NUMERICAL METHODS FOR THE SOLUTION OF VARIATIONAL PROBLEMS OF PHYSICS AND ENGINEERING

KRISHAN C. ANAND

DISSERTATION FOR
M.Sc. (MATHEMATICS)
WITH COMPUTER SCIENCE
NOVEMBER 1976

UNIVERSITY OF ZAMBIA
LUSAKA

CONTENTS

PREFACE

Most of the significant problems of Physics and Engineering admit formulations as conventional boundary value problems or equivalent problems in Variational Calculus. The great majority of physical laws governing processes in these two areas can be stated as optimization of related functions.

The solutions of Variational Problems can be approached indirectly or directly. The indirect approach leads to the solution of ordinary or partial differential equations which are seldom in practice solved in closed form and are, therefore, generally subjected to numerical attack. The Numerical Methods in the solutions of differential equations are complex. In view of this a great deal of current research in Engineering and Physics has centered around effective numerical methods for the solution of more pressing problems in Physics and Engineering in their variational form.

Ritz's paper of 1908 marks the starting point of this line of research and since then many problems that have hitherto defied solution have been effectively solved. Since then refinements and alternate numerical techniques have been devised. These methods have become specially powerful and speedy with the advent of digital computers. This dissertation stresses the understanding of types of methods or solutions that can be applied to digital computers rather than to the actual mechanics of routine programming.

By way of illustration this dissertation attempts the solution of a specific set of problems of a thin plate fixed at the edges and under load distributions of various kinds. This is formulated as a Variational Problem and solutions are constructed using one of the techniques under survey. The functionals in question are separated into two parts, (i) The plate characteristic part which remains the same for all load distributions and (ii) The load characteristic part. The minimising equations $\frac{\partial J}{\partial \alpha} = 0 = \frac{\partial J}{\partial \beta}$ etc. throw up a system of linear equations of the form:

$$M(\underline{q}) = (\underline{c})$$

where the coefficient matrix 'M' is the same for all loading distributions and the constant $\underline{c}$ is dependent on the load. 'M' is computed exactly by a systematic integration procedure which can also be adopted to cope with load distributions of many general kinds. In the distributions examined print outs of curves of constant deflection are provided for purposes of comparison.

Variational procedures described and illustrated in this dissertation are applicable to a surprisingly wide range of real problems that occur in Physics and Engineering for example: Vibration problems in electrical circuits and in machinery, deflection problems in rigid structures, equilibrium and stability problems in complex flexible and not so flexible structures, quantum mechanics and in control theory.

The questions of (a) the existence of solutions (b) the convergence of iterative numerical procedures used are central to this subject. A general theory based on these important questions presents considerable difficulties that start from the great variety of possibilities that exist. These questions are refered to in the course of this dissertation. (Ch. IV) distributions of various kinds. This is formulated as a Variational Problem and solutions are constructed using one of the techniques under survey. The functionals in question are separated into two parts, (i) The plate characteristic part which remains the same for all load distributions and (ii) The load characteristic part. The minimising equations $\frac{\partial J}{\partial \alpha} = 0 = \frac{\partial J}{\partial \beta}$ etc. throw up a system of linear equations of the form:

$$M(\underset{\sim}{q}) = (\underline{c})$$

where the coefficient matrix 'M' is the same for all loading distributions and the constant $\underline{c}$ is dependent on the load. 'M' is computed exactly by a systematic integration procedure which can also be adopted to cope with load distributions of many general kinds. In the distributions examined print outs of curves of constant deflection are provided for purposes of comparison.

Variational procedures described and illustrated in this dissertation are applicable to a surprisingly wide range of real problems that occur in Physics and Engineering for example: Vibration problems in electrical circuits and in machinery, deflection problems in rigid structures, equilibrium and stability problems in complex flexible and not so flexible structures, quantum mechanics and in control theory.

The questions of (a) the existence of solutions (b) the convergence of iterative numerical procedures used are central to this subject. A general theory based on these important questions presents considerable difficulties that start from the great variety of possibilities that exist. These questions are refered to in the course of this dissertation. (Ch. IV)

ACKNOWLEDGEMENTS

I would like to thank the teaching staff of the Department of Mathematics of the School of Natural Sciences of the University of Zambia and in particular Dr. Murthy, Dr. Ciampi, Dr. Emenalo and Dr. Mweene for the constant assistance and various facilities provided to me.

My special thanks go to my project supervisor Professor E. F. Bartholomeusz whose guidance and encouragement was invaluable and to Mrs. Bartholomeusz for the indispensable translation of the Ritz's paper of 1908 from the original version in German.

My thanks go to the operating staff of the Computer Centre of the University of Zambia for their timely and reliable service and to Mr. Kasaila for doing the typing of the manuscript.

CHAPTER - I

INTRODUCTION AND HISTORICAL SURVEY

The Calculus of Variations was founded in 18th century by Euler and Lagrange and extensively developed by mathematicians in 19th and 20th centuries. It is assuming an important role in the fields of Analysis, Physics and Engineering. The variational calculus originated as a study of certain isolated optimization problems not solvable by the techniques of elementary calculus. It provides us at present with powerful methods for the solutions of problems in dynamics and statics of rigid bodies, general elasticity, the theory of plates and shells, optics, quantum mechanics, optimization of orbits and controls, determination of natural frequencies of vibrating systems and the study of vibrations in general including electrical oscillations.

1.1 Definition

Concept of Functionals:- The calculus of variations originated from the endeavour to determine extrema or stationary values for functionals. A functional is defined as a quantity or function which depends upon the entire course or path on one or more functions rather than on a number of discrete variables. In general any operator $J(\phi)$ on a class of given functions and their derivatives in a given domain D forms a functional. Therefore, when studying a functional and its extrema we must stipulate the domain of definition of the functional, i.e. the class of functions for which the values of the functional are defined. The class of this type is a linear space consisting of functions on which linear operations are performed. These spaces are called functional spaces.

Example:- A functional of y and z subject to given boundary conditions may look like

$J(y,z) = \int F(x,y,z,y',z')dx$, where F is continous function of its five arguments x,y,z,y',z'.

1.2 Historical Variational Problems:

Some problems had considerable influence on the development of the variational calculus. An ancient problem called the problem of Dido was to find a simple rectifiable closed curve (i.e. in parametric representation $x(t)$, $y(t)$ are continuously differentiable) which encloses the largest area. This is referred to as an isoperimetric problem and the curve was found to be a circle.

The above problem consists of finding the extrema of the functional $J[x,y] = \int_{t_o}^{t_1} (x\dot{y} - y\dot{x})dt$ under the additional condition that the length of curve is constant i.e. the functional

$$L = \int_{t_o}^{t_1} \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}\, dt$$

is kept constant in the space of piecewise smooth functions $x(t)$, $y(t)$ in (t_o, t_1).

Another classical problem called the Brachistochrone problem consists in finding which curve joining two given points A and B, not lying on the same vertical line, has the property that a massive particle sliding down this curve from A to B reaches B in the shortest time. The curve of quickest descent turned out to be a cycloid.

Yet another classical problem was the general problem of Geodesics in any Metric point space. Its 3-D version is based on finding the geodesics on a given surface (x,y,z) = 0, that is, of joining two points on the surface with coordinates (x_o, y_o, z_o) and (x_1, y_1, z_1) by the shortest possible curve lying in the surface. This involved finding the minimum of the functional

$$l = \int_{x_o}^{x_1} \sqrt{1 + y'^2 + z'^2}\, dx$$

where functions y(x) and z(x) must satisfy condition $\phi(x,y,z) = 0$.

An interesting navigation problem was posed by Zermelo and is briefly described here:-

In an unbounded plane where the wind distribution is given by a vector field as a function of position and time, a ship moves with constant velocity relative to the surrounding air mass. The problem centred on the determination of the steering direction of the ship in order to come from a starting point to a given goal in the shortest time. Let x_1, x_2 denote the rectangular co-ordinates of the moving ship, U_1, U_2 the corresponding wind velocity components, x_3 the time and ϕ the 'steering direction'. ϕ is the angle which the vector of the relative velocity forms with the x_1 direction. Every motion of the ship can be represented by defining the three co-ordinates x_i as a function of a parameter 't' and the following conditions must be satisfied:-

$$\frac{\mathring{x}_1}{\dot{x}_3} = U_1(x_1, x_2, x_3) + \cos\phi \; ; \; \frac{\dot{x}_2}{\dot{x}_3} = U_2(x_1, x_2, x_3) + \sin\phi \, .$$

The necessary and sufficient condition for each point of the path to correspond to at least one steering direction ϕ , by whose observation the path is navigated, reads

$$(\dot{x}_1 - U_1\dot{x}_3)^2 + (\dot{x}_2 - U_2\dot{x}_3)^2 - \dot{x}_3^2 = 0$$

and the integral to be minimized has the form $I = \int \dot{x}_3 dt$. From the navigation formula given by Zermelo the desired relation was found to be

$$\frac{d\phi}{dt} = -\frac{\partial U}{\partial x^2}$$

The steering must always be turned toward the side according to which the wind component acting against the steering direction becomes greater.

CHAPTER II

THE GENERAL STRUCTURE OF BOUNDARY VALUE PROBLEMS

The problem of integrating a differential equation under prescribed boundary conditions is known as a Boundary Value Problem. The meaningful formulation of boundary-value-problems has dominated much of the work on partial-differential-equations of Mathematical Physics. A good deal of investigation has been carried into boundary-value-problems that yield solution, that are unique and continuously dependent on data.

2.1 Definition

Domain:- A domain G is a set of points in space which is characterized by two properties:

(i) In a 3-D Euclidian Space, if some point P belongs to the domain, then all points sufficiently close to P also belong to the domain (Openness).

(ii) Any two points of the domain can be joined by a line entirely within the domain (Connectedness).

The Boundary:- The boundary of a domain is defined as the set of points in any neighbourhood of each of which there are both points belonging to the domain and points not belonging to it.

The unknown functions f(x) which satisfy a given condition (usually a partial differential equation over a domain G of the independent variable $\underline{x}$) and prescribed boundary conditions must be defined in a specific domain G. If the prescribed domain is a surface the function sought will be a function of two free variables and if the prescribed domain is a volume the function sought will be a function of three free variables.

The formulation of 'well-posed' Boundary-Value-Problems has dominated most of the work on Partial Differential Equations of Mathematical Physics and this work solely discusses such problems i.e. Boundary-Value-Problems that yield solutions, that are unique and continuously dependent on the data.

Notation:- $\underline{X} = (X_1, X_2, X_3, X_4, \ldots..)$ in a given domain.

$\underline{X} = (X_1, X_2, X_3)$ ----- 3D

$\underline{X} = (X_1, X_2)$ --------- 2D

$\underline{X} = X$ -------------- 1D

2.2

The linear Partial Differential Equations of Mathematical Physics are classified thus:-

(A) Elliptic-: The prototype has the form

$$\nabla^2 U(\underline{x}) = f(\underline{x}) \qquad \text{(Poisson's)}$$

for $f(\underline{x}) = 0$, above condition leads to Laplace's Equations.

(B) Hyperbolic-: The protoype in 3-D has the form

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) U = \frac{1}{C^2} \frac{\partial^2 U}{x_4^2}$$

for vibration problems the above equation reduces to

$$\nabla^2 U - \frac{1}{C^2} \frac{\partial^2 U}{\partial t^2} = 0$$

(C) Parabolic-: The prototype in 3-D has the form

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) U = C \frac{\partial U}{\partial x_4}$$

For diffusion and heat conduction problems the parabolic equation becomes

$\nabla^2 U = k \frac{\partial U}{\partial t}$; where 't' denotes time.

2.3 Classification of Domain $(G + \partial G)$

(i) By a closed ∂G we refer to one (whether it is finitely bounded or extends to ∞) on which boundary conditions are specified at all points including ∞, if such points exist on ∂G.

(ii) By an open ∂G we refer to one where the boundary is infinite but the boundary conditions are not specified at ∞.

2.4 Classification of boundary conditions of simpler type and of common occurrence in natural phenomenon

Boundary Conditions on ∂G

Cauchy B.C: $U = U(s)$ and $\frac{\partial U}{\partial n} = N(\underline{s})$ are both given on ∂G.

Dirichlet B.C: $U = U(\underline{s})$ alone is specified on ∂G.

Neumann B.C: $\frac{\partial U}{\partial n} = N(\underline{s})$ alone is specified on ∂G.

Intermediate or Churchill : Some linear combination $U + \alpha \frac{\partial U}{\partial n} = C(\underline{s})$ is specified on ∂G.

In all cases if the specifying functions $U(s)$, N and C are $\equiv 0$ on ∂G, the corresponding boundary condition is said to be homogeneous otherwise they are said to be non-homogeneous.

CHAPTER III

THE GENERAL VARIATIONAL PROBLEM

Notation-:

(i) $\underline{x} = (x_1, x_2, \ldots, x_n)$

where $x_1, x_2, \ldots\ x_n$ are independent variables in a given (n-D) domain D.

(ii) F = The comparison function space $\{y(\underline{x})/\underline{x} \in D\}$ of a function $y(\underline{x})$ which together with their derivatives satisfies prescribed continuity and boundary conditions in D.

(iii) $\Phi[y(\underline{x})] = \Phi[y;\ y_{x_1}, y_{x_2} \ldots, y_{x_1x_1}, y_{x_1x_2} \ldots,]$

= a function of $y(\underline{x})$ and its derivatives w.r.t. $x_1, x_2, \ldots$.

3.1

It often happens that under given stimuli and prescribed constraints, natural systems can theoretically admit a whole class of compatible responses. Among the response of this set the particular member that constitutes the response of the system that actually occurs can often be identified by a principle that requires the minimization or maximization of a particular intrinsic characteristics (example: Potential Energy, Hamiltonian in Motion Dynamics ... etc) of the system. Such a principle when it is available is called a Variational Principle governing the process under consideration. In Mathematical terms such a principle, in most cases requires the optimization of a functional

$$J[y_1(\underline{x}), y_2(\underline{x}), \ldots] = \int_D \Phi[y_r(\underline{x})]dD, \quad (\Phi \text{ being specified}).$$

in the class of function $\{\{y_1(\underline{x})\}, \{y_2(\underline{x})\}, \ldots / \underline{x} \in D\}$ subject to specific continuity and boundary conditions on the functions and their derivatives over the working domain D .

Some of the Variational Principles of Physics are listed below.

Fermat's Principle of least Time-: Fermat's principle of least time in optics asserts that among all paths in a transparent medium from one given point 'A' to another point 'B' the natural trajectory of all light rays from 'A' and passing through 'B' is that particular path which is traversed in the least possible time.

Minimum Energy Principle-: This states that of all geometrically compatible displacements of a stable conservative structure under given constraints the particular displacement that occurs is that which minimizes the Potential Energy V.

Hamilton's Variational Principle-: For a system S passing from a given configuration '1' at time 't_0', to a second given configuration '2' at time 't_1', under specified constraints, the actual trajectory among a class of possible trajectories is that for which

$$\int_{t_0}^{t_1} (K.E - P.E)\,dt \quad \text{is stationary.}$$

K.E = Kinetic Energy
P.E = Potential Energy

Reissner's Variational Theorem of Elasticity-: It generalizes principle of stationary Potential Energy and simultaneously yields relationships between the stress and displacement components, the equilibrium equations and the boundary conditions.

Reissner's Variational Principle for elastic deformations demands the optimization of the functional

$$I = \iiint_V [\sigma_{ij} U_{i,j} - U_i F_i - \phi_o]dV - \iint_{S_1} U_i \cdot p_i dS - \iint_{S_2} (U_i - \overline{U}_i) p_i dS$$

Where U_i = displacement tensor, σ_{ij} = stress tensor. The elastic body occupies a region V in space and 'S' denotes its surface separated into S_1 and S_2, ϕ_o is the energy density and $F_i (i = 1, 2, 3)$ is the body force per unit volume. On 'S_1' the stress vector components p_i

are specified while on 'S$_2$' the displacement vector components $U_i (i = 1, 2, 3)$ are specified.

Reissner's theorem states that the equilibrium state of a body is such that $\delta I = 0$ for independent variations of U_i and σ_{ij}. From the condition $\delta I = 0$, the differential equations of equilibrium, boundary conditions and stress displacement relations are deduced.

CHAPTER IV

THE EXISTENCE AND UNIQUENESS OF SOLUTIONS TO VARIATIONAL PROBLEMS

In the discussion so far it was assumed (from the beginning) that the given equation say $AU = f$ admits **a unique solution possessing finite** energy. A solution procedure for the problem of locating the minimum of any functional was without reference to the important questions of existence and uniqueness of solutions which together are the decisive characteristics of a well posed Boundary Value Problem or a Variational Problem. The analysis of this topic is quite complex since a given Variational Problem may not have a unique solution along with another possibility that the solution found may not belong to the given function space. A solution may exist to the given Boundary Value Problem but the boundary value problem may not have an associated functional to be maximised. The existence of solution to a Variational Problem in general implies the existence of the solution to the equivalent boundary value problem. It is possible to lay down some conditions for the uniqueness and existence of the solutions to variational problems after setting up a machinery of definitions.

4.1

Defn. The Scalar Product-: In a given domain D, the scalar or inner product of two functions U(P) and V(P) is denoted as

$$(U, V) = \int_D U(P) \cdot V(P)dD .$$

The symbolic form (AU, U) stands for the scalar product of 'U' and 'AU', in the field of definition of operator A. If the boundary conditions are specified, then (AU, U) is called the energy product, symbolically denoted by [U,U], of the functional 'U' with itself. The energy product has the features of a conventional scalar product in a finite dimensional vector space.

Defn. Positive-Definite Operator-: A symmetric operator 'A' is said to be positive definite, if for any function from its field of definition, U(P) not identically zero, and the inequality

$$(AU, U) > 0 \qquad \text{is valid.}$$

Defn. Positive Regular Variational Problem-: In order that a positive definite Variational Problem whose basic function $F(x_i , \dot{x}_i)$ is uniquely defined for all line elements $(x_i , \dot{x}_i)$ of a certain domain, may be regular, it is necessary and sufficient that the function $F_1(x_i , \dot{x}_i)$ be different from zero throughout.

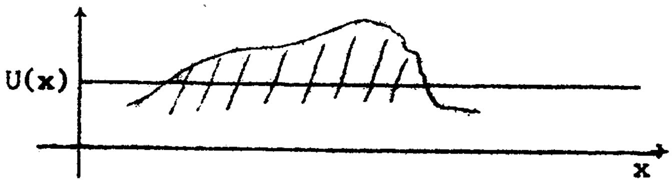
Defn. Positive-Bounded-Below Operator-: This concept is of considerable importance in the discussion of the existence of solution to a Variational Problem. A symmetric operator 'A' is said to be positive-bounded-below if for any function $U \in D_A$ the inequality

$$(AU,U) \geq \nu^2 ||U||^2 \quad \text{is valid, where } '\nu' \text{ is a +ve constant.}$$

A positive bounded below operator is always positive definite since inequality $(AU,U) > 0$ is always true for such operators.

Example-: Operators $[B]U = -\dfrac{d^2U}{dx^2}$ and $LU = -\dfrac{d}{dx}(x^3 \dfrac{dU}{dx})$ are positive bounded below.

Physical Interpretation of a Positively Bounded Below Operator-: The displacement at a point which a physical system experiences can be compared with the norm of the displacement. In the picture below it can be seen that norm of U(x) provides a scale of largeness against which U(x) at each point can be compared.



If $U(x) \geq ||U||$ at a point, then $U(x)$ is large at that point. A displacement is large if it is greater than the norm and small if it is less than the norm. (Therefore relatively large displacements which do not effect the norm are not of interest.) The inequality

$$(AU,U) \geq \nu^2||U||^2, \quad \nu \text{ a +ve constant},$$

implies that a large displacement is possible only by a great expenditure of energy. If the operator is positive definite but not positively bounded below it is possible to give the system an arbitrarily large displacement with expenditure of small amount of energy.

4.2

It has been emphasised that no unified theory exists for the existence and uniqueness of solutions to Variational Problems. This necessitates the classification of basic theorems and results for specific set of problems. The equivalence (if it exists) of Boundary Value Problems and Variational Problems is of much interest since it can allow the extension of the results for Variational Problems to boundary value problems and vice versa. A Boundary Value Broblem for homogeneous boundary conditions has the form $AU = f(P)$ where $f(P)$ is a known function assumed to be finite throughout. The uniqueness of the solution of $AU = f(P)$ is ensured if the operator 'A' is positive definite.

AN EQUIVALENCE THEOREM (For Variational and Boundary Value Problems)

A fundamental theorem based on the equivalence between Variational Problems and Boundary Value Problems is given below.

Theorem 1 If the equation $AU = f$ has a solution and operator 'A' is positive definite then of all values, which are given to the quadratic functional

$$F(U) = (AU,U) - 2(U,f) = \int_D [U(P)AU - 2U(P)f(P)]dD \; *$$

by all possible functions from the field of definition D_A of the operator A, the least is the value given to this functional by the solution of

$$AU = f(P)$$

and conversely if there exists in the field of definition D_A of the given functional, a function which gives a minimal value to the functional * , then this function is the solution of $AU = f(P)$.

It follows from the last theorem that if $AU = f$ has a solution of finite energy then the problem of the minimisation of the functional

$$F(U) = (AU,U) - 2(U,f) = [U,U] - 2(U,f)$$

has a solution also. (c.f. ref 11 pp 75)*

4.3

Existence of a Solution to a Variational Problem-: The problem of the minimization of the functional for a Variational Problem in a given function space can be viewed from a local and a global point of view.

---

* The references for the proofs of the theorems are to the Bibliography.

Some of the fundamental theorems on the existence of a local and a global minimum are listed below.

7 EXISTENCE THEOREMS

Theorem 2-: If a Variational Problem is positive regular at all points of a closed domain G, then it is always possible to join two points A and B of the interior of G with each other by a regular extremal of the problem which lies entirely in the interior of G, if for these points a minimal sequence of curves exists whose distance from the boundary of G exceeds a fixed number ε and whose lengths lie below a fixed bound.

(c.f ref 2 page 320)

Theorem 3-: If a Variational Problem is positive regular at all points of the space x_i and if with

$$\dot{x}_1^2 + \cdots + \dot{x}_n^2 = 1 \quad \text{always;}$$

$$F(x_j , \dot{x}_j) > m ; \quad m > 0$$

then it is always possible to join two arbitrary points of the space by at least one regular extremal for which the absolute minimum of the Variational Problem is realized for the given boundary conditions.

Theorem 4-: For a Variational Problem that is positive regular in the entire n-dimension space, let the following conditions be fulfilled:

(1) For every pair of points P,Q of this space different from each other

$$\varepsilon(P,Q) > 0$$

(2) If we denote by $r(0,P)$ the distance of the origin '0' of coordinates from the point P, then as $r(0,P) \to \infty$ we also always have $\varepsilon(0,P) \to \infty$.

Then we can join every pair of points A and B of the space with at least one regular extremal which represents a solution of the problem of the absolute minimum.

(c.f ref 2 pp 322)

Theorem 5-: For a Variational Problem that is positive regular and positive definite in the entire space, the Boundary Value problem always has at least one solution if the second condition of therem 4 above is fulfilled.

(c.f ref 2. pp 324)

Theorem 6-: Under assumptions of theorem 4, there corresponds to each point P of the space numbers $\delta(P)$, s.t each extremal piece that begins at P and whose diameter is not greater than $\delta(P)$ is an absolute minimal.

(c.f ref.2 pp 328)

A fairly general theorem of existence of solution for Variational Problem, characterised by a class of operator 'A' , is given below.

Theorem 7-: The minimisation of a given functional F(U) can lead to a solution in the class of functions with finite energy only if the operator 'A' in the equivalent boundary value problem is positive definite and positive bounded below.

(c.f ref.11 pp 122)

Hence for a positive definite operator if equation $AU = f$ has a solution then this solution also minimises the functional

$$F(U) = (AU,U) - (U,f) - (f,U) .$$

Illustration-: (A Positive-Bounded-Below Operator)

Consider the operator $[B]U = -\frac{d^2U}{dx^2}$

where function U(x) are defined in $0 \leqslant x \leqslant 1$ and satisfy boundary conditions $U(0) = U(1) = 0$. This operator can be shown to be positive-bounded-below since $U(0) = 0$

$U(x) = \int_0^x U'(t).dt$; on applying Cauchy's Inequality to the integral

$$U^2(x) \leq \int_0^x 1^2 dt \int_0^x U'^2(t)dt = x \int_0^x U'^2(t)dt.$$

Since $0 < x < 1$, the latter inequality by changing the upper limit of integral to unity, gives

$U^2(x) \leq x \int_0^1 U'^2(t)dt.$ Integrating this inequality over $0 < x < 1$

$$\int_0^1 U^2(x)dx \leq \tfrac{1}{2} \int_0^1 U'^2(t)dt \quad \ldots\ldots\ldots\ldots \quad 'I'$$

By definition of norm $\int_0^1 U^2(x)dx = ||U||^2$ and for operator B considered

$\int_0^1 U'^2(t).dt = (BU,U)$; and inequality 'I' gives

$(BU,U) \geq 2||U||^2$

Thus operator B is positive-bounded-below and for this operator it is possible to take $\nu = \sqrt{2}$. Most operators in Mathematical Physics like

$$LU = -\frac{d}{dx}\left(x^3 \frac{dU}{dx}\right), \qquad 0 < x < 1$$

are positive - bounded - below.

Note-: It may happen that if an operator 'A' is defined for an insufficiently wide linear set then the problem of minimisation of functional can have no solution for that set. It can exist only if linear set say M and its operator are sufficiently extended.

Example-:

Consider operator $\nabla^4 w$ for the deflection of a square plate rigidly fixed at the edges with boundary conditions.

$$w\Big|_L = 0 = \frac{\partial w}{\partial n}\Big|_L$$

where 'S' is the surface bounded by 'L' . Let the operator $\nabla^4 w$ be such that all fourth order derivatives are continuous. If a normal load distribution $q(x,y)$ is applied which varies discontinuously then the problem reduces to the integration under boundary conditions of

$$\nabla^4 w = \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q(x,y)}{D}$$

This equation cannot have a solution in the field of operator ∇^4 defined above since one of $\frac{\partial^4 w}{\partial x^4}$, $\frac{\partial^4 w}{\partial x^2 \partial y^2}$, $\frac{\partial^4 w}{\partial y^4}$ is discontinuous. The solution to the equation $\nabla^4 w = \frac{q(x,y)}{D}$ can exist only if the field of definition of operator $\nabla^4 w$ is extended to include discontinuous fourth order derivatives.

A theorem of considerable importance in the general theory of partial differential equations is the Cauchy-Kowalewski theorem. The results of this theorem can be extended to Variational Problems by virtue of theorem 1.

Defn. Analytic Function-: Let 'λ' be a complex parameter varying over a domain D, then a generalized function f_λ is defined to be an analytic function of λ in D if f_λ is differentiable in D.

Theorem 8-:

Cauchy-Kowalewski Theorem-: Let there be given the following system of equations in the partial derivatives of unknown functions $U_1, \dots, U_N$ w.r.t. independent variables $t, x_1, \dots, x_n$

$$\frac{\partial^{n_i} U_i}{\partial t^{n_i}} = F_i\left(t, x_1, \dots, x_n, U_1, \dots, U_N, \dots, \frac{\partial^k U_j}{\partial t^{k_o}\, \partial x_1^{k_1} \dots \partial x_n^{k_n}} \dots, \dots\right) \; *$$

$$(i,j = 1,2,\dots, N \; ; \; k_o + k_1 + \dots + k_n = k \leq n_j \; ; \; k_o < n_j)$$

and for some value $t = t_o$ of t, prescribe the initial values of the unknown functions and of their derivatives w.r.t 't' of orders up to $n_i - 1$. In other words for $t = t_o$

$$\frac{\partial^k U_i}{\partial t^k} = \phi_i^{(k)}(x_1, x_2, \dots, x_n)(k = 0,1,2,\dots,n_i - 1)$$

The Cauchy Problem consists in finding a solution of system '*' satisfying the above initial conditions.

Cauchy-Kowalewski theorem states that if all the function F_i are analytic in some neighbourhood of the point $(t^0, x_1^0, x_2^0, \ldots, \phi_j^0, k_o, k_1, \ldots, k_n, \ldots)$, and if all the functions $\phi_j^{(k)}$ are analytic in some neighbourhood of the point $(x_1^0, x_2^0, \ldots, x_n^o)$, then the Cauchy Problem has a unique analytic solution in some neighbourhood of the point $(t^0, x_1^0, x_2^0, \ldots, x_n^0)$.

(c.f. ref. 12, pp 15)

The above theorem implies the uniqueness of the solution under the stated conditions. A Uniqueness theorem is based on the characteristics of operator 'A' and is given below.

Theorem 9-:

2 UNIQUENESS THEOREMS

Uniqueness Theorem-: A given equation $AU = f(P)$ cannot have more than one solution if the operator 'A' is positive definite.

Proof-: Let the equation $AU = f$ have two solutions U_1 and U_2 s.t. $AU_1 = f$ and $AU_2 = f$. If $\underline{U} = U_1 - U_2$ then by linearity of operator 'A'

$A(\underline{U}) = A(U_1 - U_2) = AU_1 - AU_2 = f - f = 0$.

$A(\underline{U})$ multiplied scalarly by $\underline{U}$ gives $(A\underline{U}, \underline{U}) = 0$ which is contrary to the condition that operator 'A' is positive definite, hence $U_1 = U_2$.

Theorem 10-: If two solutions of equation $P(\frac{\partial}{\partial x})U = 0$ coincide near the boundary of V, they coincide throughout V. If a solution of $P(\frac{\partial}{\partial x}) = 0$ vanishes near the boundary of V, it vanishes identically in V.

(c.f. ref 15, pp 165)

Conversely the element of the Hilbert space which realizes the minimum of the functional F(U) satisfies equation AU = f .

4.4 Convergence of Solutions-:

It was illustrated in this Chapter that a given Variational Problem can be reduced (under certain conditions) to a Boundary Value Problem and hence even the existence of the solutions can be investigated. However, this approach is not always effective and is greatly complicated by the fact that what is needed to solve a given Variational Problem is not a solution of the corresponding differential equation in a small neighbourhood of some point, but rather a solution in some fixed region R, which satisfies prescribed boundary conditions on the boundary of R. The difficulties inherent in this approach have led to the search for Variational Methods of a different kind known as Direct Methods (Ch.VII), which do not entail the reduction of a Variational Problem to a Boundary Value Problem.

Most of the direct methods are based on the problem of locating the minimum of a functional J[y] defined in a space 'F' of admissible functions y. The solution of a given Variational Problem by direct methods involves (1) construction of a minimizing sequence $\{y_n\}$; (2) proving that $\{y_n\}$ has a limit function $\underline{y}$ and (3) checking whether $\underline{y}$ belongs to the given function space or not.

Minimizing Sequences-: If a given functional $J[y]$ defined in a space F of admissible functions y is to be minimized then it must be assured that there exist functions in F for which

$$J[y] < +\infty$$

and $\inf J[y] = k > -\infty$; where the greatest lower bound is taken over all admissible y. By definition of k there exists an infinite sequence of functions

$$\{y_n\} = y_1 , y_2 , \ldots\ldots, \quad \text{called a minimizing sequence, s.t.}$$

$$\lim_{n \to \infty} J[y_n] = k .$$

If the sequence $\{y_n\}$ has a limit function $\underline{y}$, and if

$$J[\underline{y}] = \lim_{n \to \infty} J[y_n]$$

and $$J[\lim_{n \to \infty} y_n] = \lim_{n \to \infty} J[y_n]$$

then $$J[\underline{y}] = k$$

A convergence theorem used to establish the convergence (under some conditions) is given below.

Theorem 8-: If $\{y_n\}$ is a minimizing sequence of the functional $J[y]$, with limit function $\underline{y}$, and if $J[y]$ is lower semicontinuous at $\underline{y}$, then

$$J[\underline{y}] = \lim_{n \to \infty} J[y_n]$$

(c.f ref 10, pp 194)

Note-: $J[y]$ is said to be lower semicontinuous at $\underline{y}$ if for any $\varepsilon > 0$, there exists a $\delta > 0$ s.t.

$$J[\underline{y}] - J[y] < \varepsilon \; ; \text{ if } \|y - \underline{y}\| < \delta .$$

Most direct methods discussed in this dissertation involve the choice of a sequence of functions, in a given function space F, of the form

$$\phi_1, \phi_2, \phi_3, \quad - - - - - - - - \quad \text{(i)}$$

The set of linear combinations of the above functions are chosen in the form

$$\alpha_1\phi_1 + \dots + \alpha_n\phi_n \quad - - - - - - - \quad \text{(ii)}$$

where $\alpha_1, \alpha_2, \dots \alpha_n$ are chosen in such a way so as to minimize

$$J[\alpha_1\phi_1 + \alpha_2\phi_2 + \dots + \alpha_n\phi_n]$$

Theorem 9-: If the functional $J[y]$ is continuous and if the sequence (i) is complete then

$$\lim_{n \to \infty} U_n = U$$

where $U = \inf_{y} J[y]$

(c.f ref 10, pp 196)

Note-: The sequence (i) is said to be complete (in F) if $\forall y \in F$, and any $\varepsilon > 0$, there is a linear combination y_n of the form (ii) s.t.

$$\|y_n - y\| < \varepsilon \qquad (\text{'n' depends on } \varepsilon)$$

The convergence of each direct method is discussed in Chapter VII.

CHAPTER V

EQUIVALENCE OF VARIATIONAL AND BOUNDARY VALUE PROBLEMS

The mathematical formulation of the physical laws governing most physical processes and the constraints imposed on a system take the form of a Variational Problem or a Boundary Value Problem for the variables of the process. In the latter case it is often (though not always) possible to set up an equivalent Variational Problem for a functional which can be constructed from the differential equation of the boundary value problem and its attendent constraints. It should be stressed that the large majority of Variational Problems are concerned with local optimization over the domains of the free variables as against global extrema over the same domain, although global extrema can be useful for certain purposes like stability.

The formulation of a boundary value problem as an equivalent Variational Problem has the advantage that in contrast with the conventional solutions via differential equations, it provides for the application of powerful and generally sharply converging numerical methods in the construction of solutions. It must be borne in mind that the technique is to some extent limited by the form of the functional of the equivalent Variational Problem itself. The form of the functional and its working domain must be such as would provide for the actual existence of extrema within the admissible function space.

In this dissertation while references have been made to the important problems of uniqueness and convergence its prime concern will be the actual applications of the direct methods.

5.1 The Equivalent Boundary Value Problem of a given Variational Problem - The Euler - Lagrange Equations.

Consider a fairly general Variational Problem

$$J[y] = \int_D \Phi\,[y(\underline{x})]d\underline{x} \qquad \text{for} \qquad y(\underline{x}) \in D$$

for one function of several variables.

If J is stationary at $y = y_o(\underline{x})$ in D then for a variation $\underline{\delta y}$ in $y_o(\underline{x})$ within D, the first variation $\delta J = 0$. This leads to a condition of the form

$$\int_D [\phi]_y \cdot \delta y \cdot d\underline{x} = 0 \qquad \text{for all admissible} \quad \underline{\delta y}$$

which in turn leads to a differential equation

$$[\phi]_y = 0 \,.$$

This is the Euler Lagrange equation. This differential equation with the boundary conditions defining F is a boundary value problem. Note that this equivalence is a consequence of the assumption that solutions optimizing $J[y]$, exist, and when they do the Boundary Value Problem leads to the same solution set.

For the general functional

$$J[y] = \int_D \Phi\,[y(\underline{x})]d\underline{x} \qquad \text{for} \qquad y(\underline{x}) \in D$$

of one function of several variables the Euler's equation has the form

$$\Phi - \Sigma \frac{\partial}{\partial x_1}\left(\frac{\partial \phi}{\partial y_{x_1}}\right) + \Sigma\Sigma \frac{\partial}{\partial x_i x_j}\left[\frac{\partial \phi}{\partial y_{x_i x_j}}\right] \ldots = 0 \cdot$$

The extension to functional 'J' of more than one argument function $y(\underline{x})$ is a natural one and leads to a system of Euler Equation of above form.

Some familiar examples of the boundary value problems generated by Variational Problem are given below.

<u>(i)</u> A functional of two variables is discussed in the study of the deflection of a rigid plate.

$$J = \iint_D \left[\left(\frac{\partial^2 z}{\partial x^2}\right)^2 + \left(\frac{\partial^2 z}{\partial y^2}\right)^2 + 2\left(\frac{\partial^2 z}{\partial x . \partial y}\right)^2\right] dx.dy$$

and any function that gives an extemum must satisfy the <u>biharmonic equation</u>

$$\frac{\partial^4 z}{\partial x^4} + 2\,\frac{\partial^4 z}{\partial x^2 . \partial y^2} + \frac{\partial^4 z}{\partial y^4} = 0$$

<u>(ii)</u> The following functional is obtained for the minimum surface that is enclosed in a given closed curve 'c' in the given space.

$$J(z(x,y)) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\; dx.dy$$

The corresponding Euler's equation called the Ostrogradski equation is

$$\frac{\partial^2 z}{\partial x^2}(1 + (\frac{\partial z}{\partial y})^2) \quad - \quad 2\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\frac{\partial^2 z}{\partial x.\partial y} \quad + \quad \frac{\partial^2 z}{\partial y^2}(1 + (\frac{\partial z}{\partial x})^2) \quad = \quad 0$$

<u>(iii)</u> A functional of the form

$$J \quad = \quad \iint\limits_D ((\frac{\partial^2 z}{\partial x^2})^2 \quad + \quad (\frac{\partial^2 z}{\partial y^2})^2 \quad + \quad 2(\frac{\partial^2 z}{\partial x.\partial y})^2 \quad - \quad 2zf(x,y))dx.dy$$

leads to the following Euler's equation for the extemum of the functional

$$\nabla^4 z \quad \equiv \quad \frac{\partial^4 z}{\partial x^4} \quad + \quad 2\frac{\partial^4 z}{\partial x^2 \partial y^2} + \frac{\partial^4 z}{\partial y^4} \quad = \quad f(x,y)$$

This is the deflection equation for a rigid plate under a load distribution f(x,y)/area

<u>(iv)</u>

Finally consider the functional

$$J(\phi(x,y)) \quad = \quad \iint\limits_D ((\frac{\partial\phi}{\partial x})^2 \quad + \quad (\frac{\partial\phi}{\partial y})^2 \quad + \quad 2zf(x,y))dx.dy$$

for prescribed boundary conditions in **D**.

The Euler's equation takes the form

$$\frac{\partial^2 z}{\partial x^2} \quad + \quad \frac{\partial^2 z}{\partial y^2} \quad = \quad f(x,y) \ .$$

<u>(Poisson's Equation)</u>.

CHAPTER VI

THE SOLUTION OF VARIATIONAL PROBLEMS VIA EULER'S EQUATIONS

(Indirect Methods)

It has been shown already that a given Variational Problem can be converted into a boundary value problem using Euler's derivatives. The boundary value problem may be attacked by any one or a combination of the following procedures:

A. Separation of Variables.

B. **The** Eigen Function Method.

C. **Integral Transforms.**

(Laplace, Hankel, Mellin, Fourier etc.)

D. Green's Function.

E. Numerical Procedures.

A good example of a useful numerical technique in cases of Partial Differential Equations of Hyperbolic type is the Method of Characteristics, briefly sketched below.

6.1 Notation-:

'D' is working domain.

$$r = \frac{\partial^2 u}{\partial x^2}, \quad s = \frac{\partial^2 u}{\partial x . \partial y}, \quad t = \frac{\partial^2 u}{\partial y^2}, \quad p = \frac{\partial u}{\partial x} \quad \text{and} \quad q = \frac{\partial u}{\partial y} .$$

Consider a quasi-linear $O(2)$ equation

$$Q[u] = a(x,y)r + 2b(x,y)s + c(x,y)t - e(x,y,u,p,q) = 0$$

$(x,y) \in D$ (working domain).

Most of the numerical procedures are applied to Hyperbolic problems for which $b^2 > ac$, and u = prescribed $f(s)$ and $\frac{\partial u}{\partial n}$ = prescribed $g(s)$ on an initial curve T_o in D.

Cauchy Problem-: A Cauchy problem based on $Q[u] = 0$ takes the form $Q[u] = ar + 2bs + ct - e = 0$. The compatible values for u, p and q say $u_o(x)$, $p_o(x)$ and $q_o(x)$ are prescribed on some curve c, $y = y_o(x)$ through some point of the domain. A hyperbolic Cauchy problem must generate two distinct one parameter families of curves called characteristics c_λ and c_μ spanning the domain.

These two single parameter families of curves are generated as solutions of the factorizable $O(1)$ equation given below.

$$a\left(\frac{dy}{dx}\right)^2 - 2b\frac{dy}{dx} + c = 0 .$$

The solution of hyperbolic equation is best propagated from the boundary into the given domain along the characteristics. The two roots of the equation given above say m_λ and m_μ are the gradients of c_λ and c_μ at each $(x,y) \in D$.

6.2 Computational Procedure in the Method of Characteristics.

Step 1-: Solve the characteristic gradient equation

$$am^2 - 2bm + c = 0$$

for $m_\lambda(x,y)$ and $m_\mu(x,y)$. Select a suitably fine meshed grid with selected closely spaced grid points P_o, P_1,... on the initial 'grid-front' T_o on which $u, \frac{\partial u}{\partial n}$ or u,p,q are prescribed.

Step 2-: The grid point coordinates (x,y) are computed using the (x,y) propagator $dy = mdx$ along c_λ and c_μ, starting with $P_i \rightarrow Q_i \rightarrow R_i$ by successive grid-fronts $T_o, T_1, T_2, \ldots$ to desired accuracy by using successive approximation.

Step 3-: The values of p,q from the grid points on $T_o, T_1, T_2 \ldots.$ are computed using the values of $a, c, e, m_\lambda, m_\mu$ at the points P_i, Q_i, R_i etc. and the $(p - q)$ propogator

$$amdp + cdq + edy = 0$$

along both c_λ and c_μ where each point is approached from the preceeding front.

Step 4-: The estimate of u at each grid point set P_i is obtained using the u-propogator given below.

$$du = p.dx + q.dy .$$

This is used along either of the two characteristics converging on P_i from the preceeding grid front.

Step 5-: A table is formed from the whole set of values of u,p.q at all grid points in the working domain D, from which the values at any non-grid points can be estimated by finite differences, if necessary.

CHAPTER VII

NUMERICAL METHODS FOR SOLVING VARIATIONAL PROBLEMS (or their Equivalent Boundary Value Problem).

The classical methods for solving Euler's differential equation in closed form are theoretically significant but often prove to be ineffective for practical purposes due to factors like slow convergence of series, or computational complications of integrals etc. in many problems. There are some more complex problems involving functions of several variables which do not possess such solutions. These problems can often be attacked successfully by direct methods in their Variational Form without passing into the Euler-Lagrange equations, exception being Galerkin's method and the method of Least Squares which are applied directly to Boundary Value Problems.

In a very general sense all numerical methods seek approximations to the actual solution by reducing the given Variational (or Boundary Value Problem) to an approximating minimization problem in a discrete point space where a solution is ensured. These direct methods have a wide range of application and lead to a system of linear equations for most problems of Variational form in Physics and Engineering, and are also applicable to electronic computing. Some of the direct methods in use are classified below-:

(i) The Rayleigh-Ritz method.

(ii) Galerkin's Method.

(iii) The Euler's Method of finite differences.

(iv) The Method of Least Squares.

(v) Kantorovich's method.

(vi) Trifftz's Method.

(vii) The Method of Infinitely Many Variables.

Historical Background to Direct Methods

Lord Rayleigh used the equivalence between the boundary value problems and Variational Problems to compute numerical solutions for the fundamental frequencies in connection with the vibrations of an elastic system. This discovery was first published in 1870 in the 'Philosophical Transactions of the Royal Society' and has been extensively dealt with in the theory of sound. This is referred to as the Rayleigh's Principle. W.Ritz generalised Rayleigh's work in 1908 and 1909. Ritz developed a method of substituting into the functional of a Variational Problem a sequence of minimizing functions each of which is expressed as a linear combination of basic functions (Orthogonal Polynomials of different kinds). The coefficients being obtained by maximizing or minimizing the functional. He applied at each stage this method to the determination of the natural frequencies and mode shapes of vibrating plates. Ritz chose an approximate expression of the extremal in the form

$$y = g_o(x) + \sum_{k=1}^{n} c_k g_k(x) \quad ------ \quad (i)$$

where $g_o(x), g_1(x), \ldots, g_n(x)$ are approximating functions. The coefficients c_k are determined from the condition which makes the functional stationary.

Galerkin extended the Ritz method and applied it to those problems which do not have an associated functional. He discussed an error associated with the assumed solution and referred to it as the residual error. This residual error $\varepsilon(t)$ is a measure of the imbalance in the given differential equation. The lowest value for $\varepsilon(t)$ is obtained by the integral $J = \int_a^b \varepsilon^2(t)dt$. The Trifftz method was used to attack Laplace's equation by choosing appropriate co-ordinate functions based on certain solutions of Laplace's equation. The coefficients entering into the linear combinations of coordinate functions are found from the boundary conditions.

Kantcrovich generalised the Ritz's method in 1933 and applied it to the functionals dependent on several arguments. Kantorovich substituted some unknown functions of one independent variable for constant coefficients in the coordinate functions. This linear combination was substituted in the given functional and the resultant condition for the stationary value of the functional was reduced to a Boundary Value Problem. This Boundary Value Problem could be expressed as a functional and thus leads to an auxilary Variational Problem.

Euler's direct method is based on finite differences. The values of a given functional are taken along polygonal curves which consist of a prescribed number of line segments. The functional $J[y(x)]$ turns into a function $\phi(y_1, y_2, \ldots, y_{n-1})$ of ordinates $(y_1, y_2, \ldots, y_{n-1})$ of vertices for the polygon curve and these are chosen by maximizing function ϕ.

The method of least squares is applicable to many problems and is based on the minimization of the mean square deviation which is often associated with a weight function.

The method of Infinitely Many Variables is applied if it is possible to find a general formula for the coefficients c_k in the representation (i) and to then pass to the limit for $n \to \infty$ to obtain the sought for solution in the form of an infinite series. An infinite sequence of coordinate functions is used to express the desired solution as the sum of a series with undetermined coefficients.

All these methods are based on the assumption that the problem admits a solution which is unique.

7.1 Rayleigh-Ritz Method-:

It has been illustrated in the earlier sections that the solution of Boundary-Value Problems for a number of types of ordinary and partial differential equations can be reduced to the question of minimising an integral I, i.e,, to a variational problem. It is proposed, in this dissertation, to deal with the better numerical method of solution, of which the Rayleigh-Ritz method was the first devised in which the unknown solution is expressed as a linear combination of $(n = 1,2,3 \ldots)$ arbitrary, linearly independent functions drawn in sequence from a convergent, complete polynomial set which satisfy the initial or boundary conditions. Some of the important points to be borne in mind in the choice of approximating functions are given below.

Choice of approximating functions-:

(i) The function $\phi_i(x)$ chosen should be flexible enough to satisfy boundary conditions.

(ii) The choice of functions depends on the shape of domain, type of coordinate system; in many instances, a complete set of functions for example polynomial, trigonometric functions, Bessel's functions etc. are chosen.

(iii) The functions selected should satisfy conditions like smoothness and continuity.

(iv) Linearly independent, complete orthonormal sets generally provide the best approximation since this means that any function f(x) can be approximated in the mean to any required degree of accuracy by a linear combination of a finite number of functions.

(v) Apart from the above conditions, the choice of functions is arbitrary and is largely dictated by the geometric and physical conditions of the problem.

Note-: In the Ritz's paper (1908) he has discussed special examples but has not ennumerated the considerations that should govern the choice of a family of approximating functions in a general sense.

The Systematic Ritz Procedure-:

Step 1-: The first step is the formation of an approximate expression of the form

$$y_n = g_o + \sum_{i=1}^{n} c_i g_i(x)$$

where $g_o(x)$, $g_1(x), \ldots, g_n(x)$ are coordinate functions selected on the basis described in the last section. The parameters c_i are determined by the condition that the value of the functional must be stationary. The selection of functions $g_1, \ldots, g_n$ can be regarded as a part of an infinite system of functions

$$g_1(x),\ g_2(x), \ldots,\ g_n(x),\ g_{n+1}(x),\ \ldots$$

which is linearly independent and complete.

Step 2-: For such linear combinations the functional $J[y(x)]$ becomes a function $g(c_1, c_2, c_3, \ldots, c_n)$ of the coefficients $c_1, c_2, \ldots, c_n$. The coefficients c_i are chosen to make the function $g(c_1, c_2, \ldots, c_n)$ achieve an extremum. Consequently $c_1, c_2, \ldots, c_n$ are determined by the equations.

$$\frac{\partial J}{\partial c_i} = 0\ , \qquad i = 1, 2, 3, \ldots, n. \qquad - - - - - \qquad (I)$$

In actual practice the procedure consists in obtaining a sequence of approximating solutions called a minimizing sequence, where $g_o + c_1\phi_1$ is the first approximation, $g_o + c_1g_1 + c_2g_2$ the second, $g_o + c_1g_1 + c_2g_2 + c_3g_3$ the third etc. The values of c_i's are determined at each stage by the system of equation (I).

7.2 Convergence of Ritz's Method

In practice when it is necessary to estimate the accuracy of an approximate solution found by Ritz's method, a comparison is made of the numerical results obtained for successive minimizing sequences of functions. If these results are close to each other there is a good reason to expect that a sufficiently accurate solution has been obtained.

A significant point of this method is as follows:

If $y_n(x) = g_o(x) + c_1 g_1(x) + c_2 g_2(x) + \ldots + c_n g_n(x)$

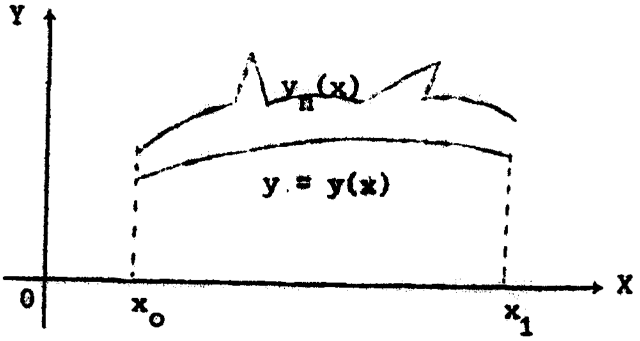
represents the n^{th} approximation to the actual solution $y = u(x)$, then

$$y_{n+1}(x) = g_o(x) + \underline{c_1} g_1(x) + \underline{c_2} g_2(x) + \ldots + \underline{c_n} g_n(x) + \underline{c_{n+1}} g_{n+1}(x)$$

which is the (n + 1) st approximation and may or may not be an improvement over $y_n(x)$ value. By comparing successive approximations, an estimate of the degree of accuracy at any stage of calculation is obtained. By convergence of the process we mean that $g_o + \sum_{k=1}^{n} c_k g_k(x)$ as $n \to \infty$ tends to function $u(x)$ which may be expected to be the actual solution. In actual practice the uniqueness and existance of solutions if possible, is established to begin with and the accuracy of the iterative procedure is tested on the basis of the order of magnitude of successive coefficients.

Some important points to note about convergence are as follows-:

(i) It is possible that $y_n(x)$ and actual solution $y(x)$ may differ significantly on a small part of the interval while being close on other parts of the interval.



(ii) The minimizing sequence $y_1, y_2, \dots, y_n, \dots$ may not have a limit in the class of admissible functions, even though $y_1, y_2, \dots, y_n, \dots$ are themselves admissible functions.

(Ritz however in his original paper (ref 19) assumes convergence if the approximating functions satisfy the boundary and continuity conditions.)

(iii) For orthonormal functions i.e. one for which

$$\int_a^b g_i(x).g_j(x)dx = 0 \; ; \quad \text{when} \quad i \neq j \quad \text{and} \quad \int_a^b g_i^2(x)dx = 1 \; ;$$

a complete set implies that there exists an 'ε' s.t

$$\int_a^b \left|f(x) - \sum_{k=1}^{n} c_k g_k(x)\right|^2 dx < \varepsilon \; .$$

If the orthonormal set is complete, then for a given $f(x)$ and positive ε, no matter how small, it is possible to find a linear combination of $g_1(x)$, $g_2(x)$ ... which is close to $f(x)$ so that the above Riemann Integral holds. The smaller the value of 'ε', the larger 'n' should be. The method of Infinitely Many Variables is a practical development of this situation.

In most problems of Mathematical Physics complete orthonormal sets like Polynomials, Sine or Cosine functions and Bessel's functions are used. Specific examples are $\sin n\pi x$ and $P_n(x)$, which are complete orthonormal systems in the space $L_2(0,1)$.

Accuracy of Results Obtained by Ritz's Method-:

To determine the exactness of results obtained by Ritz's method, or other direct methods, the following practical method can be adopted. Having computed $y_n(x)$ and $y_{n+1}(x)$, these are compared at some points of the interval (x_o, x_1). If within the degree of accuracy required for a given purpose, these values coincide, then the solution of the Variational Problem is taken as $y_n(x)$. If for some values of x, $y_n(x)$ and $y_{n+1}(x)$ differ much then $y_{n+2}(x)$ is computed and $y_{n+1}(x)$ is compared with $y_{n+2}(x)$ at all representative points of the interval (x_o, x_1)

Illustration-: Consider the functional

$$J(y) = \int_0^1 (y''^2 - kx^4 y)dx$$; where 'k' is a suitably

assigned constant, subject to the boundary conditions

$y(o) = y'(o) = y(1) = y'(1) = 0$; where y', y'', y''' are continuous and y^{iv} is Piecewise Continuous in the range (0,1).

Solutions to this problem are constructed using Ritz's method with 1, 2 and 3 terms in succession with a view to investigating the actual nature of convergence of these successive approximating solutions towards the actual solution, which can be constructed in this case. The interest of this particular example, which can be recognized as the deflection problem for a fixed ended beam under a quartic load, lies in the fact that the convergence in this case is not, as is often assumed to be the case, related to the tailing off of the unknown coefficients figuring in the sequence of minimizing functions but is significantly dependent on the polynomials used in these functions. In fact, successive coefficients are found to have comparable orders of magnitude. This important question of the sources of convergence obviously merits further systematic investigation for Variational Problems of different kinds. In the calculations presented a_o , a_1 , a_2 are in each case computed exactly to avoid effects of rounding off errors which may obscure the comparisons under survey.

Approximating Functions -: The approximating functions used are

$$a_o x^2(1 - x)^2 , \quad a_1 x^3(1 - x)^2 , \quad a_2 x^4(1 - x)^2 , \ . \ . \ .$$

each of which satisfies the stipulated Boundary Conditions. A three term solution in these terms is

$$y = x^2(1 - x)^2(a_o + a_1x + a_2x^2) , \qquad \text{and}$$

$$y'' = 30a_2x^4 + x^3(20a_1 - 40a_2) + x^2(12a_2 + 12a_o - 24a_1) - x(12a_o - 6a_1) + 2a_o$$

substituting the values of y, y'' in the given functional and performing integration using

$$\int_o^1 x^{p-1}\cdot(1-x)^{q-1}dx \; ; \quad p,q > 0 = B(p,q)$$

$$J = \frac{A^2}{9} + \frac{B^2}{7} + \frac{C^2}{5} + \frac{D^2}{3} + E^2 + 2\left[\frac{AB}{8} + \frac{AC}{7} - \frac{AD}{6} + \frac{AE}{5} + \frac{BC}{6} - \frac{BD}{5} + \frac{BE}{4} - \frac{CD}{4} + \frac{CE}{3} - \frac{DE}{2}\right] - \frac{20k}{100\times7\times8\times9\times11}(110\,a_o + 77\,a_1 + 56\,a_2)$$

Where $A = 30a_2$, $B = (20a_1 - 40a_2)$, $C = (12a_2 + 12a_o - 24a_1)$

$D = (12a_o - 6a_1)$; $E = 2a_o$ - - - - <u>(i)</u>

Choosing $k = 277.2$ for computational convenience.

$$J = \frac{A^2}{9} + \frac{B^2}{7} + \frac{C^2}{5} + \frac{D^2}{3} + E^2 + 2\left[\frac{AB}{8} + \frac{AC}{7} - \frac{AD}{6} + \frac{AE}{5} + \frac{BC}{6} - \frac{BD}{5} + \frac{BE}{4} - \frac{CD}{4} + \frac{CE}{3} - \frac{DE}{2}\right] - \frac{1}{100}(110a_o + 77a_1 + 56a_2) \text{ --- (ii)}$$

<u>I.</u> <u>The 1 term Solution</u>-: The appropriate 'J' for 1 term solution is given by (ii) with

$a_1 = a_2 = 0$, i.e $A = B = 0$, $C = D = 12a_o$, $E = 2a_o$.

Then $J = \frac{384}{5} a_o^2 + 4 a_o^2 - 80 a_o^2 - \frac{110}{100} a_o$.

For minimization, $\frac{\partial J}{\partial a_o} = 0.$ This yields $a_o = 0.6875$

giving the <u>one term solution</u> $\boxed{y_1 = 0.6875x^2 \cdot (1 - x)^2}$

<u>II.</u> <u>The 2 term solution:-</u> From (i) for $a_2 = 0$ we get

$A = 0$, $B = 20a_1$, $C = 12a_o - 24a_1$, $D - 12a_o - 6a_1$, $E = 2a_o$.

On subsutituting in (ii) $J = a_1^2(\frac{12}{35}) + a_o^2(\frac{4}{5}) + a_o a_1(\frac{4}{5}) - a_o(\frac{11}{10}) - a_1(\frac{77}{100})$

For minimization $\frac{\partial J}{\partial a_o} = 0$ and $\frac{\partial J}{\partial a_1} = 0$ which yield the system of equation:-

$$\begin{bmatrix} \frac{48}{35} & \frac{8}{5} \\ \frac{4}{5} & \frac{8}{5} \end{bmatrix} \begin{bmatrix} a_1 \\ a_o \end{bmatrix} \begin{bmatrix} \frac{77}{50} \\ \frac{11}{10} \end{bmatrix} ;$$

for a_o and a_1 giving

$a_o = 0.3025$ and

$a_1 = 0.77$

$\therefore$ <u>The 2 term solution is</u> $\boxed{y_2 = x^2(1 - x)^2(0.3025 + 0.77x)}$

III. The 3 term solution:- From (i) on substituting in (ii) we get

$$J = \frac{12}{15}a_o^2 + \frac{12}{35}a_1^2 + \frac{8}{35}a_2^2 + \frac{4}{5}a_oa_1 + \frac{16}{35}a_oa_1 + \frac{18}{35}a_1a_2 - \left(\frac{11}{10}a_o + \frac{77}{100}a_1 + \frac{56}{100}a_2\right)$$

For minimization $\frac{\partial J}{\partial a_o} = \frac{\partial J}{\partial a_1} = \frac{\partial J}{\partial a_2} = 0$; which yield the exact system of equations:

$$\begin{bmatrix} \frac{8}{5} & \frac{4}{5} & \frac{16}{35} \\ \frac{4}{5} & \frac{24}{35} & \frac{18}{35} \\ \frac{16}{35} & \frac{18}{35} & \frac{16}{35} \end{bmatrix} \begin{bmatrix} a_o \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{11}{10} \\ \frac{77}{100} \\ \frac{56}{100} \end{bmatrix}$$

for a_o, a_1, a_2 ; giving

$a_o = 0.4375$

$a_1 = 0.14$

$a_2 = 0.63$

∴ The 3 term solution is

$$\boxed{y_3 = x^2 \cdot (1 - x)^2 (0.4375 + 0.14x + 0.63x^2)}$$

The exact solution found by general theory for the Euler's equation corresponding to the given functional is

$$y_{exact} = 0.0825(x^8 - 6x^3 + 5x^2) , \qquad k = 277.2 .$$

The comparison of the 1,2,3 term solution by Ritz's method and the exact solution is given below:

x	y_1	y_2	y_3	y_{exact}
0.1	0.0055687	0.0030739	0.0037081	0.00363
0.3	0.0303187	0.0235273	0.0236464	0.0237654
0.5	0.0429687	0.0429687	0.0415625	0.0415722
0.9	0.0055687	0.0080635	0.0086977	0.0087835

The attached plotting illustrates clearly the character of the convergence of each successive Ritz solution towards the exact solution.

7.3 The Ritz method for Functionals of Functions of Several Variables.

The scheme of the Ritz method for functionals dependent on functions of several variables is essentially the same as for functions of one variable but the choice of coordinate functions can be difficult depending on the domain and the prescribed boundary conditions. In the case of two variables where the solution is sought in a domain 'G' with boundary 'Γ', with boundary conditions of the form

$$u\Big|_{(\Gamma)} = 0 .$$

the selection of a system of coordinate functions is best effected by choosing a continuous function $w(x,y)$ which is positive in the interior of (G) and equals zero at all points of (Γ). On choosing such a function the system of coordinate functions can be taken as

$$g_{ij}(x,y) = x^i y^k \, w(x,y)$$

$$(i,k = 0, 1, 2, \dots)$$

numbered with two indices.

Illustration-:

Consider the functional

$$J\{u\} = \int_{-a}^{a} dx \int_{-b}^{b} [(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 + 2Au]dy$$

where 'A' is a constant, dependent on the function $u(x,y)$ defined in the domain G s.t $-a \leq x \leq a$, $-b \leq y \leq b$ with zero Boundary Values on the contour (Γ) bounding the domain. The equivalent Boundary-Value-Problem to the above variational problem is the Poisson Problems

$$\nabla^2 u = A \quad \text{for} \quad (x,y) \in G , \qquad u\Big|_{(\Gamma)} = 0 .$$

Using a continuous function $w = (a^2 - x^2)(b^2 - y^2)$ and a system of functions with even indices i and j. Since above Boundary-Value Problem is known to have a unique solution and is invarient w.r.t the mappings $x \to -x$ and $y \to -y$, the solution must be an even function of x for any fixed y and even function of y for any fixed x. Choosing a one term approximation for the Ritz method:-

$$u = c_1(a^2 - x^2)(b^2 - y^2)$$

and substituting in the given functional yields

$$J[u] = \frac{128}{45} a^3b^3(a^2 + b^2)c_1^2 + \frac{32}{9} a^3b^3Ac_1.$$

At stationary points of 'J' , $\frac{\partial J}{\partial c_1} = 0$, this gives the relation

$$c_1 = - \frac{5A}{8(a^2 + b^2)}$$

from which the approximate solution is obtained as

$$u_{approx.} = - \frac{5A}{8(a^2 + b^2)} (a^2 - x^2)(b^2 - y^2) .$$

7.4 Further Applications.

Rayleigh's Energy method for Vibration and Stability problems.

Most important problems of vibrations that occur in engineering practice involve the determination of the natural frequencies of a vibrating system. The lowest or fundamental frequency is important and the Rayleigh's energy method with Ritz's procedure is used for its determination.

Procedure -: The procedure consists in assuming a mode shape or configuration of the system when it is in the maximum displacement position corresponding to a fundamental mode. For accuracy a mode shape which satisfies boundary conditions is chosen even though one or more boundary conditions are violated. The statical deflection multiplied by an unknown constant is often taken as the mode shape in analysis of the structural elements.

An equation is then set up equating maximum K.E and maximum gain in Potential Energy. This is used in determining the fundamental frequency approximately. The Principle of Stationary Potential Energy by the Rayleigh Ritz procedure is also used for the approximate determination of critical or buckling loads on elastic structures. The illustrations given below are for the determination of the fundamental frequency of a cantilevered beam.

Illustration Using an Approximating Function :

$$y(x,t) = \sum_{n=1,3...}^{2m+1} An(\cos\frac{n\pi x}{2L} - 1) \cos\omega t \; ; \qquad 0 \le x \le L$$

determine an upper bound for the fundamental frequency of a beam of length L encastered at $x = 0$ and pinjointed at $x = L$ with, constant flexural rigidity 'EI' and constant mass 'm' per unit length. Considering a one term approximation, the approximate deflection is $y(x,t) = A(\cos\frac{\pi x}{2L} - 1)\cos\omega t = f(x)\cos\omega t$ where 'ω' is the angular frequency.

The boundary conditions are satisfied for $y(0,t) = y_x(0,t) = 0$ at the built in end and $y_{xx}(L,t) = 0$ at the free end.
It is essential to satisfy the boundary conditions at the fixed end to obtain an upper bound for the fundamental frequency.

Maximum K.E $\quad T_{max} = \int_0^L \frac{m\dot{y}^2}{2} dx = \frac{m\omega^2 A^2}{2} \int_0^L (\cos \frac{\pi x}{2L} - 1)^2 dx$

$$= \frac{m\omega^2 A^2 L}{2} (\frac{3}{2} - \frac{4}{\pi})$$

Maximum P.E $\quad V_{max} = \frac{EI}{2} \int_0^L (f_{xx})^2 dx = \frac{\pi^4 EIA^2}{64L^3}$

Equating $\quad T_{max} = V_{max}$

$$\omega^2 = \frac{\pi^4 EI}{(\frac{3}{2} - \frac{4}{\pi})\, 32mL^4} = 13.142 \frac{EI}{mL^4}$$

or $\quad \omega = 3.66 \sqrt{\frac{EI}{mL^4}}$ $\quad$ whereas the exact solution

$\omega_{exact} = 3.515 \sqrt{\frac{EI}{mL^4}}$ $\quad$ giving a relative error of 4.1% .

A better approximation can be made with more terms in the approximate solution.

Another Illustration of the Rayleigh-Ritz Method for The Determination of Fundamental Frequencies.

This problem determines the fundamental frequency of a Cantilevered Beam of length 'L' , constant flexural rigidity 'EI' and constant mass 'm' per unit length. Assume that during vibration the deflection curve of the centroidal line is the same as one produced by a uniformly distributed load s.t

$$y(x,t) = \frac{A}{24EI} \cdot x^2(x^2 - 4Lx + 6L^2)\cos\omega t.$$

The above approximation satisfies the boundary conditions exactly as

$$y(0,t) = y_x(0,t) = 0$$ at built in end and

$$y_{xx}(L,t) = 0$$ at the free end.

The condition of zero shear at the free end is

$$y_{xxx}(L,t) = 0, \quad \text{if} \quad A \neq 0.$$

Therefore all boundary conditions are specified.

<u>The Maximum K.E</u> $\underline{T\max} = \int_0^L \frac{m\dot{y}^2}{2}\,dx = \frac{1}{2}m\omega^2\int_0^L f^2.dx$

where $f = \frac{A}{24EI} \cdot (x^2) \cdot (x^2 - 4Lx + 6L^2)$, since $\cos\omega t = 1$

for maximum value.

$$\therefore \underline{T\max} = \frac{m\omega^2}{24 \times 48E^2I^2} \int_0^L x^4(x^2 - 4Lx + 6L^2)^2dx$$

<u>The Maximum P.E.</u> $\underline{V\max} = \frac{EI}{2}\int_0^L (f_{xx})^2dx$

where if
$$f = \frac{A}{24EI} \cdot x^2(x^2 - 4Lx + 6L^2)$$
$$= \frac{A}{24EI} \cdot (x^4 - 4Lx^3 + 6L^2x^2)$$

then
$$\frac{\partial f}{\partial x} = \frac{A}{24EI}(4x^3 - 12Lx^2 + 12L^2x)$$

and
$$\frac{\partial^2 f}{\partial x^2} = \frac{A}{24EI}(12x^2 - 24Lx + 12L^2)$$

$$\underline{V\max} = \frac{EIA^2 \times 144}{E^2I^2 \times 24 \times 24} \int_0^L (x^2 - 2Lx + L^2)^2 dx$$

On performing the necessary integration

$$\underline{V\max} = \frac{A^2 L^5}{40EI}$$

$$\underline{T\max} = \frac{m\omega^2 A^2}{24 \times 48 \; E^2I^2} \int_0^L x^4(x^4 + 16L^2x^2 + 36L^4 - 8Lx^3 + 12x^2L^2 - 48L^3x)dx$$

On performing integration and rearranging terms.

$$\underline{T\max} = \frac{m\omega^2 \cdot A^2 \cdot L^9 \times 2.3111}{24 \times 48 \times E^2I^2}$$

$$\underline{V\max} = \frac{A^2 . L^5}{40EI}$$

Equating Vmax and Tmax

$$\frac{m\omega^2 A^2 L^9 \times 2.3111}{24 \times 48 \times E^2I^2} = \frac{A^2 . L^5}{40.E.I}$$

$$\therefore \quad \omega^2 = \frac{EI \times 24 \times 48}{mL^4 \times 40 \times 2.311}$$

from which the fundamental frequency

$$\omega = \sqrt{\frac{EI}{mL^4}} \times \sqrt{\frac{24 \times 48}{40 \times 2.311}}$$

$$= \sqrt{\frac{EI}{mL^4}} \times \sqrt{\frac{1152}{92.444}} = \sqrt{12.4615} \times \sqrt{\frac{EI}{mL^4}}$$

$$\therefore \quad \boxed{\omega = 3.530 \sqrt{\frac{EI}{mL^4}}}$$

is an approximation for the fundamental frequency with a relative error of 0.5% .

7.5 Galerkin's Method-:

Galerkin in 1915 suggested a direct method which has a wider scope than the Rayleigh-Ritz method. The Rayleigh-Ritz method can be used for solving Boundary-Value-Problems by forming an associated functional in the form of a definite integral which is then made stationary. There are problems in mathematical physics however which do not have a variational equivalent. It is for such problems that Galerkin's method attains its importance. Galerkin's procedure supplies a useful procedure for not only solving Boundary Value Problems based on differential equations but also in the direct solution of variational problems.

In certain categories of problems (for example in linear differential equations) the Rayleigh-Ritz method and Galerkin's method are equivalent. The equivalence is demonstrated by showing that the system of equations in one method can be transformed, using integration by parts, to yield the system of equations obtained by the other.

Systematic Procedure.

Step 1-: The search for an approximate solution for a given differential equation is done by examining an error associated with this solution, called the residual error.

A general non-linear equation may have the form $f(D, x, t) = 0$. Where $f(D,x,t)$ is some general non-linear function of operator $D = \frac{d}{dt}$ and of dependent variable 'x' and independent variable 't' .

An approximate solution containing a suitable number of free constants and satisfying the given initial conditions is chosen in the form of a linear combination of certain functions, just as is done in the Ritz method (see 7.1). The exact solution $x(t)$ is approximated by a function $X(t)$ of the form $\phi_o(t) + c_1\phi_1(t) + \ldots + c_m\phi_m(t)$ where $\phi_o, \phi_1, \ldots \phi_m$ are appropriate linearly independent functions.

Step 2 -: The approximate solution is substituted into the given differential equation and since the solution is not in general exact, it will not satisfy the equation identically. The residual error

$\epsilon(t) = f[D, X(t), t]$ which is a measure of the imbalance in the differential equation is obtained. This residual error is not directly the difference between approximate and exact solution as the exact solution is assumed.

Since ϵ may be positive or negative, its mean square $\int_a^b \epsilon^2 dt$ is a convenient measure of overall error and is arranged to approach zero over range of interest.

$$J = \int_a^b \epsilon^2(t)dt \qquad \text{should be minimum.}$$

(This criterion follows from principle of least squares). Constants $c_1, c_2, \ldots, c_n$ appear in the residual $\epsilon(t)$ and in integral J and in order to minimise J it is necessary that

$$\left.\begin{array}{l} \dfrac{\partial J}{\partial c_1} = \displaystyle\int_a^b 2\epsilon(t)\,\frac{\partial \epsilon(t)}{\partial c_1}.dt = 0 \\ - - - - - - - - - - - - - - - - - - \\ - - - - - - - - - - - - - - - - - - \\ \dfrac{\partial J}{\partial c_m} = \displaystyle\int_a^b 2\epsilon(t)\,\frac{\partial \epsilon(t)}{\partial c_m}.dt = 0 \end{array}\right] \quad \ldots\ldots \text{ I}$$

The above computations are quite difficult for non-linear differential equations. If the original differential equation is linear and if ϕ_i's are orthogonal over $a \leq t \leq b$ the conditions to minimise 'J' become

$$\left.\begin{array}{l} \displaystyle\int_a^b \epsilon(t)\,\phi_1(t).dt = 0 \\ - - - - - - - - - - - - - - \\ - - - - - - - - - - - - - - \\ \displaystyle\int_a^b \epsilon(t)\,\phi_m(t).dt = 0 \end{array}\right] \quad \ldots\ldots \text{ II}$$

In the form II Galerkin's method matches with Ritz's method.

Galerkin's method imposes the condition that error function $\epsilon(t)$ must be orthogonal to each of functions ϕ_i. The advantage of this method is that it is directly applicable to Boundary Value Problems whether or not an energy type functional can be found.

7.6

Illustration-:

Non-linear spring under a load

A kind of spring can be imagined in which force is proportional only to square of deflection and in opposition to it the differential equation is:-

$$\frac{d^2x}{dx^2} + a^2x|x| = 0$$

Let initial conditions be $x = A$, $\dot{x} = 0$ at $t = 0$.

Assume the approximate solution

$$x = \phi_o(t) = A\cos\omega t$$

which when substituted in above differential equation gives the residual error

$$\epsilon = -\omega^2 A\cos\omega t \pm a^2A^2\cos^2\omega t$$

With the '+' sign in the range $-\frac{\pi}{2} \leq \omega t \leq \frac{\pi}{2}$

and the '-' sign in the range $\frac{\pi}{2} \leq \omega t \leq \frac{3\pi}{2}$.

The condition to be met is

$$\int_{-\pi/2}^{+\pi/2} \epsilon\left(\frac{\partial\epsilon}{\partial A}\right) d(\omega t) = 0$$

$$\Longrightarrow \int_{-\pi/2}^{+\pi/2} (-\omega^2 A\cos\omega t + a^2A^2\cos^2\omega t)(-\omega^2\cos\omega t + 2a^2A\cos^2\omega t)d(\omega t) = 0$$

which is satisfied trivially if $A = 0$

or if $\omega^4 - \frac{8}{\pi} a^2A\omega^2 + \frac{3}{2} a^4A^2 = 0$

The result may be written $\omega^2 = ka^2A$

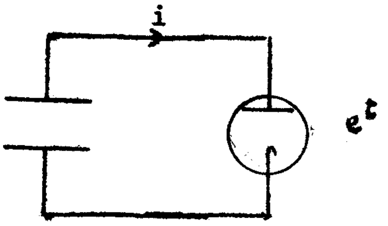
and k may be 1.63 or 0.92.

The Minimum value for J corresponds $\omega^2 = 0.92a^2A$.

Illustration

Capacitor discharge through a non-linear diode.

The governing equations for e are



$$\frac{de}{dt} + Ae + Be^2 = 0, \quad (0 < t < \infty)$$

with $e(0) = E$

Using Galerkin's reduction to Ritz's procedure assume an approximate solution upto 2 terms

$$e = \phi_o + k\phi_1 = E\,e^{-At} + kt\,e^{-At} .$$

This satisfies given boundary conditions.

The residual error is

$$\varepsilon = ke^{-At} + B(E + kt)^2 e^{-2At}$$

and the minimum error condition is

$$\int_0^\infty \varepsilon\phi_1 dt = \int_0^\infty [kte^{-2At} + Bt(E + kt)^2 e^{-3At}]dt = 0$$

i.e. $\frac{2B}{27A^2} k^2 + (\tfrac{1}{4} + \frac{4BE}{27A})k + \frac{BE^2}{9} = 0$. In the case $\frac{BE}{A} = \tfrac{1}{2}$ and $k = \frac{k}{AE}$;

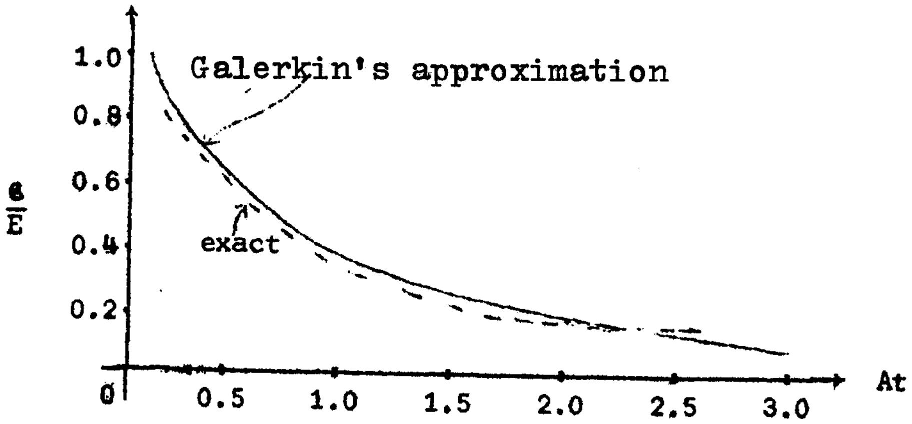
$$4k^2 + 35k + 6 = 0$$

which gives roots $\dfrac{k = -8.6}{\text{too large}}$ and $k = -0.172$

The appropriate $k = -0.172AE$ and

the approximate solution is $e = E(1 - 0.172At)e^{-At}$

which is graphed against the exact solution of this problem for purposes of comparison.



<u>Exact Solution</u>-:

$$\epsilon = Ee^{-At}\left[1+\frac{\mu BE}{A}(e^{-At}-1) + \left(\frac{\mu BE}{A}\right)^2 (e^{-At}-1)^2\right]$$

'μ' is some constant.

7.7 Method of L.V. Kantorovich.

This method was devised in 1933 by L.V. Kantorovich and is a generalization of the Ritz method to functions dependent on several arguments. Its basic idea is the replacement of constants in the Ritz representation by unknown functions of one independent variable. After a linear combination of coordinate functions with these coefficients is substituted into the functional the condition of 'stationariness' of the resultant expression is reduced either to a Boundary Value Problem with a system of ordinary differential equations or to an auxiliary variational problem. The solution of either auxiliary problem leads to an approximate solution of the original problem.

The Procedure:

Step 1-: Suppose a functional

$$J[u] = \iint_A F(x,y,u,u_x,u_y)dx.dy$$

is to be extremized subject to certain homogeneous boundary conditions. The coefficients to be selected viz. c_i's are not treated as constants but unknown functions of x, say $c_i(x)$. To accommodate nonhomogeneous boundary conditions a term $\phi_o(x,y)$ may be introduced.

giving a minimizing sequence

$$u_n = \phi_o(x,y) + \sum_{k=1}^{n} c_k(x)\phi_k(x,y)$$

the boundary conditions are satisfied if ϕ_o satisfies boundary conditions on c and for each k

$$c_k(x).\phi_k(x,y) = 0 \quad \text{on} \quad c .$$

Step 2-: Substituting u_n into J(u) and integrating w.r.t 'y'

$$J(u_n) = \int_{x_2}^{x_1} G(c_1, c_1', c_2, c_2', \ldots, c_n, c_n', x)dx$$

To make J stationary, the functions $c_k(x)$ need to satisfy the system of ordering differential equations

$$G_{c_k} - \frac{d}{dx} G_{c_k'} = 0 , \qquad k = 1,2,\ldots, n$$

subject to some given boundary conditions.

Advantages of Kantorovich's Method-:

The reason why approximations made by Kantorovich's method are effective is that the class of functions

$$u_m = \sum_{k=1}^{m} c_k(x_i)\phi_k(x_i) \qquad i = 1, 2, \ldots, m.$$

is much wider when $c_k(x_i)$ are not constants, but functions of one variable x_i .

An approximate solution obtained in this way is therefore better than that obtained by the Ritz's method with the same coordinate functions and the same number of terms m. These remarks are illustrated in an example in Chapter IX.

CHAPTER VIII

EULER'S METHOD FOR FINITE DIFFERENCES

Euler's method is based on finite differences. The values of a given functional are considered along polygonal curves which consist of a prescribed number of line segments.

8.1 Systematic Procedure-:

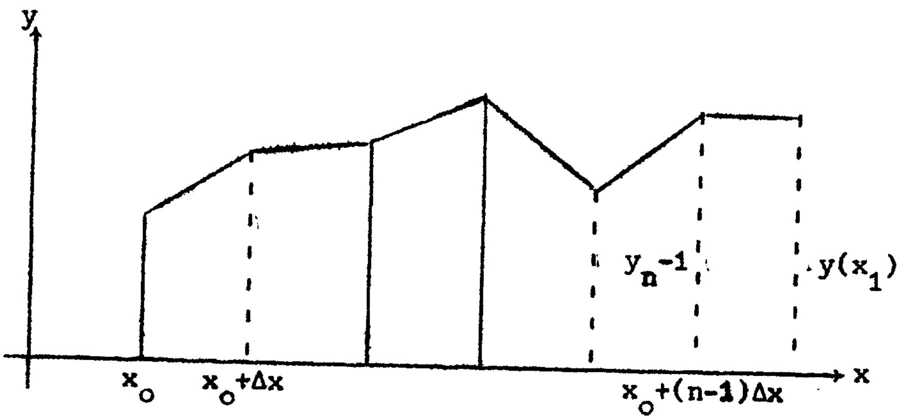
Step 1-: Consider a simple functional

$$J(y) = \int_{x_o}^{x_1} F(x, y, y')dx , \qquad y(x_o) = a$$
$$y(x_1) = b$$

Let the line segments with fixed abscissa of vertices be

$x_o + \Delta x$, $x_o + 2\Delta x$, . . . , $x_o + (n - 1)\Delta x$

where $\Delta x = \frac{x_1 - x_o}{n}$



The functional $J[y(x)]$ reduces to a function $\phi(y_1, y_2, \ldots, y_{n-1})$ of ordinates $y_1, y_2, \ldots, y_{n-1}$ of vertices for a polygon curve which is completely determined by these ordinates. Choose these ordinates $y_1, y_2, \ldots, y_{n-1}$ so that the function $\phi(y_1, y_2, \ldots, y_{n-1})$ has an extremum i.e. we determine $y_1, y_2, \ldots, y_{n-1}$ by equations

$$\frac{\partial\phi}{\partial y_1} = 0, \quad \frac{\partial\phi}{\partial y_2} = 0, \ldots, \frac{\partial\phi}{\partial y_{n-1}} = 0$$

and then pass to the limit with $n \to \infty$. In practice the functional $J[y(x)]$ along polygon curves is found approximately.

In the simple case

$$\int_{x_0}^{x_1} F(x, y, y')dx = \sum_{k=0}^{n-1} \int_{x_0 + k\Delta x}^{x_0+(k+1)dx} F(x, y, \frac{y_{k+1} - y_k)}{\Delta x} dx$$

This can be replaced by a finite sum

$$\sum_{i=1}^{n} F(x_i, y_i, \frac{\Delta y_i}{\Delta x}).\Delta x$$

i.e.

$$J[y(x)] \approx \phi(y_1, y_2, \ldots y_{n-1}) = \sum_{i=0}^{n-1} F(x_i, y_i, \frac{y_{i+1} - y_i}{\Delta x})\Delta x$$

along polygonal curve.

Since there are only two terms that depend on y_i, namely the i^{th} and the $(i - 1)^{th}$ terms

$$F(x_i, y_i, \frac{y_{i+1} - y_i}{\Delta x})\Delta x \quad \text{and} \quad F(x_{i-1}, y_{i-1}, \ldots, \frac{y_i - y_{i-1}}{\Delta x})\Delta x$$

The equations $\frac{\partial \phi}{\partial y_i} = 0$; $i = 1, 2, \ldots, n-1$

take the form

$$F_{y_i}(x_i, y_i, \frac{y_{i+1} - y_i}{\Delta x})\Delta x + F_{y_i'}(x_i, y_i, \frac{y_{i+1} - y_i}{\Delta x}) \times (-\frac{1}{\Delta x})\Delta x$$
$$+ F_{y'}(x_{i-1}, y_{i-1}, \frac{y_i - y_{i-1}}{\Delta x})\frac{1}{\Delta x}.\Delta x = 0$$
$$\left(i = 1,2,\ldots, n-1\right)$$

$$\text{i.e } F_{y_i}(x_i, y_i, \frac{\Delta y_i}{\Delta x}) - \frac{F_{y_i'}(x_i, y_i, \frac{\Delta y_i}{\Delta x}) - F_{y_i'}(x_{i-1}, y_{i-1}, \frac{\Delta y_{i-1}}{\Delta x})}{\Delta x} = 0$$

$$\text{i.e } F_{y_i}(x_i, y_i, \frac{\Delta y_i}{\Delta x}) - \frac{\Delta F y_i'}{\Delta x} = 0 \quad - - - - - - - - \quad \langle I \rangle$$

As $n \to \infty$ these yield the Euler equations

$$F_y - \frac{d}{dx} F_{y}' = 0 .$$

If the limiting process is not carried out in full then we can determine $y_1 y_2 - y_{n-1}$ from the equations

$$\frac{\partial\phi}{\partial y_i}, \quad i = 1,2,\ldots, n-1$$

A polygon curve is then obtained which is an approximation to the solution of the given variational problem.

It is important to note that the system 'I' may be linear or non-linear. The amount of labour to find $y_i (i = 1,2,\ldots, n)$ increases for large 'n' and for greater accuracy.

8.2

Illustration -: Determine a polygon line approximation to the extemal of

$$\int_0^2 [(y')^2 + 6x^2 y]dx \quad ; \qquad y(0) = 0 , \qquad y(2) = 4$$

(i) Let $n = 2$ then $\Delta x = 1, \quad x_o = 0, x_1 = 1, \quad x_2 = 2$

$y_o = 0, y_2 = 4$

$y_1 = y(x_1) = y(1)$ is unknown.

Take the approximation

$$\phi(y_1) = \sum_{i=0}^{2} [6x_i^2 . y_i + (\frac{y_{i+1} - y_i}{\Delta x})^2]\Delta x$$ to the given functional.

which becomes

$$\phi(y_1) = 2y_1^2 - 2y_1 + 16$$

for stationary ϕ, $\frac{d\phi}{dy_1} = 0$. This gives $y_1 = \frac{1}{2}$.

(ii) Let $n = 3$, then $\Delta x = \frac{2}{3}$, $x_o = 0$, $x_1 = \frac{2}{3}$, $x_2 = \frac{4}{3}$, $x_3 = 2$, $y_o = 0$ and $y_3 = 4$. The unknown ordinates are $y_1 = y(\frac{2}{3})$ and $y_2 = y(\frac{4}{3})$, and

$$\phi(y_1, y_2) = \left[\frac{9}{2} y_1^2 + \frac{8}{3} y_1 + \frac{9}{2} y_2^2 - \frac{9}{2} y_1 y_2 - \frac{22}{3} y_2 + 36\right] \frac{2}{3}$$

The equations for determination of y_1 and y_2 are

$$9y_1 - \frac{9}{2} y_2 + \frac{8}{3} = 0$$

$$-\frac{9}{2} y_1 + 9 y_2 - \frac{22}{3} = 0$$

which yields $y_1 = \frac{4}{27}$ and $y_2 = \frac{24}{27}$.

For $n = 4$

$x_o = 0$, $x_1 = \frac{1}{2}$, $x_2 = 1$, $x_3 = \frac{3}{2}$, $y_o = 0$ and $y_4 = 4$

$$\phi(y_1, y_2, y_3) = \sum_{i=0}^{3} \left(6x_i^2 y_i + \left(\frac{y_{i+1} - y_i}{\Delta x}\right)^2\right)\Delta x$$

$$= \frac{1}{2}\left[8(y_1^2 + y_2^2 + y_3^2 - y_1 y_2 - y_2 y_3) + \frac{3}{2}y_1 + 6y_2 - \frac{37}{2}y_3 + 64\right]$$

, and $\frac{\partial \phi}{\partial y_i} = 0$; $(i = 1,2,3)$ yields

$$\left[\begin{array}{l} 16y_1 - 8y_2 + \frac{3}{2} = 0 \\ 16y_2 - 8(y_1 + y_2) + 6 = 0 \\ 16y_3 - 8y_2 - \frac{37}{2} = 0 \end{array}\right]$$

and solution yields $y_1 = \frac{1}{16}$; $y_2 = \frac{5}{16}$ and $y_3 = \frac{21}{16}$.

Exact Solution -: From the given functional

$F(x, y, y') = (y')^2 + 6x^2 y$, the Euler-Lagrange equation is $3x^2 - y'' = 0$, which on fitting the boundary conditions gives

$$y \doteq \frac{x^4}{4} .$$

The results obtained are more accurate for larger values of 'n' and the relative errors are large for 'x' close to zero. The exact solution to this problem itself is highly non-linear and **this explains** the slowness of convergence. Euler's method is particularly suited to electronic computing where 'n' can be made quite large.

8.3

The Method of Least Squares

(For the solution of Boundary Value Problems)

The method of least squares is based on the minimization of the mean square deviation which is sometimes computed with an appropriate weight function. This method can be applied not only to differential equations but to other types of equations involving one or several unknown functions.

Definitions

Linear Operator-: An operator 'A' is called linear if its field of definition is a linear set and

$$A(a_1\phi_1 + a_2\phi_2) = a_1A\phi_1 + a_2A\phi_2 \qquad (a_1, a_2 \text{ are constants}).$$

Bounded Operator-:

An operator 'A' is said to be bounded if

$$||A\phi|| \leq C||\phi|| , \qquad C = \text{constant.}$$

The least of the constants C satisfying above inequality is known as norm of the bounded operator A and is denoted by the symbol $||A||$,

$$\therefore \quad ||A\phi|| \leq ||A|| \, ||\phi|| \, .$$

Systematic Procedure-:

Step 1-: Let A be a linear differential operator defined for some linear set D_A which is dense in a given Hilbert space H.

If the equation $AU = f$ is to be solved where 'f' is a given element from H. choose a sequence of linearly independent coordinate elements $\{\phi_n\}$, $\{\phi_n\} \in D_A$ and construct an appropriate solution in the form

$$u_n = \sum_{k=1}^{n} a_k \phi_k$$

Let the given Hilbert space be complex, then the mean square deviation has the form

$$||Au_n - f||^2 = \left[\sum_{k=1}^{n} a_k A\phi_k - f \, , \quad \sum_{m=1}^{n} a_m A\phi_m - f\right]$$

$$= \sum_{k,m=1}^{n} a_k \bar{a}_m (A\phi_k \; A\phi_m) - \sum_{k=1}^{n} a_k (A\phi_k f)$$

$$- \sum_{k=1}^{n} \bar{a}_k (f, \, A\phi_k) + (f, \, f).$$

Put $a_k = \alpha_k + i\beta_k$, then the equations giving the minimum values of a_k are

$$\frac{1}{2}\left\{\frac{\partial||Au_n - f||^2}{\partial\alpha_m} + \frac{i\partial||Au_n - f||^2}{\partial\beta_m}\right\} = \frac{\partial||Au_n - f||^2}{\partial\bar{a}_m} = 0 .$$

The desired system of equations is

$$\sum_{k=1}^{n} a_k(A\phi_k, A\phi_m) = (f, A\phi_m) ; \qquad m = 1,2,\ldots, n .$$

If the given equation **has** the homogeneous form $Au = 0$, then it admits only one trivial solution $u = 0$; and $A\phi_1, A\phi_2,\ldots, A\phi_m$ are linearly independent. In the opposite case if a_k are found s.t $a_k \neq 0$ and $A \sum_{k=1}^{n} a_k\phi_k = 0$, then it would contradict the linear independence of coordinate elements. Thus if the homogeneous equation $Au = 0$ has only a trivial solution then approximate solutions can be constructed by the method of least squares for any n and they are determined uniquely.

Conditions of Convergence of the Method of least squares.

The method of least squares gives a sequence of approximating functions which converge to the exact solution if

(i) The sequence of coordinate elements is A-complete.

(ii) Equation $Au = f$ has a solution.

(iii) There exists a constant k, such that for every $u \in D_A$

$$||u|| \leq k||Au||$$

Note-: A-completeness follows from ordinary completeness if the operator A is bounded. If system $\{\phi_n\}$ is complete and operator 'A' is bounded i.e. $||A\phi|| \leq C||\phi||$; then it is possible to find $n, c_1, c_2 \ldots, c_n$ such that

$$||u - u_n|| < \frac{\varepsilon}{||A||} .$$

It follows that

$$||Au - Au_n|| = ||A(u - u_n)|| \leq ||A|| \; ||u - u_n|| < \varepsilon$$

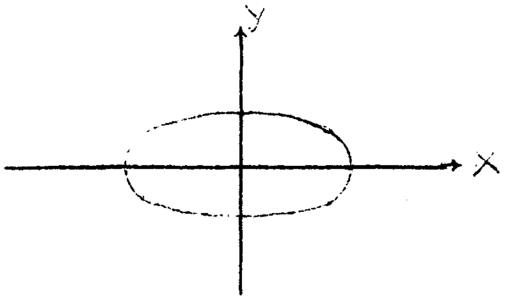
8.4

Illustration -:

The Dirich let Problem for an Ellipse

It is required to determine a function $u(x,y)$ which is harmonic within the ellipse $x = a \cos t$; $y = b \sin t$; $0 \leq t \leq 2\pi$ along with the boundary condition

$$u\Big|_s = 4(\sigma) \text{ , where 's' is the edge of ellipse.}$$



Consider $u(x,y)$ to be the real part of some analytic function $f(z)$.

$$u(x,y) = \text{Re}\{ f(z) \}; \quad z = x + iy \text{ .}$$

Adopting approximating functions

$$\phi_0(z) = 1 \ ; \quad \phi_k(z) = [z + \sqrt{z^2 - c^2}]^k + [z - \sqrt{z^2 - c^2}]^k \ ; \quad k \geq 1$$

as the coordinate functions $(c = \sqrt{a^2 - b^2})$. This system forms a complete linear set of functions which are holomorphic within the ellipse and continuous along the edge.

The Approximate Solution Sequence can be expressed in the form

$$f_n(z) = \sum_{k=0}^{n} c_k\phi_k(z) \; ; \quad \text{Imaginary } (a_o) = 0.$$

Denoting the length of arc of ellipse by $d\sigma$, then

$$d\sigma = \surd(a^2\sin^2 t + b^2\cos^2 t)dt.$$

The fundamental Hilbert space for this problem is $L_2(p, s)$ where

$$p(\sigma) = \frac{1}{\surd(a^2\sin^2 t + b^2\cos^2 t)} .$$

By definition of the scalar product of the functions ϕ and ψ equals

$$(\phi,\psi) = \int_0^{2\pi} \phi(\sigma)\, \overline{\psi(\sigma)}dt .$$

By the method of least squares the coefficients c_k can be determined from the condition

$$||\text{Re} \sum_{k=0}^{n} c_k\phi_k(z) - u(\sigma)||^2 = \text{minimum.}$$

On the edge of ellipse $z = a\cos t + ib\sin t$ and consequently.

$$z^2 - c^2 = (b\cos t + i\sin t)^2.$$

Hence

$$\phi_k(z) = (a + b)^k e^{ikt} + (a - b)^k e^{-ikt} \; ; \quad k > 0.$$

Using the notation $c_k = \alpha_k - i\beta_k$, then on ellipse

$$\text{Re } \{c_k\phi_k(z)\} = [(a + b)^k + (a - b)^k]\alpha_k \cos kt$$
$$+ [(a + b)^k - (a - b)^k]\beta_k \sin kt \; ; \quad k > 0 .$$

The coefficients above are the Fourier coefficients of function $u(\sigma)$; hence

$$\left.\begin{aligned}\alpha_o &= c_o = \frac{1}{2\pi}\int_o^{2\pi} u(\sigma)dt\\ \alpha_k &= \frac{1}{\pi[(a+b)^k + (a-b)^k]}\int_o^{2\pi} u(\sigma)\cos ktdt\\ \beta_k &= \frac{1}{\pi[(a+b)^k - (a-b)^k]}\int_o^{2\pi} u(\sigma)\sin ktdt\ .\end{aligned}\right| \quad ---- \quad III$$

The coefficients $\alpha_o, \alpha_k, \beta_k$ do not depend on number n, therefore the limiting process as $n \to \infty$ leads to $f_n(z) = \sum_{k=0}^{n} c_k\phi_k(z)$ yielding the exact solution as

$$f(z) = c_o + \sum_{k=1}^{\infty} c_k[(z + \sqrt{z^2 - c^2})^k + (z - \sqrt{z^2 - c^2})^k]$$

where coefficients c_k are determined from equations III given above.

The coefficients above are the Fourier coefficients of function $u(\sigma)$; hence

$$\left.\begin{aligned}
\alpha_o &= c_o = \frac{1}{2\pi}\int_o^{2\pi} u(\sigma)dt \\
\alpha_k &= \frac{1}{\pi[(a+b)^k + (a-b)^k]}\int_o^{2\pi} u(\sigma)\cos kt\,dt \\
\beta_k &= \frac{1}{\pi[(a+b)^k - (a-b)^k]}\int_o^{2\pi} u(\sigma)\sin kt\,dt \quad .
\end{aligned}\right| \quad ---- \quad \text{III}$$

The coefficients α_o, α_k, β_k do not depend on number n, therefore the limiting process as $n \to \infty$ leads to $f_n(z) = \sum_{k=0}^{n} c_k\phi_k(z)$ yielding the exact solution as

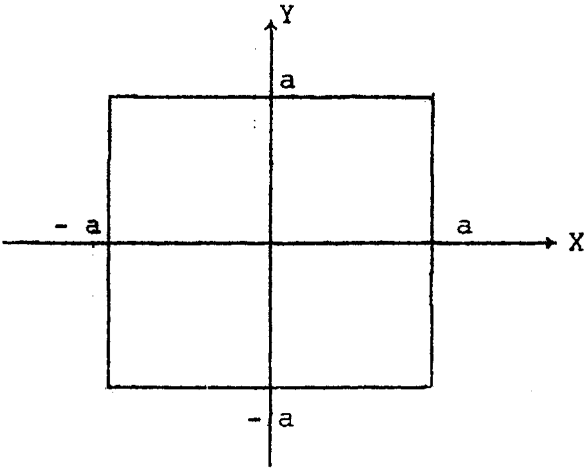
$$f(z) = c_o + \sum_{k=1}^{\infty} c_k[(z + \sqrt{z^2 - c^2})^k + (z - \sqrt{z^2 - c^2})^k]$$

where coefficients c_k are determined from equations III given above.

CHAPTER IX

BENDING OF A CLAMPED THIN SQUARE PLATE UNDER VARYING LOAD DISTRIBUTIONS

(PROJECT OF THE DISSERTATION)



This Chapter consists of the theory of the deflection of a thin square plate when it is subjected to uniform and nonuniform load distributions of various kinds. The solutions are constructed using the direct methods of Ritz and Kantorovich. The integrals and calculations at various stages have been performed on the IBM1130 computer of the University of Zambia Computer Center and the results for various load distributions have been plotted on the IBM 1627 plotter. (ref. Appendix).

Consider a thin plate of planform dimensions 2a and 2b in the x and y directions respectively and subject to a normal load 'q' per unit area. It is assumed that the edges of the plate are clamped i.e. deflection and rotation about the edges are prevented by constraining couples and forces along the edges.

The problem consists of integrating the biharmonic equation

$$\nabla^4\omega = \frac{q}{D} = p$$

where 'D' is the rigidity of the plate, under the boundary conditions

$$\omega\Big|_L = 0 \; ; \qquad \frac{\partial\omega}{\partial n}\Big|_L = 0$$

Choose the coordinate axes parallel to the sides of the plate and place the origin of coordinates at its centre.

The Strain Energy for the bending of an initially flat plate is given by

$$u = \frac{D}{2}\iint_A [(\omega_{xx} + \omega_{yy})^2 - 2(1 - \nu)(\omega_{xx}\omega_{yy} - \omega_{xy}^2)]dxdy$$

where D = plate flexural rigidity $= \dfrac{Eh^3}{12(1-\nu^2)}$

and h = plate thickness

ω = plate deflection.

ν = Poisson's ratio.

A = plate planform,

E = Young's Modulus.

The expression $k \equiv \omega_{xx}\omega_{yy} - \omega_{xy}^2$ is an approximation for the Gaussian curvature of the deflected middle surface. This expression can be neglected if plate planform is polygonal and edges remain straight. The Potential Energy 'V' of the system consisting of the plate and external load is obtained by subtracting the gain in potential energy of the external load 'q' from the strain energy.

$$V = \iint_A [\tfrac{D}{2}(\nabla^2\omega)^2 - q\omega]dx.dy \quad - - - - \quad (i)$$

The fourth order biharmonic equation is reduced to the determination of the minimum of the above functional. The origin of the coordinates is taken at the plate centre and the Gaussian term is omitted.

9.1 Setting up a One term solution by Ritz's method

Consider a one term solution of the form

$$\omega = cX(x)\cdot Y(y)$$

where $X(x) = (x^2 - a^2)^2$ and $Y(y) = (y^2 - a^2)^2$

This choice of coordinate functions automatically satisfies the boundary conditions at the four edges of the clamped plate.

$$\omega\Big|_{x = \pm a} = \frac{\partial\omega}{\partial x}\Big|_{x = \pm a} = 0$$

and $$\omega\Big|_{x = \pm a} = \frac{\partial\omega}{\partial y}\Big|_{y = \pm a} = 0$$

Substituting a one term approximating solution for 'ω' into the functional form for potential energy V.

$$V = J(\omega) = \iint_A \left[\frac{C^2 D}{2}(X''.Y + Y''.X)^2 - CqX.Y\right]dx.dy$$

The constant 'C' is determined by the condition $\frac{dV}{dC} = 0$

i.e $$C = \frac{q \int_{-a}^{a}\int_{-a}^{a} XY dx.dy}{D \int_{-a}^{a}\int_{-a}^{a} (X''Y + XY'')^2 dx.dy}$$

On performing repeated integration after substituting $X = (x^2 - a^2)^2$ and $Y = (y^2 - a^2)^2$

$$C \doteq \frac{q}{Da^4}\left(\frac{0.285}{10.4 + 2.97}\right) = \frac{q}{Da^4} \cdot 0.0213$$

The <u>one term</u> solution for the deflection of a square plate under a uniform load $p = \frac{q}{D}$ is

$$\boxed{\omega(x, y) = \frac{q(x^2 - a^2)^2(y^2 - a^2)^2}{Da^4} \cdot 0.0213}$$

If a = 1, then the above solution reduces to

$$\omega(x,y) = 0.0213.p.(x^2 - 1)^2.(y^2 - 1)^2$$

or $$\underline{\omega(x,y) = 0.0213p\,(1 - x^2)^2.(1 - y^2)^2}$$

The bending of a thin Square Plate contd.

One term Solution by Kantorovich's Method.

For purposes of comparison a one term solution of the same problem is constructed by Kantorovich's method.

Let the approximation for 'ω' be

$$\omega = A(x)(y^2 - a^2)^2$$

with boundary conditions $\omega = \omega_y = 0$ on $y = \pm a$.

Substituting this in the relation for 'V' and integrating w.r.t 'y' yields

$$J[\omega] = V = \frac{16a^5}{315}\int_{-a}^{+a} \left[\frac{D}{2} (16a^4(A'')^2 - 96a^2A.A'' + 504A^2) - 21q.A\right]dx.$$

If the integrand is denoted by F, then for stationary J, the Euler equation

$$F_A - \frac{d}{dx} F_{A'} + \frac{d^2}{dx^2} F_{A'} = 0$$

must be satisfied. On substituting values of F_A and $F_{A'}$ the above differential equation becomes

$$2a^4A'''' - 12b^2A'' + 63A = \frac{21q}{8D} \qquad - - - - - \qquad \text{(i)}$$

The complete solution of (i) is found to be

$$\begin{aligned} A = \; & c_1 \sinh(2.075 \tfrac{x}{a}).\sin(1.143 \tfrac{x}{a}) \\ & + c_2 \cosh(2.075 \tfrac{x}{a}).\cos(1.143 \tfrac{x}{a}) \\ & + c_3 \sinh(2.075 \tfrac{x}{a}).\cos(1.143 \tfrac{x}{a}) \\ & + c_4 \cosh(2.075 \tfrac{x}{a}).\sin(1.143 \tfrac{x}{a}) \\ & + \frac{q}{24D} \end{aligned}$$

Since $A(x)$ is an even function of 'x' i.e. $A(-x) = A(x)$; it follows that $c_3 = c_4 = 0$. The constants c_1 and c_2 can be found from the boundary conditions

$$\omega(a,y) = \omega_x(a,y) = 0 \; ; \quad \text{which implies}$$

$$A(a) = A'(a) = 0$$

and yields c_1 and c_2 for given $\frac{q}{D}$.

The full solution for 'ω' is thus given by

$$\omega = A(x).(y^2 - a^2)^2$$

$$\begin{aligned} \text{where} \quad A(x) = \; & c_1 \sinh(2.075\tfrac{x}{a}).\sin(1.143\tfrac{x}{a}) \\ & + c_2 \cosh(2.075\tfrac{x}{a}).\cos(1.143\tfrac{x}{a}) \\ & + \frac{q}{24D} \end{aligned}$$

The comparison of the results from the Ritz's method and Kantorovich's method can be observed from the table given below-:

a	$\omega(0,0)/qa^4/D$; Poisson's Ratio $v = 0.3$		
	RITZ	KANTOROVICH	EXACT
1	0.0213	0.0200	0.0202

For one term solution, Kantorovich's method is clearly superior to the Ritz's method.

9.2

A 3-term approximating solution by Ritz's Method.

Using the substitution $p = \frac{q}{D}$ for the load per unit area, the functional form for the potential energy of the system consisting of the plate and external load is given by

$$V = J[\omega] = \int_{-a}^{a} dx \int_{-a}^{a} [(\nabla^2\omega)^2 - 2p\omega]dy.$$

For an approximating solution of more than one term consider polynomials of the form

$$(x^2 - a^2)^2(y^2 - a^2)^2(a_1 + a_2(x^2 + y^2) + . . .)$$

This form omits odd powers of x and y since $\omega(x,y)$ is symmetric about the coordinate axes. The approximating functions for the 3-term solution can be taken as

$$\phi_1 = (x^2 - a^2)^2(y^2 - a^2)^2 , \quad \phi_2 = x^2\phi_1 , \quad \phi_3 = y^2\phi_1$$

and $\omega_3 = a_1\phi_1 + a_2\phi_2 + a_2\phi_3$. Since it is a square plate in the three term approximation for ω, a_2 is equal to a_3 .

In the given functional

$$J = \iint_A((\nabla^2\omega)^2 - 2p\omega)dx.dy$$

the '$(\nabla^2\omega)^2$' term is independent of the load distribution and is therefore a characteristic of the plate while the term '$2p\omega$' is a load dependent factor.

$$\nabla^2\omega = \phi^2\cdot\psi^2(-4\beta) + 2[-4\phi\psi(\phi\underline{v} + \psi\underline{u}) \times (-2\beta)(\underline{u} + \underline{v})]$$

$$+ [\alpha + \beta(\phi + \psi)][-4(3\phi\psi(\phi + \psi) - 2(\phi^2 + \psi^2))]$$

$$= -4\beta\phi^2\cdot\psi^2 + 16\beta\phi\psi(\phi\underline{v} + \psi\underline{u})(\underline{u} + \underline{v})$$

$$- 4[\alpha + \beta(\phi + \psi)][3\phi\psi(\phi + \psi) - 2(\phi^2 + \psi^2)]$$

$$\frac{\nabla^2\omega}{-4} = \beta\phi^2\cdot\psi^2 - 4\beta\phi\psi(\phi + \psi - 2\phi\psi)$$

$$+ (\alpha + \beta(\phi + \psi))(3\phi\psi(\phi + \psi) - 2(\phi^2 + \psi^2))$$

$$\left(\frac{\nabla^2\omega}{-4}\right)^2 = \frac{(\nabla^2\omega)^2}{16} = \beta^2\phi^4\cdot\psi^4 + 16\beta^2\cdot\phi^2\cdot\psi^2(\phi + \psi - 2\phi\psi)^2$$

$$+ (\alpha + \beta(\phi + \psi))^2(3\phi\psi(\phi + \psi) - 2(\phi^2 + \psi^2))^2$$

$$- 8\beta^2\phi^2\cdot\psi^3(\underline{\phi + \psi} - 2\phi\psi)$$

$$+ 2\beta\phi^2\psi^2(\alpha + \beta(\phi + \psi))(3\phi\psi(\phi + \psi) - 2(\phi^2 + \psi^2))^2$$

$$- 8\beta\phi\psi(\phi + \psi - 2\phi\psi)(\alpha + \beta(\phi + \psi))$$

$$\times (3\phi\psi(\phi + \psi) - 2(\phi^2 + \psi^2))^2$$

INTEGRALS

The following integrals are applicable in the computation of 'J' .

$$\int_{-1}^{1} \phi^n dx = \int_{-1}^{1} \psi^n dy = A_n = \frac{n!(n + 1)!}{(2n + 2)!} \cdot 2^{2n+2} \quad \text{and}$$

$$\int_{-1}^{1} \phi^m\cdot\psi^m(\phi + \psi)^n dx.dy = I_{mn}.$$

(i) 'n' odd-:

$$I_{mn} = 2[A_{m+n}A_m + {}^nc_1 A_{m+n-1}A_{m+1} + {}^nc_2 A_{m+n-2}\cdot A_{m+2} \; . \; . \; . \; \frac{n+1}{2} \text{ terms}]$$

(ii) 'n' even-:

$$I_{mn} = 2[A_{m+n}A_m + {}^nc_1 A_{m+n-1}A_{m+1} + \ldots \frac{n}{2} \text{ terms}] + {}^nc_{n/2} A^2_{m+n/2}$$

In terms of above integrals

$$\int_{-1}^{1}\int_{-1}^{1} \frac{(\nabla^2\omega)^2}{16}\, dx.dy = \left[\begin{array}{l} \alpha^2(4I_{o4} - 16I_{12} - 12I_{13} + 24I_{21} \\ \qquad + 9I_{22} + 16A_2^2) \\ \\ + 2\alpha\beta(4I_{o5} - 8I_{13} - 12I_{14} - 6I_{22} + 9I_{23} \\ \qquad + 27I_{31} + 36A_3^2) \\ \\ + \beta^2(4I_{o6} - 12I_{15} - 36I_{23} + 9I_{24} + 54I_{32} \\ \qquad + 81A_4^2) \end{array}\right]$$

Using the tables for I_{mn} and A_n (ref Appendix.)

$$\int_{-1}^{1}\int_{-1}^{1} \frac{(\nabla^2\omega)^2}{16}\, dx.dy = 3.3436734\alpha^2 + 12.1588122\alpha\beta + 12.31466\beta^2 \; .$$

Details of the Computations:

The actual computations of 'J' for symmetric load distributions of different kinds can prove quite tedious, but is greatly simplified and facilitated by the introduction of the functions

$$\phi = (1 - x^2) \; ; \qquad \psi = (1 - y^2)$$

and the integrals

$$A_n = \int_{-1}^{+1} \phi^n dx = \int_{-1}^{+1} \psi^n dy$$

$$I_{mn} = \int_{-1}^{+1} \phi^m \psi^m (\phi + \psi)^n dxdy \; .$$

The integrals $\int_{-1}^{+1} (\nabla^2 \omega)^2 dxdy$ can clearly be expressed in these terms. Their applicability extends, however, to most of the functionals and load distributions that are likely to be encountered over square and other domains, and to functionals associated with many Partial Differential Equations of orders higher than 4. As a preliminary step therefore, a programme for the computation of these integrals was prepared and the computations actually executed up to m,n = 10 (c.f Appendix C). The modification necessitated by rectangular domains and non symmetric loadings is fairly straight forward. In fact using the tables provided it is possibl to compute the functionals representing Potential Energy for elliptic, rectangular, circular thick or thin plates with ease. The tables of these integrals computed for the purpose of this dissertation are appended for their wide usefulness in other contexts.

Case (1)

Consider a uniform load distribution 'p' ; for simplicity let $p = 1$; then

$$- \frac{1}{16} \int_{-1}^{1} \int_{-1}^{1} 2p\omega dx.dy = - \frac{1}{8} \iint_A (\phi^2\psi^2)[\alpha + \beta(\phi + \psi)]dx.dy$$

$$= - \frac{1}{8} (\alpha A_2^2 + \beta I_{21})$$

$$- \frac{1}{16} \int_{-1}^{1} \int_{-1}^{1} 2p\omega dx.dy = - \frac{1}{8} (1.1377776\alpha + 1.950476\beta)$$

$$= -0.1422222\alpha - 0.2438095\beta$$

$$\therefore J = 3.3436734\alpha^2 + 12.1588122\alpha\beta + 12.31466\beta^2$$

$$- 0.1422222\alpha - 0.2438095\beta$$

Minimizing 'J'

$$\frac{\partial J}{\partial \alpha} = 6.6873468\alpha + 12.1588122\beta - 0.1422222 = 0$$

$$\frac{\partial J}{\partial \beta} = 12.1588122\alpha + 24.62932\beta - 0.2438095 = 0$$

In matrix representation

$$\underbrace{\begin{bmatrix} 6.6873468 & 12.1588122 \\ 12.1588122 & 24.62932 \end{bmatrix}}_{\text{Plate Matrix}} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \underbrace{\begin{bmatrix} 0.1422222 \\ 0.2438095 \end{bmatrix}}_{\text{Load Vector}}$$

On solving

$$\alpha + 1.8181817\beta = 0.0212673$$

$$\alpha + 2.0256353\beta = 0.020052$$

$$- 0.2074536\beta = 0.0012153$$

$$\beta = - 0.0058581$$

$$a_2 = -\beta = 0.0058581$$

$$\alpha = 0.0212673 + 1.8181817 \times 0.0058581$$

$$= 0.0319183$$

and $a_1 = \alpha - 2a_2$

$$= 0.0319183 - 2 \times 0.0058581$$

$$a_1 = 0.0319183 - 0.0117162$$

$$\begin{array}{|l} a_1 = 0.0202021 \\ a_2 = 0.0058581 \end{array}$$

The three term approximating solution for uniform load 'p'.

$$\underline{\omega = (1 - x^2)^2(1 - y^2)^2[0.0202021 + 0.0058581 (x^2 + y^2)]}$$

The approximate value of the central deflection

$\underline{\omega(0,0) = 0.0202021}$. This approximation is better than Kantorovich Method's one term approximation. The curves of equal deflection for the uniform load are plotted (ref. plotting 'A'.)

Case (2) Non Uniform Load Distribution

$$p = (1 - x^2)(1 - y^2) = \phi \cdot \psi$$

$$- \frac{1}{16} \int_{-1}^{1} \int_{-1}^{1} 2p \cdot \omega dx \cdot dy = - \frac{1}{8} \iint_{A} \phi\psi(\phi^2\psi^2)[\alpha + \beta(\phi + \psi)]dx \cdot dy$$

$$= - \frac{1}{8} \int_{-1}^{1} \int_{-1}^{1} [\alpha\phi^3\psi^3 + \beta\phi^3\psi^3(\phi + \psi)]dx \cdot dy$$

$$= - \frac{1}{8} (\alpha A_3^2 + \beta I_{31})$$

$$- \frac{1}{16} \int_{-1}^{1} \int_{-1}^{1} 2p.\omega dx.dy = - \frac{1}{8} [0.8359183\alpha + 1.486077\beta]$$

$$J = 3.3436734\alpha^2 + 12.1588122\alpha\beta + 12.31466\beta^2 - 0.1044897\alpha - 0.1857596\beta$$

$$\frac{\partial J}{\partial \alpha} = 6.6873468\alpha + 12.1588122\beta = 0.1044897$$

$$\frac{\partial J}{\partial \beta} = 12.1588122\alpha + 24.62932\beta = 0.1857596$$

$$\underbrace{\begin{bmatrix} 6.6873468 & 12.1588122 \\ 12.1588122 & 24.62932 \end{bmatrix}}_{\text{Plate Characteristic}} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \underbrace{\begin{bmatrix} 0.1044897 \\ 0.1857596 \end{bmatrix}}_{\text{Load Factor}}$$

On solving

$$\alpha + 1.8181817\beta = 0.0156249$$

$$\alpha + 2.0256353\beta = 0.0152777$$

Subtracting

$$-0.2074536\beta = 0.0003472$$

$$\beta = -0.0016736$$

$$a_2 = 0.0016736$$

$$\alpha = 0.0156249 + 1.8181817 \times 0.0016736$$

$$= 0.0186678$$

$$a_1 = \alpha - 2a_2 = 0.0186678 - 0.0033472$$

$$a_1 = 0.0186678 - 0.0033472$$

$$= 0.0153206$$

$$\underline{a_2 = 0.0016736}$$

The three term approximate solution by Ritz's Technique under the load

$$p = (1 - x^2)(1 - y^2) \quad \text{is}$$

$$\underline{\omega = (1 - x^2)(1 - y^2)(0.0153206 + 0.0016736(x^2 + y^2))}.$$

The central deflection of $\omega(x,y)$ for $x = 0$, $y = 0$ is given by

$$\underline{\omega(0,0) = 0.0153206}$$

The plottings of curves of equal deflection differ in their characteristics. The deflection at the centre is maximum and reduces non-uniformly with the increase in (x,y) coordinates.

(ref. plotting B) .

Case (3) Non Uniform Load Distribution.

$$p = (x^2 + y^2) = (1 - \phi + 1 - \psi)$$

$$p = (2 - (\phi + \psi))$$

$$-\frac{1}{16}\int_{-1}^{1}\int_{-1}^{1} 2p\omega dx.dy = -\frac{1}{8}\int_{-1}^{1}\int_{-1}^{1}(2 - (\phi + \psi))(\phi^2\psi^2)[\alpha + \beta(\phi + \psi)]dxd$$

$$= -\frac{1}{8}\int_{-1}^{1}\int_{-1}^{1}(2\phi^2\psi^2 - \phi^2\psi^2\ \phi + \psi))[\alpha + \beta(\phi + \psi)]\,dxdy$$

$$= -\frac{1}{8}[2\alpha\ A_2^2 + 2\beta I_{21} - \alpha I_{21} - \beta I_{22}]$$

$$-\frac{1}{16}\int_{-1}^{1}\int_{-1}^{1} 2p\omega dx.dy = -\frac{1}{8}[\alpha(2A_2^2 - I_{21}) + \beta(2I_{21} - I_{22})]$$

$$= -\frac{1}{8}[0.3250792\alpha + 0.4953588\beta]$$

$$-\frac{1}{16}\int_{-1}^{1}\int_{-1}^{1} 2p\omega dx.dy = -0.0406349\alpha - 0.0619198\beta$$

Minimising 'J'

$$\begin{vmatrix} 6.6873468\alpha & 12.1588122\beta \\ 12.1588122\alpha & 24.62932\beta \end{vmatrix} \begin{vmatrix} \alpha \\ \beta \end{vmatrix} = \begin{vmatrix} 0.0406349 \\ 0.0619198 \end{vmatrix}$$

Plate Matrix — Load Vector

$$\alpha + 1.8181814\beta = 0.0060763$$

$$\alpha + 2.0256353\beta = 0.0050925$$

On subtracting

$$-0.2074536\beta = 0.0009838$$

$$\therefore \beta = 0.0047422 \quad ; \quad a_1 = 0.0047422$$

$$\alpha = 0.0060763 + 1.8181814 \times 0.0047422$$

$$\alpha = 0.0060763 + 0.0086221$$

$$\alpha = 0.0146984$$

$$a_1 = \alpha - 2a_2$$

$$= 0.0146984 - 0.0094844$$

$$a_1 = 0.005214$$

$$a_2 = 0.0047422$$

Three term Approximate Solution
by Ritz's Technique under a load $p=(x^2 + y^2)$

$$\omega = (1 - x^2)(1 - y^2)(0.005214 + 0.0047422(x^2 + y^2))$$

The deflection at the centre is given by

$$\underline{\omega(0,0) = 0.005214} \; .$$

Since a_1 and a_2 are of the same order the solution is untrustworthy as is also borne out by the curves of equal deflection in this case. Therefore, it is necessary to consider a 4-term approximation, using the integral tables just as before.

9.3

Discussion of the results with possible extensions of the Project

The attatched program computes the deflection at various points of the plate. It can be seen from the print outs that for each load distribution, the deflection is maximum at the centre of the plate, and reduces to zero along the edges. The plottings of the curves of equal deflection for uniform and nonuniform load distributions have been done with the maximum plotter precision available. The successive curves of equal deflection are drawn by changing the deflection by a fixed amount. The comparison of the results for a uniform load '$p = c$' and for non-uniform load $p = (1 - x^2)^2 \cdot (1 - y^2)^2$ can be done by observing the plottings. There is a difference in the central deflection along with the changes in the points of inflexion and the curves of accumulation.

The nonuniform load distribution $p = (x^2 + y^2)$ creates problems of numerical unstability since both a_1 and a_2 are of the same order and small. The error in computations was minimized by performing the calculations in the best possible way but the extended precision of the main line program could not be extended to the plotter since it involves rewriting of the package subroutines for the IBM1627 plotter operation. An extended plotter precision is desired in this case since the curves of equal deflection turned out to be asymmetrical about the origin.

Most of the results obtained for this problem can be easily extended to rectangular and circular plates. Thick plates can be included in the discussion by taking into account the Gaussian Quadrature term. The plottings at each stage can help to locate the possible areas of maximum stress and the points where the plate can crack. An extension is also possible to plates clamped at two edges only which can allow the computation of the deflection of bridges under symmetric and nonsymmetric load distributions. It should be emphasised that for non-symmetric loads a three term approximation may not be enough and more terms may be needed in the approximating solution.

BIBLIOGRAPHY

1 AKHIEZER,N. I,
The Calculus of Variations, translated by A.H. Frink from original Russian, Blaisdell Publishing Company, New York, 1962.

2. CARATHEODORY,C.,
Calculus of Variations and Partial Differential Equations of first order, vol 12, translated by R.B. Dean, Holden-Day, Inc., San Francisco, California.

3. CUNNINGHAM,W.J.,
Introduction to Non-linear Analysis, McGraw-Hill, New York, 1958.

4. COURANT,R AND HILBERT,D.,
Methods of Mathematical Physics, vol II, Partial Differential Equations, Interscience Publishers, New York, 1962.

5. CRAGGS,I.W.,
Calculus of Variations, George Allen & Union Limited, London, 1973.

6. DORN,S.W AND McCRACKEN,D.,
Numerical Methods with FORTRAN-IV, case studies John Wiley & Sons, Inc., New York, 1972.

7. ELSGOLC,L.E.,
Calculus of Variations, English edition, Published by Addison-Wesley Publishing Company Inc., Reading, Mass., 1962.

8. FORRAY,J.M.,
Variational Calculus in Science and Engineering, McGraw-Hill Book Company, New York, 1968.

9. GARABEDIAN,P.R.,
Partial Differential Equations, John Wiley and Sons Inc., New York, 1964.

10. GELFAND,I.M. AND FOMIN,S.V.,
Calculus of Variations, Translated from the Russian by R.A. Silverman, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1963.

11 MIKHLIN,S.G.,
Variational Methods in Mathematical Physics Translated by T. Boddington, Pergamon Press, Inc., New York, 1964.
(Translation from the original Russian, published in 1957 by Gostekhizdat, Moscow).

12 PETROVSKY,I.G.,
Lectures on Physical Differential Equations, Translated from Russian by A. Shenitzer, Interscience Publishers, Inc., New York, 1964.

13 RITZ,W.,
Uber eine neue Methode Zur L'o'sung gewisser Variations probleme der Mathematischen Physik, J.reine angew, Mathematik, 13s, 1(1908) and W.Ritz, Gesammelte Werke, Paris (1911).

14 SAATY,
Non-linear Mathematics, International Student edition, McGraw-Hill, Inc., Tokyo, 1964.

15 SHILOV,E.G.,
Generalized Functions and Partial Differential Equations, translated from the Russian by B.Seckler, Gordon and Breach, New York, 1968.

16 SMIRNOV,V.I.,
A Course for Higher Mathematics vol-iv, 5th Edition, Gostekhizdat (1958) English translation published by Pergamon Press, Oxford (1964).

17 VOLOSOV,V.M.,
Advanced Mathematics for Engineers, English Translation published by Mir Publishers, Moscow, 1975.