, q), then we have that

$$q(u, W(x, y, \lambda)) \leq \lambda q(u, x) + (1 - \lambda) q(u, y) \tag{4.13}$$

and

$$q(W(x, y, \lambda), u) \leq \lambda q(x, u) + (1 - \lambda) q(y, u). \tag{4.14}$$

Also, since W is a TCS on (X, q^{-1}) we have that

$$q^{-1}(u, W(x, y, \lambda)) \leq \lambda q^{-1}(u, x) + (1 - \lambda) q^{-1}(u, y) \tag{4.15}$$

$$q^{-1}(W(x, y, \lambda), u) \leq \lambda q^{-1}(x, u) + (1 - \lambda) q^{-1}(y, u). \tag{4.16}$$

Now, adding (4.13) and (4.15) we obtain

$$\begin{aligned} q^+(u, W(x, y, \lambda)) &= q(u, W(x, y, \lambda)) + q^{-1}(u, W(x, y, \lambda)) \\ &\leq \lambda \left(q(u, x) + q^{-1}(u, x) \right) + (1 - \lambda) \left(q(u, y) + q^{-1}(u, y) \right) \\ &= \lambda q^+(u, x) + (1 - \lambda) q^+(u, y). \end{aligned}$$

Hence we have $q^+(u, W(x, y, \lambda)) \leq \lambda q^+(u, x) + (1 - \lambda) q^+(u, y)$.
Similarly, adding (4.14) and (4.16) we obtain $q^+(W(x, y, \lambda), u) \leq \lambda q^+(x, u) + (1 - \lambda) q^+(y, u)$ and so W is a TCS with the metric q^+. □

Proposition 4.3.6. *([19, Remark 4]) If $W(x,y,\lambda)$ is a TCS on a T_0-quasi-metric space (X,q), then $W^{-1}(x,y,\lambda) := W(y,x,1-\lambda)$ whenever $x,y \in X$ and $\lambda \in [0,1]$ is a TCS on (X,q).*

Proof. Let $x,y \in X$ and $\lambda \in [0,1]$. Since $W(x,y,\lambda)$ is a TCS on X, then

$$\begin{aligned} q\left(u, W^{-1}(x,y,\lambda)\right) &= q\left(u, W(y,x,1-\lambda)\right) \\ &\leq (1-\lambda)q(u,y) + (1-(1-\lambda))q(u,x) \\ &= \lambda q(u,x) + (1-\lambda)q(u,y), \end{aligned}$$

and similarly,

$$\begin{aligned} q\left(W^{-1}(x,y,\lambda), u\right) &= q\left(W(y,x,1-\lambda), u\right) \\ &\leq (1-\lambda)q(y,u) + (1-(1-\lambda))q(x,u) \\ &= \lambda q(x,u) + (1-\lambda)q(y,u), \end{aligned}$$

whenever $u \in X$. Hence, $W^{-1}(x,y,\lambda)$ is a TCS on X. □

The next two results give some general properties of a Takahashi convexity structure in T_0-quasi-metric spaces:

Proposition 4.3.7. *(Compare with Proposition 3.3.11) Let (X,q) be a T_0-quasi-metric space with TCS W on X. Then we have the following for all $x,y \in X$ and $\lambda \in [0,1]$:*

(i) $W(x,x,\lambda) = x$ and

(ii) $W(y,x,0) = x$ and $W(y,x,1) = y$.

Proof. We note that the proof of these properties relies on a T_0-property:

(i) To see this, let $x,y \in X$ and $\lambda \in [0,1]$, then $q(x, W(x,x,\lambda)) \leq \lambda q(x,x) + (1-\lambda)q(x,x) = 0$. Similarly, $q(W(x,x,\lambda),x) \leq \lambda q(x,x) + (1-\lambda)q(x,x) = 0$ and so $W(x,x,\lambda) = x$ by the T_0-property of (X,q).

(ii) Let $x,y \in X$ and $\lambda \in [0,1]$, then we have that $q(x, W(y,x,0)) \leq 0.q(x,y) + (1-0)q(x,x) = 0$ and also, $q(W(y,x,0),x) \leq 0.q(y,x) + (1-0)q(x,x) = 0$ and so by T_0-property we have $W(y,x,0) = x$. Similarly, $q(y, W(y,x,1)) \leq 1.q(y,y) + (1-1)q(y,x) = 0$ and also $q(W(y,x,1),y) \leq 1.q(y,y) + (1-1)q(x,y) = 0$. Thus by T_0-property we have $W(y,x,1) = y$.

□

Proposition 4.3.8. *(Compare with Proposition 3.3.10) Let (X,q) be a T_0-quasi-metric space with TCS W. For any $x,y \in X$ and $\lambda \in [0,1]$, we have*

$$q(x,y) = q(x, W(x,y,\lambda)) + q(W(x,y,\lambda), y)$$

and

$$q(y,x) = q(y, W(x,y,\lambda)) + q(W(x,y,\lambda), x).$$

Proof. Let $x, y \in X$ and $\lambda \in [0,1]$. Since W is a TCS on T_0-quasi-metric space (X, q), then by the triangle inequality we have

$$\begin{aligned} q(x,y) &\leq q(x, W(x,y,\lambda)) + q(W(x,y,\lambda), y) \\ &\leq \lambda q(x,x) + (1-\lambda)q(x,y) + \lambda q(x,y) + (1-\lambda)q(y,y) \\ &= (1-\lambda)q(x,y) + \lambda q(x,y) \\ &= q(x,y) \end{aligned}$$

and thus $q(x,y) = q(x, W(x,y,\lambda)) + q(W(x,y,\lambda), y)$.
Similar argument shows that $q(y,x) = q(y, W(x,y,\lambda)) + q(W(x,y,\lambda), x)$. □

From the conclusions of the above Proposition 4.3.7, we can also derive the following results.

Proposition 4.3.9. *(Compare with Lemma 3.3.12) Let (X, q) be a T_0-quasi-metric space with TCS W. For any $x, y \in X$ and $\lambda \in [0,1]$, we have*

(i) $q(x, W(x,y,\lambda)) = (1-\lambda)q(x,y)$ *and* $q(W(x,y,\lambda), y) = \lambda q(x,y)$.

(ii) $q(y, W(x,y,\lambda)) = \lambda q(y,x)$ *and* $q(W(x,y,\lambda), x) = (1-\lambda)q(y,x)$.

Proof. (i) Since X is a T_0-quasi-metric space with TCS W, for any $x, y \in X$ and $\lambda \in [0,1]$, we have

$$q(x, W(x,y,\lambda)) \leq \lambda q(x,x) + (1-\lambda)q(x,y) = (1-\lambda)q(x,y)$$

which implies

$$q(x, W(x,y,\lambda)) \leq (1-\lambda)q(x,y) \tag{4.17}$$

Using the triangle inequality and the fact that W is a TCS, we have that

$$\begin{aligned} q(x,y) &\leq q(x, W(x,y,\lambda)) + q(W(x,y,\lambda), y) \\ &\leq q(x, W(x,y,\lambda)) + \lambda q(x,y) + (1-\lambda)q(y,y) \\ &= q(x, W(x,y,\lambda)) + \lambda q(x,y) \end{aligned}$$

which gives

$$(1-\lambda)q(x,y) \leq q(x, W(x,y,\lambda)) \tag{4.18}$$

Hence, combining 4.17 and 4.18 gives $q(x, W(x,y,\lambda)) = (1-\lambda)q(x,y)$ whenever $x, y \in X$ and $\lambda \in [0,1]$. A similar argument shows that $q(W(x,y,\lambda), y) = \lambda q(x,y)$ for all $x, y \in X$ and $\lambda \in [0,1]$.

(ii) Similarly, for any $x, y \in X$ and $\lambda \in [0,1]$ we have that

$$q(y, W(x,y,\lambda)) \leq \lambda q(y,x) + (1-\lambda)q(y,y) = \lambda q(y,x)$$

which gives

$$q(y, W(x, y, \lambda)) \leq \lambda q(y, x). \tag{4.19}$$

Using the triangle inequality and the fact that W is a TCS, we have that

$$\begin{aligned} q(y, x) &\leq q(y, W(x, y, \lambda)) + q(W(x, y, \lambda), x) \\ &\leq q(y, W(x, y, \lambda)) + \lambda q(x, x) + (1 - \lambda) q(y, x) \\ &= q(y, W(x, y, \lambda)) + (1 - \lambda) q(y, x) \end{aligned}$$

which implies

$$\lambda q(y, x) \leq q(y, W(x, y, \lambda)). \tag{4.20}$$

Hence, combining 4.19 and 4.20 we have that $q(y, W(x, y, \lambda)) = \lambda q(y, x)$ for all $x, y \in X$ and $\lambda \in [0, 1]$. A similar argument shows that $q(W(x, y, \lambda), x) = (1 - \lambda) q(x, y)$ whenever $x, y \in X$ and $\lambda \in [0, 1]$.

□

Takahashi [28] does not require any continuity in his definition of convex structure for metric spaces. However, we can sometimes make the assumption that W satisfies some additional properties of continuity. We next give a result for a general convexity structure of T_0-quasi-metric space that is analogous to a well known result in metric spaces.

Proposition 4.3.10. *(Compare with Theorem 3.3.18) Let W be a TCS on a T_0-quasi-metric space (X, q). Then for any $x \in X$ and $\lambda \in [0, 1]$, W is continuous at (x, x, λ) in $X \times X \times [0, 1]$ where X carries the topology $\tau(q)$ or $\tau(q^{-1})$.*

Proof. Let $((x_n, y_n, \lambda_n))$ be a sequence in $X \times X \times [0, 1]$ converging to (x, x, λ) where X is equipped with the topology $\tau(q)$. By Proposition 4.3.7, we have that $W(x, x, \lambda) = x$, so it suffices to show that $W(x_n, y_n, \lambda_n)$ converges to x. Since the sequence (x_n) converges to x with respect to the topology $\tau(q)$, for any $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that $q(x, x_n) < \varepsilon$ for all $n \geq N_1$. Also, since (y_n) converges to x with respect to the topology $\tau(q)$, for any $\varepsilon > 0$, there exists $N_2 \in \mathbb{N}$ such that $q(x, y_n) < \varepsilon$ for all $n \geq N_2$. Now, for all $n \geq N := \max\{N_1, N_2\}$ and using condition (i) of Definition 4.3.1 we have

$$\begin{aligned} q(x, W(x_n, y_n, \lambda_n)) &\leq \lambda_n q(x, x_n) + (1 - \lambda_n) q(x, y_n) \\ &< \lambda_n \varepsilon + (1 - \lambda_n) \varepsilon \\ &= \lambda_n \varepsilon + \varepsilon - \lambda_n \varepsilon \\ &= \varepsilon. \end{aligned}$$

Hence, we have that $W(x_n, y_n, \lambda_n)$ converges to x with respect to topology $\tau(q)$. Similarly, by condition (ii) of Definition 4.3.1 the analogous results holds if we work with the topology $\tau(q^{-1})$ on X. □

We now look at the standard convexity structure in an asymmetric normed real vector space X.

Definition 4.3.11. ([19, Definition 2]) Let (X, q) be a T_0-quasi-metric space with TCS W on X, then W is said to be synchronised if $W^{-1}(x, y, \lambda) = W(x, y, \lambda)$ whenever $x, y \in X$ and $\lambda \in [0, 1]$.

Definition 4.3.12. (Compare with Definition 3.3.26) We say that a TCS W on a T_0-quasi-metric space (X, q) has condition (C) if $W(x, y, \lambda) = W(y, x, 1-\lambda)$ whenever $x, y \in X$ and $\lambda \in [0, 1]$.

Example 4.3.13. ([19, Example 4]) Let A be a convex subset of a real vector space X equipped with the asymmetric norm $||.|$. Then $S(x, y, \lambda) = x\lambda + (1-\lambda)y$ for all $x, y \in A$ and $\lambda \in [0, 1]$ defines a synchronised convex structure for the T_0-quasi-metric space (A, q) where $q(x, y) = ||x - y|$ whenever $x, y \in A$.

Proof. For any $x, y \in A$ and $\lambda \in [0, 1]$ we have

$$\begin{aligned} q(S(x, y, \lambda), u) &= ||S(x, y, \lambda) - u| \\ &= ||x\lambda + (1-\lambda)y - u| \\ &= ||x\lambda + (1-\lambda)y - (u\lambda + (1-\lambda)u)| \\ &= ||\lambda(x-u) + (1-\lambda)(y-u)| \\ &\leq \lambda||x-u| + (1-\lambda)||y-u| \\ &= \lambda q(x, u) + (1-\lambda)q(y, u) \end{aligned}$$

Hence, condition (i) of Definition 4.3.1 is satisfied by $S(x, y, \lambda)$. Similarly, for any $x, y, u \in A$ and $\lambda \in [0, 1]$,

$$\begin{aligned} q(u, S(x, y, \lambda)) &= |u - S(x, y, \lambda)|| \\ &= |u - x\lambda - (1-\lambda)y|| = |(u\lambda + (1-\lambda)u) - (x\lambda + (1-\lambda)y)|| \\ &= |\lambda(u-x) + (1-\lambda)(u-y)|| \leq \lambda|u-x|| + (1-\lambda)|u-y|| \\ &= \lambda q(u, x) + (1-\lambda)q(u, y) \end{aligned}$$

Hence, condition (ii) of Definition 4.3.1 is also satisfied. Therefore, $S(x, y, \lambda)$ is a TCS on A. To show that $S(x, y, \lambda)$ is synchronised convexity structure on A, we need to show that $S(x, y, \lambda) = S(y, x, 1-\lambda)$ whenever $x, y \in A$ and $\lambda \in [0, 1]$. To see this, we have

$$\begin{aligned} S(x, y, \lambda) &= \lambda x + (1-\lambda)y \\ &= (1 - 1 + \lambda)x + (1-\lambda)y \\ &= (1 - (1-\lambda))x + (1-\lambda)y \\ &= S(y, x, 1-\lambda) \end{aligned}$$

Hence, indeed $S(x, y, \lambda)$ whenever $x, y \in A$ and $\lambda \in [0, 1]$ is a synchronised convexity structure on A. $\square$

Remark 4.3.14. ([19, Remark 7]) We note that the crucial property in the preceding result is the condition that $S(x+u, y+u, \lambda) = S(x, y, \lambda) + u$ whenever $x, y, u \in X$ and $\lambda \in [0,1]$. Such convexity structure W in a real vector space X is said to be translation-invariant.

Proposition 4.3.15. *([19, Remark 8]) Suppose that $W(x, y, \lambda)$ and $W'(x, y, \lambda)$ are convexity structures in an asymmetric normed space $(X, ||.|)$, then for each $\alpha \in [0,1]$, $W_\alpha(x, y, \lambda) = \alpha W(x, y, \lambda) + (1-\alpha)W'(x, y, \lambda)$ is also a convexity structure for a T_0-quasi-metric space (X, q) where $q(x, y) = ||x - y|$ whenever $x, y \in X$.*

Proof. We show that $W_\alpha(x, y, \lambda)$ satisfies the two conditions of Definition 4.3.1. For any $x, y \in X$ and $\lambda \in [0,1]$, we have

$$\begin{aligned} q(W_\alpha(x, y, \lambda), u) &= ||W_\alpha(x, y, \lambda) - u| \\ &= ||\alpha W(x, y, \lambda) + (1-\alpha)W'(x, y, \lambda) - u| \\ &= ||\alpha W(x, y, \lambda) + (1-\alpha)W'(x, y, \lambda) - (\alpha u + (1-\alpha)u)| \\ &\leq \alpha||W(x, y, \lambda) - u| + (1-\alpha)||W'(x, y, \lambda) - u| \\ &= \alpha q(W(x, y, \lambda), u) + (1-\alpha)q(W'(x, y, \lambda), u) \\ &\leq \alpha\lambda q(x, u) + \alpha(1-\lambda)q(y, u) + \lambda(1-\alpha)q(x, u) + (1-\alpha)(1-\lambda)q(y, u) \\ &= \alpha\lambda q(x, u) + \lambda(1-\alpha)q(x, u) + \alpha(1-\lambda)q(y, u) + (1-\alpha)(1-\lambda)q(y, u) \\ &= \lambda q(x, u)(\alpha + 1 - \alpha) + (1-\lambda)q(y, u)(\alpha + 1 - \alpha) \\ &= \lambda q(x, u) + (1-\lambda)q(y, u), \end{aligned}$$

proving (i) of Definition 4.3.1. A similar argument shows that $W_\alpha(x, y, \lambda)$ satisfies inequality (ii) of Definition 4.3.1. Therefore, $W_\alpha(x, y, \lambda)$ is a TCS on X. □

Definition 4.3.16. (Compare with Definition 3.3.13) Let (X, q) be a T_0-quasi-metric space. We say that (X, q) is said to be strictly convex if for each $x, y \in X$ and $\lambda \in [0,1]$, there exists a unique $w(x, y, \lambda) \in X$ such that

(i) $q(x, w(x, y, \lambda)) = (1-\lambda)q(x, y)$ and $q(w(x, y, \lambda), y) = \lambda q(x, y)$

(ii) $q(y, w(x, y, \lambda)) = \lambda q(y, x)$ and $q(w(x, y, \lambda), x) = (1-\lambda)q(y, x)$.

From Proposition 4.3.9, we observe that, any TCS W satisfies the conditions of Definition 4.3.16, and see that a strictly convex T_0-quasi-metric space admits at most one convex structure.

In the following example, we show that a T_0-quasi-metric space (X, q) may be strictly convex but (X, q^s) is not strictly convex. Therefore, we can conclude that strict convexity on (X, q) does not necessarily imply strict convexity on (X, q^s).

Example 4.3.17. ([19, Example 5]) Let $\mathbb{R}^2$ be equipped with the asymmetric norm $||x| = \max\{||x_1|, ||x_2|\}$ where $x = (x_1, x_2)$ and the associated T_0-quasi-metric q. Let us consider the points $x = (0,0)$ and $y = (2,0)$ in $\mathbb{R}^2$. Then we have that

$$q(x, y) = ||x - y| = ||(0,0) - (2,0)| = ||(-2,0)| = \max\{||-2|, ||0|\} = 0.$$

Also, we have

$$q(y,x) = ||y - x| = ||(2,0) - (0,0)| = ||(2,0)| = \max\{||2|, ||0|\} = 2.$$

Thus, $q(x,y) = 0$ and $q(y,x) = 2$. Therefore, we obtain the system of equation at (x,y): $q(x, W(x,y,\lambda)) = 0$, $q(W(x,y,\lambda), y) = 0$, $q(y, W(x,y,\lambda)) = 2\lambda$ and $q(W(x,y,\lambda), x) = 2(1-\lambda)$ with $\lambda \in [0,1]$ and $W(x,y,\lambda) = ((W(x,y,\lambda))_1, (W(x,y,\lambda))_2)$. We now show that $W(x,y,\lambda)$ is unique. Since $q(x, W(x,y,\lambda)) = 0$, then

$$\begin{aligned} 0 = q(x, W(x,y,\lambda)) &= ||(0,0) - ((W(x,y,\lambda))_1, (W(x,y,\lambda))_2))| \\ &= ||(-(W(x,y,\lambda))_1, -(W(x,y,\lambda))_2)| \\ &= \max\{|| - (W(x,y,\lambda))_1|, || - (W(x,y,\lambda))_2|\}. \end{aligned}$$

Now, for $\max\{||-(W(x,y,\lambda))_1|, ||-(W(x,y,\lambda))_2|\} = 0$, we must have that $||(W(x,y,\lambda))_1| \geq 0$ and $||(W(x,y,\lambda))_2| \geq 0$. Also, since $q(W(x,y,\lambda), y) = 0$, then

$$\begin{aligned} 0 = q(W(x,y,\lambda), y) &= ||W(x,y,\lambda) - y| \\ &= ||((W(x,y,\lambda))_1, (W(x,y,\lambda))_2) - (2,0)| \\ &= ||((W(x,y,\lambda))_1 - 2, (W(x,y,\lambda))_2)| \\ &= \max\{||(W(x,y,\lambda))_1 - 2|, ||(W(x,y,\lambda))_2|\}. \end{aligned}$$

Now, for $\max\{||(W(x,y,\lambda))_1 - 2|, ||(W(x,y,\lambda))_2|\} = 0$, we must have $(W(x,y,\lambda))_1 \leq 2$ and $(W(x,y,\lambda))_2 \leq 0$. Hence, we see that $0 \leq ||(W(x,y,\lambda))_1| \leq 2$ which can be written as $(W(x,y,\lambda))_1 = 2 - 2\lambda$ for all $\lambda \in [0,1]$ and also $(W(x,y,\lambda))_2 = 0$ and therefore, we have $W(x,y,\lambda) = (2-2\lambda, 0)$ for all $\lambda \in [0,1]$. Therefore, $W(x,y,\lambda)$ is unique for each $\lambda \in [0,1]$ and so $(\mathbb{R}^2, q)$ is strictly convex T_0-quasi-metric space.
Let us now look at the symmetrised metric space $(\mathbb{R}^2, q^s)$ with the same points x and y. Then we have two equations $q^s(y, W(x,y,\lambda)) = 2t\lambda$ and $q^s(W(x,y,\lambda), x) = 2(1-\lambda)$. Now,

$$\begin{aligned} 2\lambda = q^s(y, W(x,y,\lambda)) &= ||y - W(x,y,\lambda)| \\ &= ||(2,0) - ((W(x,y,\lambda))_1, (W(x,y,\lambda))_2)| \\ &= ||(2 - (W(x,y,\lambda))_1, -(W(x,y,\lambda))_2)| \\ &= \max\{||2 - (W(x,y,\lambda))_1|, || - (W(x,y,\lambda))_2|\}. \end{aligned}$$

Now, for $\max\{||2 - (W(x,y,\lambda))_1|, || - (W(x,y,\lambda))_2|\} = 2\lambda$, implies that $||(W(x,y,\lambda))_1| \geq 2(1-\lambda)$ and $||(W(x,y,\lambda))_2| \geq 2\lambda$ for all $\lambda \in [0,1]$. Also, similarly

$$\begin{aligned} 2(1-\lambda) = q^s(W(x,y,\lambda), y) &= ||W(x,y,\lambda) - y| \\ &= ||((W(x,y,\lambda))_1, (W(x,y,\lambda))_2) - (2,0)| \\ &= ||((W(x,y,\lambda))_1 - 2, (W(x,y,\lambda))_2)| \\ &= \max\{||(W(x,y,\lambda))_1 - 2|, ||(W(x,y,\lambda))_2|\}. \end{aligned}$$

Now, for $\max\{||(W(x,y,\lambda))_1-2|, ||(W(x,y,\lambda))_2|\} = 2(1-\lambda)$, implies that $||(W(x,y,\lambda))_1| \leq 2(1-\lambda)$ and $||(W(x,y,\lambda))_2| \leq 2(1-\lambda)$ for all $\lambda \in [0,1]$. This gives us $(W(x,y,\lambda))_1 = 2(1-\lambda)$ and $2\lambda \leq ||(W(x,y,\lambda))_2| \leq 2(1-\lambda)$ for all $\lambda \in [0,1]$. Hence, we have $W(x,y,\lambda) = (2-2\lambda, 2\lambda)$ if $\lambda \leq \frac{1}{2}$ and $W(x,y,\lambda) = (2-2\lambda, 2-2\lambda)$ if $\lambda > \frac{1}{2}$ as the solution to these equations. Also, we have that $W(x,y,\lambda) = (2-2\lambda, 0)$ with $\lambda \in [0,1]$ also solves the system of equation. Thus, the system do not have a unique solution. Therefore, (X, q^s) is not necessarily a strictly convex T_0-quasi-metric space.

We now introduce another condition related to strict convexity in T_0-quasi-metric spaces in the spirit of an analogous condition due to Talman [29] in metric spaces.

Definition 4.3.18. (Compare with Definition 3.3.14) Let W be a TCS on a T_0-quasi-metric space (X,q). We say that W is a unique TCS if for any $w \in X$ such that there exists $(x,y,\lambda) \in X \times X \times [0,1]$ with

$$q(z,w) \leq \lambda q(z,x) + (1-\lambda)q(z,y) \quad \text{and} \quad q(w,z) \leq \lambda q(x,z) + (1-\lambda)q(y,z)$$

whenever $z \in X$, we have that $w = W(x,y,\lambda)$.

Lemma 4.3.19. *(Compare Lemma 3.3.15) Let W be a unique TCS on a T_0-quasi-metric space (X,q). Then for every $x,y \in X$ and $\alpha,\beta \in [0,1]$, we have*

$$W(W(x,y,\beta),y,\alpha) = W(x,y,\alpha\beta).$$

Proof. Let $x,y \in X$ and $\alpha,\beta \in [0,1]$. Then

$$\begin{aligned} q(z, W(W(x,y,\beta),y,\alpha)) &\leq \alpha q(z, W(x,y,\beta)) + (1-\alpha)q(z,y) \\ &\leq \alpha\left(\beta q(z,x) + (1-\beta)q(z,y)\right) + (1-\alpha)q(z,y) \\ &= \alpha\beta q(z,x) + \alpha(1-\beta)q(z,y) + (1-\alpha)q(z,y) \\ &= \alpha\beta q(z,x) + (\alpha - \alpha\beta + 1 - \alpha)q(z,y) \\ &= \alpha\beta q(z,x) + (1-\alpha\beta)q(z,y). \end{aligned}$$

Similarly,

$$\begin{aligned} q(W(W(x,y,\beta),y,\alpha),z) &\leq \alpha q(W(x,y,\beta),z) + (1-\alpha)q(y,z) \\ &\leq \alpha\left(\beta q(x,z) + (1-\beta)q(y,z)\right) + (1-\alpha)q(y,z) \\ &= \alpha\beta q(x,z) + \alpha(1-\beta)q(y,z) + (1-\alpha)q(y,z) \\ &= \alpha\beta q(x,z) + (\alpha - \alpha\beta + 1 - \alpha)q(y,z) \\ &= \alpha\beta q(z,x) + (1-\alpha\beta)q(y,z). \end{aligned}$$

Thus by uniqueness of W, we have that $W(W(x,y,\beta),y,\alpha) = W(x,y,\alpha\beta)$. □

The following Proposition is as a result of a theorem in metric setting:

Proposition 4.3.20. *(Compare with Theorem 3.3.19) Let (X,q) be a T_0-quasi-metric space with a unique TCS W such that X is $\tau(q^s)$-compact. Then the mapping $W : X \times X \times [0,1] \longrightarrow X$ is continuous, with respect to $\tau(q^s)$ topology.*

Proof. Let $x, y \in X$ and $\lambda \in [0,1]$. Consider any sequence $((x_n, y_n, \lambda_n))$ in $X \times X \times I$ converging to (x, y, λ) with respect to $\tau(q^s)$ topology $\tau(q^s)$, and let W be any $\tau(q^s)$-accumulation point of the sequence $W(x_n, y_n, \lambda_n)$. Then for any $z \in X$ we have,

$$q(z, W(x_{n_k}, y_{n_k}, \lambda_{n_k})) \leq \lambda_{n_k} q(z, x_{n_k}) + (1 - \lambda_{n_k}) q(z, y_{n_k})$$

for all $n \in \mathbb{N}$. By continuity of a T_0-quasi-metric q, we have

$$q(z, W) \leq \lambda q(z, x) + (1 - \lambda) q(z, y).$$

Similarly, we get that $q(W, z) \leq q(x, z) + (1 - \lambda) q(y, z)$. By uniqueness of convex structure in Definition 4.3.18, we get $W = W(x, y, \lambda)$. We therefore conclude that $W(x, y, \lambda)$ is the only $\tau(q^s)$-accumulation point of the sequence $W(x_n, y_n, \lambda_n)$ for all $n \in \mathbb{N}$. Since X is $\tau(q^s)$-compact, we have $W(x_n, y_n, \lambda_n)$ converges to $W(x, y, \lambda)$. □

It should be noted that the given condition(I) in Definition 3.3.5 is not suitable for a T_0-quasi-metric space (X, q) that is not a metric. To see this, if q is a T_0-quasi-metric with property (C) in Definition 4.3.12 and Property (I) in Definition 3.3.5, then we have

$$\begin{aligned} |1-0| q(x, y) &= q(W(x, y, 1), W(x, y, 0)) \quad \text{by property (I)} \\ &= q(W(y, x, 1-1), W(y, x, 1-0)) \quad \text{by Property (C)} \\ &= q(W(y, x, 0), W(y, x, 1)) = |0-1| q(y, x) \quad \text{by property(I)} \end{aligned}$$

for all $x, y \in X$. This implies that $q(x, y) = q(y, x)$ and so q must be a metric. Hence, for a T_0-quasi-metric space (X, q) we suggest that property (I) be formulated to property(I') as follows:

Definition 4.3.21. ([19, p.10]) A T_0-quasi-metric space (X, q) with TCS W is said to have property(I') if for every $x, y \in X$ and $\lambda_1, \lambda_2 \in [0,1]$, we have

$$q(W(x, y, \lambda_1), W(x, y, \lambda_2)) = (\lambda_1 - \lambda_2) q(x, y) \quad \text{if} \quad \lambda_1 \geq \lambda_2$$

and

$$q(W(x, y, \lambda_1), W(x, y, \lambda_2)) = (\lambda_2 - \lambda_1) q(y, x) \quad \text{if} \quad \lambda_2 > \lambda_1.$$

The following Proposition is a generalisation of a well known result in metric spaces [29]:

Proposition 4.3.22. *(Compare with Theorem 3.3.24) If a TCS W on a T_0-quasi-metric space (X, q) is unique, then for every $x, y \in X$ with $x \neq y$ the function $h : I \longrightarrow X$ defined by $h(\lambda) = W(x, y, \lambda)$ satisfies property (I') of Definition 4.3.21.*

Proof. Let $\lambda_1, \lambda_2 \in [0,1]$. We show that the function $h : [0,1] \longrightarrow X$ defined by $h(\lambda) = W(x, y, \lambda)$ for all $x, y \in X$ satisfies property (I') in Definition 4.3.21. Suppose that $\lambda_1 < \lambda_2$.

By Lemma 4.3.19 and Proposition 4.3.9, we have

$$\begin{aligned}
q(W(x,y,\lambda_1),W(x,y,\lambda_2)) &= q(W(W(x,y,\lambda_2\frac{\lambda_1}{\lambda_2}),y,\frac{\lambda_1}{\lambda_2}),W(x,y,\lambda_2)) \quad \text{by Lemma 4.3.18}\\
&= (1-\frac{\lambda_1}{\lambda_2})q(y,W(x,y,\lambda_2)) \quad \text{by Proposition 4.3.9}\\
&= (1-\frac{\lambda_1}{\lambda_2})\lambda_2 q(y,x) \quad \text{by Proposition 4.3.9}\\
&= (\lambda_2-\lambda_1)q(y,x).
\end{aligned}$$

Thus, the second equality is established.
Similarly, we note that if TCS W on a T_0-quasi-metric space (X,q) is unique, then it satisfies condition (C) in Definition 4.3.12. Now, for $\lambda_1 \geq \lambda_2$, we have

$$q(W(x,y,\lambda_1),W(x,y,\lambda_2)) = q(W(x,y,1-\lambda_1),W(x,y,1-\lambda_2)) = (\lambda_1-\lambda_2)q(x,y),$$

which proves the first equality. Hence $h = W(x,y,\lambda)$ satisfies property (I') in Definition 4.3.21. □

We next introduce and discuss briefly an interesting property of some convexity structures that will turn out to be very useful in our subsequent investigations. We start with the following definition of a well-known property (S).

Definition 4.3.23. (Compare with Definition 3.3.16) Let (X,q) be a T_0-quasi-metric space with a TCS W. We say that (X,q) has property (S) provided that

$$q(W(x,y,\lambda),W(x',y',\lambda)) \leq \lambda q(x,x') + (1-\lambda)q(y,y')$$

whenever $x,y,x',y' \in X$ and $\lambda \in [0,1]$.

Proposition 4.3.24. *[19, Remark 11]*

(i) If a T_0-quasi-metric space with a TCS W has property (S) in Definition 4.3.23 together with the condition that for any $x \in X$ and $\lambda \in [0,1]$, $W(x,x,\lambda) = x$ then conditions (i) and (ii) of Definition 4.3.1 hold.

(ii) If we replace q by q^{-1}, then we get a condition equivalent to property (S) of Definition 4.3.23.

(iii) The standard convex structure on a convex set of an asymmetric normed real vector space X has property (S) of Definition 4.3.23.

(iv) Property (S) of Definition 4.3.23 together with property (I') of Definition 4.3.21 for a TCS W on a T_0-quasi-metric space (X,q) imply continuity of $W : (X,\tau(q)) \times (X,\tau(q)) \times ([0,1],\tau(u^s)) \longrightarrow (X,\tau(q))$.

Proof. We only prove (iii) and (iv) as (i) and (ii) follows directly.
(iii) To see this, let $W(x,y,\lambda) = \lambda x + (1-\lambda)y$ and $W(x',y',\lambda) = \lambda x' + (1-\lambda)y'$ for any

$x, y, x', y' \in X$ and $\lambda \in [0,1]$ then

$$\begin{aligned} q(W(x,y,\lambda), W(x',y',\lambda)) &= ||\lambda x + (1-\lambda)y - (\lambda x' + (1-\lambda)y')| \\ &= ||\lambda(x-x') + (1-\lambda)(y-y')| \\ &\leq \lambda||x-x'| + (1-\lambda)||y-y'| \\ &= \lambda q(x,x') + (1-\lambda)q(y,y') \end{aligned}$$

(iv) Let $((x_n, y_n, \lambda_n))$ be a sequence in $X \times X \times [0,1]$ converging to (x, y, λ) with respect to the topology $\tau(q) \times \tau(q) \times \tau(u^s)$. By using the triangle inequality, property(I') of Definition 4.3.21 and property(S) of Definition 4.3.23, we obtain

$$\begin{aligned} q(W(x,y,\lambda), W(x_n,y_n,\lambda_n)) &\leq q(W(x,y,\lambda), W(x,y,\lambda_n)) + q(W(x,y,\lambda_n), W(x_n,y_n,\lambda_n)) \\ &\leq |\lambda - \lambda_n| q(x,y) + \lambda_n q(x,x_n) + (1-\lambda_n)q(y,y_n) \end{aligned}$$

for all $n \in \mathbb{N}$. Since (x_n) and (y_n) converge to x and y with respect to the topology $\tau(q)$ respectively and $\lambda_n \longrightarrow \lambda$ with respect to the topology $\tau(u^s)$, we have that $q(W(x,y,\lambda), W(x_n,y_n,\lambda_n)) \longrightarrow 0$ for all $n \in \mathbb{N}$ and so $(W(x_n,y_n,\lambda_n)$ converges to $W(x,y,\lambda)$ with respect to the topology $\tau(q)$. Hence, continuity of W follows. □

In the following we assume that the T_0-quasi-metric space has a TCS W that satisfies property (S) in Definition 4.3.23. Furthermore, we shall work on the sub-collection $\mathcal{CB}_0(X)$ of bounded convex elements of $\mathscr{P}_0(X)$ (all nonempty subsets of X).

Proposition 4.3.25. *([19, p.13]) If W is a TCS on a T_0-quasi-metric space (X,q) which satisfies property (S) in Definition 4.3.23, then for any $A, B \in \mathcal{CB}_0(X)$ the set $W(A,B,\lambda) = \{W(x,y,\lambda) : x \in A, y \in B\}$ is bounded.*

Proof. Let $x, x' \in A$ and $y, y' \in B$. We show that $q(W(x,y,\lambda), W(x',y',\lambda)) < \infty$. Now, since $A, B \in \mathcal{CB}_0(X)$, they are bounded i.e the diameter of A and that of B are finite ($diam(A) < \infty$ and $diam(B) < \infty$). Further, since W has property (S) in Definition 4.3.23, we have that

$$q(W(x,y,\lambda), W(x',y',\lambda)) \leqslant \lambda q(x,x') + (1-\lambda)q(y,y') \leqslant \lambda diam(A) + (1-\lambda)diam(B) < \infty.$$

Hence, $W(A,B,\lambda)$ is bounded. □

We now investigate an additional property that allows us to define a TCS on an appropriate subspace of $\mathcal{CB}_0(X)$, which is induced by W. We start with the following definition:

Proposition 4.3.26. *([19, Lemma 3]) Let (X,q) be a T_0-quasi-metric space with a TCS Wwhich satisfies property (S) in Definition 4.3.23. If $A \in \mathcal{CB}_0(X)$, then its double closure $cl_{\tau(q)}A \cap cl_{\tau(q^{-1})}A$ also belongs to $\mathcal{CB}_0(X)$.*

Proof. We show that $cl_{\tau(q)}A \cap cl_{\tau(q^{-1})}A$ is bounded and convex. Suppose that the diameter of A is less than m (i.e $diam(A) \leq m$) and $x, y \in cl_{\tau(q)}A \cap cl_{\tau(q^{-1})}A$. Then for any $\varepsilon > 0$, there exists $a, a' \in A$ such that $q(x,a) < \varepsilon$ and $q(a',y) < \varepsilon$ and thus by the triangle inequality we have that

$$q(x,y) \leq q(x,a) + q(a,a') + q(a',y) = q(a,a') + 2\varepsilon \leq diam(A) + 2\varepsilon < m + 2\varepsilon.$$

As $\varepsilon \to 0, \quad q(x,y) \leq m$. Hence m is the upper bound of diameter of the double closure of A. It therefore, follows that the double closure of A is bounded.
We now show that $cl_{\tau(q)}A \cap cl_{\tau(q^{-1})}A$ is convex. Fix $\lambda \in [0,1]$. For $a, b \in cl_{\tau(q)}A \cap cl_{\tau(q^{-1})}A$, there exists sequences (a_n) and (b_n) in A such that $q(a,a_n) \longrightarrow 0$ and $q(b,b_n) \longrightarrow 0$. Furthermore, by property (S) in Definition 4.3.23 we have

$$q(W(a,b,\lambda), W(a_n,b_n,\lambda)) \leq \lambda q(a,a_n) + (1-\lambda)q(b,b_n) \longrightarrow 0$$

which implies that $(W(a_n,b_n,\lambda))$ converges to $W(a,b,\lambda)$ with respect to the topology $\tau(q)$. Hence, $W(a,b,\lambda) \in cl_{\tau(q)}A$, since $(W(a_n,b_n,\lambda)) \in A$ whenever $n \in \mathbb{N}$ by convexity of A. Thus $cl_{\tau(q)}A$ is convex. Similarly, $cl_{\tau(q^{-1})}A$ is convex. Hence the proof is complete. □

We end this section by now showing that Takahashi convexity implies Menger convexity in T_0-quasi-metric spaces. The proof of the following follows in the same way as that of Proposition 3.3.20.

Proposition 4.3.27. *(Compare with Proposition 3.3.20) Let (X,q) be a T_0-quasi-metric space. If (X,q) is convex with TCS W, then it is Menger convex.*

Proof. Let $x, y \in X$ and $0 \leq t \leq 1$. We need to show that conditions (i) and (ii) of Definition 4.2.1 are satisfied. Let $z = W(x,y,1-t)$, then by proposition 4.3.9,

$$\begin{aligned} q(x,z) &= q(x, W(x,y,1-t)) \\ &= (1-(1-t))q(x,y) \\ &= tq(x,y) \end{aligned}$$

and

$$\begin{aligned} q(z,y) &= q(W(x,y,1-t), y) \\ &= (1-t)q(x,y). \end{aligned}$$

Hence condition (i) of Definition 4.2.1 is satisfied. Also,

$$\begin{aligned} q(y,z) &= q(y, W(x,y,1-t)) \\ &= (1-t)q(x,y). \end{aligned}$$

and

$$\begin{aligned} q(z,x) &= q(W(x,y,1-t), x) \\ &= (1-(1-t))q(y,x) \\ &= tq(y,x). \end{aligned}$$

Hence condition (ii) of Definition 4.2.1 is satisfied. □

In this section, we generalise M-convexity to the framework of T_0-quasi-metric spaces. We begin by introducing the concept of strong convexity in T_0-quasi-metric spaces. We notice that this concept was introduced in [19] for T_0-quasi-metric with $TCS\ W$ where it was called strict convexity.

Definition 4.4.1. (Compare with Definition 3.4.1) A T_0-quasi-metric space (X, q) is said to be strongly convex if for each pair of points $x, y \in X$ and every $t \in [0, 1]$, there exists a unique point $z \in X$ which satisfies the following two conditions:

(i) $q(x, z) = tq(x, y)$ and $q(z, y) = (1 - t)q(x, y)$.

(ii) $q(y, z) = (1 - t)q(y, x)$ and $q(z, x) = tq(y, x)$.

We notice that condition(ii) of Definition 4.2.1 is formulated for the dual T_0-quasi-metric. We observe that if q has symmetry, then we have that $q(x, z) = q(z, x) = tq(x, y) = tq(y, x)$ which implies $q(x, z) = tq(x, y)$. Similarly, $q(z, y) = (1 - t)q(x, y) = q(y, z) = (1 - t)q(y, x)$ which implies $q(z, y) = (1-t)q(x, y)$. Hence, obtaining the conditions (i) and (ii) of Definition 3.4.1.

We now generalise the concept of M-convexity, introduced in [11], to the framework of T_0-quasi-metric spaces.

Definition 4.4.2. (Compare with Definition 3.4.6) A T_0-quasi-metric space (X, q) is called M-convex if for any x and y in X with $q(x, y) = \lambda_1$ and $q(y, x) = \lambda_2$ and for every $r \in [0, \lambda_1]$ and $s \in [0, \lambda_2]$ there exists a unique point $z_{rs} \in X$ such that

$$C_q(x, r) \cap C_{q^{-1}}(x, s) \cap C_q(y, \lambda_1 - r) \cap C_{q^{-1}}(y, \lambda_2 - s) = \{z_{rs}\}.$$

Remark 4.4.3. If $r = \frac{\lambda_1}{2}$ and $s = \frac{\lambda_2}{2}$, then z_{rs} is a midpoint of x and y. Also, if (X, q) is M-convex, then (X, q^{-1}) is M-convex too. However, M-convexity of (X, q) does not necessarily imply M-convexity of (X, q^s) (see Example 4.4.5).

We now show that for any T_0-quasi-metric space (X, q), M-convexity and strong convexity are equivalent.

Proposition 4.4.4. *(Compare with Proposition 3.4.8) Let (X, q) be a T_0-quasi-metric space. Then (X, q) is M-convex if and only if it is strongly convex.*

Proof. Suppose (X, q) is M-convex. Let $x, y \in X$ with $q(x, y) = \lambda_1$ and $q(y, x) = \lambda_2$ and $0 \leq t \leq 1$. Then $0 \leq t\lambda_1 \leq \lambda_1$ and $0 \leq t\lambda_2 \leq \lambda_2$. Let $r_1 = t\lambda_1$ and $r_2 = \lambda_1 - t\lambda_1$. Then we have that $r_1 + r_2 = \lambda_1 = q(x, y)$. Also, let $s_1 = t\lambda_2$ and $s_2 = \lambda_2 - t\lambda_2$. Then we have $s_1 + s_2 = \lambda_2 = q(y, x)$. Since X is M-convex, we have

$$C_q(x, r_1) \cap C_{q^{-1}}(x, s_1) \cap C_q(y, r_2) \cap C_{q^{-1}}(y, s_2) = \{z_t\}.$$

Since $z_t \in C_q(x, r_1)$, we have $q(x, z_t) \leq r_1 = t\lambda_1 = tq(x, y)$. Also, $z_t \in C_q(y, r_2)$ implies that

$q(z_t, y) \leq r_2 = (1-t)\lambda_1 = (1-t)q(x, y)$. By the triangle inequality

$$q(x, y) \leq q(x, z_t) + q(z_t, y) \leq tq(x, y) + (1-t)q(x, y) = q(x, y)$$

and so $q(x, y) = q(x, z_t) + q(z_t, y)$. Therefore, we have

$$q(x, z_t) = tq(x, y) \quad \text{and} \quad q(z_t, y) = (1-t)q(x, y).$$

One can use the same argument to show that

$$q(y, z_t) = (1-t)q(y, x) \quad \text{and} \quad q(z_t, x) = tq(y, x).$$

For uniqueness, suppose $z_t^{'} \in X$ also exists such that $q(x, z_t') = tq(x, y), \quad q(z_t', y) = (1-t)q(x, y)$ and $q(y, z_t') = (1-t)q(y, x), \quad q(z_t', x) = tq(x, y)$. Then

$$z_t' \in C_q(x, r_1) \cap C_{q^{-1}}(x, s_1) \cap C_q(y, r_2) \cap C_{q^{-1}}(y, s_2) = \{z_t\}$$

and so $z_t = z_t'$. Hence, (X, q) is a strongly convex T_0-quasi-metric space.
Conversely, suppose that (X, q) is strongly convex. Then we need to show that it is M-convex. To do this, let $x, y \in X$ with $q(x, y) = \lambda_1, q(y, x) = \lambda_2, r \in [0, \lambda_1]$ and $s \in [0, \lambda_2]$. To show that

$$C_q(x, r) \cap C_{q^{-1}}(x, s) \cap C_q(y, \lambda_1 - r) \cap C_{q^{-1}}(y, \lambda_2 - s) = \{z_{rs}\},$$

let $t = \dfrac{r}{\lambda_1}$ and $t = \frac{s}{\lambda_2}$, that is $t \in [0, 1]$. Then by strong convexity of (X, q), there exists a unique z_{rs} such that $q(x, z_{rs}) = tq(x, y) = r, \quad q(z_{rs}, y) = (1-t)q(x, y) = \lambda_1 - r$ and $q(y, z_{rs}) = (1-t)q(y, x) = \lambda_2 - s, \quad q(z_{rs}, x) = tq(x, y) = s$. That is,

$$z_{rs} \in C_q(x, r) \cap C_{q^{-1}}(x, s) \cap C_q(y, \lambda_1 - r) \cap C_{q^{-1}}(y, \lambda_2 - s).$$

For uniqueness, suppose $z_{rs}' \in X$ also exist such that $z_{rs}' \in C_q(x, r) \cap C_{q^{-1}}(x, s) \cap C_q(y, \lambda_1 - r) \cap C_{q^{-1}}(y, \lambda_2 - s)$. Then $q(x, z_{rs}') \leq r, \quad q(z_{rs}', y) \leq \lambda_1 - r$ and $q(y, z_{rs}') \leq \lambda_2 - s, \quad q(z_{rs}', x) \leq s$. Now

$$q(x, y) \leq q(x, z_{rs}') + q(z_{rs}', y) \leq r + \lambda_1 - r = \lambda_1 = q(x, y),$$

which implies that $q(x, y) = q(x, z_{rs}') + q(z_{rs}', y)$ and so $q(x, z_{rs}') = r \quad$ and $\quad q(z_{rs}', y) = \lambda_1 - r$. Similarly, it can be shown that $q(y, x) = q(y, z_{rs}') + q(z_{rs}', x)$. Thus $q(y, z_{rs}') = \lambda_2 - s, \quad$ and $\quad q(z_{rs}', x) = s$.
Since $q(x, z_{rs}') = r = q(x, z_{rs}), \quad q(z_{rs}', y) = \lambda_1 - r = q(z_{rs}, y) \quad$ and $\quad q(y, z_{rs}') = \lambda_2 - s = q(y, z_{rs}), \quad q(z_{rs}', x) = s = q(z_{rs}, x)$. Therefore, $z_{rs}' = z_{rs}$ and so

$$C_q(x, r_1) \cap C_{q^{-1}}(x, s_1) \cap C_q(y, r_2) \cap C_{q^{-1}}(y, s_2) = \{z_{rs}\}.$$

□

In the following example, we show that a T_0-quasi-metric space (X, q) may be strongly convex but (X, q^s) is not strongly convex. Since M-convexity and strong convexity are equivalent on any T_0-quasi-metric space, we conclude that M-convexity on (X, q) does not necessarily imply M-convexity on (X, q^s).

Example 4.4.5. ([19, Example 5]) Let $\mathbb{R}^2$ be equipped with the asymmetric norm $||x| = \max\{||x_1|, ||x_2|\}$ where $x = (x_1, x_2)$ and the associated T_0-quasi-metric q. Let us consider the point $x = (0, 0)$ and $y = (2, 0)$ in $\mathbb{R}^2$. Then we have that

$$q(x, y) = ||x - y| = ||(0, 0) - (2, 0)| = ||(-2, 0)| = \max\{|| - 2|, ||0|\} = 0.$$

Also, we have

$$q(y, x) = ||y - x| = ||(2, 0) - (0, 0)| = ||(2, 0)| = \max\{||2|, ||0|\} = 2.$$

Thus, $q(x, y) = 0$ and $q(y, x) = 2$. Therefore, we obtain the system of equation at (x, y): $q(x, y) = 0$, $q(z, y) = 0$, $q(y, z) = 2t$ and $q(z, x) = 2(1 - t)$ with $t \in I$ and $z = (z_1, z_2)$. We now show that z is unique. Since $q(x, z) = 0$, then

$$0 = q(x, z) = ||(0, 0) - (z_1, z_2)| = ||(-z_1, -z_2) = \max\{|| - z_1|, || - z_2|\}.$$

Now, for $\max\{|| - z_1|, || - z_2|\} = 0$, we must have that $||z_1| \geq 0$ and $||z_2| \geq 0$. Also, since $q(z, y) = 0$, then

$$0 = q(z, y) = ||z - y| = ||(z_1, z_2) - (2, 0)| = ||(z_1 - 2, z_2)| = \max\{||z_1 - 2|, ||z_2|\}.$$

Now, for $\max\{||z_1 - 2|, ||z_2|\} = 0$, we must have $||z_1| \leq 2$ and $||z_2| \leq 0$. Hence, we see that $0 \leq ||z_1| \leq 2$ which can be written as $z_1 = 2-2t$ for all $t \in I$ and also $||z_2| = 0$ and therefore, we have $z = (2 - 2t, 0)$ for all $t \in I$. Therefore, z unique for each $t \in I$ and so $(\mathbb{R}^2, q)$ is strongly convex T_0-quasi-metric space.

Let us now look at the symmetrised metric space $(\mathbb{R}^2, q^s)$ with the same points x and y. Then we have two equations $q^s(y, z) = 2t$ and $q^s(z, x) = 2(1 - t)$. Now,

$$2t = q^s(y, z) = ||y - z| = ||(2, 0) - (z_1, z_2)| = ||(2 - z_1, -z_2)| = \max\{||2 - z_1|, || - z_2|\}.$$

Now, for $\max\{||2 - z_1|, || - z_2|\} = 2t$, implies that $||z_1| \geq 2(1 - t)$ and $||z_2| \leq 2t$ for all $t \in I$. Similarly

$$2(1 - t) = q^s(z, y) = ||z - y| = ||(z_1, z_2) - (2, 0)| = ||(z_1 - 2, z_2)| = \max\{||z_1 - 2|, ||z_2|\}.$$

Now, for $\max\{||z_1 - 2|, ||z_2|\} = 2(1 - t)$, implies that $||z_1| \leq 2(1 - t)$ and $||z_2| \leq 2(1 - t)$ for all $t \in I$. This gives us $z_1 = 2(1 - t)$ and $2t \leq ||z_2| \leq 2(1 - t)$ for all $t \in I$. Hence, we have $z = (2 - 2t, 2t)$ if $t < \frac{1}{2}$ and $z = (2 - 2t, 2 - 2t)$ if $t > \frac{1}{2}$ as the solution to these equations. Also, we have already seen that the other solution to these equations is $z = (2 - 2t, 0)$ with $t \in I$. Thus, the system does not have a unique solution. Therefore, (X, q^s) is not necessarily a strong convex T_0-quasi-metric space.

Definition 4.4.6. (Compare with Definition 3.4.9) Let (X, q) be a T_0-quasi-metric space and $A \subset X$. Then A is said to be convex if for every $x, y \in A$,

$$C_q(x,(1-t)\lambda_1) \cap C_{q^{-1}}(x,(1-t)\lambda_2) \cap C_q(y,t\lambda_1) \cap C_{q^{-1}}(y,t\lambda_2) \subseteq A$$

for all $t \in (0,1)$, $\lambda_1 = q(x,y)$ and $\lambda_2 = q(y,x)$.

Definition 4.4.7. (Compare with Definition 3.4.10) Let (X, q) be a Menger convex T_0-quasi-metric space. Then (X, q) is said to be strictly convex if for any $x, y \in C_q(z,r_1) \cap C_{q^{-1}}(z,r_2)$ with $z \in X$ we have that

$$C_q(x,(1-t)\lambda_1) \cap C_{q^{-1}}(x,(1-t)\lambda_2) \cap C_q(y,t\lambda_1) \cap C_{q^{-1}}(y,t\lambda_2) \subseteq C_q(z,r_1) \cap C_{q^{-1}}(z,r_2)$$

where $q(x,y) = \lambda_1, \quad q(y,x) = \lambda_2$ and all $t \in (0,1)$.

Theorem 4.4.8. (Compare with Theorem 3.4.11) Let (X, q) be a T_0-quasi-metric space. If (X, q) is strictly convex, then (X, q) is M-convex.

Proof. Let $x, y \in X$ with $q(x,y) = \lambda_1$ and $q(y,x) = \lambda_2$. By strictly convexity of (X, q), we have

$$E(t) = C_q(x,(1-t)\lambda_1) \cap C_{q^{-1}}(x,(1-t)\lambda_2) \cap C_q(y,t\lambda_1) \cap C_{q^{-1}}(y,t\lambda_2) \neq \emptyset$$

for all $t \in (0,1)$. Suppose that $z_1, z_2 \in E(t)$, then

$$q(x,z_1) \leq (1-t)\lambda_1,\ q(z_1,x) \leq (1-t)\lambda_2,\ q(y,z_1) \leq t\lambda_2,\ q(z_1,y) \leq t\lambda_1$$

and

$$q(x,z_2) \leq (1-t)\lambda_1,\ q(z_2,x) \leq (1-t)\lambda_2,\ q(y,z_2) \leq t\lambda_2,\ q(z_2,y) \leq t\lambda_1.$$

Now

$$\begin{aligned} q(x,y) &\leq q(x,z_1) + q(z_1,y) \\ &\leq (1-t)\lambda_1 + t\lambda_1 \\ &= \lambda_1 = q(x,y). \end{aligned}$$

Thus $q(x,y) = q(x,z_1) + q(z_1,y)$ and so $q(x,z_1) = (1-t)\lambda_1$ and $q(z_1,y) = t\lambda_1$
Similarly, we have $q(y,x) = q(y,z_1) + q(z_1,x)$ and so $q(z_1,x) = (1-t)\lambda_2$ and $q(y,z_1) = t\lambda_2$. And also, for the point z_2, we have $q(x,y) = q(x,z_2) + q(z_2,y)$ and so $q(x,z_2) = (1-t)\lambda_1$ and $q(z_2,y) = t\lambda_1$ and $q(y,x) = q(y,z_2) + q(z_2,x)$ and so $q(y,z_2) = t\lambda_2$ and $q(z_2,x) = (1-t)\lambda_2$. Then by strict convexity of (X, q), we have

$$\begin{aligned} &C_q(z_1,(1-s)k_1) \cap C_{q^{-1}}(z_1,(1-s)k_1) \cap C_d(z_2,sk_1) \cap C_d(z_2,sk_2) \\ &\subseteq C_q(x,(1-t)\lambda_1) \cap C_{q^{-1}}(x,(1-t)\lambda_2) \cap C_q(y,t\lambda_1) \cap C_{q^{-1}}(y,t\lambda_2) \end{aligned}$$

for all $s, t \in (0,1)$, where $q(z_1,z_2) = k_1$ and $q(z_1,z_2) = k_2$. Since the points z_1 and z_2 are q, q^{-1}-between x and y, then we have

$$q(x,y) = q(x,z_1) + q(z_1,z_2) + q(z_2,y) \quad \text{and} \tag{4.21}$$

$$q(y,x) = q(y,z_1) + q(z_1,z_2) + q(z_2,x), \tag{4.22}$$

and

$$q(x,y) = q(x,z_2) + q(z_2,z_1) + q(z_1,y) \quad \text{and} \tag{4.23}$$

$$q(y,x) = q(y,z_1) + q(z_2,z_1) + q(z_1,x), \tag{4.24}$$

Now, from 4.21 we obtain, $q(z_1,z_2) = q(x,y) - q(x,z_1) - q(z_2,y)$, but $q(x,y) = q(x,z_1) + q(z_1,y)$. Hence, $q(z_1,z_2) = |q(z_1,y) - q(z_2,y)| = |t\lambda_1 - t\lambda_1| = 0$.
Also from 4.23, we get $q(z_2,z_1) = |q(z_2,y) - q(z_2,y)| = |(1-t)\lambda_2 - (1-t)\lambda_2| = 0$.
Since X is a T_0-quasi-metric space and $q(z_1,z_2) = 0 = q(z_2,z_1) \implies z_1 = z_2$ then we have that (X,q) is an M-convex T_0-quasi-metric space. □

4.5. Best approximation in T_0-quasi-metric spaces

In this section, we introduce the concept of best approximations in T_0-quasi-metric space, this notion was introduced in metric spaces by Khalil [11], in 1988:
Let (X,q) be a T_0-quasi-metric space and $A \subset X$. Due to asymmetry, we consider two distances from x to A:

$$q(x,A) = \inf\{q(x,y) : y \in A\}$$

$$q(A,x) = \inf\{q(y,x) : y \in A\}$$

Observe that $q^{-1}(x,A) = q(A,x)$. Also, let the set valued maps $P_A : X \longrightarrow 2^A$ and $P_A^{-1} : X \longrightarrow 2^A$ be defined as follows:

$$P_A(x) = \{y \in A : q(x,y) = q(x,A)\}$$

$$P_A^{-1}(x) = \{y \in A : q(y,x) = q(A,x)\}$$

denote a metric projection on A. An element $y \in P_A(x)$ is called a q-nearest point to x in A. While an element $\bar{y} \in P_A^{-1}(x)$ is called a q^{-1}-nearest point to x in A. The set A is called

(i) q-proximinal if $P_A(x) \neq \emptyset$ for every $x \in X$.

(ii) q-semi-Chebyshev if $n(P_A(x)) \leq 1$ for every $x \in X$, that is every $x \in X$ has at most one q-nearest point in A where $n(P_A(x))$ is the number of elements of the set $P_A(x)$.

(iii) q-Chebyshev if $n(P_A(x)) = 1$ for every $x \in X$, that is every $x \in X$ has exactly one q-nearest point in A where $n(P_A(x))$ is the number of elements in the set $P_A(x)$.

The corresponding notion for the conjugate T_0-quasi-metric q^{-1} are defined similarly.

We now give some properties of the T_0-quasi-metrics q and q^{-1} in terms of proximinality and Chebyshevity of some set A (compare with properties in [11]).
Theorem 4.5.1. Let (X,q) be a Menger convex T_0-quasi-metric space. Then the following are equivalent.

(i) (X,q) is M-convex.

(ii) $C_q(z, r_1) \cap C_{q^{-1}}(z, r_2)$ is q-Chebyshev and q^{-1}-Chebyshev for all $z \in X$ and $r_1, r_2 \geq 0$.

(iii) $P_A(x) \cap P_A(y) = \emptyset$ and $P_A^{-1}(x) \cap P_A^{-1}(y) = \emptyset$ for all $x \neq y$ and all double balls A in X.

Proof. $(i) \implies (ii)$ Let $C_q(z, r_1) \cap C_{q^{-1}}(z, r_2)$ where $z \in X, r_1, r_2 > 0$ be double closed balls. Let $x \in X \setminus C_q(z, r_1) \cap C_{q^{-1}}(z, r_2)$ and

$$q(x, z) = s = r_1 + \lambda_1$$

and

$$q(z, x) = m = r_2 + \lambda_2.$$

Then

$$q(x, C_q(z, r_1) \cap C_{q^{-1}}(z, r_2)) = \lambda_1 \quad \text{and} \quad q(C_q(z, r_1) \cap C_{q^{-1}}(z, r_2), x) = \lambda_2.$$

Since (X, q) is M-convex, we have that

$$C_q(x, \lambda_1) \cap C_{q^{-1}}(z, \lambda_2) \cap C_q(x, r_1) \cap C_{q^{-1}}(z, r_2) = \{y\}$$

for some $y \in X$. Thus $P_A(x) = P_A^{-1}(y) = \{y\}$ where $A = C_q(z, r_1) \cap C_{q^{-1}}(z, r_2)$ for $z \in X, r_1, r_2 > 0$ is a double ball. This implies that $C_d(z, r_1) \cap C_{q^{-1}}(z, r_2)$ is q-Chebshev and q^{-1}-Chebyshev for all $z \in X, r_1, r_2 > 0$.
$(ii) \implies (iii)$ Suppose that $P_A(x) \cap P_A(y) \neq \emptyset$ for some $A = C_q(z, r) \cap C_{q^{-1}}(z, r_2)$ for some $z \in X, r_1, r_2 > 0$ and $x, y \in A$. Then $\omega \in P_A(x) \cap P_A(y)$,

$$q(\omega, x) = q(\omega, y) = dist(\omega, A).$$

Then we have that x and y are q-nearest element of A and therefore, contradicting (ii). Hence, $P_A(x) \cap P_A(y) = \emptyset$ for some $x, y \in X$ and $x \neq y$. A similar argument also shows that if (ii) holds, then $P_A^{-1}(x) \cap P_A^{-1}(y) = \emptyset$.
$(iii) \implies (i)$ Let $x, y \in X$ and $q(x, y) = \lambda_1$ and $q(y, x) = \lambda_2$. Then by convexity of (X, q), there exists a $t \in [0, 1]$ such that

$$C_q(x, (1-t)\lambda_1) \cap C_{q^{-1}}(y, (1-t)\lambda_2) \cap C_q(x, t\lambda_1) \cap C_{q^{-1}}(y, t\lambda_2) = E(t) \neq \emptyset$$

If $z_1, z_2 \in E(t)$ and $z_1 \neq z_2$, then

$$x \in P_A(z_1) \cap P_A(z_2) \neq \emptyset$$

where $A = C_q(z, r_1) \cap C_{q^{-1}}(z, r_2)$ for $z \in X, r_1, r_2 > 0$. However this contradicts (iii). Hence, (X, q) is M-convex. □

Definition 4.5.2. (Compare with Definition 3.4.15) Let (X, q) be a T_0-quasi-metric space and X be M-convex, then for every $x, y \in X$ such that $q(x, y) = \lambda_1, \quad q(y, x) = \lambda_2$ and each $t \in [0, 1]$, we define

$$\begin{aligned} L(x, y) &= \bigcup_{0 \leq t \leq 1} (C_q(x, (1-t)\lambda_1) \cap C_q(x, (1-t)\lambda_2) \cap C_{q^{-1}}(y, t\lambda_1) \cap C_{q^{-1}}(y, t\lambda_2)) \\ &= \bigcup_{0 \leq r \leq \lambda} \Big(C_q(x, r) \cap C_q(y, \lambda_1 - r) \Big) \cap \bigcup_{0 \leq s \leq \lambda_2} \Big(C_{q^{-1}}(x, s) \cap C_{q^{-1}}(y, \lambda_2 - s) \Big). \end{aligned}$$

Theorem 4.5.3. (Compare with Theorem 3.5.5) Let (X, q) be a T_0-quasi-metric space. Suppose that (X, q) is M-convex in which all double closed balls are convex. If A is a double closed convex set in (X, q) and $x \notin A$, then $P_A(x)$ and $P_A^{-1}(x)$ are convex subset of X.

Proof. We prove that $P_A(x)$ is convex, the proof of $P_A^{-1}(x)$ is similar. Suppose that A is a double closed convex set in (X, q) and $x \notin A$. If $P_A(x) = \emptyset$, then it is trivially convex. Suppose that $P_A(x) \neq \emptyset$, and let $z_1, z_2 \in P_A(x)$ and $q(z_1, z_2) = \lambda_1$ and $q(z_2, z_1) = \lambda_2$. If $q(x, A) = r_1$ and $q(A, x) = r_2$, then $z_1, z_2 \in C_q(x, r_1) \cap C_{q^{-1}}(x, r_2)$. Since we have assumed that closed balls are convex, then $L[z_1, z_2] \subset C_q(x, r_1) \cap C_{q^{-1}}(x, r_2)$. Also since A is convex, we get $L[z_1, z_2] \subseteq A$. Consequently, $L[z_1, z_2] \subseteq S_q(x, r_1) \cap S_{q^{-1}}(x, r_2)$. Hence $L[z_1, z_2] \subseteq P_A(x)$. Therefore, $P_A(x)$ is convex. □

Theorem 4.5.4. (Compare with Theorem 3.5.6) Let (X, q) be a T_0-quasi-metric space. If (X, q) is M-convex in which every q-proximinal convex set is q-Chebyshev, then $C_q(z, r_1) \cap C_{q^{-1}}(z, r_2)$ is convex for all $z \in X$ and $r_1, r_2 > 0$.

Proof. Let $x, y \in X$, $q(x, y) = \lambda_1$ and $q(y, x) = \lambda_2$. Let

$$\{C_q(x, (1-t)\lambda_1) \cap C_{q^{-1}}(x, (1-t)\lambda_2) \cap C_q(y, t\lambda_1) \cap C_{q^{-1}}(y, t\lambda_2)\} \notin C_q(z, r_1) \cap C_{q^{-1}}(z, r_2)$$

for some $t \in [0, 1]$. Suppose that there exists distinct points $z_1, z_2 \in S_q(z, r_1) \cap S_{q^{-1}}(z, r_2)$ such that $\gamma = L(z_1, z_2)$ is not contained in $C_q(z, r_1) \cap C_{q^{-1}}(z, r_2)$. Since γ is compact (being continuous image of $[0, q(z_1, z_2)]$), then γ is q-proximinal. Bet $z_1, z_2 \in dist(z, \gamma)$, contradicting the q-Chebyshevity of γ, since γ is convex. Hence $C_q(z, r_1) \cap C_{q^{-1}}(z, r_2)$ is convex. □

The following theorem is analogous to Theorem 3.5.7 in metric settings, but here we use the property of double balls.

Theorem 4.5.5. Let (X, q) be a strictly convex T_0-quasi-metric space. Then every q-proximinal convex set in (X, q) is q-Chebyshev.

Proof. Let $G \subseteq X$ be q-proximinal and convex. Let $z \in X \setminus G$ such that $P_G(z)$ contains more than one element. consider $\{z_1, z_2\} \subseteq P_G(z)$ and $q(z_1, z_2) = \lambda_1$ and $q(z_2, z_1) = \lambda_2$. Since $\{z_1, z_2\} \subseteq P_G(z)$, then we have $q(z, z_1) = q(z, z_2) = q(z, G) = r_1$ and $q(z_1, z) = q(z_2, z) = q(G, z) = r_2$. Thus $\{z_1, z_2\} \subseteq C_q(z, r_1) \cap C_{q^{-1}}(z, r_2)$. Since $\{z_1, z_2\} \subseteq C_q(z, r_1) \cap C_{q^{-1}}(z, r_2)$, it follows from the strict convexity of (X, q) that

$$w(t) = C_q(x, (1-t)\lambda_1) \cap C_{q^{-1}}(x, (1-t)\lambda_2) \cap C_q(y, t\lambda_1) \cap C_{q^{-1}}(y, t\lambda_2) \in C_q(z, r_1) \cap C_{q^{-1}}(z, r_2).$$

The convexity of G implies that $w(t) \in G$. Since every strictly convex T_0-quasi-metric space is M-convex, we have that $w(t)$ is a singleton set. Hence, $q(z_1, z_2) = \lambda_1 = 0$ and $q(z_2, z_1) = \lambda_2 = 0$ and so $z_1 = z_2$. Hence G is q-Chebyshev. □

CHAPTER 5

DISCUSSION

In this thesis, several results concerning convexities, namely; Menger convexity [21], Takahashi convexity [19] and M-convexity [11] were extended from metric spaces to general T_0-quasi-metric settings with minor or no modifications to both the assumptions and proofs. We observed that all these convexities are a generalization of convexities in a linear spaces, but the converse is not true. We refer the reader to Example 3.2.5, Example 3.3.4 and Example 3.4.3.

In chapter four, we started our own investigations. Since these convexities rely on the concept of betweenness, a fundamental concept to the study of axiomatic geometry. Therefore, we started our discussion with the concept of betweenness and midpoint in T_0-quasi-metric spaces which was introduced by Blumenthal [4]. We discussed that q-betweenness does not necessarily imply q^{-1}-betweenness. Thereafter, we also discussed that q, q^{-1}-betweenness implies q^{+}-betweenness in T_0-quasi-metric space. We finalised this section by discussing a well known result [4, Theorem 12.1] for the relation of betweenness in metric spaces, to the setting of T_0-quasi-metric spaces.

In the second section, we discussed the concept of Menger convexity [21] from metric setting to the framework of T_0-quasi-metric spaces. We extended Proposition 3.2.6 to T_0-quasi-metric setting with minor modification to both the assumption and the proof. We failed to extend Theorem 3.2.14 to T_0-quasi-metric spaces due to the fact that it does not use symmetry in its proof.

In the third section, we recalled the convexity structure in the sense of Takahashi in T_0-quasi-metric spaces. We discussed various important results about convexity structures in metric spaces can generalised to the quasi-metric settings. We also showed that convexity structures naturally occur in asymmetric normed real vector spaces. We ended this section by discussing the relationship between Takahashi and Menger convexity in T_0-quasi-metric spaces.

In the fourth section, we discussed the concept of M-convexity [11] from the metric setting to the framework of T_0-quasi-metric spaces. We started by introducing strong convexity in T_0-quasi-metric spaces, and thereafter, we showed that a T_0-quasi-metric space is strongly convex if and only if it is M-convex (see Proposition 4.4.4). We also showed that a T_0-quasi-metric space (X, q) may be strongly convex but (X, q^s) is not strongly convex. Since M-convexity and strong convexity are equivalent on any T_0-quasi-metric space, we concluded that M-convexity on (X, q) does not necessarily imply M-convexity on (X, q^s) (see Example 4.4.5). We ended this section by showing that if (X, q) is strictly convex T_0-quasi-metric space, then it is M-convex. In the fifth section, we generalised the concept of best approximations in M-convex metric spaces. Then we showed that if (X, q) is an M-convex T_0-quasi-metric space, then the double closed ball $C_d(x, \delta)$ is convex if and only if there is a closed, convex subset G of X and $x \notin G$ such that q-proximinal and q^{-1}-proximinal sets are convex. We ended this section by showing that, if (X, q) is an M-convex T_0-quasi-metric space in which double closed balls are convex, and A is a double closed convex set in X and $x \notin A$, then $P_A(x)$ and $P_A^{-1}(x)$ are convex subsets of X.

In this dissertation we did not manage to answer the following questions:

Problem 1. *Is it possible to generalize Menger convexity [21] which was introduced by Karl Menger to asymmetric normed spaces?*

Problem 2. *Is it possible to generalize M-convexity [11] which was introduced by Rashido Khalil to asymmetric normed spaces?*

Problem 3. *Under what conditions can the Fundamental Theorem of Menger convexity be generalised to T_0-quasi-metric spaces?*

Problem 4. *Under what conditions can we say that a T_0-quasi-metric space (X, q) is M-convex by considering a unique curve of length $q(x, y)$?*

CHAPTER 6

CONCLUSION

This study has revealed that a number of results on convexities in metric spaces can be readily generalized to T_0-quasi-metric spaces with minor or no modifications to both the assumptions and the proofs. There are, however, some results which could not be generalized meaningfully from metric setting to the general context of T_0-quasi-metric spaces (see Theorem 3.4.15, Theorem 3.4.16 and Theorem 3.2.14). We notice that these theorems do not depend on symmetry, hence making them impossible for us to extend them to the quasi-metric setting. In conclusion, we find a surprising fact that many classical results about convexities do not make essential use of the symmetry of the metric and, therefore, making more interesting observations in quasi-metric setting.

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