THE WIND FLOW OVER A HILL

Master of Science Thesis submitted to the Univesiry of Zambia in partial requirement to the degree of Master of Science (Mathematics)

by

SALVATOR MANDA, B.Sc. (Nairobi), F.R. Met. S.

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Salvator Bernard Manda

DR E.M. Lungu
THESIS SUPERVISOR

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Chapter One

1.0 Introduction

The presence of a hill or obstruction in an air stream causes distortion in the structure of the air stream. In meteorology and other sciences this effect is of great interest. The nature of the airflow over mountains is significant for the following reasons:-

(a) It contributes substantially to the moisture consentrations and cloud formations in the vicinity of hills and mountains. See for Example [1].

(b) It influences the distribution of rainfal, the concentration of pollutants and the extent of vegetation on mountain slopes.

(c) It affects the movement of glider and powered aircraft and even the flight of birds and locusts to mention a few.

Many aircraft hazards in mountain areas are known to be due to the effect of airflow over such terrain. This Thesis attempts to set up a mathematical model from which the main effects of the presence of hills in airflow may be ascertained.

We shall consider a hill of specified shape with gentle incline, i.e where the angles of slope are small so that the streamlines of the air currents are practically parallel to the underlying surface. The problem is to find expressions for the changes in streamline pattern and velocity distribution caused by the presence of a hill.

It is evident, see for example [1], [2] that the problem is mathematically and physically complex. In view of this the following assumptions are made:-

(1) consider motion in the x - y plane which is effectively homogeneous, incompressible and steady.

(2) The effect of pressure gradient on the boundary layer $\frac{\partial p}{\partial x}$ is ignored; this effectively means that the term $\frac{\partial u}{\partial x} = 0$, i.e the flow stream velocity across the surfac does not change.

It may be noted that:-

(i) These assumptions make the mathematics tractable.

(ii) They do not substantially alter the overall physical nature of the effects of the presence of a hill or obstruction in an air stream and provide a basis for further investigations.

1.1 Formulation of the Problem

The Governing Equations

Let $q = (u(x,y), y(x,y))$ represent the velocity field of motion, $\phi(x,y)$ the velocity potential and $\psi(x,y)$ the stream function.

Then $u = \dfrac{\partial\phi}{\partial x} = \dfrac{\partial\psi}{\partial y}$ -- (1.1)

and $v = \dfrac{\partial\phi}{\partial y} = -\dfrac{\partial\psi}{\partial x}$ -- (1.2)

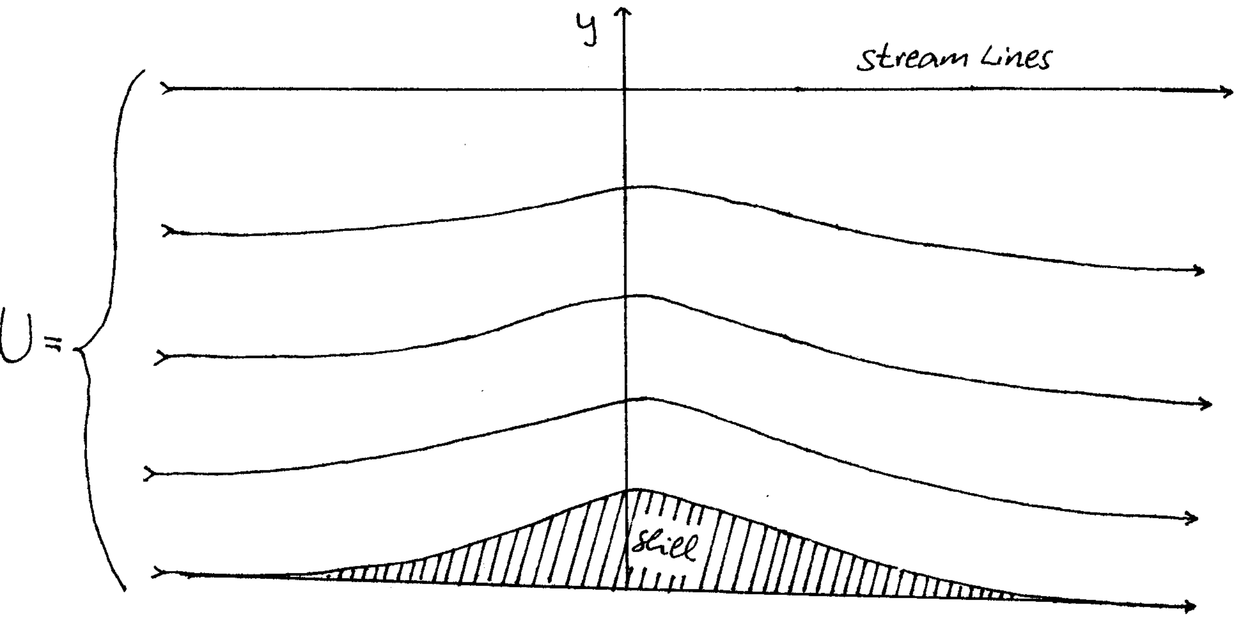
where $\Delta^2\phi = 0$ -- (1.3)

and $\Delta^2\psi = 0$ -- (1.4)

over the working domain $(x - y)$, see for example Milne-Thomson [3]. From equations (1.1), (1.2), (1.3) and (1.4) it is obvious that the stream function ψ can be calculated if ϕ is known; similarly also the velocity potential ϕ can be calculated if ψ is known.

The Two Dimensional Source - Sink Model

A Model of a Symmetric Ridge



Figure(1)

Consider for definiteness the situation sketched in figure (1). An incoming stream with velocity (U, o) approaches a source - sink combination $+m$ at $(-x_o, 0)$ and $-m$ at $(x_o, 0)$. Following Lamb [4] and Milne-Thomson [3], the stream function ψ at $x = \pm \infty$ is given by

$$\psi = Uy \quad \text{------------} \quad (1.5)$$

or in polar co-ordinates

$$\psi = Ur \sin \theta \quad \text{------------} \quad (1.6)$$

The stream function for a two dimensional source is given by

$$\psi_1 = \frac{m\theta_1}{2\pi} \quad \text{------------} \quad (1.7)$$

and the stream function for a two dimensional sink being

$$\psi_2 = -\frac{m\theta_2}{2\pi} \quad \text{------------} \quad (1.8)$$

where m is the strength of the source or sink and θ_1 θ_2 are the angles made by the x - axis and the lines joining any point P in the $x - y$ plane to the source point and sink point.

The stream function representing the combined flow of a source of strength m at the point $(x_o, 0)$ and a sink of equal strength m at $(-x_o, 0)$ with the uniform flow of velocity $(U, 0)$ is therefore given by

$$\psi = \frac{m}{2\pi} \left\{ \tan^{-1} \frac{y}{x - x_o} - \tan^{-1} \frac{y}{x + x_o} \right\} + Uy \quad \text{------------} (1.9)$$

Figure (2)

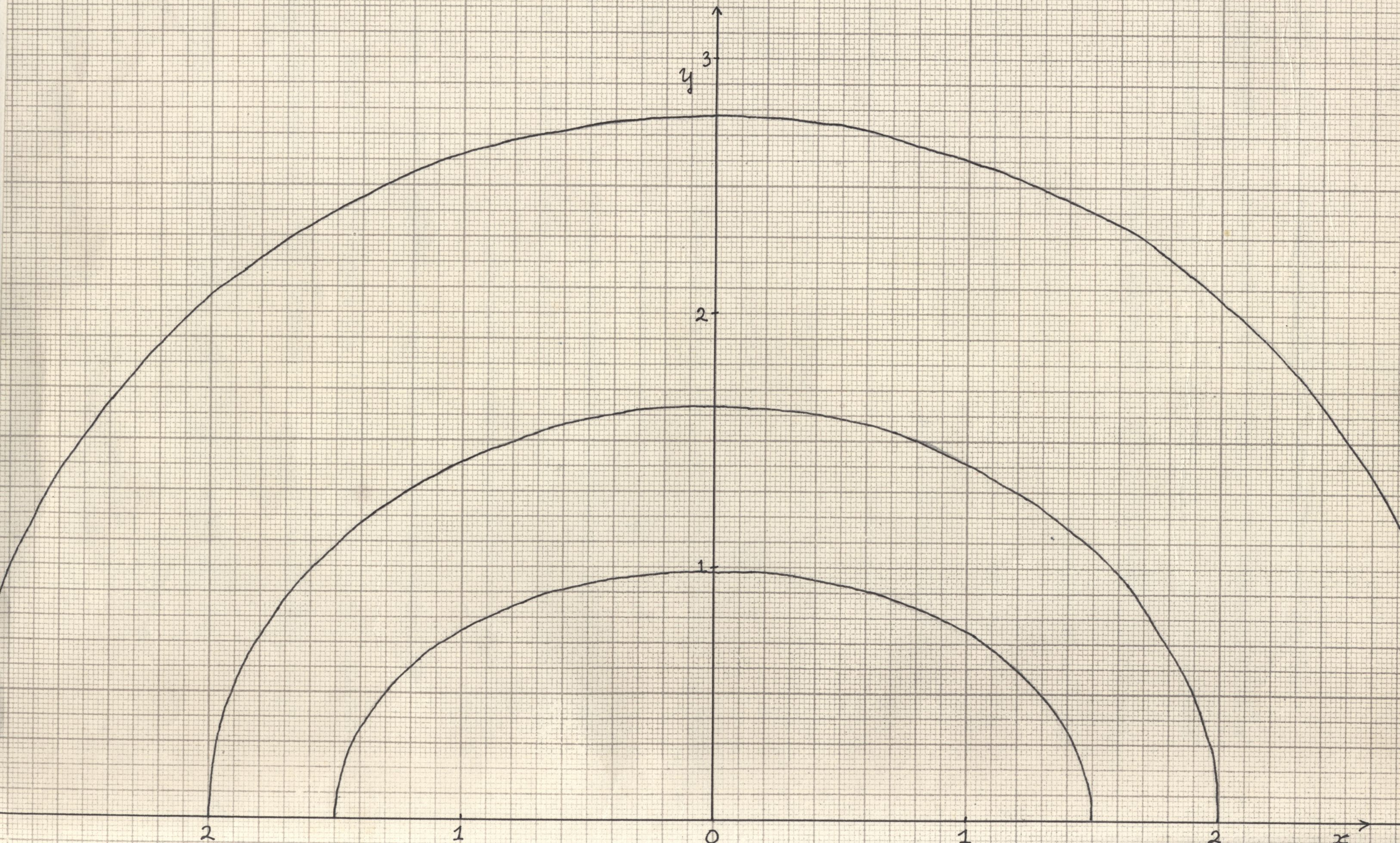
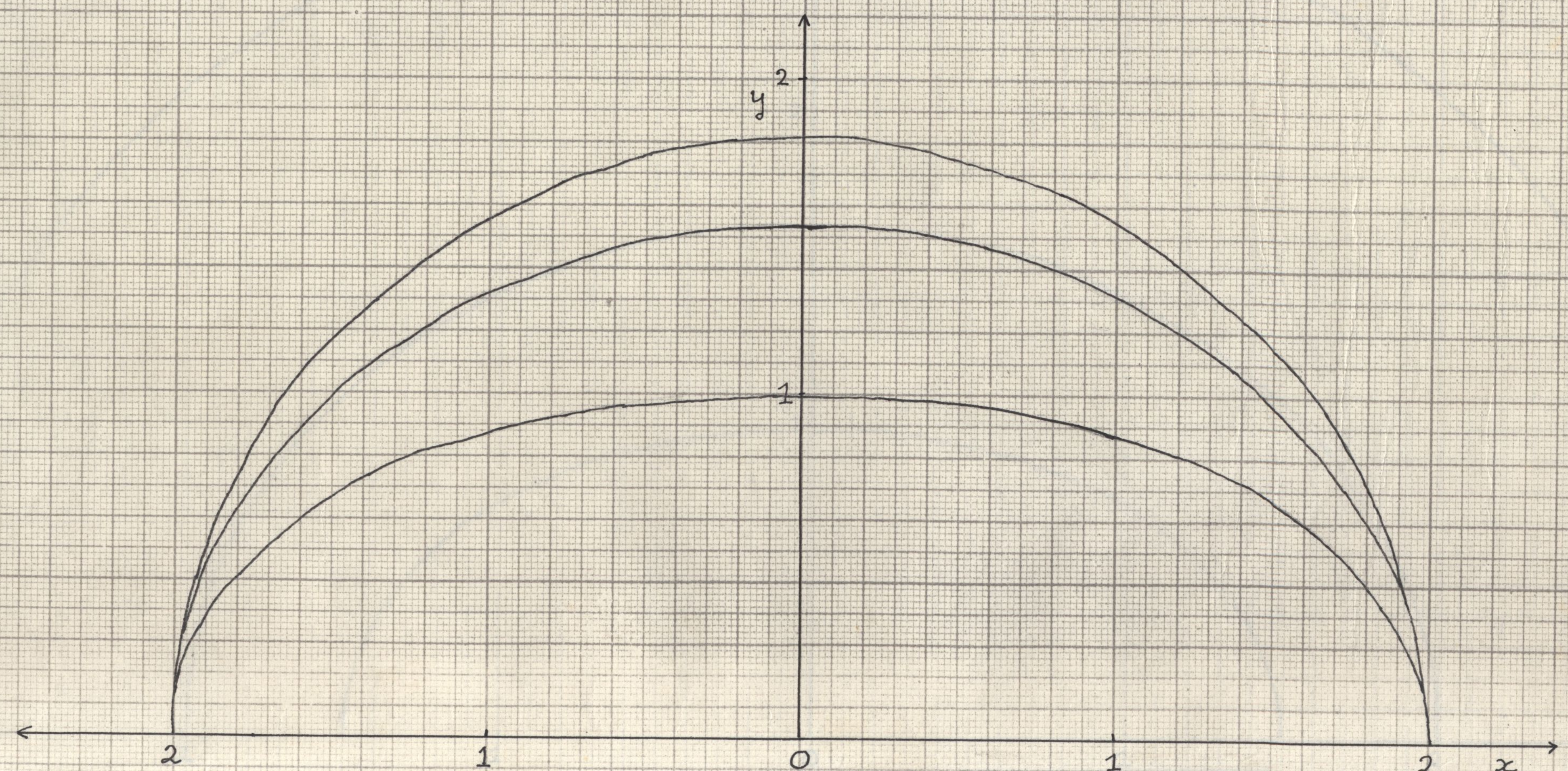


Figure (3)



Differentiating Ψ with respect to y at constant x, we obtain the velocity components

$$u = \frac{\partial \Psi}{\partial y}$$

$$= U + \frac{m}{2\pi}\left\{\frac{x - x_o}{[(x-x_o)^2+y^2]} - \frac{x + x_o}{[(x+x_o)^2+y^2]}\right\} \text{---------} \quad (1.10)$$

Similary differentiating Ψ with respect to x at constant y we obtain

$$v = \frac{\partial \Psi}{\partial x}$$

$$= \frac{my}{2\pi}\left[\frac{1}{(x+x_o)^2+y^2} - \frac{1}{(x-x_o)^2+y^2}\right] \text{----------------} \quad (1.11)$$

The velocity profile at the top of the hill is found by taking x to be zero in equation (1.10). This gives

$$u = U - \frac{2mx_o}{2\pi(x_o^2+y^2)^{3/2}} \text{---------------------------} \quad (1.12)$$

From the dividing steamline, $\Psi = 0$, we obtain

$$Uy + \frac{m}{2\pi}\left[\tan^{-1}\frac{y}{x-x_o} - \tan^{-1}\frac{y}{x+x_o}\right] = 0. \text{---------} \quad (1.13)$$

Let $\tan\alpha = \dfrac{y}{x-x_o}$, and $\tan\beta = \dfrac{y}{x+x_o}$,

then $$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

$$= \frac{2x_oy}{x^2+y^2-x_o^2}, \text{----------------------} \quad (1.14)$$

so that

$$\alpha - \beta = \tan^{-1}\left(\frac{2x_oy}{x^2+y^2-x^2}\right) \text{--------------------------} \quad (1.15)$$

and equation (1.13) becomes

$$Uy + \frac{m}{2\pi}\left[\tan^{-1}\left\{\frac{2x_o y}{x^2+y^2-x_o^2}\right\}\right] = 0 \quad \text{------------------------} \quad (1.16)$$

It is clear from equation (1.16) that

$$x^2 = x_o^2 - y^2 - \frac{2x_o y}{\tan\left(\frac{2\pi Uy}{m}\right)} \quad \text{------------------------} \quad (1.17)$$

A plot of y as a function of x for different values of x_o and fixed m is shown in figure (2). A plot of y as a function of x for different values of m and fixed x_o using equation (1.17) is shown in figure (3). It is found that the stream function $\Psi = 0$ represents flow over the surface of a hill.

1.2 The Three Dimensional Source-Sink Model

Following Milne-Thomson [3], the ~~steam~~ stream function ψ, for a three dimensional source at a point x_o from the axis, is represented by

$$\psi = \frac{M}{4\pi} \frac{x - x_o}{\sqrt{(x-x_o)^2+z^2}} \quad \text{------------------------} \quad (1.18)$$

and the velocity potential ϕ by

$$\phi = \frac{M}{4\pi\sqrt{(x-x_o)^2+z^2}} \quad \text{------------------------} \quad (1.19)$$

where $M = 4\pi m$ is the output of a source or sink of strength m. Combining a source of strength m at the point $(x_o, 0,0)$ and a sink of strength $-m$ at the point $(-x_o,0,0)$ with a uniform main stream velocity $u = (U,0,0)$ in the negative direction of the x-axis, the combined flow is then represented by the stream function Ψ given by

$$\psi = \frac{Uz^2}{2} - \frac{M}{2\pi}\left\{\frac{x+x_o}{\sqrt{(x+x_o)^2+z^2}} - \frac{x-x_o}{\sqrt{(x-x_o)^2+z^2}}\right\} \text{------} (1.20)$$

It is clear from the analysis of section (1.1) that $\psi = 0$ gives the equation of the dividing steamline. In this case the cross-section of the Rankine's body is given by

$$z^2 = \frac{M}{2U\pi}\left\{\frac{x+x_o}{\sqrt{(x+x_o)^2+z^2}} - \frac{x-x_o}{\sqrt{(x-x_o)^2+z^2}}\right\} \text{------} (1.21)$$

Taking $x = 0$ the equation for the height, $h=z$, of the body is given by

$$h^2 = z^2 = \frac{2Mx_o}{U\pi\sqrt{x_o^2+h^2}} \text{------} (1.22)$$

Let $x = \pm \ell$ be the stagnation points. The analysis of Milne-Thomson [3] gives

$$\frac{M}{4\pi(\ell - x_o)^2} - \frac{M}{4\pi(\ell + x_o)^2} = U, \text{------} (1.23)$$

where 2ℓ the length of the body can now be determined.

From equation (1.20) the velocity components u and v are given by

$$u = \frac{\partial\psi}{\partial z}$$

$$= Uz + \frac{M}{4\pi}\left\{\frac{x+x_o}{[(x+x_o)^2+z^2]^{3/2}} - \frac{x-x_o}{[(x-x_o)^2+z^2]^{3/2}}\right\} \text{----} (1.24)$$

and $$v = \frac{\partial\psi}{\partial x}$$

$$= \frac{M}{4\pi} z\left\{\frac{1}{[(x+x_o)^2+z^2]^{3/2}} - \frac{1}{[(x-x_o)^2+z^2]^{3/2}}\right\} \text{------} (1.25)$$

Along $x = 0$, the velocity profile for u takes the form

$$u = U + \frac{Mx_o}{2\pi(x_o^2+z^2)^{3/2}} \text{------} (1.26)$$

For the source-sink model of perfect wind flow over a hill to be of great use we require that the three dimensional model be reducable to the two dimensional version. However comparing equation (1.12) with equation (1.26) and comparing equation (1.17) with equation (1.21), we find that the attempt to reduce the three dimensional model to a two dimensional version leads to a different solution for u although the analysis carried out in this chapter is correct.

Chapter Two

2.0 Ideal Flow Over Ellipsoids

The difficulties mentioned above can be overcome by considering the flow about two dimensional elliptical bodies and three dimensional ellipsoidal bodies. The flow past the circle is used to find the solution past the ellipse by using the Jacobian Transformation.

$$\text{Let} \quad Z = \zeta + \frac{a^2 - c^2}{4} / \zeta \quad \text{------------------------------------- (2.1)}$$

represent the Jacobian Transformation where Z and ζ are the complex functions given by

$$Z = x + iz \quad \text{and} \quad \zeta = \xi + i\eta \quad \text{------------------------------- (2.2)}$$

Note that this transformation has the following properties:-

(i) It is 1 - 1.

(ii) as $\zeta \to \infty$, $Z \to \infty$.

(iii) The x - Axis transforms into the ξ-axis and the z-axis transforms into the η - axis.

(iv) The uniform main stream velocity, $u=(U,0,0)$, in the Z-plane corresponds to a uniform main stream velocity in the ζ-plane in the ζ-direction.

(v) The circle $|\zeta| = \zeta^2 + \eta^2 = r^2$ in the ζ-plane transforms into the ellipse $\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$ in the Z - plane.

The complex potential ω is defined by

$$\omega = \phi + i\psi, \quad \text{-- (2.3)}$$

where ϕ is the velocity potential and ψ the stream function.

For uniform mainstream flow $u = (U,0,0)$, the complex potential ω is given by

$$\omega = U\zeta \qquad (2.4)$$

If a circular cylinder given by $|\zeta| = r$ is introduced into the main stream the complex potential representing the flow is given by the circle theorem as

$$\omega = U(\zeta + \frac{r^2}{\zeta}) \qquad (2.5)$$

Velocities in the Z-plane are given by

$$\frac{d\omega}{dz} = \frac{d\omega}{d\zeta} \cdot \frac{d\zeta}{dz} = u - iv. \qquad (2.6)$$

From equation (2.5), we obtain

$$\frac{d\zeta}{d\zeta} = U[1 - (\frac{a + c}{2})^2/\zeta^2]$$

and

$$\frac{d\zeta}{dz} = \frac{1}{1 - (\frac{a^2 - c^2}{4})/\zeta^2}$$

and equation (2.6) becomes

$$u - iv = U[\frac{4\zeta^2 - (a + c)^2}{4\zeta^2 - (a^2 - c^2}] . \qquad (2.7)$$

Substituting (2.7) in (2.1), gives

$$2\zeta = Z \pm \sqrt{Z^2 - (a^2 - c^2)} . \qquad (2.8)$$

Let $\zeta' = 2\zeta$ so that $\zeta' = Z \pm \sqrt{Z^2 - (a^2 - c^2)}$ (2.9)

Then equation (2.7) becomes

$$u - iv = U[\frac{(\zeta')^2 - (a + c)^2}{(\zeta') - (a^2 - c^2)}] \qquad (2.10)$$

It is clear from equation (2.10) that if ζ' is known then the velocities u and v at any general point (x,z) in two dimensional space are also known.

Let z' be the height above the surface of the hill such that

$$z'\ z = -c\sqrt{(1 - \frac{x^2}{a^2})} \text{ if } |x| < a$$

$$= z \text{ if } |x| > a \text{ ------------------------------------- } (2.11)$$

Note that $z' = 0$ is the ground surface.

At the point $(x, c\sqrt{[1 - \frac{x^2}{a^2}]})$, i.e at the surface

$$Z^2 = x^2 - c^2(1 - \frac{x^2}{a^2}) + 2ixz \text{ ------------------------- } (2.12)$$

and substituting this values of Z in equation (2.9) gives

$$\zeta' = (a + c)(\frac{n}{\varepsilon} - i\ \frac{z}{c}) \text{ ---------------------------------- } (2.13)$$

Subsituting equation (2.13) in equation (2.10) gives

$$u = u_2 = \frac{a(a + c)(a^2 - x^2)U}{a^4 - x^2(a^2 - c^2)} \text{ ------------------------- } (2.14)$$

and $$v = v_2 = \frac{-a(a + c)xzU}{a^4 - x^2(a^2 - c^2)} \text{ -------------------------- } (2.15)$$

Thus for any point $(0,z)$ vertically above the top of the hill $v_2 = 0$. Therefore to compare velocities in two dimensional case with velocities in the three dimensional case we consider the limit of such velocities as $x \to 0$.

Thus taking $x = \delta$ where $\delta < 1$,

$Z = \delta + iz$

$$2\delta = \delta' = \delta + iz + \frac{1}{(z^2+a^2-c^2)^{\frac{1}{2}}}[\delta z + i(z^2 + a^2 - c^2)].$$

Inserting this value of ζ' in equation (2.10) gives

$$v_2 = \frac{-c(a + c)\delta U}{(Z^2+a^2-c^2)^{3/2}}, \quad \text{------------------------------} \quad (2.16)$$

to first order accuracy.

2.1 Streamline Analysis

Equation (2.1) and equation (2.5), in section (2.0), give

$$\omega = U[\zeta + (\frac{a+c}{2})^2/\zeta] \quad \text{and} \quad Z = [\zeta + (\frac{a^2 - c^2}{4})/\zeta], \quad \text{-------} \quad (2.17)$$

from which

$$\omega = \phi + i\psi$$

$$= U[\frac{a+c}{a-c} Z - \frac{2c}{a-c} \zeta]. \quad \text{------------------------------} \quad (2.18)$$

Using equation (2.2), in section (2.0), gives

$$\text{Im}\omega = \psi$$

$$= U[\frac{a+c}{a-c} z - \frac{2c}{a-c} \eta] \quad \text{------------------------------} (2.19)$$

Using equaion (2.2) and equation (2.1), we find that Im ζ gives

$2\eta = z + \hat{z}$, where

$\hat{z} = \text{Im}\sqrt{Z^2 - a^2 + c^2}$, so that

$$\hat{z}^2 = -f(x,z) + \sqrt{f^2(x,z) + x^2 z^2}, \quad \text{----------------------} \quad (2.20)$$

where $f(x,z) = \dfrac{x^2 - z^2 - a^2 + c^2}{2}$. ------------------------------(2.21)

Substituting equation (2.20) in equation (2.19) gives

$$\psi = \frac{U}{a-c}(az - c\hat{z}) \quad \text{------------------------------} (2.22)$$

Note that on a streamline ψ = constant, i.e.

$$az - c\hat{z} = \ell^2 = \text{constant, gives}$$

$$\hat{z} = \frac{az - \ell^2}{c}. \quad \text{------------------------------------} (2.23)$$

Combining equations (2.19), (2.20) and (2.23), the equation for the streamlines is given by

$$x^2 = [\frac{az - \ell^2}{c}]^2 \; [\frac{(a^2 - c^2)(z^2 - c^2) + \ell^2(\ell^2 - 2az)}{z^2(c^2 - a^2) - \ell^2(\ell^2 - 2az)}]. \quad \text{----} (2.24)$$

When $\ell = 0$

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1, \quad \text{------------------------------------} (2.25)$$

which is the surface streamline. This result is in agreement with the initial assumption that the steamlines of air currents are almost parallel to the underlying surface.

When $x = 0$, equation (2.24) reduces to

$$(a^2 - c^2)(z^2 - c^2) + \ell^2(\ell^2 - 2az) = 0 \quad \text{--------------} (2.26)$$

while when $x = \infty$, equation (2.24) yields

$$z^2(c^2 - a^2) - \ell^2(\ell^2 - 2az) = 0 \quad \text{----------------------} (2.27)$$

Let z_o denote the height of a streamline at $x = 0$ and z_∞ denote the height of the same streamline at $x = \infty$. Then z_o satisfies equation (2.26) and z_∞ satisfies equation (2.27). It can be noted that when $\ell = 0$, $z_o = c$ and $z_\infty = 0$.

Solving equation (2.26) and equation (2.27) for z_o and z_∞

yields

$$z_o = \frac{a\ell^2}{a-c} + \sqrt{\frac{c^2\ell^4}{(a^2-c^2)^2} + c^2}$$

and $z_\infty = \dfrac{\ell^2}{a-c}$

The maximum shift in the streamlines,

$z_o - z_\infty$, is given by

$$z_o - z_\infty = c\sqrt{\left(1 + \left[\frac{z_\infty}{a+c}\right]^2\right)} - \frac{cz_\infty}{a+c} \text{ . } \qquad (2.28)$$

It may be noted that as $z_\infty \to \infty$, $z_o - z_\infty = 0$ This result is **intuitively correct.**

2.2 Three Dimensional Flow

Introduction

To simplify our work ellipsoidal co-ordinates are used. Consider the confocal quadrics given by the equations

$$\frac{x^2}{a^2+\theta} + \frac{y^2}{b^2+\theta} + \frac{z^2}{c^2+\theta} - 1 = 0; \quad a>b>c \qquad (2.29)$$

which can be written as

$$\begin{aligned} f(\theta) = {} & x^2(b^2+\theta)(c^2+\theta) \\ & + y^2(c^2+\theta)(a^2+\theta) \\ & - (a^2+\theta)(b^2+\theta)(c^2+\theta) = 0 \end{aligned} \qquad (2.30)$$

Clearly $f(\theta)$ is a cubic equation and therefore admits three roots (λ,μ,ν).

Thus

$$(\lambda - \theta)(\mu - \theta)(\nu - \theta) = x^2(b^2+\theta)(c^2+\theta) + y^2(c^2+\theta)(a^2+\theta) + z^2(a^2+\theta)(b^2+\theta) - (a^2+\theta)(b^2+\theta)(c^2+\theta) = 0 \qquad (2.31)$$

Using the ideas from solid geometry it can easily be shown that given a point $p(x,y,z)$ there are three orthogonal surfaces given by λ = constant, μ = constant and ν = constant passing through the point p.

Let $-\infty < \lambda < -c^2$

$-c^2 < \mu < -b^2$

$-b^2 < \nu < -a^2$

Then the surfaces

λ = constant

μ = constant

and ν = constant

are correspondingly ellipsoids, hyperboloids of one sheet and hyperboloids of two sheets.

Substituting $\theta = -a^2$, $-b^2$ and $-c^2$, in turn, in equation (2.31) we obtain

$$x^2 = \frac{(a^2+\lambda)(a^2+\mu)(a^2+\gamma)}{(a^2-b^2)(a^2-c^2)}$$

$$y^2 = \frac{(b^2+\lambda)(b^2+\mu)(b^2+\nu)}{(b^2-c^2)(b^2-a^2)} \qquad (2.32)$$

$$z^2 = \frac{(c^2+\lambda)(c^2+\mu)(c^2+\gamma)}{(c^2-b^2)(c^2-a^2)}$$

Thus if λ , μ and ν are known the position (x, y, z) is also known. λ , μ and ν define orthogonal curvilinear co-ordinates.

Logarithmic differentiation of equations (2.32) with respect to λ gives the following results:-

$$\frac{\partial x}{\partial \lambda} = \frac{1}{2} \frac{x}{(a^2+\lambda)} \qquad (2.33)$$

$$\frac{\partial y}{\partial \lambda} = \frac{1}{2} \frac{y}{(b^2+\lambda)} \qquad (2.34)$$

$$\frac{\partial z}{\partial \lambda} = \frac{1}{2} \frac{z}{(c^2+\lambda)} \qquad (2.35)$$

Following Milne-Thomson [3] or Murray R. Spiegel [illegible], Laplace's equation in general orthogonal curvilinear co-ordinates can be written as

$$\nabla^2 \phi = \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial \lambda}\left(\frac{h_2 h_3}{h_1} \frac{\partial \phi}{\partial \lambda}\right) + \frac{\partial}{\partial \mu}\left(\frac{h_3 h_1}{h_2} \frac{\partial \phi}{\partial \mu}\right) + \frac{\partial}{\partial \nu}\left(\frac{h_1 h_2}{h_3} \frac{\partial \phi}{\partial \nu}\right)\right\} \qquad (2.36)$$

where

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 = h_1^2 (d\lambda)^2 + h_2^2 (d\mu)^2 + h_3^2 (d\nu)^2$$

i.e

$$\left(\frac{\partial x}{\partial \lambda} d\lambda + \frac{\partial x}{\partial \mu} d\mu + \frac{\partial x}{\partial \nu} d\nu\right)^2$$

$$+ \left(\frac{\partial y}{\partial \lambda} d\lambda + \frac{\partial y}{\partial \mu} d\mu + \frac{\partial y}{\partial \gamma} d\nu\right)^2$$

$$+ \left(\frac{\partial z}{\partial \lambda} d\lambda + \frac{\partial z}{\partial \mu} d\mu + \frac{\partial z}{\partial \nu} dz\right)^2$$

$$= h_1^2 (d\lambda)^2 + h_2^2 (d\mu)^2 + h_3^2 (d\nu)^2 \qquad (2.37)$$

along the λ-coordinate $d\mu = 0$ and $d\nu = 0$ and therefore

$$h_1^2 = (\frac{\partial x}{\partial \lambda})^2 + (\frac{\partial y}{\partial \lambda})^2 + (\frac{\partial z}{\partial \lambda})^2 \quad \text{------} \quad (2.38)$$

$$h_2^2 = (\frac{\partial x}{\partial \mu})^2 + (\frac{\partial y}{\partial \mu})^2 + (\frac{\partial z}{\partial \mu})^2 \quad \text{------} \quad (2.39)$$

$$h_3^2 = (\frac{\partial x}{\partial \nu})^2 + (\frac{\partial y}{\partial \nu})^2 + (\frac{\partial z}{\partial \nu})^2 \quad \text{------} \quad (2.40)$$

substituting equation (2.33) in equation (2.38) gives

$$h_1^2 = \tfrac{1}{4}\left\{ \frac{x^2}{(a^2 + \lambda)^2} + \frac{y^2}{(b^2 + \lambda)^2} + \frac{z^2}{(c^2 + \lambda)^2} \right\}$$

$$= \tfrac{1}{4}\, \frac{(\lambda - \mu)(\lambda - \nu)}{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)} \quad \text{------} \quad (2.41)$$

similarly **substituting** equation (2.34) in equation (2.39) and equation (2.35) in equation (2.40) gives the following results

$$h_2^2 = \tfrac{1}{4}\, \frac{(\mu - \nu)(\mu - \lambda)}{(a^2 + \mu)(b^2 + \mu)(c^2 + \mu)} \quad \text{------} \quad (2.42)$$

$$h_3^2 = \tfrac{1}{4}\, \frac{(\nu - \lambda)(\nu - \mu)}{(a^2 + \nu)(b^2 + \nu)(c^2 + \nu)} \quad \text{------} \quad (2.43)$$

Using equations (2.41), (2.42) and (2.43)

Laplace's equation in Ellipsoidal co-ordinates becomes

$$\nabla^2\phi = -\frac{4}{(\mu-\nu)(\nu-\lambda)(\lambda-\mu)} [(\mu-\nu)\{(a^2+\lambda)^{\frac{1}{2}}(b^2+\lambda)^{\frac{1}{2}}(c^2+\lambda)^{\frac{1}{2}} \frac{\partial}{\partial\lambda}\}^2$$

$$+ (\nu-\lambda)\{(a^2+\mu)^{\frac{1}{2}}(b^2 + \mu)^{\frac{1}{2}}(c^2 + \mu)^{\frac{1}{2}} \frac{\partial}{\partial\mu}\}^2$$

$$+(\lambda-\mu)\{(a^2+\nu)^{\frac{1}{2}}(b^2+\nu)^{\frac{1}{2}}(c^2+\nu)^{\frac{1}{2}} \frac{\partial}{\partial\nu}\}^2]\phi$$

$$= -\frac{4}{(\mu-\nu)(\nu-\lambda)(\lambda-\mu)} [(\mu-\nu)k_\lambda \frac{\partial}{\partial\lambda}(k_\lambda \frac{\partial\phi}{\partial\lambda})$$

$$+ (\nu-\lambda)k_\mu \frac{\partial}{\partial\mu}(k_\mu \frac{\partial\phi}{\partial\mu})$$

$$+ (\lambda-\mu)\, k_\nu\frac{\partial}{\partial\nu}(k_\lambda \frac{\partial\phi}{\partial\nu})] = 0 \quad \text{------------------------------}(2.44)$$

where

$$(k_\lambda)^2 = (a^2 +\lambda)(b^2 +\lambda)(c^2+\lambda) \quad \text{------------------------------}(2.45)$$

$$(k_\mu)^2 = (a^2 +\mu)(b^2 +\mu)(c^2 +\mu) \quad \text{------------------------------}(2.46)$$

$$(k_\nu)^2 = (a^2 +\nu)(b^2 +\nu)(c^2 +\nu) \quad \text{------------------------------}(2.47)$$

Equation (2.44) can be rewritten in the form

$$(\mu-\lambda)\ (k_\lambda \frac{\partial}{\partial\lambda})^2\phi + (\nu-\lambda)(k_\mu \ \frac{\partial}{\partial\mu})^2 \ \phi + (\lambda-\mu)(k_\lambda \frac{\partial}{\partial\gamma})^2\phi=0 \quad \text{----}(2.48)$$

We lŏok for a solution of equation (2.48) of the form

$$\phi = \alpha\chi(\lambda) \quad \text{--}(2.49)$$

where α is a parameter to be determined and $\chi(\lambda)$ is a function of λ only.

Differentiating (2.49) with respect to λ we obtain

$$k_\lambda \frac{\partial}{\partial\lambda}(\alpha\chi) = k_\lambda \chi \frac{\partial\alpha}{\partial\lambda} + k\alpha \frac{\partial\chi}{\partial\lambda}$$

$$\frac{\partial}{\partial\lambda}\{k_\lambda \frac{\partial}{\partial\lambda}(\alpha\chi) = \chi\frac{\partial}{\partial\lambda}(k_\lambda \frac{\partial\alpha}{\partial\lambda}) + 2k_\lambda \frac{\partial\alpha}{\partial\lambda}\frac{\partial\chi}{\partial\lambda}$$

$$+ \alpha \frac{\partial k_\lambda}{\partial\lambda}\frac{\partial\chi}{\partial\lambda} + \alpha k \frac{\partial^2\chi}{\partial\lambda^2}$$

$$= \chi \frac{\partial}{\partial\lambda}(k_\lambda \frac{\partial\alpha}{\partial\lambda}) + 2k \frac{\partial\alpha}{\partial\lambda}\frac{\partial\chi}{\partial\lambda} + \alpha \frac{\partial k_\lambda}{\partial\lambda}\frac{\partial\chi}{\partial\lambda}$$

$$+ \alpha k_\lambda \frac{\partial^2\chi}{\partial\lambda^2}, \quad \text{------------------} (2.50)$$

and subsitituting equation (2.50) in equation (2.48) yields

$$2k_\lambda \frac{\partial\alpha}{\partial\lambda}\frac{\partial\chi}{\partial\lambda} + \alpha \frac{\partial k_\lambda}{\partial\lambda}\frac{\partial\chi}{\partial\lambda} + \alpha k_\lambda \frac{\partial^2\chi}{\partial\lambda^2} = 0, \quad \text{------------} (2.51)$$

from which

$$-\frac{2}{\alpha}\frac{\partial\alpha}{\partial\lambda} = \frac{\frac{\partial k_\lambda}{\partial\lambda}\frac{\partial\chi}{\partial\lambda} k_\lambda \frac{\partial^2\chi}{\partial\lambda^2}}{k_\lambda \frac{\partial\chi}{\partial\lambda}}$$

$$= \frac{1}{k_\lambda}\frac{\partial k_\lambda}{\partial\lambda} + \frac{\partial^2\chi}{\partial\lambda^2} \Big/ \frac{\partial\chi}{\partial\lambda}$$

$$= \frac{\partial}{\partial\lambda}\ln(k_\lambda \frac{\partial\chi}{\partial\lambda}). \quad \text{------------------} (2.52)$$

We note that the right handside of equation (2.52) is a function of λ only. Thus we represent α in the form

$$\alpha = \alpha_\lambda f(\mu,\nu), \quad \text{------------------} (2.53)$$

so that equation (2.52) becomes

$$\frac{d}{d\lambda}\,\ell n\left(k_\lambda \frac{d\chi}{d\lambda}\right) = \frac{d}{d\lambda}\,\ell n \frac{1}{\alpha^2_\lambda} \qquad (2.54)$$

integrating equation (2.54) gives

$$\chi = A \int \frac{d\lambda}{\alpha^2_\lambda k_\lambda} + B \qquad (2.55)$$

Following Milne-Thomson [3], we find that since x, y and z are solutions of Laplace's Equation in Carterian Co-ordinates, they are also solutions of equation (2.48). So that using equation (2.32) and noting that α corresponding to x is of the form

$$\alpha = (a^2 + \lambda)^{\frac{1}{2}} (a^2 + \mu)^{\frac{1}{2}} (a^2 + \nu)^{\frac{1}{2}}$$

Equation (2.49) becomes

$$\phi = cx \int_\lambda^\infty \frac{d\lambda}{(a^2 + \lambda)k_\lambda} \qquad (2.56)$$

2.3 Flow around an Ellipsoid

Consider the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad (2.57)$$

which is from a family of the confocal quadrics

$$\frac{x^2}{a^2+\theta} + \frac{y^2}{b^2+\theta} + \frac{z^2}{c^2+\theta} = 1, \qquad (2.58)$$

for $\theta = \lambda = 0$, moving with velocity $-U$ in the positive x direction in a fluid at rest, with boundary conditions

$$\frac{\partial\phi}{\partial\lambda} = U\frac{\partial x}{\partial\lambda}; \quad \lambda = 0 \qquad \text{(2.59)}$$

and $\phi \to \infty$, $\lambda \to \infty$

It is evident from equations (2.59) that the motion described here is axisymmetrical with no pertubation at infinity and zero normal velocity at the sphere. The velocity potential ϕ is given by

$$\phi = C\, x \int_{\lambda}^{\infty} \frac{d\lambda}{(a^2+\lambda)k_{\lambda}} \qquad \text{(2.60)}$$

Differentiating equation (2.60) with respect to λ gives

$$\frac{\partial\phi}{\partial\lambda} = U\frac{\partial x}{\partial\lambda}$$

$$= -C\frac{\partial x}{\partial\lambda}\int_{\lambda}^{\infty}\frac{d\lambda}{(a^2+\lambda)k_{\lambda}} + C\frac{x}{a^2(abc)}; \qquad \text{(2.61)}$$

for $\lambda = 0$.

From equation (2.33), we note that

$$\frac{\partial x}{\partial\lambda} = \frac{x}{2a^2}; \qquad \text{(2.62)}$$

for $\lambda = 0$.

Using equation (2.61), the constant C is given by

$$C = -\frac{ab\ U}{2-\alpha_o}$$

$$= \frac{ab\ U}{\alpha_o-2},$$

where

$$\alpha_o = abc \int_o^\infty \frac{d\lambda}{(a^2+\lambda)k_\lambda}. \quad \text{------------(2.63)}$$

On the ellipsoid $(\lambda = 0)$ the velocity potential ϕ is given by

$$\phi = \frac{\alpha_o x\ U}{\alpha_o-2} \quad \text{------------(2.64)}$$

If on this ellipsoid which is moving in the positive x direction with velocity -U we superimpose a flow with velocity U in the positive x direction, we get the flow pattern due to a stationary ellipsoid in a flow field U. The resultant velocity potential ϕ is then given by

$$\phi = Ux + \frac{2abUx}{\alpha_o-2} \int_\lambda^\infty \frac{d\lambda}{(a^2+\lambda)^{3/2}(b^2+\lambda)^{\frac{1}{2}}(c^2+\lambda)^{\frac{1}{2}}} \quad \text{------------(2.65)}$$

$$= Ux[1+c\ I(\lambda)] \quad \text{------------(2.66)}$$

where

$$I(\lambda) = \ (\lambda)$$

$$= \int_o^\infty \frac{abc\ d\lambda}{(a^2+\lambda)k_\lambda} \quad \text{------------(2.67)}$$

We note in particular that the velocity potential ϕ for the flow round a spherical hill can be determined from equation (2.66). In the case of a sphere, a=b=c. The equation for the confocal quadrics becomes

$$\frac{x^2 + y^2 + z^2}{a^2 + \theta} = 1 \quad \text{------}(2.68)$$

sice $\theta = \lambda$ is a solution, then

$$a^2+\lambda = x^2 + y^2 + z^2$$

$$= R^2,$$ the polar distance in (x,y,z) plane

For a sphere, equation (2.67) yields

$$I(\lambda) = 2/3 \frac{a^3}{(a^2+\lambda)^{3/2}} \quad \text{------}(2.69)$$

Differentiating equation (2.69) with respect to λ gives

$$I'(\lambda) = - \frac{a^3}{(a^2+\lambda^2)^{5/2}} \quad \text{------}(2.70)$$

Differentiating equation (2.66) with respect to λ gives

$$\frac{\partial \phi}{\partial \lambda} = U[1 + C\, I(\lambda)] \frac{\partial x}{\partial \lambda} + U\, Cx\, I'(\lambda) \quad \text{------}(2.71)$$

For $\lambda = 0$, i.e when there is no flow across the surface

$$U[1 + c\, I(o)] \frac{x}{2a^2} - Uc \frac{x}{a^2} = 0 \quad \text{------}(2.72)$$

solving for C yields

$$C = \frac{1}{2 - I(o)} \quad \text{------}(2.73)$$

In the case of a sphere

$$c = 3/4 \qquad \text{------------------------------(2.74)}$$

Substituting equation (2.74) and equation (2.69) in equation (2.66) gives the velocity potential for flow around a spherical hill as

$$\phi = Ux[1 + \frac{a^3}{2R^3}] \qquad \text{------------------------------(2.75)}$$

This result is in agreement with Milne-Thomson [3] although the approach used here is different.

2.4 Velocities in three dimentional Flow

Differentiating equation (2.66) with respect to x keeping z constant and then with respect z keeping x constant gives the velocity components

$$u = \frac{\partial\phi}{\partial x}$$

$$= U[1 + I(\lambda) + Cx\, I'(\lambda)\, \frac{\partial\lambda}{\partial x}] \qquad \text{------------------------------(2.76)}$$

and $v = \frac{\partial\phi}{\partial z}$

$$= UCx\, I'(\lambda)\, \frac{\partial\lambda}{\partial z} \qquad \text{------------------------------(2.77)}$$

where

$$\frac{\partial\lambda}{\partial x} = \frac{2x(b^2+\lambda)(c^2+\lambda)}{(\lambda-\mu)(\lambda-\nu)} \qquad \text{------------------------------(2.78)}$$

and

$$\frac{\partial\lambda}{\partial z} = \frac{2z\,(a^2+\lambda)(b^2+\lambda)}{(\lambda-\mu)(\lambda-\nu)} \qquad \text{------------------------------(2.79)}$$

Let $\nu = -b^2$ on $y = 0$, i.e on the centre line. Then using equation (2.67), equation (2.76) and equation (2.77) degenerate to

$$u = u_3 = U[1 + C\ I(\lambda) - 2Cx^2 \frac{abc(c^2+\lambda)}{(a^2+\lambda)(\lambda-\mu)k_\lambda}] \text{------------(2.80)}$$

and
$$v = -\frac{2\ UC\ x\ z\ abc}{(\lambda - \mu)\ k_\lambda} \text{------------(2.81)}$$

The expression for the velocity component v does not contain $I(\lambda)$. It is therefore possible to find analytical expressions for v.

Equation (2.57) gives

$$x^2 = \frac{(a^2 + \lambda)(a^2 + \mu)}{a^2 - c^2} \text{------------(2.82)}$$

and
$$z^2 = \frac{(c^2 + \lambda)(c^2 + \mu)}{c^2 - a^2} \text{------------(2.83)}$$

Equation (2.82) and equation (2.83) yield

$$\lambda + \mu = x^2 + z^2 - a^2 - c^2 \text{------------(2.84)}$$

and
$$\lambda\ \mu = a^2c^2 - c^2x^2 - a^2z^2 \text{------------(2.85)}$$

Let
$$\Sigma = \lambda + \mu = x^2 + z^2 - a^2 - c^2 \text{------------(2.86)}$$

$$\Pi = \lambda\ \mu = a^2c^2 - c^2x^2 - a^2z^2 \text{------------(2.87)}$$

so that λ and μ are the roots of

$$\theta^2 - \Sigma\ \theta + \Pi = 0 \text{------------(2.88)}$$

$$\lambda = \frac{\Sigma}{2} + \sqrt{\left(\frac{\Sigma^2}{4} - \Pi\right)} \quad \text{and} \quad \mu = \frac{\Sigma}{2} - \sqrt{\left(\frac{\Sigma^2}{4} - \Pi\right)}$$

On the surface $\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$, i.e $z^2 = c^2 - \frac{c^2 x^2}{a^2}$ ---------- (2.89)

so that on the surface, i.e for $\lambda = 0$,

$$\lambda + \mu = \mu = x^2 + z^2 - a^2 - c^2$$

$$= -\frac{1}{a^2}\,[a^4 - x^2(a^2 - c^2)] \qquad (2.90)$$

Therefore in the three demensional case surface velocities are given by:-

(a) $$u = [1 + C\,I(o) - \frac{2C\,x\,c}{a^4 - x^2(a^2 - c^2)}]$$

$$= a^4 - a^2x^2 + C\,I(o)a^4$$

$$- C\,I(o)x^2a^2 + c^2x^2(1 + C\,I(o) - 2c) \qquad (2.91)$$

Equation (7.23) gives

$$c\,x\,(1 + C\,I(o) - 2c) = 0$$

so that equation (2.91) becomes

$$u = U(1 + C\,I(o))\,\frac{a^2(a^2 - x^2)}{a^4 - x^2(a^2 - c^2)} \qquad (2.92)$$

similarly

(b) $$v = -U(1 + C\,I(o))\,\frac{a^2 x\,z}{a^4 - x^2(a^2 - c^2)} \qquad (2.93)$$

For velocities at any point vertically above the top of the hill, i.e along $x = 0$, the vertical velocity v is as given in the two dimensional case equal to zero.

Therefore to compare three dimensional and two dimensional velocities along the top of the hill i.e along $x = 0$, we need to consider the limit of such velocities as $x \to 0$.

Let $x = \delta$ so that

$$\lambda + \mu \quad \delta^2 + z^2 - a^2 - c^2$$

and $$\lambda \mu = a^2 c^2 - a^2 z \quad c \, \delta^2$$

working to first order accuracy, $x = \delta \to 0$,

$$\lambda + \mu = z^2 - a^2 - c^2$$

and $$\lambda\mu = a^2 c^2 - c^2 z^2$$

so that $$\lambda = a^2(c^2 - z^2) \quad \text{and} \quad \mu = -a^2$$

$$k_\lambda = [a^2 + z^2 - c^2)(b^2 + z^2 - c^2)]^{\frac{1}{2}}$$

substituting these values of λ and k_λ in equation (2.31) yields

$$V_3 = \frac{-2abc \, \delta \, C \, U}{[(b \quad + z \quad - c \quad)(a \quad + z \quad - c \quad)]^{\frac{1}{2}}} \quad \text{-----------------(2.94)}$$

$$= \frac{-2abc \, \delta \, [1 - C \, I \, (o)]U}{[(b^2 + z^2 - c^2)(a^2 + z^2 - c^2)]^{\frac{1}{2}}} \quad \text{-------------------(2.95)}$$

2.5 Stream Surfaces

In the two demensional a case, the stream potential is given by

$$\omega = \phi + i\psi \ , \qquad (2.96)$$

where

ϕ is the velocity potential

ψ is the stream function,

and ϕ and ψ are orthogonal i.e $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$ and $\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$. From equation (2.96), the equation of the stream lines is given by

$$\psi = \text{constant} \qquad (2.97)$$

In three demensional space we have stream surfaces rather than stream lines and we consider the two dimensional intersections of these surfaces with the plane $y = 0$. In this case we require that

$$\psi = \text{constant} \quad \text{and} \quad \frac{dz}{dx} = \frac{v}{u} \qquad (2.98)$$

The stream function for axisymmetrical inviscid flow around a sphere is given as in Milne-Thomson [3] by

$$\psi = z^2\left(1 - \frac{a^3}{R^3}\right)$$

$$= z^2 f(R)$$

$$= z^2 f(\lambda) \qquad (2.99)$$

On stream surfaces ψ = constant, so that

$$\frac{d\psi}{dx} = 2z\frac{dz}{dx}f(\lambda) + z^2 f'(\lambda)\left[\frac{\partial\lambda}{\partial x} + \frac{\partial\lambda}{\partial z}\frac{dz}{dx}\right]$$

$$= \frac{dz}{dx}\left[2z\, f(\lambda) + z^2 f'(\lambda)\frac{\partial\lambda}{\partial z}\right] + z^2 f'(\lambda)\frac{\partial\lambda}{\partial x} = 0$$

and

$$\frac{dz}{dx} = -\frac{z^2 f'(\lambda)\dfrac{\partial\lambda}{\partial xx}}{2zf(\lambda) + z^2 f'(\lambda)\dfrac{\partial\lambda}{\partial z}}$$

$$= -\frac{z\, f'(\lambda)\dfrac{\partial\lambda}{\partial x}}{2f(\lambda) + zf'(\lambda)\dfrac{\partial\lambda}{\partial x}} \qquad (2.100)$$

Using equation (1.1) and equation (1.2), i.e

$\frac{d\phi}{dx} = v$ and $\frac{d\phi}{\partial z} = u$ equation (2.98) follows

$$\frac{dz}{dx} = \frac{v}{u} \qquad (2.101)$$

Substituting for u and v from equation (2.76) and equation (2.77), we obtain

$$\frac{U\, C\, x\, I'(\lambda)\dfrac{\partial\lambda}{\partial z}}{U[1 + C\, I(\lambda) + C\, x\, I'(\lambda)\dfrac{\partial\lambda}{\partial x}]} = \frac{-z\, f'(\lambda)\dfrac{\partial\lambda}{\partial x}}{2f(\lambda) + zf'(\lambda)\dfrac{\partial\lambda}{\partial x}} \qquad (2.102)$$

Simplifying equation (2.102) yields

$$[2f(\lambda) + zf'(\lambda)\frac{\partial\lambda}{\partial z}]\, UCx\, I'(\lambda)\frac{\partial\lambda}{\partial z} = -\, zf'(\lambda)\frac{\partial\lambda}{\partial x}\, [U(1 + C\, I(\ )+CxI'(\lambda)\frac{\partial\lambda}{\partial x}$$

$$\left[\frac{2f(\lambda)}{zf'(\lambda)\frac{\partial\lambda}{\partial x}} + \frac{\partial\lambda}{\partial z}\Big/\frac{\partial\lambda}{\partial x}\right] = \frac{-\,U[1 + C\, I(\lambda) + CxI'(\lambda)\frac{\partial\lambda}{\partial x}]}{UCxI'(\lambda)\frac{\partial\lambda}{\partial z}}$$

$$= \frac{1 + C\, I\,(\lambda)}{CxI'(\lambda)\frac{\partial\lambda}{\partial z}} + \frac{\partial\lambda}{\partial x}\Big/\frac{\partial\lambda}{\partial z} \qquad \text{------------(2.103)}$$

Finally substituting for $\frac{\partial\lambda}{\partial x}$ and $\frac{\partial\lambda}{\partial z}$ from equation (2.78) and equation (2.79) gives

$$\frac{-\,f}{f'(c^2+\lambda)} = \frac{1+ C\, I\,(\lambda)}{2CI'(\lambda)(a^2+\lambda)}$$

$$+ \frac{(a^2+\lambda)\,(c^2+\lambda)}{(\lambda - \mu)}\left[\frac{x^2}{(a^2+\lambda)^2} + \frac{z^2}{(c^2+\lambda)^2}\right] \qquad \text{------------(2.104)}$$

comparing with the two dimensional case when $y = 0$ and $\nu = -b^2$ yields

$$\frac{x^2}{(a^2+\lambda)^2} + \frac{z}{(c^2+\lambda)^2} = \frac{\lambda - \mu}{(a^2+\lambda)(c^2+\lambda)} \qquad \text{------------(2.105)}$$

where
$$\frac{f}{f'(c^2+\lambda)} + \frac{1 + C\, I(\lambda)}{2C\, I'(\lambda)(a^2+\lambda)} = -\,1 \qquad \text{------------(2.106)}$$

and equation (2.84) becomes

$$\frac{f'}{f} = -\left(\frac{a^2+\lambda}{c^2+\lambda}\right) \frac{2C\, I'(\lambda)}{[1 + C\, I(\lambda) + 2CI'(\lambda)(a^2+\lambda)} \qquad \text{------------(2.107)}$$

To solve this equation we need to use numerical methods for the evaluation of the integral $I(\lambda)$. Thus for a general triad (a,b,c) calculation of f and therefore $\psi = z^2 f$ is laborious and time consuming. However for some specific values of a,b and c equation (2.106) has been evaluated and the results are used in chapter 3.

Chapter Three

3.0 Comparison of Velocities

To compare horizontal velocity profiles around two dimensional hill shapes and horizontal velocity profiles around three dimensional hill shapes, the ratio

$$r(x,z') = \frac{\text{horizontal velocity at } (x,z') \text{ in three dimensional case}}{\text{horizontal velocity at } (x,z') \text{ in two dimensional case}} ,$$

is used where z' is the height above the surface of the hill as defined by equation (2.11). At the surface, i.e. $z' = 0$, the expression for

$$\frac{U_3}{U_2} = r(x,0) = r_o = U[1 + C\, I(0)] \frac{a^2(a^2 - x^2}{a^4 - x^2(a^2 - c^2)} - \frac{a(a+c)(a^2-x^2)U}{a^4 - x^2(a^2 - c^2)}$$

$$= [1 + C\, I(0)] \frac{a}{a+c} \text{ ------------------------------------ } (3.1)$$

It may be noted from equation (3.1) that r_o is independent of x. Similarly from equation (2.15) and equation (2.93) the ratio $\frac{v_3}{v_2}$ of the surface vertical velocities may be obtained as

$$\frac{v_3}{v_2} = [1 + C I(0)] \frac{a}{a+c} \text{ ------------------------------------ } (3.2)$$

We note that the right hand side of equations (3.1) is equal ot the right hand side of equation (3.2). Thus

$$\frac{v_3}{u} = \frac{v_2}{u_2} = \text{ the slope of the surface ---------------------- } (3.3)$$

This is in agreement with the initial assumption that the flow is parallel to the surface of the hill.

$$\text{Let} \quad \beta = \frac{a}{c} ; \ \gamma = \frac{b}{c} ; \ \gamma' = \frac{b}{a} = \frac{\gamma}{\beta} \text{ -------------------------- } (3.4)$$

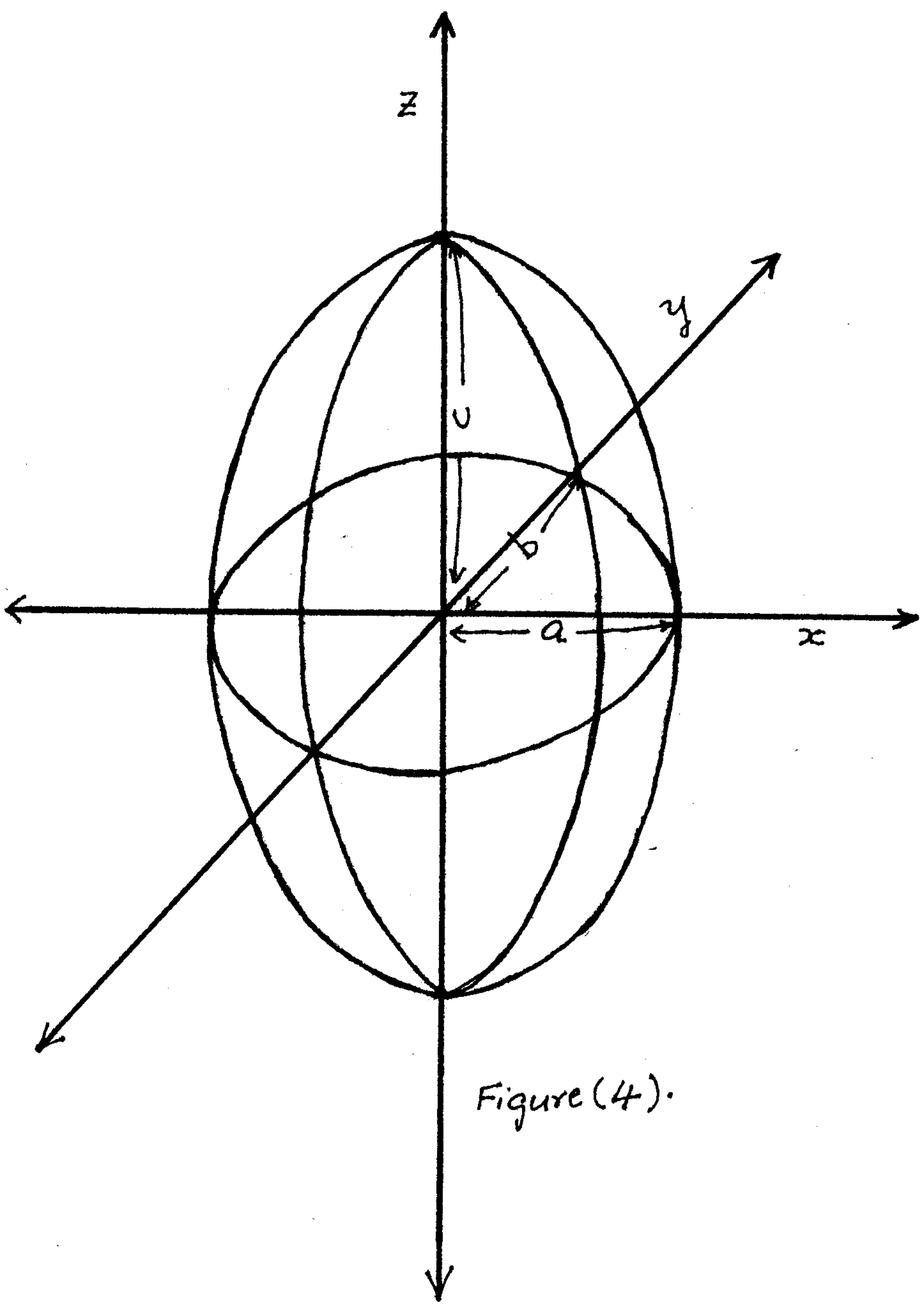


Figure (4).

c is a length and therefore β_1 and γ^1 are then dimensionless quantities. Vertically above the hills i.e at all points $(0,z')$ the expression of $\frac{v_3}{v_2}$ may be obtained from equation (2.95) and equation (2.16). This is given by

$$\frac{v_3}{v_2} = r(0,z) = \frac{a}{a+c}(1 + C\, I(o)\, [\frac{b}{(b^2+z^2+c^2)^{\frac{1}{2}}}] \quad \text{-----}\ (3.5)$$

at $z = c$,

$$r(0,z) = (1 + C\, I(o))\frac{a}{a+c}$$

$$= r_o$$

This is one of the special values mentioned in the previous chapter.

As $z \to \infty$, $r(0,z) \to 0$. It is desirable however that as $z \to \infty$, $r(0,z) \to 1$ since the mainstream velocity U is appreached in this limit for both two dimensional and three dimensional cases.

In general it is desirable that the expression for r satisfies the following:-

(i) $r(x,0) = r_o$ if $|X| < a$

i.e r constant over the surface

(ii) $\lim_{z' \to \infty} r(x,z') = 1$

(iii) from equation (3.1) we require $I(o) \to 0$ as $b \to 0$ and $r_o \to \frac{a}{a+c} = \frac{\beta}{1+\beta}$

(iv) as $b \to \infty$, $u_3 \to u_2$ implying $r \to 1$,

i.e a flat plate, see figure (4)

From the expressions for the surface ratio r_o and the ratio of vertical velocities the expression

$$r = 1 - Q(x, \beta, \gamma')\sigma(\beta_1, z')$$

where $Q(\beta, \gamma') = \dfrac{1}{1 + \gamma' + [1 + 8/3\,\gamma' + \frac{1}{3}(\gamma')^2]\beta}$

$$\sigma(\beta_1 \gamma', z') = e^{-\alpha(x, \beta, \gamma')z'}$$

and $\alpha(\beta, \gamma') = \dfrac{1}{\frac{1}{2} + 4/5\beta + 2\gamma'(\frac{1}{4} + \frac{1}{3}\beta)}$, was proposed by Deaves [6] and found to agree to within 1.5% and 2% even though the shapes were not ellipses.

3.1 Comparison of Streamlines

To compare streamlines in the two dimensional case with stream-surfaces in the three dimensional case the ratio

$$\gamma_\psi = \frac{z_o^{(3)} - z_\infty}{z_o^{(2)} - z_\infty} \qquad (3.6)$$

is used where $z_o^{(3)}$ denotes the height of a stream surface over the centre line and $z_o^{(2)}$ denotes the height of a streamline along $x = 0$. z_∞ being the height of the same streamsurface and streamline respectively at inifinity, i.e away from the hill. The difference $z_o^{(i)} - z_\infty$, $(i = 2, 3)$, is the displacement in streamlines in the two dimensional hill and the displacement in streamsurfaces in the case of a three dimensional hill.

Let us now consider an ellipse with parameters a and c, such that $\frac{a}{c} > 1$ say; the corresponding three dimensional shapes with parameters a, b and c, $0 < b < \infty$, are the following cases:

case (1) Taking $b = 0$, the ellipsoid reduces to a flat elliptical plate. In this case there is no displacement and $z_o^{(3)} = z_\infty$. Equation (3.6) reduces to

$$\gamma_\psi = 0 \quad\text{--} \quad (3.7)$$

case (2) when there is axial symmetry. Taking $b = c$ the ellipsoid reduces to a circular shape in the $y - z$ plane. From equation (3.4) we obtain $\gamma = 1$.

case (3) $b = a$; this gives a circular mound in the x-y plane and using equation (3.4) we have $\gamma^1 = 1$.

case (4) As $b \to \infty$; we have a limiting case $z_o^{(3)} \to z_o^{(2)}$ and $\gamma_\psi \to 1$.

It may be noted from the cases above that $0 < \gamma_\psi < 1$. For measured circular hill shapes $b = a$, $\gamma^1 \simeq 1$ and $\beta \geq 5$, so that $\gamma_\psi > \frac{1}{2}$. The displacement of streamlines for the three dimensional case will tend to those obtained in the two dimensional case. The computed results of Deaves [6] show that in the upper boundary the displacement in streamlines is smaller in the three dimensional case than in the two dimesional case and that there is a smaller departure of the streamlines in three dimensional case from their two dimensional counterpart in turbulent boundary layer type flow than in potential flows.

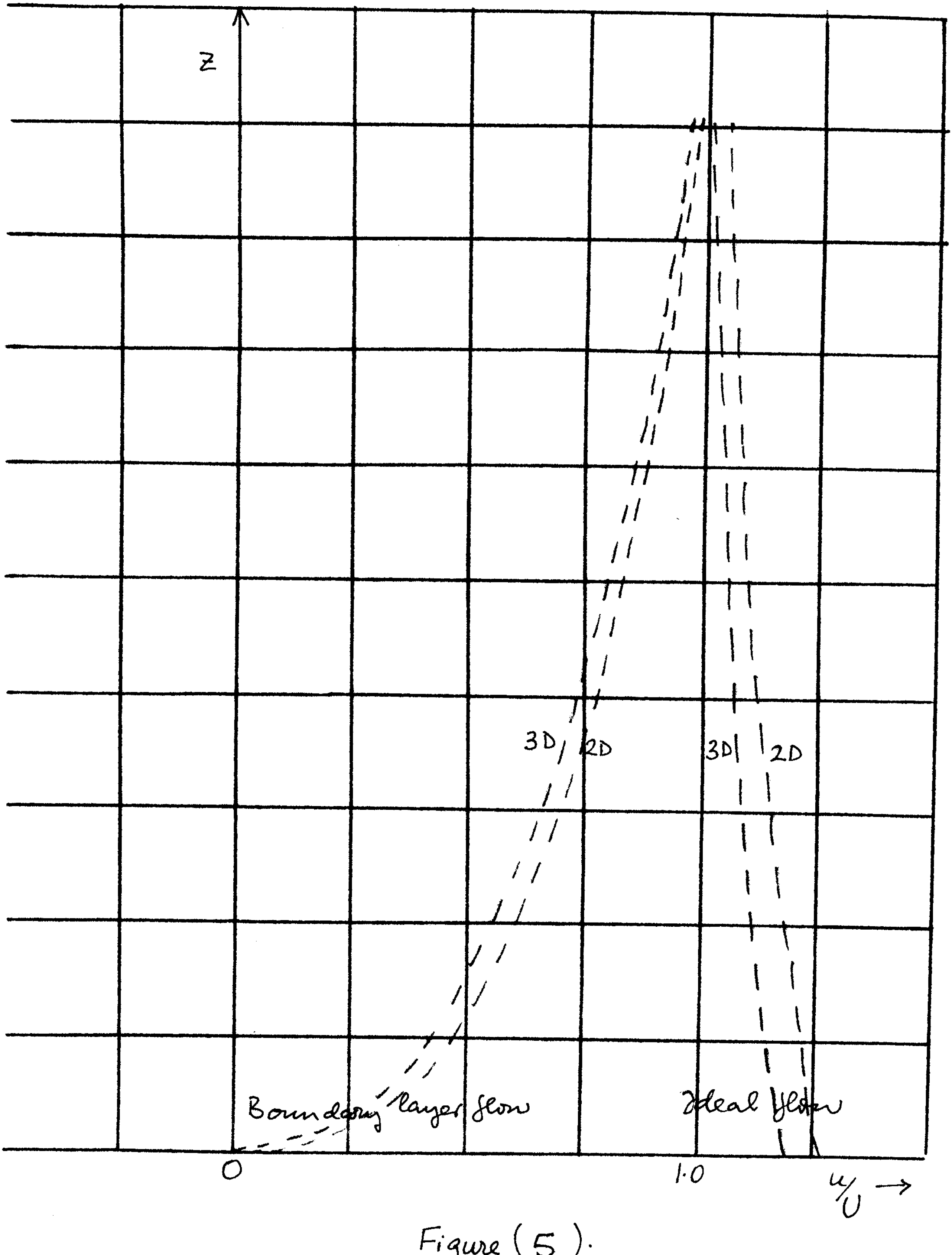


Figure (5).

3.2 Practical Applications

Let us suppose that the ratios

$$r = \frac{u_3}{u_2} \quad \text{--} \quad (3.8)$$

which have been obtained for ideal flow can be applied to the boundary layer. We note, however,, that:-

(i) The velocity at the surface in viscous flow is zero but the velocity at the surface in ideal flow is not.

(ii) The velocity in boundary layer flow is a maximum at the point which represents the top of the hill.

(iii) In ideal flow the velocity defect between the three dimensional and the two dimensional profiles is a maximum at the surface and slowly approaches zero at infinity.

(iv) In boundary layer flow the velocity defect between three dimensional and two dimensional profiles is zero at the surfaces maximal at a low height and then slowly tends to zero at infinity. See figure (5) after Deaves.

The points (i) - (iv) must be taken into account when extending the results obtained in this thesis to boundary layer flow.

In Chapter 2 the streamlines in three dimensional ideal flow have been shown to lie below the corresponding two dimensional ideal flow. The shift in streamlines in three dimensional boundary layer flow compared to the two dimensional

dimensional boundary layer flow is examined by writting the stream function $\psi(z)$ as

$$\psi(z) = \int_{z_o}^{z} u\,dz \qquad (3.9)$$

where $\psi = 0$ at z_o for continuity of flow.

Let u represent the horizontal velocity in boundary layer flow and U represent the horizontal velocity in ideal flow at the same point. Thus

$$\delta = \int_0^\infty \left\{ \frac{U - u}{U} \right\} dz \qquad ((3.10)$$

is a measure of the shift in streamlines between the ideal and boundary layer flows.

Let

$$\delta_2 = \int_0^\infty \left\{ \frac{U_2 - u_2}{U_2} \right\} dz \qquad (3.11)$$

represent the shift in the two dimensional case and

$$\delta_3 = \int_0^\infty \left\{ \frac{U_3 - u_3}{U_3} \right\} dz \qquad (3.12)$$

represent the shift in the three dimensional case, then

$$\delta_3 - \delta_2 = \int_0^\infty \frac{u_2}{U_3} \left\{ \frac{U_3}{U_2} - \frac{u_3}{u_2} \right\} dz. \qquad (3.13)$$

Using equation (3.8), we have

$$\frac{U_3}{U_2} - \frac{U_3}{u_2} = 0$$

and $\delta_3 = \delta_2$ --(3.14)

The physical meaning of equation (3.14) is that there is the same shift in streamlines in boundary layer flow as in ideal flow, that is the same **lowering** of streamlines whereby the three dimensional boundary layer streamlines will lie below the corresponding two dimensional boundary layer streamlines. A very notable difference is that the atmospheric boundary layer extends to several hill heights although here the boundary layer thickness has been assumed small compared with the dimensions of the obstacle so that the velocity at the outside edge of the boundary layer is assumed equal to the velocity at the surface.

Conclusion

In nature clouds provide the most visible evidence of disturbances to the airflow caused by mountainous terrain lying across an airstream. These disturbances are not always accompanied by clouds. The behaviour and movement of smoke and dust can also reveal disturbances to the airflow caused by the presence of mountains. In an aircraft the rate-of climb indicator and the altimeter can provide evidence of disturbances to the airflow.

Mathematical expressions for the changes in streamline pattern and velocity distribution caused by the presence of a hill of specified shape with gentle incline, i.e, where the streamlines of the air currents are practically parallel to the underlying surface, have been deduced first using the source - sink model and then secondly using the flow over ellipsoids. The source - sink model was found unsuitable because the three dimensional case does not reduce to the two dimensional model easily. However using the known flow around a sphere and circular cylinder the flow around the ellipse and ellipsoids was found. In this case however the three dimensional model reduces to the two dimensional model as shown in appendix I.

Difficulties in the practical applications of the model to the atmospheric boundary layer have been mentioned. The assumptions of a basic wind independent of time, having constant density everywhere and the motion being in two dimensions, i.e barotropic motion, is mainly introduced in order to make the mathematics simple. The nature of the airflow over a mountain is dependent on the dynamic properties of the airstream. It is however clear from this study that the changes in streamline pattern and velocity distribution in ideal flow resemble remarkably well the flow caused by the presence of a hill of specified shape with gentle incline in the atmospheric boundary layer where the streamlines of the air currents are almost parallel to the underlying surface.

A P P E N D I X

Appendix I

To show that the three dimensional potential reduces to the two dimensional potential i.e to show that

$$\lim_{b \to \infty} \quad Ux[1 + C\, I(\lambda)] = Ux + \frac{Uc}{2}(a+c)\, R\left(\frac{1}{\zeta}\right),$$

where

$$I(\lambda) = \int_{\lambda}^{\infty} \frac{abc\, dt}{(a^2 + t)[(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)]^{\frac{1}{2}}}\,.$$

As $b \to \infty$, i.e for a two dimensional hill,

$$I(\lambda) \to \int_{\lambda}^{\infty} \frac{ac\, dt}{(a^2 + t)^{3/2}(c^2 + \lambda)^{\frac{1}{2}}}\,. \quad \text{------------------(i)}$$

Let $\quad \tan^2\theta = \dfrac{c^2 + t}{a^2 - c^2}$ and $\sec^2\theta = \dfrac{a^2 + t}{a^2 - c^2}$

Then

$$I(\lambda) = \int_{\lambda}^{\infty} \frac{ac\, dt}{(a^2+ t)^{3/2}(c^2+ t)^{\frac{1}{2}}}$$

$$= \int_{\theta=\theta_o}^{\theta=\pi/2} \frac{2ac\, \cos\theta\, d\theta}{a^2 - c^2}$$

$$= \frac{2ac}{a^2 - c^2}[\sin\theta]^{\pi/2}$$

$$I(\lambda) \to \int_{\lambda}^{\infty} \frac{ac\, dt}{(a^2+t)^{3/2}(c^2+\lambda)^{\frac{1}{2}}} = \frac{2ac}{a^2 - c^2}\left[1 - \sqrt{\frac{c^2+\lambda}{a^2+\lambda}}\right]$$

From section (2.3)

$$C = \frac{1}{2 - I(o)}$$

$$= \frac{a + c}{2a}$$

Then

$$\text{limit } \phi = Ux + \frac{Ucx}{a-c}\left(1 - \sqrt{\frac{c^2 + \lambda}{a^2 + \lambda}}\right) \text{ ----------------(ii)}$$

In the two dimensional case the expression for ϕ if given by

$\phi = Ux + \frac{Uc(a+c)}{2} R\left(\frac{1}{\zeta}\right)$, where ζ is as defined in section ( ).

It now remains to show that

$$Ux + \frac{Ucx}{a-c}\left(1 - \sqrt{\frac{c^2+\lambda}{a^2+\lambda}}\right) = Ux + \frac{Uc}{2}(a + c) \; R\left(\frac{1}{\zeta}\right),$$

i.e to show that

$$\frac{x}{a - c}\left(1 - \sqrt{\frac{c^2+\lambda}{a^2+\lambda}}\right) = \frac{a + c}{2} \; R\left(\frac{1}{\zeta}\right) \text{ ------------------(iii)}$$

Let $\rho = \rho e^{i\sigma}$ then

$$R\left(\frac{1}{\zeta}\right) = \frac{1}{\rho} \cos \sigma$$

$$= \frac{x}{\rho^2 + \left(\frac{a^2 - c^2}{4}\right.}$$

so that

$$x = \left(\rho + \frac{a^2 - c^2}{4\rho}\right)\cos \sigma \quad \text{and} \quad z = \left(\rho - \frac{a^2 - c^2}{4\rho}\right) \sin\sigma, \text{ -----(iv)}$$

From equation (iii) and equation (iv) it is now required to prove that

$$\sqrt{\frac{c^2+\lambda}{a^2+\lambda}} = \frac{\rho^2 - \frac{a^2-c^2}{4}}{\rho^2 + \frac{a^2-c^2}{4}}$$

i.e to show that

$$\lambda = \rho^2 + \frac{a^2+c^2}{2} + \left(\frac{a^2-c^2}{4\rho^2}\right) \text{-------------------------} (v)$$

From section (2.4)

$$\lambda = \frac{\Sigma}{2} + \sqrt{\left(\frac{\Sigma^2}{4} + \Pi\right)}$$

where

$$\Sigma = x^2 + z^2 - a^2 - c^2 \quad \text{and} \quad \Pi = a^2c^2 - c^2x^2 - a^2z^2$$

Equation (v) then becomes

$$\frac{x^2+z^2}{2} + \sqrt{\left(\frac{\Sigma^2}{4} - \Pi\right)} = \rho^2 + \left(\frac{a^2-c^2}{4\rho^2}\right) \text{----------------} (vi)$$

From equation (iv)

$$X + Y = x + z$$

and $XY = \left(\frac{a^2-c^2}{2}\right)\left[x^2 - z^2 - \left(\frac{a^2-c^2}{2}\right)\right]$

where $X = \rho^2 + \left(\frac{a^2-c^2}{4}\right)^2 \frac{1}{\rho^2}$

and $Y = \frac{a^2-c^2}{2} \cos 2\sigma$

From which we get

$$X = \frac{x^2 + z^2}{2} + \{(\frac{x^2 + z^2}{2}) - \frac{a^2 - c^2}{2})[x^2 - z^2 - (\frac{a^2 - c^2}{2})]\}^{\frac{1}{2}}$$

$$= \frac{x^2 + z^2}{2} + \surd(\frac{\Sigma^2}{4} - \pi) \quad \text{---------(Q.E.D.}$$

This is the same as equation (vi).

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