

ORDINARY REPRESENTATIONS OF THE GENERALISED

SYMMETRIC GROUP, B_n^m

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DECLARATION

I declare that this is indeed my own work and it has not been submitted before for any qualification and at any University.

DEDICATIONS

I dedicate this dissertation to my mother Tifwilemo (Chima) Zimba for being patient with my unending studies, I expect more of this behaviour from her. And I also dedicate this dissertation to my lovely wife Beatrice (Kapande) Zimba whose moral support and encouragement assisted in seeing me through the difficulties I experienced while I was doing this work.

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CONTENTS

	<u>Page</u>
INTRODUCTION -----	(viii)
LIST OF SYMBOLS AND NOTATION-----	(x)
<u>CHAPTER I: ORDINARY REPRESENTATION THEORY</u> <u>OF FINITE GROUPS</u> -----	<u>1</u>
1.1 BASIC CONCEPTS -----	1
1.2 CONSTRUCTION OF NEW REPRESENTATIONS OF FINITE GROUPS-----	24
<u>CHAPTER II: REPRESENTATION THEORY OF THE</u> <u>SYMMETRIC GROUP</u> -----	<u>34</u>
2.1 THE SYMMETRIC GROUP, S_n -----	34
2.2 REPRESENTATION THEORY OF SYMMETRIC GROUPS	37
2.3 REPRESENTATIONS OF THE YOUNG SUBGROUPS	51
<u>CHAPTER III: THE GENERALISED SYMMETRIC GROUP AND</u> <u>ITS CONJUGACY CLASSES</u> -----	<u>53</u>
3.1 THE GENERALISED SYMMETRIC GROUP, B_n^m .-----	53
3.2 THE CONJUGACY CLASSES OF B_n^m -----	57
<u>CHAPTER IV: REPRESENTATION THEORY OF THE GENERALISED</u> <u>SYMMETRIC GROUP</u> -----	<u>66</u>
4.1 GENERALISED YOUNG SUBGROUPS AND BASIC REPRESENTA- TIONS -----	67
4.2 REPRESENTATIONS OF THE GENERALISED YOUNG SUB- GROUPS -----	71
4.3 REPRESENTATIONS OF THE GENERALISED SYMMETRIC GROUP -----	73

	Page
<u>CHAPTER V: APPLICATIONS TO THE REPRESENTATION</u>	
<u>THEORY OF THE HYPEROCTAHEDRAL GROUP.</u> -----	77
5.1 THE HYPEROCTAHEDRAL GROUP, B_n^2 -----	77
5.2 REPRESENTATIONS OF THE GROUP B_n^2 -----	81
5.3 ORDINARY REPRESENTATIONS OF B_3^2 AND B_4^2 -----	84
5.3.1 REPRESENTATIONS OF B_3^2 -----	85
5.3.2 REPRESENTATIONS OF B_4^2 -----	88
APPENDIX -----	94
REFERENCES -----	97

INTRODUCTION

Pioneering work in the determination of ordinary linear representations of the generalised symmetric group B_n^m was first carried out independently by Osima [13] and Puttaswamaiah [14] by adhoc means. More recently Kerber [10], Read [16] and Hughes [8] determined the irreducible ordinary representations of B_n^m by using Clifford's theory applied to Wreath products.

The results of Osima, Puttaswamaiah, Kerber, Read and Hughes have recently been obtained in a much easier and more elegant way by M. Saeed-Ul-Islam [20], and without using Clifford's theory.

Our object here is to give a detailed account of the inequivalent irreducible ordinary representations of B_n^m by using Saeed-Ul-Islam's approach. We also apply the results obtained to the construction of the irreducible ordinary representations of the hyperoctahedral group B_n^2 which is isomorphic to the Weyl group of type B_n .

The following is a description of the arrangement of the chapters. In chapter I, we give the theory of ordinary representations of finite groups which will be used in the sequel. We describe the ordinary representation theory of the symmetric group S_n and its Young subgroups in chapter II. The basic concepts of the generalised symmetric group B_n^m and a description of its conjugacy classes are given in chapter III.

The main results of the ordinary representations of the generalised symmetric group B_n^m are given in chapter IV. As a direct application of this work, we give an explicit construction of the ordinary representations of the hyperoctahedral group, B_n^2 in chapter V. We also give an illustration of the construction of the ordinary representations of the groups B_3^2 and B_4^2 . The character tables of these groups are given in the appendix. In all that follows, results will be numbered X.Y.Z, where X is the chapter number, y is the section number and Z is the item number in section Y. The end of a proof will be marked by the symbol .

LIST OF SYMBOLS AND NOTATION

We shall use the following notation without further reference.

G = a finite group with identity e .

K = an algebraically closed field of characteristic zero.

\mathbb{N} = the set of natural numbers.

$\mathbb{N}^* = \mathbb{N} \cup \{0\}$

\mathbb{C} = the field of complex numbers

$\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ = the multiplicative group of \mathbb{C} .

V = a finite dimensional vector space over \mathbb{C} .

$\text{Hom}_{\mathbb{C}}(V, V)$ = the set of all endomorphisms of V over \mathbb{C} .

$GL(V)$ = the general linear group, contained in $\text{Hom}_{\mathbb{C}}(V, V)$.

$GL_n(\mathbb{C})$ = the full group of invertible $n \times n$ matrices over \mathbb{C}

$GL_n(\mathbb{C})$ is isomorphic to $GL(V)$ if the dimension (V:C) of V over \mathbb{C} is n .

Ω = the set containing n objects here represented by

$\{1, 2, 3, \dots, n\}$.

CHAPTER I

ORDINARY REPRESENTATION THEORY OF FINITE GROUPS

Our aim in this chapter is to establish much of the notation used in the ordinary representation theory of finite groups, and to give some basic results on the subject. For most of the material in this chapter we follow closely Burrow [1] and Curtis and Reiner [6]. All representations considered here are over the field \mathbb{C} of complex numbers, though the general theory used applies equally to any algebraically closed field K of characteristic zero.

1.1 BASIC CONCEPTS

DEFN 1.1.1

Let $\text{Hom}_{\mathbb{C}}(V, V)$ be the set of all endomorphisms T of a finite dimensional vector space V over the field \mathbb{C} of complex numbers. Then T is said to be reducible if V contains a non-trivial subspace U which is invariant under T (i.e. $Tu = u, \forall u \in U$). T is said to be irreducible if no non-trivial subspace of V is invariant under T . T is said to be completely reducible if whenever U is a non-trivial invariant subspace of V , there exists

another non-trivial invariant subspace W of V such that V is a direct sum of U and W (i.e. $V = U \oplus W$).

DEFN 1.1.2

An ordinary (linear) representation of G with representation space V is a homomorphism T of G into $GL(V)$ (i.e. $T(gh) = T(g)T(h)$ and $T(e) = I_V \forall g, h, e \in G$ where I_V is the identity element of V).

Two ordinary representations T and T' of G with representation spaces V and V' respectively are said to be equivalent if there exists a \mathbb{C} - isomorphism S of V into V' such that $T'(g)S = ST(g)$ for all $g \in G$. The dimension $(V:\mathbb{C})$ of V over \mathbb{C} is called the degree of T denoted by $\text{deg } T$.

An ordinary representation T is said to be reducible or completely reducible if its representation space is so.

The following is an alternative definition to definition 1.1.2.

DEFN 1.1.3

A matrix representation of G of degree n is a homomorphism A of the group G into the full matrix group $GL_n(\mathbb{C})$ (i.e. $A(gh) = A(g)A(h)$ and $A(e) = I_n \forall g, h, e \in G$ where I_n is the identity element of $GL_n(\mathbb{C})$).

Two matrix representations A and A' are equivalent if there exists a fixed invertible matrix P in $GL_n(\mathbb{C})$ such that $A'(g)P = PA(g)$ for all $g \in G$. Equivalent matrix representations have the same degree.

The concepts of reducibility, irreducibility and complete reducibility given above are immediately applicable to matrix representations of G .

REMARK 1.1.4

Let a matrix representation A of G be equivalent

to

$$A'(g) = \begin{pmatrix} A_1(g) & B(g) \\ 0 & A_2(g) \end{pmatrix}$$

for all $g \in G$. Then

- (i) A is reducible
- (ii) A is completely reducible if $B(g) = 0$.

In all that follows, a representation of G shall mean an ordinary representation (here abbreviated by o.r.) of G . An irreducible ordinary representation of G shall be abbreviated by i.o.r.

DEFN 1.1.5

Let G be a group. Let $\mathbb{C}G = \{ \sum_{g \in G} \alpha_g g \mid \alpha_g \in \mathbb{C} \}$ be a set of formal sums of elements of G . Define addition and scalar multiplication on $\mathbb{C}G$ component-wise and multiplication by

$\left(\sum_{g \in G} \alpha_g g \right) \left(\sum_{g' \in G} \alpha_{g'} g' \right) = \sum_{g, g' \in G} \alpha_g \alpha_{g'} (gg')$. Then $\mathbb{C}G$ is an associative algebra called the group algebra of G .

In what follows a representation of a group algebra $\mathbb{C}G$ shall mean an algebra homomorphism of $\mathbb{C}G$.

LEMMA 1.1.6

There is a bijection between the o.r.'s of a group G and those of its group algebra $\mathbb{C}G$. Furthermore there exists a bijection between the o.r.'s of $\mathbb{C}G$ and finite dimensional left $\mathbb{C}G$ -modules M .

PROOF

Let T' be an o.r. of $\mathbb{C}G$ with representation space M . Define $T: G \rightarrow GL(M)$ by $T(g) = T'(\alpha_g g) \forall g \in G$, i.e. a restriction of T' to G .

Then

$T(gh) = T'(\alpha_{gh} gh) = T'(\alpha_g g)T'(\alpha_h h) = T(g)T(h)$ i.e T is a homomorphism. Also $T(e) = T'(\alpha_e e) = I_M$. Thus T is an o.r. of G corresponding to T' . Conversely let T be an o.r. of G with representation space M . Define

$$T' : \mathbb{C}G \rightarrow GL(M) \text{ by } T'(a) = \sum_{g \in G} \alpha_g T(g),$$

where $a = \sum_{g \in G} \alpha_g g$, $\alpha_g \in \mathbb{C}$. Then if $a' = \sum_{h \in G} \alpha'_h h$,

we have

$$\begin{aligned} T'(aa') &= \sum_{gh \in G} \alpha_{gh} T(gh), \text{ where } \alpha_{gh} = \alpha_g \alpha_h \\ &= \sum_{g, h \in G} \alpha_g \alpha_h T(g)T(h) \\ &= \left(\sum_{g \in G} \alpha_g T(g) \right) \left(\sum_{h \in G} \alpha_h T(h) \right) = T'(g)T'(h). \end{aligned}$$

If now this T' is restricted to G we get back T . Thus T' corresponds to T .

Further, consider an o.r. T of G with representation space M . Let $r = \sum_{g \in G} \alpha_g g \in \mathbb{C}G$. Let $m \in M$. Define rm by $rm = \left(\sum_{g \in G} \alpha_g T(g)m \right) \in M$. Then since M is an additive abelian group it is easily seen to be a $\mathbb{C}G$ -module corresponding to T .

Conversely let $\{m_i / i=1, \dots, n\}$ be a \mathbb{C} -basis of a left $\mathbb{C}G$ -module M and $r \in \mathbb{C}G$. Let $G = \{g_j / j=1, \dots, n\}$.

We have

$$\begin{aligned}
 rm_i &= \left(\sum_{g_j \in G} \alpha_{g_j} g_j \right) \left(\sum_{i=1}^n a_i m_i \right) \\
 &= \sum_{i=1}^n \left(\sum_{g_j \in G} (\alpha_{g_j} g_j) a_i m_i \right) \\
 &= \sum_{i=1}^n \left(\sum_{g_j \in G} \begin{pmatrix} a_i & \alpha_{g_j} \end{pmatrix} \right) g_j m_i \\
 &= \sum_{i=1}^n \left(\sum_{g_j \in G} (a_{i,j}) \right) g_j m_i
 \end{aligned}$$

Now take $T(g_j) = (a_{i,j})$

Then T is an o.r. of G with representation space M .

■

Thus by Lemma 1.1.6, the problem of classifying o.r's of G is equivalent to the problem of classifying finite dimensional left $\mathbb{C}G$ -modules up to isomorphism. We have the following theorem on the reducibility of representations of a group G .

THEOREM 1.1.7 (MASCHKES)

Let G be a group, then every $\mathbb{C}G$ -module M is completely reducible.

PROOF

Let M_1 be a non-trivial $\mathbb{C}G$ -submodule of M . We construct a non-trivial $\mathbb{C}G$ -submodule M_2 of M such that $M=M_1 \oplus M_2$, a direct sum. Considering M as a vector space there exists a $\mathbb{C}G$ -subspace U such that $M=M_1 \oplus U$. For if $\phi:M \rightarrow M$ is a linear map given by $\phi(m) = m_1$, $m \in M$, $m_1 \in M_1$. Then $\phi(M)=M_1 \subset M$ and $\phi^2=\phi$ by definition of ϕ .

Now define $T:M \rightarrow M$ by

$$T(m) = 1/|G| \sum_{g \in G} g\phi g^{-1}(m).$$

Then $\forall h \in G$, we have

$$\begin{aligned} hTh^{-1} &= 1/|G| \sum_{g \in G} hg\phi g^{-1}h^{-1} \\ &= 1/|G| \sum_{hg \in G} hg\phi(hg)^{-1} \\ &= T, \end{aligned}$$

as hg runs through G as g does. Now since T is a linear combination of ϕ , it follows that $T^2(m)=T(T(m))=T(m_1)=m_1=T(m) \forall m \in M$ that is $T^2=T$ on M .

We have

$$T((1_M - T)(m)) = T(m - T(m)) = T(m) - T^2(m) = m_1 - m_1 = 0$$

that is $\ker T = (1_M - T)(M)$. We also have

$$(1_M - T)(T(m)) = T(m) - T^2(m) = m_1 - m_1 = 0 \quad \text{that is}$$

$$T(m) = \ker(1_M - T). \text{ Hence } \ker(T) \cap T(M) = \{0\}$$

$$\text{and } M = T(M) \oplus \ker(T). \text{ Now let } M_1 = T(M)$$

and $M_2 = \ker(T)$ which gives the result $M = M_1 \oplus M_2$.

■

The above result has the following consequence in terms of o.r.'s of G .

COROLLARY 1.1.8

Let $T:G \longrightarrow GL_n(\mathbb{C})$ be an o.r of G with representation space M . Then there exists an invertible matrix S in $GL_n(\mathbb{C})$ such that $\forall g \in G$, we have

$$S^{-1}T(g)S = \begin{pmatrix} T_1(g) & & & 0 \\ & T_2(g) & & \\ & & \ddots & \\ 0 & & & T_r(g) \end{pmatrix}$$

where T_i ($i = 1, \dots, r$) are i.o.r's of G . We write $T \cong T_1 \dot{+} \dots \dot{+} T_r$.

Let $M_{n_i}(\mathbb{C})$, where $i = 1, 2, \dots, s$ be the full matrix algebra of $n_i \times n_i$ matrices over \mathbb{C} . Since $\mathbb{C}G$ is completely reducible as a $\mathbb{C}G$ -module, it is a direct sum: $\mathbb{C}G \cong M_{n_1}(\mathbb{C}) \oplus \dots \oplus M_{n_s}(\mathbb{C})$ and this decomposition of $\mathbb{C}G$ is unique (see[6]). Since matrix multiplication is non-abelian, the summands $M_{n_i}(\mathbb{C})$ of $\mathbb{C}G$ are non-abelian so that $\mathbb{C}G$ is semisimple. Furthermore each $M_{n_i}(\mathbb{C})$ is completely reducible by theorem 1.1.7. Thus there exists inequivalent $\mathbb{C}G$ -modules M_i ($i = 1, \dots, s$) such that $M_{n_i}(\mathbb{C})$ is isomorphic to a direct sum of m_i copies of M_i , where $m_i = (M_i : \mathbb{C})$, that is $M_{n_i}(\mathbb{C}) = M_i \oplus \dots \oplus M_i$ (m_i copies). we need the following definition.

DEFN 1.1.9

An element φ of $\mathbb{C}G$ is called an idempotent if $\varphi^2 = \varphi$. An idempotent φ is called a primitive idempotent of $\mathbb{C}G$ if $\varphi \neq \varphi_1 + \varphi_2$, where $\varphi_1^2 = \varphi_1, \varphi_2^2 = \varphi_2$ and $\varphi_1\varphi_2 = \varphi_2\varphi_1 = 0$. It is clear that an idempotent φ of $\mathbb{C}G$ is primitive if and only if the ideal generated by φ is minimal (see [1] or [6]).

The following combinatorial lemma is due to John Von Neumann (see Burrow [1]).

LEMMA 1.1.10

Let H_1 and H_2 be two subgroups of G with representations T_1 and T_2 respectively, both of degree one. Suppose that for all $g \in G$, $g \notin H_2H_1$ if and only if there exists $h_1 \in H_1$, $h_2 \in H_2$ such that $gh_1g^{-1} = h_2$ and $T_1(h_1) \neq T_2(h_2)$. Then $\varphi = PN$ is a scalar multiple of a primitive idempotent, where

$$P = \sum_{h_1 \in H_1} h_1 T_1(h_1) \text{ and } N = \sum_{h_2 \in H_2} h_2 T_2(h_2).$$

PROOF

$$\text{Note that } Ph'_1 T_1(h'_1) = \sum_{h'_1, h_1 \in H_1} h_1 T_1(h_1) h'_1 T_1(h'_1)$$

$$= \sum_{h_1 h'_1 \in H_1} h_1 h'_1 T_1(h_1 h'_1)$$

$$= P, \text{ for } h'_1 \in H_1.$$

Similarly we can show that $Nh'_2 T_2(h'_2) = N$. Now consider $PNgPN$. If $g \in H_2H_1$, $g = h_2h_1$ say, then we have $PNgPN = PNh_2h_1PN = T_1^{-1}(h_1)T_2^{-1}(h_2)P(Nh_2T_2(h_2))(h_1T_1(h_1)P)N$

$$= T_1^{-1}(h_1)T_2^{-1}(h_2)(PN)^2,$$

where $T_i^{-1}(h_i)$ is the reciprocal of the scalar $T_i(h_i)$, T_i being of degree one, $i=1,2$. If $g \notin H_2H_1$, then

$$\begin{aligned} \text{PNgPN} &= T_1(h_1) \text{PNgh}_1 g^{-1} \text{gPN} = T_1(h_1) \text{PN } h_2 \text{gPN} \\ &= T_1(h_1) T_2^{-1}(h_2) \text{PNgPN}. \end{aligned}$$

Hence $\text{PNgPN}(1 - T_1(h_1) T_2^{-1}(h_2)) = 0$. This implies $\text{PNgPN} = 0$ since $T_1(h_1) \neq T_2^{-1}(h_2)$, and this with $\varphi = \text{PN}$ yields

$$\varphi \mathbb{C}G\varphi = \mathbb{C}\varphi^2$$

Now $\varphi \neq 0$ since the coefficient of the identity

I in PN is

$$k = \sum_{\substack{h_2 \in H_2 \\ h_1 \in H_1}} T_1(h_1) T_2(h_2)$$

$\forall h_1, h_2$ such that $h_1 h_2 \in I$, that is all $h_1, h_2 \in H_1 \cap H_2$. But $Ih I_1^{-1} = h_2^{-1}$ and since $I \in H_2 H_1$, we have $T_1(h_1) = T_2(h_2^{-1})$. So that $k = |H_1 \cap H_2| \neq 0$. Also $\varphi^2 \neq 0$ otherwise $\mathbb{C}G\varphi\mathbb{C}G\varphi = 0$, implying $\mathbb{C}G\varphi$ is a non-zero abelian ideal of $\mathbb{C}G$ contradicting the fact that $\mathbb{C}G$ is semi-simple.

Similarly on considering Pg^{-1}N we obtain $\text{PCGN} = \mathbb{C}\text{PN} = \mathbb{C}\varphi$. Since $\text{PCGN} \supset \text{PNCGPN}$, we get $\mathbb{C}\varphi \supset \varphi\mathbb{C}G\varphi$. Now this with $\varphi\mathbb{C}G\varphi = \mathbb{C}\varphi^2$ yields $\varphi^2 = k\varphi$, where $k \in \mathbb{C}$. Now φ is primitive (otherwise $\varphi = \varphi_1 + \varphi_2$ with $\varphi_1 \varphi_2 = 0$. But $\varphi_1 \neq 0$ and by $\varphi^2 = k\varphi$, we have $\varphi_2 = 0$, a contradiction). Thus if G satisfies the conditions of the lemma, then $\varphi = \varphi^2 = k\varphi$ is a primitive idempotent of $\mathbb{C}G$.

We now wish to determine the number of inequivalent i.o.r's of G over the field \mathbb{C} of complex numbers. First we prove some preliminary lemmas.

LEMMA 1.1.11

Let G be a group. Let \mathbb{C} be the complex field and $Z(\mathbb{C}G)$ be the centre of $\mathbb{C}G$. Then

$$Z(\mathbb{C}G) = Z(M_{n_1}(\mathbb{C})) \otimes \dots \otimes Z(M_{n_s}(\mathbb{C})) \text{ and } (Z(\mathbb{C}G) : \mathbb{C}) = s.$$

PROOF

From Corollary 1.1.8 $\mathbb{C}G \cong M_{n_1}(\mathbb{C}) \otimes \dots \otimes M_{n_s}(\mathbb{C})$.

Hence $Z(\mathbb{C}G) = Z(M_{n_1}(\mathbb{C})) \otimes \dots \otimes Z(M_{n_s}(\mathbb{C}))$. Further the only matrices which commute with all matrices $M_{n_i}(\mathbb{C})$ are scalar multiples of I_{n_i} . Thus $(Z(M_{n_i}(\mathbb{C})) : \mathbb{C}) = 1$. Hence

$$(Z(\mathbb{C}G) : \mathbb{C}) = s$$

LEMMA 1.1.12

Let $\{h_1, h_2, \dots, h_t\}$ be the set of conjugacy classes of G . Define $C_i = \sum_{g \in h_i} g$ ($i=1, \dots, t$). Then $\{C_1, \dots, C_t\}$ is a \mathbb{C} -basis of $Z(\mathbb{C}G)$.

PROOF

First we show that the elements of $\{C_1, \dots, C_t\}$ belong to $Z(\mathbb{C}G)$. For all $g \in G$, $gC_1g^{-1} = \sum_{x \in h_1} gxg^{-1} = C_1$. Furthermore since the C_i 's are sums of group elements of disjoint sets h_i , the set $\{C_1, \dots, C_t\}$ is linearly independent over \mathbb{C} . Now let $h = \sum_{g \in G} \alpha_g g \in Z(\mathbb{C}G)$. Then for each $y \in G$, $\sum \alpha_g g = h = yhy^{-1} = \sum \alpha_g ygy^{-1}$. Therefore $\alpha_{y^{-1}gy} = \alpha_g$ for all $y \in G$. Hence $\alpha_g = \alpha_{g'}$, if and only if g and g' belong to the same conjugacy class h_i of G . Therefore h is a \mathbb{C} -linear combination of the set $\{C_1, \dots, C_t\}$.

■

We now prove the following theorem.

THEOREM 1.1.13

Keeping the notation above, the number of inequivalent irreducible $\mathbb{C}G$ - modules M is equal to the number of conjugacy classes of G .

PROOF

By lemma 1.1.11 $(Z(\mathbb{C}G) : \mathbb{C}) = s$. And the number of elements of the \mathbb{C} -basis of $Z(\mathbb{C}G)$ is t , which is equal to the number of conjugacy classes of G . Therefore $t = s$.

LEMMA 1.1.14 (SCHUR'S LEMMA).

Let T and T' be irreducible matrix representations of G of degree f and f' respectively. Let P be an $f \times f'$ matrix such that $T(g)P = PT'(g)$, for all $g \in G$. Then either $P = 0$ and T and T' are not equivalent or $f' = f$ and $|P| \neq 0$ in which case T and T' are equivalent.

PROOF

Suppose that $P \neq 0$. Then there exists non-singular matrices A and B such that $APB = N$ where

$$N = \left(\begin{array}{c|c} I_{rr} & O_{rt} \\ \hline O_{sr} & O_{st} \end{array} \right)_{f \times f'}$$

Now let $S(g) = AT(g)A^{-1}$ and $S'(g) = B^{-1}T'(g)B$ for all $g \in G$. Then since $T(g)P = PT'(g)$, we have

$$A^{-1}S(g)AP = PBS'(g)B^{-1}$$

if and only if $S(g)N = NS'(g)$, where $N=APB$, i.e $S(g)$ is of the form

$$\left(\begin{array}{c|c} S_{rr}(g) & S_{rs}(g) \\ \hline O_{sr}(g) & S_{ss}(g) \end{array} \right)$$

and $S'(g)$ is of the form

$$\left(\begin{array}{c|c} S'_{rr}(g) & O_{rt}(g) \\ \hline S'_{tr}(g) & S'_{tt}(g) \end{array} \right) \bullet$$

Hence S and S' are reducible representations, which contradicts the irreducibility of T and T' unless $r=f=f'$.

Therefore $\text{rank } P = f$, so that $|P| \neq 0$, and we can therefore write

$$T'(g) = P^{-1}T(g)P, \text{ for all } g \in G$$

i.e T and T' are equivalent.

COROLLARY 1.1.15

Let T be an irreducible representation of G of degree f over \mathbb{C} such that $P^{-1}T(g)P = T(g)$ for all $g \in G$, where P is an invertible matrix. Then $P = \lambda I_f$, where $\lambda \in \mathbb{C}^*$.

PROOF

Since we are working in the field of complex numbers, P has an eigenvalue, λ say. i.e $|P - \lambda I| = 0$.

Hence by hypothesis we have

$$T(g)(P-\lambda I_f) = (P-\lambda I_f)T(g) \quad \forall g \in G.$$

The result now follows from theorem 1.1.14.

We need the following definition.

DEFN 1.1.16

Let A be a matrix representation of G of degree n . Define χ_A by $\chi_A(g) = \text{trace}(A(g))$ for all $g \in G$. Then χ_A is called a character of G afforded by A . For brevity this shall be known simply as the character of A .

We now prove the following result.

THEOREM 1.1.17

Let χ_T denote the character of a representation T of G . Then

- (i) χ_T is a class function on G
- (ii) if S and T are equivalent matrix representations of G then $\chi_S(g) = \chi_T(g)$ for all $g \in G$.

PROOF

(i) Since similar matrices have the same characteristic polynomial, they have the same trace. Now since $T(x^{-1}gx)$ as a matrix is similar to $T(g)$, we have

$$\chi_T(x^{-1}gx) = \text{Trace}T(x^{-1}gx) = \text{Trace}T(g) = \chi_T(g)$$

i.e χ_T is a class function.

(ii) If S and T are equivalent representations, then by definition $S(g)$ and $T(g)$ are similar matrices for all $g \in G$. The result is now immediate from (i) above.

THEOREM 1.1.18 (SCHUR'S RELATIONS)

Let T_1 and T_2 be irreducible matrix representations of G , where

$$T_1(g) = (\alpha_{ij}(g))_{n \times n} \text{ and } T_2(g) = (B_{ij}(g))_{m \times m}.$$

Then if T_1 and T_2 are inequivalent,

$$(i) \sum_{g \in G} \alpha_{is}(g^{-1}) B_{tj}(g) = 0 \quad \forall i, s, t, j$$

Otherwise if T_1 and T_2 are equivalent,

$$(ii) \sum_{g \in G} \alpha_{is}(g^{-1}) \alpha_{tj}(g) = \frac{|G|}{n} \delta_{ij} \delta_{st} \quad \forall i, j, s, t.$$

PROOF

Let S be any $n \times m$ matrix and define an $n \times m$ matrix

$$f(S) \text{ by } f(S) = \sum_{g \in G} T_1(g^{-1}) S T_2(g)$$

Then $T_1(h) f(S) = f(S) T_2(h)$, $\forall h \in G$. Thus if in particular we set $S = E_{st}$, the $n \times m$ matrix with 1 in the (s,t) - position and 0 elsewhere.

Then

$$f(E_{st}) = \sum_{g \in G} (\alpha_{ij}(g^{-1}) (E_{st}) (\beta_{ij}(g))).$$

Therefore by Schurs lemma

- (i) if T_1 and T_2 are not equivalent then $f(E_{st}) = 0$ so that

$$\sum_{g \in G} \alpha_{is}(g^{-1}) B_{tj}(g) = 0, \quad \forall i, j, s, t.$$

- (ii) If T_1 and T_2 are equivalent, then $f(E_{st})$ commutes with T_1 , so that by corollary 1.1.15

$$\begin{aligned} f(E_{st}) &= \lambda_{st} I_n, \text{ for some } \lambda_{st} \in \mathbb{C}^* \\ &= \lambda \delta_{st} \delta_{ij}. \end{aligned}$$

Now $(\alpha_{ij}(g^{-1})) (\alpha_{ij}(g)) = I_n$ if and only if

$$\sum_{s=1}^n \alpha_{is}(g^{-1}) \alpha_{sj}(g) = \delta_{ij} \quad \text{and}$$

$$n\lambda = \sum_{s=1}^n \lambda = \sum_{s=1}^n \left(\sum_{g \in G} \alpha_{is}(g^{-1}) \alpha_{si}(g) \right) = \sum_{g \in G} 1 = |G|$$

Therefore $\lambda = \frac{|G|}{n}$ so that $f(E_{st}) = \frac{|G|}{n} \delta_{st} \delta_{ij}$. giving the result.

DEFN 1.1.19

Let T_1 and T_2 be o.r's of G with characters χ_{T_1} and χ_{T_2} respectively. Then the inner product (χ_{T_1}, χ_{T_2}) , of the characters of G is defined by

$$(\chi_{T_1}, \chi_{T_2}) = \frac{1}{|G|} \sum_{g \in G} \chi_{T_1}(g) \chi_{T_2}(g^{-1}).$$

we have the following lemma.

LEMMA 1.1.20

Let T_1 and T_2 be i.o.r's of G with characters χ_{T_1} and χ_{T_2} respectively. Then

- (i) $(\chi_{T_1}, \chi_{T_2})_G = 0$, if T_1 is not equivalent to T_2 .
- (ii) $(\chi_{T_1}, \chi_{T_2})_G = 1$, if T_1 is equivalent to T_2 .

PROOF

Let $T_1(g) = (\alpha_{ij}(g))_{n \times n}$ and $T_2(g) = (\beta_{ij}(g))_{m \times m}$.
Then $\chi_{T_1}(g) = \sum_{i=1}^n \alpha_{ii}(g)$ and $\chi_{T_2}(g) = \sum_{i=1}^m \beta_{ii}(g)$

(i) if T_1 is not equivalent to T_2 , then

$$(\chi_{T_1}, \chi_{T_2})_G = \frac{1}{|G|} \sum_{g \in G} \chi_{T_1}(g) \chi_{T_2}(g^{-1}) = 0$$

by theorem 1.1.18.

(ii) Now suppose T_1 is equivalent to T_2 then $\chi_{T_1} = \chi_{T_2}$

and we have

$$\begin{aligned}
 (\chi_{T_1}, \chi_{T_2}) &= \frac{1}{|G|} \sum_{g \in G} \chi_{T_1}(g) \chi_{T_1}(g^{-1}) \\
 &= \frac{1}{|G|} \sum_{i=1}^n \sum_{i=1}^n \left(\sum_{g \in G} \alpha_{i1}(g) \alpha_{i1}(g^{-1}) \right) \\
 &= \frac{1}{|G|} \sum_{i=1}^n \frac{|G|}{n} = \sum_{i=1}^n \frac{1}{n} = 1.
 \end{aligned}$$

We now prove the following theorem.

THEOREM 1.1.21

Let T be an o.r of G with character χ_T . Then T is an irreducible representation of G if and only if

$$(\chi_T, \chi_T)_G = 1.$$

PROOF

If T is irreducible then lemma 1.1.20 gives the result.

If $(\chi_T, \chi_T)_G = 1$, then by Maschke's theorem T is completely reducible, being a direct sum of irreducible representations

$$\text{i.e. } T(g) \cong \begin{pmatrix} T_1(g) & & & 0 \\ & T_2(g) & & \\ & & \ddots & \\ 0 & & & T_s(g) \end{pmatrix}$$

and we write $T(g) \cong T_1(g) + T_2(g) + \dots + T_s(g)$. Let χ_{T_i} be a character of T_i of G .

$$\begin{aligned} \text{Then } (\chi_T, \chi_T)_G &= \frac{1}{|G|} \sum_{g \in G} \chi_T(g) \chi_T(g)^{-1} \\ &= \frac{1}{|G|} \sum_{g \in G} \left(\sum_{i=1}^s (a_i \chi_{T_i}(g)) \right) \left(\sum_{i=1}^s a_i \chi_{T_i}(g)^{-1} \right) \\ &= \sum_{i=1}^s a_i^2 \left(\frac{1}{|G|} \sum_{i=1}^s \chi_{T_i}(g) \chi_{T_i}(g^{-1}) \right) \\ &= \sum_{i=1}^s a_i^2, \text{ since the inner sum is 1 by} \end{aligned}$$

lemma 1.1.20.

But $(\chi_T, \chi_T) = 1 = \sum_{i=1}^s a_i^2$ implies that $a_i = 1$ for only one i such that $1 \leq i \leq s$ and $a_j = 0, \forall j \neq i$. By lemma 1.1.20, T has one irreducible constituent, ie. T is irreducible.

LEMMA 1.1.22

Every irreducible representation of G is a component of the regular representation of G .

PROOF

Let r_G be the regular character of G . Since for $g' \neq e, g \in G, eg = g, gg' \neq g, r_G(e) = |G|$ and $r_G(g) = 0 \forall g(\neq e) \in G$.

Let χ_T be the character of any irreducible representation T of G, then

$$\begin{aligned}
 (r_G, \chi_T) &= \frac{1}{|G|} \left(\sum_{g \in G} r_G(g) \chi_T(g^{-1}) \right) \\
 &= \frac{1}{|G|} \left(r_G(e) \chi_T(e) + \sum_{\substack{g \in G \\ g \neq e}} r_G(g) \chi_T(g^{-1}) \right) \\
 &= \chi_T(e) + 0 \\
 &= \text{Trace}(T(e)) = \text{Trace } I_f = f
 \end{aligned}$$

where f is the degree of T. We have $f = (r_G, \chi_T)$ is the number of times χ_T appears as a component of r_G .

We now obtain an alternative to theorem 1.1.18.

THEOREM 1.1.23 (SECOND ORTHOGONALITY RELATIONS)

Let $\{T_1, T_2, \dots, T_s\}$ be a complete set of i.o.r's of G with characters $\chi_{T_1}, \dots, \chi_{T_s}$ of degrees n_1, \dots, n_s respectively. Let Cl_i denote the ith conjugacy class of G, and Cl_i^* be the class containing the inverse of $g \in Cl_i$. Let $h_i = |Cl_i|$ be the order of Cl_i . Let χ_i^j denote the value of χ_i on Cl_j . Then

(i) if Cl_j is not inverse to Cl_k ,

$$\sum_{i=1}^s \chi_i^j \chi_i^k = 0$$

(ii) if Cl_j is inverse to Cl_k ,

$$\sum_{i=1}^s \chi_i^j \chi_i^k = \frac{|G|}{h_j}$$

PROOF

Let $C_i = \sum_{g \in Cl_i} g$. Then $C_j g = g C_j \quad \forall g \in G$ by lemma 1.1.12. Therefore we have $T_i(C_j g) = T_i(g C_j)$ if and only if $T_i(C_j) T_i(g) = T_i(g) T_i(C_j)$, $\forall g \in G$. Thereby by Schur's lemma, we have $T_i(C_j) = w_i^j I_{n_i}$. And on computing the traces we have $h_j \chi_i(g) = n_i w_i^j$ (where $n_i = \text{trace } I_{n_i}$). If now χ_i^j denotes the value of χ_i on an

element g of Cl_j , then we have

$$w_i^j = \frac{h_j \chi_i^j}{n_i}.$$

Now since $\{C_1, \dots, C_s\}$ is a \mathbb{C} -basis for $Z(G)$, then

$$C_j C_k = \sum_{l=1}^s \alpha_{jkl} C_l, \text{ where } \alpha_{jkl} \in \mathbb{C}.$$

Hence $T_i(C_j) T_i(C_k) = \sum_{l=1}^s \alpha_{jkl} T_i(C_l)$ if and only if

$$w_i^j w_i^k = \sum_{l=1}^s \alpha_{jkl} w_i^l \text{ if and only if}$$

$$\frac{h_j \chi_i^j}{n_i} \cdot \frac{h_k \chi_i^k}{n_i} = \sum_{l=1}^s \alpha_{jkl} \frac{h_l \chi_i^l}{n_i}$$

Now if r_G is the regular character of G then lemma

1.1.22 implies that $r_G^1 = \sum_{i=1}^s n_i \chi_i^1$.

$$\begin{aligned} \text{Hence } h_j h_k \sum_{i=1}^s \chi_i^j \chi_i^k &= \sum_{i=1}^s \alpha_{jki} h_i r_G^1 \\ &= \alpha_{jki} |G|, \end{aligned}$$

$$\text{since } r_G^1 = \begin{cases} |G| & \text{if } Cl_1 = \{e\} \\ 0 & \text{otherwise.} \end{cases}$$

Now $|Cl_j^*| = |Cl_j| = h_j$., we have

$$\alpha_{j k 1} = \begin{cases} h_j & \text{if } Cl_j = Cl_j^* \\ 0, & \text{other wise.} \end{cases}$$

Therefore $\sum_{i=1}^s \chi_i^j \chi_i^k = 0$ if Cl_j is not inverse to Cl_k .

$$\frac{h_j}{|G|} \sum_{i=1}^s \chi_i^j \chi_i^k = 1 \text{ if } Cl_k \text{ is inverse to } Cl_j.$$

1.2 CONSTRUCTION OF NEW REPRESENTATIONS OF FINITE GROUPS

In this section we describe the methods of constructing new o.r's of a finite group from old ones to be used here.

DEFN 1.2.1

Let $A = (a_{ij})$ and $B = (b_{ij})$ be two matrices of degree s and r respectively. Then the tensor (Kronecker) product of matrices A and B is defined by

$$A \otimes B = \begin{bmatrix} a_{11} B & a_{12} B & \dots & a_{1s} B \\ a_{21} B & a_{22} B & \dots & a_{2s} B \\ \vdots & \vdots & \ddots & \vdots \\ a_{s1} B & a_{s2} B & \dots & a_{ss} B \end{bmatrix} .$$

Now let T_1 and T_2 be representations of G of degree s and r respectively. Define T by

$$T(g) = T_1(g) \otimes T_2(g) \in GL(sr, \mathbb{C}).$$

Then T as defined is a representation of G of degree sr called the tensor (Kronecker) product representation of G . We shall write $T = T_1 \otimes T_2$.

The representation $T = T_1 \otimes T_2$ is in general reducible. Its reduction may yield new i.o.r's of G . The following construction gives rise to new i.o.r's of G . We have a definition.

DEFN 1.2.2

Let G be a group of form HxH' , where H and H' are groups. Let T and T' be o.r's of H and H' respectively. Then the map $T\#T'$ defined by

$$T\#T' (h, h') = T(h) \otimes T'(h')$$

is a representation of G called the outer tensor product representation.

LEMMA 1.2.3

Let $G = H \times H'$ be a group with outer tensor product representation $T \# T'$. Then $T \# T'$ is irreducible if and only if T and T' are both irreducible. Further if $\{T_1, \dots, T_r\}$ and $\{T'_1, \dots, T'_s\}$ are complete sets of inequivalent i.o.r's of H and H' respectively then $\{T_i \# T'_j \mid i = 1, \dots, r; j = 1, \dots, s\}$ is a complete set of inequivalent i.o.r's of G .

PROOF

It suffices to show the second part. Let $\chi_{T_i \# T'_j}$ be the character of $T_i \# T'_j$. Then

$$\begin{aligned} & \frac{1}{|H \times H'|} \sum_{(h, h') \in G} \chi_{T_i \# T'_j}(h, h') \chi_{T_k \# T'_l}(h, h')^{-1} \\ &= \frac{1}{|H| |H'|} \sum_{h \in H} \sum_{h' \in H'} \chi_{T_i}(h) \chi_{T'_j}(h') \chi_{T_k}(h^{-1}) \chi_{T'_l}(h')^{-1} \end{aligned}$$

where χ_{T_i} is the character of T_i and $\chi_{T'_j}$ is that of T'_j ($i, k = 1, \dots, r$), ($j, l = 1, \dots, s$)

$$\begin{aligned} &= \left(\frac{1}{|H|} \sum_{h \in H} \chi_{T_i}(h) \chi_{T_k}(h^{-1}) \right) \left(\frac{1}{|H'|} \sum_{h' \in H'} \chi_{T'_j}(h') \chi_{T'_l}(h')^{-1} \right) \\ &= \begin{cases} 1 & \text{if } i = k, \quad j = l \\ 0 & \text{otherwise,} \end{cases} \quad \text{by theorem 1.1.18 and theorem 1.1.21.} \end{aligned}$$

We now consider the relationship between the o.r.'s of G and those of its subgroup H . Consider i.o.r's T and t of degree n and m of G and H respectively. We may limit consideration to those matrices of the i.o.r T of G associated with the elements of H . By such a restricting process we obtain an o.r. of H which we denote by T_H or $T \downarrow H$. Similarly, we may obtain an o.r of G from that t of a subgroup H of G as follows.

DEFN 1.2.4

Let H be a subgroup of G such that $G = \bigcup_{i=1}^s Hg_i$ is a decomposition of G into right cosets modulo H . Define

T by

$$T(x) = \begin{bmatrix} t(g_1 x g_1^{-1}) & t(g_1 x g_2^{-1}) \dots t(g_1 x g_s^{-1}) \\ t(g_2 x g_1^{-1}) & t(g_2 x g_2^{-1}) \dots t(g_2 x g_s^{-1}) \\ \vdots & \\ t(g_s x g_1^{-1}) & t(g_s x g_2^{-1}) \dots t(g_s x g_s^{-1}) \end{bmatrix}$$

$$= \dot{t}(g_i x g_j^{-1}), \quad i, j \in (1, \dots, s), \text{ where}$$

$$\dot{t}(g_i x g_j^{-1}) = \begin{cases} t(g_i x g_j^{-1}) & \text{if } g_i x g_j^{-1} \in H \\ 0 & \text{if } g_i x g_j^{-1} \notin H. \end{cases}$$

Then T is easily seen to be a representation of G of degree m s called a representation induced from t or the induced representation of G . We denote the induced representation of G by t^G or $t \uparrow G$.

Now let θ and χ be characters of H and G respectively. We shall write θ^G or $(\theta \uparrow G)$ for the character of t^G (or $t \uparrow G$) and χ_H (or $\chi \downarrow H$) for the character of G restricted to H . For the formula for the character of G induced from that of H we have

$$\begin{aligned} \theta^G(g) &= \text{trace}(\dot{t}(x_i g x_j^{-1})) \\ &= \sum_{i=1}^s \dot{\theta}(x_i g x_j^{-1}) \end{aligned}$$

Where $\dot{\theta}(g') = \theta(g')$ if $g' \in H$, or 0 if $g' \notin H$.

Now if $g \in G$, then g has the form Hg_j for some $j \in \{1, \dots, s\}$, where $s = [G:H]$.

Thus $g = hg_j$, for some $h \in H$, and we have $\dot{\theta}(x_i g x_j^{-1}) = \dot{\theta}(g_j g g_j^{-1})$

$$\text{Therefore } \theta^G(g) = \frac{1}{|G|} \sum_{x \in G} \dot{\theta}(x g x^{-1})$$

Now as x ranges over G , $x g x^{-1}$ are in the conjugacy class Cl_1 of G containing g . Thus as x ranges over G , $x g x^{-1}$ ranges over Cl_1 and each element $y \in Cl_1$ appears $\frac{|G|}{h_1}$ times, where $h_1 = |Cl_1|$.

Thus alternatively we have

$$\theta^G(g) = \frac{|G|}{|H|h_1} \sum_{y \in Cl_1 \cap H} \theta(y), \quad g \in Cl_1$$

The representation t^G is in general reducible.

The following results are associated with induced o.r's of G .

THEOREM 1.2.5 (FROBENIUS RECIPROCITY)

Let $H < G$ with characters θ and χ respectively.
Then $(\chi, \theta^G)_G = (\chi_H, \theta)_H$.

PROOF

$$\begin{aligned} (\chi, \theta^G)_G &= \frac{1}{|G|} \sum_{g \in G} \chi(g) \theta^G(g^{-1}) \\ &= \frac{1}{|G|} \sum_{t \in G} \left(\frac{1}{|H|} \sum_{g \in G} \chi(g) \dot{\theta}(t^{-1} g^{-1} t) \right) \end{aligned}$$

where $\dot{\theta}(t^{-1} g^{-1} t) = 0$ if $t^{-1} g^{-1} t \notin H$. Since

$\chi(t^{-1} g t) = \chi(g)$, $\forall t \in G$, we have

$$(\chi, \theta^G)_G = \frac{1}{|G||H|} \sum_{t \in G} \left(\sum_{g \in G} \chi(\bar{t}^{-1} g t) \dot{\theta}(\bar{t}^{-1} \bar{g}^{-1} t) \right)$$

Now for a fixed $t \in G$, g varies through G as $t^{-1}gt$ varies in a conjugacy class containing g , so that

$$\begin{aligned} (\chi, \theta^G)_G &= \frac{1}{|G||H|} \sum_{t \in G} \sum_{g' \in G} \chi(g') \dot{\theta}(g'^{-1}) \\ &= \frac{1}{|H|} \sum_{g' \in G} \chi(g') \dot{\theta}(g'^{-1}) \\ &= \frac{1}{|H|} \sum_{g' \in H} \chi_H(g') \theta(g'^{-1}) = (\chi_H, \theta)_H. \end{aligned}$$

Hence $(\chi, \theta^G)_G = (\chi_H, \theta)_H$.

LEMMA 1.2.6 (MACKEYS SUBGROUP FORMULA)

Let H and K be subgroups of G with characters χ_1 and χ_2 respectively. Let X be the set of representatives for the double cosets KxH in G , (i.e. $x \in X$).

Then
$$(\chi_1, (\chi_2^G)_H)_H = \sum_{x \in KxH} \left(\chi_1, ((\chi_2^x)_{xHx^{-1} \cap K})^H \right)_H$$

where $\chi_2^x(g) = \chi_2(x^{-1}gx)$.

PROOF (see [11] or [6]) ■

The following result for the calculation of characters constructed in a special way, will be used in the sequel.

LEMMA 1.2.7 (BRAUER)

If a group G has a pair of subgroups H_1 and H_2 with linear characters χ_1 and χ_2 respectively, such that the induced characters χ_1^G and χ_2^G have exactly one irreducible constituent in common in each case with multiplicity one, then this irreducible constituent character can be calculated using the formula

$$\chi (= \chi_1^G \wedge \chi_2^G, \text{ say}) (g) = \frac{n}{|\text{Ccl}(g)| |H_1 \cap H_2|} \sum \chi_1(h_1) \chi_2(h_2)$$

where $|\text{ccl}(g)|$ is the order of the conjugacy class of g , n is the degree of the irreducible constituent character and the summation is taken over all $h_1 \in H_1$, $h_2 \in H_2$ such that $h_1 h_2 \in \text{ccl}(g)$.

PROOF (see [2])

We consider yet another application of the subgroup structure of G to the construction of o.r.'s of G .

DEFN 1.2.8

Let H be a normal subgroup of index s in G . Let $X = \{g_i, i = 1, \dots, s\}$ be the set of representatives for the right cosets Hg_i in G . Let $T' : G/H \rightarrow GL_n(\mathbb{C})$ be an o.r. of the quotient group G/H of G modulo H .

Define

the composition $T = T' \circ h$ of the homomorphism $h: G \rightarrow G/H$ and T' by $T(g) = T'(Hg_1) \in GL_n(\mathbb{C})$, where g_1 represents g in X . Then $T = T' \circ h$ is an o.r. of G called a representation of G lifted from T' . we will denote $T = T' \circ h$ by $(T')^*$. Since $(T')^*$ takes the same values on G as T' does on G/H , the number of o.r.'s $(T')^*$ of G is equal to the number of the o.r.'s T' of G/H .

LEMMA 1.2.9

Let $(T')^*$ be a representation of G lifted from T' . Then $(T')^*$ is irreducible if and only if T' is irreducible.

PROOF

Let $\chi_{T'}$ and $\chi_{(T')^*}$ be characters of T' and $(T')^*$ respectively. Then $\chi_{(T')^*}(g) = \chi_{T'}(Hg)$ by definition. We show that $(\chi_{(T')^*}, \chi_{(T')^*})_G = 1$ if and only if $(\chi_{T'}, \chi_{T'})_{G/H} = 1$.

$$\begin{aligned} \text{Now } (\chi_{(T')^*}, \chi_{(T')^*})_G &= \frac{1}{|G|} \sum_{g \in G} \chi_{(T')^*}(g) \chi_{(T')^*}(g^{-1}) \\ &= \frac{1}{|G|} \sum_{g \in G} \chi_{T'}(Hg) \chi_{T'}(Hg^{-1}) \\ \left(\chi_{(T')^*}, \chi_{(T')^*} \right)_G &= \frac{1}{|G/H|} \sum_{Hg \in G/H} \chi_{T'}(Hg) \chi_{T'}(Hg^{-1}) \end{aligned}$$

$= \left(\chi_{T'} , \chi_{T'} \right)_{G/H}.$

Thus if $\left(\chi_{(T')^*} , \chi_{(T')^*} \right)_G = 1$, then $\left(\chi_{T'} , \chi_{T'} \right)_{G/H} = 1$
 and conversely.

CHAPTER II

REPRESENTATION THEORY OF THE SYMMETRIC
GROUP, S_n

Here we give a brief outline of the ordinary representation theory of the symmetric group, S_n which will be used in the sequel. The method of constructing the ordinary representations of S_n discussed here is due to A. Young [21]. First we describe the symmetric group.

2.1 THE SYMMETRIC GROUP, S_n DEFN 2.1.1

A bijective mapping of a set Ω onto itself is called a permutation of Ω . The set of all permutations of Ω together with the composition multiplication is a group called the symmetric group, S_n of degree n .

A permutation $\theta \in S_n$ of the form

$$\left(\begin{array}{cccc} 1 & 2 & \dots & k & k+1 & \dots & n \\ 2 & 3 & \dots & 1 & k+1 & \dots & n \end{array} \right) \quad (2.1)$$

is called a cycle of length k (or a k -cycle, to emphasize the number of symbols of Ω moved by θ) and is written $(123\dots k)$. Each permutation $\theta \in S_n$ can be written as a product of uniquely determined disjoint

cycles up to their order of occurrence. Such an expression of θ is called the cycle-notation of θ . The set of positive integers $\alpha_i, i=1, \dots, k$ representing the lengths of cycles in the cycle-notation of θ is called the cycle-type or structure of θ .

Taking $r_i = (i, i+1), (i=1, \dots, n-1)$, then it is well known that the group S_n has a presentation given by

$$\begin{aligned} S_n = \langle r_i (i=1, \dots, n-1) \mid r_i^2 = 1 (i=1, \dots, n-1), \\ (r_i r_{i+1})^3 = 1 (i=1, \dots, n-2), (r_i r_j)^2 = 1, \\ j \neq i, i+1, (i=1, \dots, n-1) \rangle \end{aligned} \quad (2.2)$$

S_n has order $|S_n| = n!$

DEFN 2.1.2

Let $(\alpha) = (t_1, \dots, t_k)$ be a set of non-negative integers such that $\sum_{i=1}^k t_i = n$ and $t_i \geq t_{i+1}$. Then (α) is called a partition of n . Here, we shall sometimes write (α_n) to emphasize the number being partitioned. To each partition (α) of n there corresponds a partition $(\alpha)' = (t'_1, t'_2, \dots, t'_k)$ where $t'_j = \sum_{j: j \geq i} 1$, called the associate or conjugate partition of n .

The ordered lengths $\alpha_1, \alpha_2, \dots, \alpha_k$ ($\alpha_i \geq \alpha_{i+1}$) of the cyclic factors in the cycle-notation of θ (including 1-cycles) form a partition $(\alpha) = (\alpha_1, \dots, \alpha_k)$ of n . If

θ and λ are conjugate elements in S_n then the cycle notation of λ is easily determined from that of θ .

LEMMA 2.1.3

Let $\theta, \mu \in S_n$. Then we obtain the conjugate $\mu\theta\mu^{-1}$ of θ by an application of μ to the symbols of Ω in θ .

PROOF

$$\begin{aligned} \text{This is clear since } \mu\theta\mu^{-1} &= \begin{pmatrix} i \\ \mu(i) \end{pmatrix} \begin{pmatrix} i \\ \theta(i) \end{pmatrix} \begin{pmatrix} \mu(i) \\ i \end{pmatrix} \\ &= \begin{pmatrix} \mu(i) \\ \mu\theta(i) \end{pmatrix}. \end{aligned}$$

■

Thus conjugate elements in S_n have the same cycle-structure. Furthermore, it is well known that the conjugacy classes of S_n are in one-one correspondence with the partitions of n (see [10] or [19]).

In constructing the o.r's of S_n we use certain representations of a special class of subgroups of S_n , called young subgroups in honour of A. Young for his pioneering work in this area.

DEFN 2.1.4

Let $(\alpha_1, \alpha_2, \dots, \alpha_k)$ be a partition of n . Define $P_0 = 0, P_i = \sum_{j=1}^i \alpha_j, (i=1, \dots, k)$. Denote by S_{α_i} the symmetric group on the α_i symbols $\{P_{i-1}+1, \dots, P_i\}$. Then the direct product $S_{\alpha_1} \times S_{\alpha_2} \dots \times S_{\alpha_k}$ is a subgroup of S_n called the young subgroup. Here we

shall denote the young subgroup of S_n by $S_{(\alpha_1, \dots, \alpha_k)}$ or simply $S_{(\alpha)}$.

We have the following analogous definition associated with the conjugate partition $(\alpha)'$ of n .

DEFN 2.1.5

Let $(\alpha)'$ be a conjugate partition of n . Define $P'_0 = 0$, $P'_i = \sum_{j=1}^i \alpha'_j$ ($i=1, \dots, k$). Denote by $S_{\alpha'_i}$ the

symmetric group on the α'_i symbols $\{P'_{i-1} + 1, \dots, P'_i\}$.

Then the direct product $S_{\alpha'_1} \times S_{\alpha'_2} \times \dots \times S_{\alpha'_k}$ is a subgroup of S_n called the conjugate young subgroup. Here we shall denote the conjugate young subgroup of S_n by $S_{(\alpha'_1, \alpha'_2, \dots, \alpha'_k)}$ or simply $S_{(\alpha)'}.$

2.2. REPRESENTATION THEORY OF SYMMETRIC GROUPS

We require the following definition.

DEFN 2.2.1

(i) A young diagram associated with a partition (α) of n (here written (α) -diagram) is a diagram consisting of n nodes in k -rows with α_i nodes in the i th row and all rows starting in the same column. An (α) -diagram shall be denoted by $[\alpha]$. For example for $n = 3$ and $(\alpha) = (21)$, then $[21] = \begin{matrix} \circ \circ \\ \circ \end{matrix}$ is a (21) -diagram.

The conjugate young diagram $[\alpha]'$ (here written $(\alpha)'$ -diagram) is the diagram obtained by interchanging the rows and columns in the (α) -diagram $[\alpha]$.

Now A young has shown that for algebraically closed fields of characteristic zero, each (α) -diagram $[\alpha]$ corresponds to a uniquely determined i.o.r. of S_n . This fact follows from definition 2.2.1 and our earlier remark that there is a bijection between the conjugacy classes of S_n and the partitions of n . Thus here we may denote by $[\alpha]$, also the o.r. of S_n associated with the (α) -diagram $[\alpha]$. We proceed to show how this correspondence between the i.o.r.'s of S_n and the (α) -diagrams $[\alpha]$ actually arises. First the (α) -diagram leads to the following definition.

DEFN 2.2.2

We keep the notation above. A young-tableau associated with a partition (α) (here written (α) -tableau) is one of the $n!$ arrays of symbols in Ω obtained by replacing each node in the (α) -diagram $[\alpha]$ by one of the symbols in Ω with no repetitions. An (α) -tableau is said to be a standard (α) -tableau if the symbols in it increase from left to right along each row and from top to bottom down each column. For example for $[21] = \overset{\circ\circ}{\circ}$,

1	2
3	

 and

2	1
3	

 are the standard and non-standard (21) -tableaux respectively.

If $f_{(\alpha)}$ is the number of the standard (α) -tableaux, then the standard (α) -tableaux will be denoted by

$$T_1^{[\alpha]}, \dots, T_{f_{(\alpha)}}^{[\alpha]}.$$

As we shall see later, the number $f_{(\alpha)}$ of standard (α) -tableaux gives the degree of the i.o.r. $[\alpha]$ of S_n (see eg. [9] or [19]). Now consider any of the standard (α) -tableaux, say $T_1^{[\alpha]}$. Then it is seen that the young subgroup $S_{(\alpha)}$ permutes only the symbols within the rows of $T_1^{[\alpha]}$ and the conjugate young subgroup $S_{(\alpha)'}$ permutes only the symbols within the columns of $T_1^{[\alpha]}$ (see eg. [9]). Thus each standard (α) -tableau of an (α) -diagram yields a pair of young-subgroups $S_{(\alpha)}$ and $S_{(\alpha)'}$ of S_n . For example, the group S_3 has three (α) -diagrams with standard (α) -tableaux given by

$$T_1^{[3]} =: \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array}, T_1^{[21]} =: \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}, T_1^{[1^3]} =: \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$$

(2.3)

whose pairs of young subgroups are $S_{(3)} = S_3$, $S_{(3)'} = S_1$; $S_{(21)} = \{e, (12)\}$, $S_{(21)'} = \{e, (13)\}$ and $S_{(1^3)} = S_1$, $S_{(1^3)'} = S_3$ respectively. Thus S_3 has three i.o.r.'s $[3]$ and $[1^3]$ each of degree 1 and $[21]$ of degree 2 (since

$$T_2^{[21]} = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} \text{ is another standard } (21)\text{-tableau}.$$

Since our aim is to determine i.o.r's of S_n associated with (α) -diagrams, following from the remarks after lemma 1.1.6, we now determine the representation spaces or modules which afford the i.o.r's $[\alpha]$. To this end it is enough to determine a \mathbb{C} -basis of the representation space of the i.o.r $[\alpha]$. We have the following lemma.

LEMMA 2.2.3

Let $\mathbb{C}S_n = A_1 \oplus \dots \oplus A_s$ be a decomposition of the group algebra $\mathbb{C}S_n$ of S_n into a unique direct sum of ideals $A_{(\alpha)}$ ($\alpha=1, \dots, s$). Then $A_{(\alpha)} = R_{(\alpha_1)} \oplus \dots \oplus R_{(\alpha_{f(\alpha)})}$ where the $R_{(\alpha_i)}$'s are the representation spaces of the i.o.r $[\alpha]$ of S_n . Furthermore the $R_{(\alpha_i)}$'s are generated by $\varphi_{ii}^{(\alpha)}$ ($i=1, \dots, f(\alpha)$).

PROOF

From corollary 1.1.8, since $\mathbb{C}S_n$ is completely reducible, it follows that $\mathbb{C}S_n = A_1 \oplus \dots \oplus A_s$ and this decomposition is unique with each $A_{(\alpha)}$ being expressed as a direct sum of $f_{(\alpha)}$ isomorphic minimal left ideals of $\mathbb{C}S_n$ of degree $f_{(\alpha)}$ over \mathbb{C} . Thus there are $f_{(\alpha)}^2$ elements $\varphi_{ij}^{(\alpha)}$ of $A_{(\alpha)}$ ($i=1, \dots, f_{(\alpha)}$) such that $\varphi_{ij}^{(\alpha)} \varphi_{kl}^{(\alpha)} = \delta_{jk} \varphi_{il}^{(\alpha)}$ where δ_{jk} is the Kronecker delta, and for any $a \in A_{(\alpha)}$, $a = \sum_{i,j=1}^{f_{(\alpha)}} k_{ij} \varphi_{ij}^{(\alpha)}$ where $k_{ij} \in \mathbb{C}$.

Also

$\varphi_{ij}^{(\alpha)} \varphi_{kl}^{(\beta)} = 0$ for distinct ideals $A_{(\alpha)}$ and $A_{(\beta)}$ of

$\mathbb{C}S_n$. We see that $\sum_{(\alpha)=1}^s \left(\sum_{i=1}^{f(\alpha)} \varphi_{ii}^{(\alpha)} \right) = 1 \in \mathbb{C}S_n$ where

$$\sum_{i=1}^{f(\alpha)} \varphi_{ii}^{(\alpha)} = 1 \in A_{(\alpha)}. \text{ Let } R_{(\alpha_i)} = \mathbb{C}S_n \varphi_{ii}^{(\alpha)} \text{ (} i=1, \dots, f(\alpha) \text{).}$$

Then the $R_{(\alpha_i)}$'s are left ideals of $\mathbb{C}S_n$. We have

$$\begin{aligned} A_{(\alpha)} &= \sum_{i=1}^{f(\alpha)} A_{(\alpha)} \varphi_{ii}^{(\alpha)} = \sum_{i=1}^{f(\alpha)} \left(\sum_{(\beta)=1}^s A_{(\beta)} \right) \varphi_{ii}^{(\alpha)} \\ &= \sum_{i=1}^{f(\alpha)} \mathbb{C}S_n \varphi_{ii}^{(\alpha)} \\ &= R_{(\alpha_1)} \oplus \dots \oplus R_{(\alpha_{f(\alpha)})}. \end{aligned}$$

By the uniqueness of the decomposition of $\mathbb{C}S_n$, the $R_{(\alpha_i)}$'s ($i=1, \dots, f(\alpha)$) are isomorphic minimal left ideals of $\mathbb{C}S_n$. Now by the remarks following lemma 1.1.9, the $\varphi_{ij}^{(\alpha)}$ generating $R_{(\alpha_i)}$ are primitive elements of $\mathbb{C}S_n$. From lemma 1.1.12 we only need the central primitive idempotents of the two sided-ideals $A_{(\alpha)}$ of $\mathbb{C}S_n$. The elements $\varphi_{ii}^{(\alpha)}$ ($i=1, \dots, f(\alpha)$) (i.e. $\varphi_{ij}^{(\alpha)}$, such that $i=j$) are the central primitive idempotents of $A_{(\alpha)}$ and any one of these say $\varphi_{11}^{(\alpha)}$ generates a minimal left ideal $R_{(\alpha_1)} = \mathbb{C}S_n \varphi_{11}^{(\alpha)}$ which is the representation space of the i.o.r $[\alpha]$ of S_n . Furthermore $\{\varphi_{11}^{(\alpha)}, \dots, \varphi_{1f(\alpha)}^{(\alpha)}\}$ is a \mathbb{C} -basis of $R_{(\alpha_1)}$.

This confirms that the number $f_{(\alpha)}$ is the degree of the i.o.r $[\alpha]$.

Now from above and the remarks following lemma 1.1.6, the problem of constructing i.o.r's of S_n thus reduces to that of finding primitive idempotents for each two sided ideal $A_{(\alpha)}$ of $\mathbb{C}S_n$.

In the following result we show that the primitive idempotents of $A_{(\alpha)}$ are obtained from the young subgroups $S_{(\alpha)}$ and $S_{(\alpha)'}^*$ of S_n .

LEMMA 2.2.4

Let $IS_{(\alpha)}$ be the identity representation on $S_{(\alpha)}$ and let $AS_{(\alpha)'}^*$ be the Alternating representation on $S_{(\alpha)'}^*$, let $f_{(\alpha)}$ be the number of standard (α) -tableaux. Then

$$\varphi = \frac{f_{(\alpha)}}{n!} \sum_{\substack{\theta_1 \in S_{(\alpha)} \\ \theta_2 \in S_{(\alpha)'}^*}} IS_{(\alpha)}(\theta_1) AS_{(\alpha)'}^*(\theta_2) \theta_1 \theta_2$$

is a primitive idempotent of the ideal $A_{(\alpha)}$ of $\mathbb{C}S_n$.

PROOF

It suffices to show that S_n with $T_1 = IS_{(\alpha)}$ on $S_{(\alpha)}$ and $T_2 = AS_{(\alpha)'}^*$ on $S_{(\alpha)'}^*$, satisfies the conditions of lemma 1.1.10.

Now suppose $S_{(\alpha)} \cap \lambda S_{(\alpha)}, \lambda^{-1} \neq \{e\}$, then $\lambda \notin S_{(\alpha)} S_{(\alpha)}$, (otherwise $\lambda \in S_{(\alpha)} S_{(\alpha)}$, will imply that there exists some $\theta_1 \in S_{(\alpha)}$ and $\theta_2 \in S_{(\alpha)}$, such that for $\lambda = \theta_{11} \theta_{21} \in S_{(\alpha)} S_{(\alpha)}$, $\theta_{11} \theta_{21} \theta_2 \theta_{21}^{-1} \theta_{11}^{-1} = \theta_1 \neq e$ and $\theta_{21} \theta_2 \theta_{21}^{-1} = \theta_{11}^{-1} \theta_1 \theta_{11}$ which is a contradiction since $\theta_{21} \theta_2 \theta_{21}^{-1} \in S_{(\alpha)}$, while $\theta_{11}^{-1} \theta_1 \theta_{11} \in S_{(\alpha)}$ and $S_{(\alpha)} \cap S_{(\alpha)} = \{e\}$). Conversely we can similarly show that $S_{(\alpha)} \cap \lambda S_{(\alpha)}, \lambda^{-1} \neq \{e\}$ if $\lambda \notin S_{(\alpha)} S_{(\alpha)}$. Now suppose that $S_{(\alpha)} \cap \lambda S_{(\alpha)}, \lambda^{-1} \neq \{e\}$. Let $\lambda \theta_2 \lambda^{-1} = \theta_1 \neq e$ and let (12...k) be a disjoint cycle of θ_1 . Then there is a disjoint cycle (1'2'...k') of θ_2 such that $\lambda i = i'$ ($i=1, \dots, k$). By lemma 2.1.3, for say $\theta_2 = (1'2')$, $\lambda (1'2') \lambda^{-1} = (12) = \theta_1$. In this case $AS_{(\alpha)}, (\theta_2) = -1$ and $IS_{(\alpha)}(\theta_1) = 1$. Thus we have $AS_{(\alpha)}, (\theta_2) \neq IS_{(\alpha)}(\theta_1)$ which satisfies the conditions of lemma 1.1.10. The multiple k of the primitive idempotent φ defined in lemma 1.1.10 is easily shown to be equal to $\frac{f_{(\alpha)}}{n!}$ (see [1] or [10]).

As we have seen above, each (α) -tableau of an (α) -diagram $[\alpha]$ gives rise to a double coset $S_{(\alpha)} \lambda S_{(\alpha)}$, containing λ such that $S_{(\alpha)} \cap \lambda S_{(\alpha)}, \lambda^{-1} = \{e\}$ if the conditions of lemma 1.1.10 are to be satisfied.

Now as in lemma 2.2.4, above if we denote by $IS_{(\alpha)}$ the identity representation on $S_{(\alpha)}$ (and that on $S_{(n)}$ by $[n]$), similarly if we denote the alternating representation on $S_{(\alpha)}$, by $AS_{(\alpha)}$, (and that on $S_{(n)}$ by $[1^n]$), then denoting the Induced o.r's of S_n by $IS_{(\alpha)} \uparrow S_n$ and $AS_{(\alpha)} \uparrow S_n$ respectively, we have the following lemma.

LEMMA 2.2.5

The o.r's $IS_{(\alpha)} \uparrow S_n$ and $AS_{(\alpha)} \uparrow S_n$ have one irreducible constituent in common in each case with multiplicity 1.

PROOF

We show that $\left(\chi_{IS_{(\alpha)} \uparrow S_n}, \chi_{AS_{(\alpha)} \uparrow S_n} \right) = 1$.

By lemma 1.2.6, the above inner product of characters of S_n is equal to

$$\sum_{\lambda \in S_{(\alpha)} \lambda S_{(\alpha)}} \left(\chi_{IS_{(\alpha)}} \downarrow S_\lambda, \chi_{(AS_{(\alpha)})^\lambda} \downarrow S_\lambda \right)$$

where $S_\lambda = S_{(\alpha)} \cap \lambda S_{(\alpha)} \lambda^{-1}$ and $S_{(\alpha)} \lambda S_{(\alpha)}$, is a double

coset of S_n and $(AS_{(\alpha)})^\lambda$ is an o.r of $\lambda S_{(\alpha)} \lambda^{-1}$ conjugate to $AS_{(\alpha)}$, defined by

$(AS_{(\alpha)})^\lambda (\lambda \theta_2 \lambda^{-1}) = AS_{(\alpha)} (\theta_2)$, for all $\theta_2 \in S_{(\alpha)}$. The

sum above is now equal to

$$\sum_{\lambda \in S_{(\alpha)} \lambda S_{(\alpha)}} \left(\chi_{IS_\lambda}, \chi_{AS_\lambda} \right) .$$

Since $S_{(\alpha)} \cap \lambda S_{(\alpha)}, \lambda^{-1} = \{e\}$ if $\lambda \in S_{(\alpha)} \setminus S_{(\alpha)}$,
(see Lemma 2.2.4), we have

$\chi_{I(S_{(\alpha)} \cap \lambda S_{(\alpha)}, \lambda)^{-1}} = \chi_{A(S_{(\alpha)} \cap \lambda S_{(\alpha)}, \lambda^{-1})}$ and the above
sum equals 1.

We remark that when the conditions of lemma 2.2.5 hold then the conditions of lemma 1.2.7 hold; and hence the characters $\chi_{I S_{(\alpha)}} \uparrow_{S_n}$ and $\chi_{A S_{(\alpha)}} \uparrow_{S_n}$ give rise to an irreducible character of S_n . We shall denote the common constituent in lemma 2.2.5 by

$$[\alpha] = (I S_{(\alpha)} \uparrow_{S_n}) \cap (A S_{(\alpha)} \uparrow_{S_n}). \quad (2.4)$$

Now since $S_{(1^n)}$ is a direct product of n copies of S_1 which is isomorphic to S_1 , we have $I S_{(1^n)} = I(e) = A S_{(1^n)}$. Thus the induced o.r.'s of $S_n, I S_{(1^n)} \uparrow_{S_n}$ and $A S_{(1^n)} \uparrow_{S_n}$ are the regular representation $r S_n$ of S_n . We have

$$\begin{aligned} (I S_{(1^n)} \uparrow_{S_n}) \cap (A S_{(n)} \uparrow_{S_n}) &= r S_n \cap A S_n = A S_n = [1^n] \\ (I S_{(n)} \uparrow_{S_n}) \cap (A S_{(1^n)} \uparrow_{S_n}) &= I S_n \cap r S_n = I S_n = [n] \end{aligned} \quad (2.5)$$

Thus certain o.r.'s of S_n may be obtained as above. Now denoting the identity representation on $S_{(\alpha)}$, by $I S_{(\alpha)}$, and the alternating representation on $S_{(\alpha)}$ by $A S_{(\alpha)}$, we can similarly obtain an o.r. of S_n which is given by $[\alpha]' = (I S_{(\alpha)} \uparrow_{S_n}) \cap (A S_{(\alpha)} \uparrow_{S_n})$. Since $[n] = [1^n] \otimes [1^n]$ we see that $[\alpha]$ and $[\alpha]'$ differ only on the odd permutations by a sign.

We have

$$[\alpha]' = [\alpha] \otimes [1^n] \quad (2.6)$$

The next result, which is our main result in this chapter, shows that for all partitions (α) of n the pairs of o.r's $\{[\alpha], [\alpha]' = [\alpha] \otimes [1^n]\}$ are the only inequivalent i.o.r's of S_n (see [10] page 70).

THEOREM 2.2.6

The representations, $\{[\alpha] = (IS_{(\alpha)} \uparrow S_n) \cap (AS_{(\alpha)} \uparrow S_n), [\alpha]' = [\alpha] \otimes [1^n] / (\alpha)'\}$ is a partition of n conjugate to $(\alpha)\}$ form a complete system of pair-wise inequivalent i.o.r's of the symmetric group, S_n .

PROOF

We show that for distinct partitions (α) and (β) of n , the tableaux $T^{[\alpha]}$ and $T^{[\beta]}$ provide two distinct primitive idempotents, which thus generate non-isomorphic representation spaces of the i.o.r's $[\alpha]$ and $[\beta]$ respectively. Let $\varphi_1 = k_1 P_1 N_1$ and $\varphi_2 = k_2 P_2 N_2$ be the associated primitive idempotents of the Tableaux $T^{[\alpha]}$ and $T^{[\beta]}$ respectively. Consider $P_1 N_1 \lambda P_2 N_2$, $\lambda \in S_n$. If for all $\theta'_2 = (ij) \in S_{(\beta)}$, there is no $\theta_1 \in S_{(\alpha)}$ such that $\lambda \theta'_2 \lambda^{-1} = \theta_1$, then no pair of symbols i, j from any column of $T^{[\beta]}$ can be taken by λ into symbols within a row of $T^{[\alpha]}$ (otherwise for $(ij) \in S_{(\beta)}$, we would have $\lambda (ij) \lambda^{-1} = (\lambda i \lambda j) \in S_{(\alpha)}$ which implies that the rows

of $T^{[\beta]}$ have the same lengths as those of $T^{[\alpha]}$, that is the diagrams $[\alpha]$ and $[\beta]$ are the same, which is a contradiction). This implies that there exists some $\theta'_2 = (ij) \in S_{(\beta)}$, such that

$$\lambda \theta'_2 \lambda^{-1} = \theta_1 \in S_{(\alpha)}.$$

Now

$$\begin{aligned} P_1 N_1 \lambda P_2 N_2 &= \chi_{IS_{(\alpha)}}(\theta_1) P_1 N_1 \lambda \theta'_2 \lambda^{-1} \lambda P_2 N_2 \\ &= P_1 N_1 \theta_1 \lambda P_2 N_2, \text{ since } \chi_{IS_{(\alpha)}}(\theta_1) = 1 \\ &= \left(\chi_{IS_{(\alpha)}}(\theta_1) \right)^{-1} P_1 N_1 \lambda P_2 N_2 \end{aligned}$$

Hence $\left[1 - \left(\chi_{IS_{(\alpha)}}(\theta_1) \right)^{-1} \right] P_1 N_1 \lambda P_2 N_2 = 0$. Since θ_1 is a transposition $\left(\chi_{IS_{(\alpha)}}(\theta_1) \right)^{-1} \neq 0$, so that $P_1 N_1 \lambda P_2 N_2 = 0$ for all $\lambda \in S_n$. Hence $\varphi_1 CS_n \varphi_2 = 0$. But φ_1 and φ_2 are primitive idempotents so that $\varphi_1 CS_n \varphi_2 = 0$ only if φ_1 and φ_2 belong to non-isomorphic summands $A_{(\alpha)}$ and $A_{(\beta)}$ of CS_n respectively. Thus the tableaux $T^{[\alpha]}$ and $T^{[\beta]}$ yield primitive idempotents φ_1 and φ_2 from non-isomorphic summands of CS_n and are associated with distinct i.o.r's $[\alpha]$ and $[\beta]$ respectively. Hence the (α) -tableaux of distinct (α) -diagrams provide a complete system of i.o.r's of S_n .

For an example of the application of the above result we give an explicit construction of the i.o.r's of the group S_4 .

STANDARD (α) -TABLEAUX	$f_{(\alpha)}$	STANDARD $(\alpha)'$ -TABLEAUX	$f_{(\alpha)'}$
$T^{[4]}$	1	$T^{[1^4]}$	1
$T^{[31]}$	3	$T^{[21^2]}$	3
$T^{[2^2]}$	2	$T^{[2^2]}$	2
$T^{[21^2]}$	3	$T^{[31]}$	3
$T^{[1^4]}$	1	$T^{[4]}$	1

(2.9)

That the above is a complete set of i.o.r's of S_4 follows from considering the sum of the squares of the degrees $f_{(\alpha)}$.

The young subgroups of S_4 are easily obtained from the (α) -tableaux to give:

YOUNG SUBGROUP, $S_{(\alpha)}$	CONJUGATE YOUNG SUBGROUP $S_{(\alpha)'}$
$S_{(4)} \cong S_4$	$S_{(1^4)} \cong S_1$
$S_{(31)} \cong S_3$	$S_{(21^2)} \cong S_2$
$S_{(2^2)} \cong S_2 \times S_2$	$S_{(2^2)} = S_2 \times S_2$
$S_{(21^2)} \cong S_2$	$S_{(31)} \cong S_3$
$S_{(1^4)} \cong S_1$	$S_{(4)} \cong S_4$

(2.10)

By lemma 2.2.4 the primitive idempotents of the two sided ideals of $\mathbb{C}S_n$ can be calculated using the young subgroups of S_4 given above.

The identity representations on $S_{(\alpha)}$ and the alternating representations on $S_{(\alpha)}$, are given by

$$\begin{aligned}
 IS_{(4)} &=: [4] & AS_{(1^4)} &=:[1] \otimes [1] \otimes [1] \otimes [1] \cong [1] \\
 IS_{(31)} &=:[3] \otimes [1] \cong [3] & AS_{(21^2)} &=:[1^2] \otimes [1] \otimes [1] \cong [1^2] \\
 IS_{(2^2)} &=:[2] \otimes [2] & AS_{(2^2)} &=:[1^2] \otimes [1^2] \\
 IS_{(21^2)} &=:[2] \otimes [1][1] \otimes [2] & AS_{(31)} &=:[1^3] \otimes [1] \cong [1^3] \\
 IS_{(1^4)} &=:[1] \otimes [1] \otimes [1] \otimes [1] \cong [1] & AS_{(4)} &=:[1^4]
 \end{aligned}
 \tag{2.11}$$

And the induced representations of S_4 are given by

$$\begin{aligned}
 IS_{(4)} \uparrow S_4 &=:[4] \uparrow S_4, & AS_{(1^4)} \uparrow S_4 &=:[1] \uparrow S_4 = rS_4 \\
 IS_{(31)} \uparrow S_4 &=:[3] \uparrow S_4, & AS_{(21^2)} \uparrow S_4 &=:[1^2] \uparrow S_4 \\
 IS_{(2^2)} \uparrow S_4 &=:[2] \otimes [2] \uparrow S_4, & AS_{(2^2)} \uparrow S_4 &=:[1^2] \otimes [1^2] \uparrow S_4 \\
 IS_{(21^2)} \uparrow S_4 &=:[2] \uparrow S_4, & AS_{(31)} \uparrow S_4 &=:[1^3] \uparrow S_4 \\
 IS_{(1^4)} \uparrow S_4 &=:[1] \uparrow S_4 = rS_4, & AS_{(4)} \uparrow S_4 &=:[1^4] \uparrow S_4
 \end{aligned}
 \tag{2.12}$$

By lemma 2.2.5 the above pairs of representations $IS_{(\alpha)} \uparrow S_4$ and $AS_{(\alpha)} \uparrow S_4$ have one irreducible constituent in common in each case with multiplicity 1. Using the notation above we have a complete set of i.o.r's of S_4 given below:

$$\begin{aligned}
 [1^4] &= (IS_{(1^4)} \uparrow S_4) \cap (AS_{(4)} \uparrow S_4) = rS_4 \cap AS_4 = AS_4 \\
 [4] &= (IS_{(4)} \uparrow S_4) \cap (AS_{(1^4)} \uparrow S_4) = IS_4 \cap rS_4 = IS_4 \\
 [31] &= (IS_{(31)} \uparrow S_4) \cap (AS_{(21^2)} \uparrow S_4) \\
 [21^2] &= (IS_{(21^2)} \uparrow S_4) \cap (AS_{(31)} \uparrow S_4) \\
 [2^2] &= (IS_{(2^2)} \uparrow S_4) \cap (AS_{(2^2)} \uparrow S_4)
 \end{aligned}
 \tag{2.13}$$

PROOF

This is a consequence of definition 1.2.2 and theorem 2.2.6. ■

From our earlier remarks and theorem 2.2.6 the number of inequivalent i.o.r's of each $S_{(\alpha_1)}$ is $P(\alpha_1)$. It follows that the number of inequivalent i.o.r's of $S_{(\alpha)}$ is given by $P(\alpha_1)P(\alpha_2)\dots P(\alpha_k)$.

CHAPTER III

THE GENERALISED SYMMETRIC GROUP, B_n^m ,
AND ITS CONJUGACY CLASSES

In this chapter we examine the basic results on the generalised symmetric group B_n^m (elsewhere written as $C_m \sim S_n$ or $G(m,1,n)$) and give the conjugacy classes of B_n^m . We first describe the group B_n^m and state some of the relevant concepts to be used in the sequel.

3.1 THE GENERALISED SYMMETRIC GROUP, B_n^m

The group B_n^m has a presentation given by

$$B_n^m = \langle r_i, w_j, (i=1, \dots, n-1, j=1, \dots, n) / r_i^2 = 1;$$

$$(r_i r_{i+1})^3 = 1 = (r_i r_j)^2, \quad |i-j| \geq 2; \quad w_j^m = 1,$$

$$w_j w_k = w_k w_j, \quad r_i w_i = w_{i+1} r_i, \quad r_i w_j = w_j r_i, \quad j \neq i, i+1 \rangle.$$

(3.1)

The group B_n^m is called the generalised symmetric group because we can identify r_i ($i=1, \dots, n-1$) with the transposition $(i, i+1)$ and w_j ($j=1, \dots, n$) with the map $j \rightarrow \xi j$, where ξ is some primitive m^{th} root of unity. The group generated by the r_i 's is the symmetric group S_n . Denoting by $C_m^{(j)}$ the group generated by w_j , then the group C_m^n generated by the w_j 's is the direct product $C_m^n = C_m^{(1)} \times \dots \times C_m^{(n)}$ of n cyclic groups each of

order m . Therefore B_n^m is a group permuting the symbols in Ω as well as multiplying arbitrarily subsets of Ω by powers of ξ . The group B_n^m may also be considered as the semi-direct product of C_m^n and S_n (see [8], [10] or [14]). The order of B_n^m is given by

$$\left| B_n^m \right| = m^n n! \quad (3.2)$$

On ignoring the multiples by powers of ξ , we may consider each member of B_n^m as a permutation θ of $\Omega = \{1, 2, \dots, n\}$, which is simply a member of S_n . Therefore by definition 2.1.1 we can express θ uniquely as a product of disjoint cycles. Such an expression of θ is called the cycle notation of θ . Thus if $\theta \in B_n^m$, then $\theta = \theta_1 \theta_2 \dots \theta_t$, where θ_i is a cycle given by

$$\theta_i = \left[\begin{array}{cccc} a_{i_1} & a_{i_2} & \dots & a_{i_{t_i}} \\ \xi^{k_{i_1}} a_{i_2} & \xi^{k_{i_2}} a_{i_3} & \dots & \xi^{k_{i_{t_i}}} a_{i_1} \end{array} \right]$$

where $a_{i_j} \in \Omega = \{1, \dots, n\}$, $k_{i_j} \in \{1, \dots, m\}$ and t_i is the length of θ_i . The multiplication of elements in B_n^m is defined in the usual way as in S_n .

The following result is crucial in our method of constructing representations of B_n^m .

LEMMA 3.1.1

Let $\theta \in B_n^m$ be of the form

$$\left[\begin{array}{cccc} 1 & & & n \\ & 2 & & \\ & & & \\ \xi^{k_1} a_1 & & \xi^{k_2} a_2 & \dots & \xi^{k_n} a_n \end{array} \right]$$

Define $\Phi : B_n^m \rightarrow S_n$ by

$$\Phi(\theta) = \left(\begin{array}{cccc} 1 & 2 & \dots & n \\ a_1 & a_2 & \dots & a_n \end{array} \right).$$

Then Φ a surjective homomorphism.

PROOF

Let

$$\lambda = \left[\begin{array}{cccc} 1 & 2 & \dots & n \\ \xi^{k'_1} b_1 & \xi^{k'_2} b_2 & \dots & \xi^{k'_n} b_n \end{array} \right] \in B_n^m.$$

We show that $\Phi(\theta\lambda) = \Phi(\theta)\Phi(\lambda)$

$$\begin{aligned} \theta\lambda &= \left[\begin{array}{cccc} 1 & 2 & \dots & n \\ \xi^{k_1} a_1 & \xi^{k_2} a_2 & \dots & \xi^{k_n} a_n \end{array} \right] \left[\begin{array}{cccc} 1 & 2 & \dots & n \\ \xi^{k'_1} b_1 & \xi^{k'_2} b_2 & \dots & \xi^{k'_n} b_n \end{array} \right] \\ &= \left[\begin{array}{cccc} 1 & 2 & \dots & n \\ \xi^{k''_1} c_1 & \xi^{k''_2} c_2 & \dots & \xi^{k''_n} c_n \end{array} \right] \end{aligned}$$

where $c_i = \text{some } b_i$ if λ maps a_i onto b_i ,
 $k''_i = k_i + k'_i$, otherwise leave as disjoint cycles.
 we have

$$\begin{aligned}\Phi(\theta\lambda) &= \Phi \left(\begin{array}{cccc} 1 & 2 & \dots & k \\ \xi^{k_1} c_1 & \xi^{k_2} c_2 & \dots & \xi^{k_n} c_n \end{array} \right) = \left(\begin{array}{cccc} 1 & 2 & \dots & n \\ c_1 & c_2 & \dots & c_n \end{array} \right) \\ &= \left(\begin{array}{cccc} 1 & 2 & \dots & n \\ a_1 & a_2 & \dots & a_n \end{array} \right) \left(\begin{array}{cccc} 1 & 2 & \dots & n \\ b_1 & b_2 & \dots & b_n \end{array} \right) = \Phi(\theta)\Phi(\lambda)\end{aligned}$$

Furthermore for $\theta \in S_n$ considered as an element of B_n^m , $\Phi(\theta)=\theta$, proving Φ is surjective.

From the first isomorphism theorem and lemma 3.1.1, the group $B_n^m/\ker \Phi$ is isomorphic to S_n . We determine the $\ker \Phi$ in the following result.

LEMMA 3.1.2

Let Φ be as defined above. Then $\ker \Phi = C_m^n$.

PROOF

If $\theta \in \ker \Phi$, then $\Phi(\theta) = e \in S_n$. Thus θ must

be of the form
$$\left(\begin{array}{c} 1 \\ \xi^{k_1} 1 \end{array} \right) \left(\begin{array}{c} 2 \\ \xi^{k_2} 2 \end{array} \right) \dots \left(\begin{array}{c} n \\ \xi^{k_n} n \end{array} \right)$$

which lies in C_m^n . Thus

$$\theta = (w_1)^{k_1} (w_2)^{k_2} \dots (w_n)^{k_n}$$

Now if $\theta \in C_m^n$, then θ is generated by the w_j 's so that $\theta = (w_1)^{k_1} (w_2)^{k_2} \dots (w_n)^{k_n}$. Hence $\Phi(\theta) = e \in S_n$ so that $\theta \in \ker \Phi$. Thus $C_m^n \subseteq \ker \Phi$.

Therefore $\ker \phi = C_m^n$.

We end this section with a somewhat obvious result on conjugation in B_n^m .

LEMMA 3.1.3

Let $\theta, \lambda \in B_n^m$. Then θ and $\lambda\theta\lambda^{-1}$ have equal cyclic lengths.

PROOF

Regarding θ and λ as permutations with powers of ξ ignored, we have by lemma 2.1.3

$$\lambda\theta\lambda^{-1} = \begin{pmatrix} i \\ \lambda(i) \end{pmatrix} \begin{pmatrix} i \\ \theta(i) \end{pmatrix} \begin{pmatrix} \lambda(i) \\ i \end{pmatrix} = \begin{pmatrix} \lambda(i) \\ \lambda\theta(i) \end{pmatrix},$$

that is we obtain $\lambda\theta\lambda^{-1}$ from θ by applying λ to the symbols of θ . This retains the cyclic lengths of θ since each symbol has its own image under λ . Hence θ and $\lambda\theta\lambda^{-1}$ have equal cyclic lengths.

3.2 THE CONJUGACY CLASSES OF B_n^m

To describe the conjugacy classes of the generalised symmetric group. B_n^m , we shall need some preliminary results.

DEFN 3.2.1

Let $\theta \in B_n^m$. We fix its expression in terms of disjoint cycles as $\theta = \theta_1 \theta_2 \dots \theta_t$. Define a map $f: B_n^m \rightarrow \mathbb{N}$ by $f(\theta_i) = \sum_{j=1}^{k_i} k_{ij}$, where the K_{ij} 's have the same meaning as in lemma 3.1.1 and set

$f(\theta) = \sum_{i=1}^t f(\theta_i)$. Let $a_{pq}(\theta)$ be the number of cycles θ_i of θ of length q such that $f(\theta_i) \equiv p \pmod{m}$, $1 \leq p \leq m$, $1 \leq q \leq n$. Then the $m \times n$ matrix $(a_{pq}(\theta))$ is called the Type of θ . Here we shall denote the type of θ by $\text{Typ}(\theta)$. We make this definition precise as follows.

LEMMA 3.2.2

Let $\theta = \lambda\mu$ be a product of elements λ, μ in B_n^m . Then $\text{Typ}(\theta) = \text{Typ}(\lambda) + \text{Typ}(\mu)$ if and only if λ and μ are disjoint.

PROOF

Suppose λ and μ are disjoint. Then $f(\theta) = f(\lambda) + f(\mu)$ by definition of f . And θ has the same number of disjoint cycles of the same length as $\lambda\mu$. Thus we have $\text{Typ}(\theta) = \text{Typ}(\lambda) + \text{Typ}(\mu)$.

Conversely, suppose λ and μ have some symbols in common. Let $\theta = \theta_1 \theta_2 \dots \theta_t$ be a decomposition of θ into disjoint cycles. Then multiplication of the cycles of λ with cycles of μ will in general disturb

the cyclic lengths of the cycles in θ which will be different from the lengths of the cycles of λ and μ , and $f(\theta) \neq f(\lambda) + f(\mu)$. Hence $\text{Typ}(\theta) \neq \text{Typ}(\lambda) + \text{Typ}(\mu)$.

COROLLARY 3.2.3

- (a) Keeping the notation above, $f(\theta) = f(\lambda) + f(\mu)$.
- (b) We obtain the type of a product of disjoint permutations (any number of them) by adding their respective types.
- (c) Two permutations of B_n^m will have the same type if they have the same numbers of disjoint cycles of the same lengths; hence if they have the same lengths.

The following result gives a necessary and sufficient condition for the conjugation of elements in B_n^m .

LEMMA 3.2.4

Let $\theta_1, \theta_2 \in B_n^m$. Then θ_1 is conjugate to θ_2 in B_n^m if and only if $\text{Typ}(\theta_1) = \text{Typ}(\theta_2)$.

PROOF

Let θ_1 be conjugate to θ_2 in B_n^m . Then some λ exists in B_n^m such that $\lambda\theta_1\lambda^{-1} = \theta_2$. Let $\theta_1 = \theta_{11}\theta_{12}\dots\theta_{1t}$ be the decomposition of θ_1 as a product of disjoint cycles.

Then

$$\lambda\theta_1\lambda^{-1} = \lambda\theta_{11}\theta_{12}\dots\theta_{1t}\lambda^{-1} = \lambda\theta_{11}\lambda^{-1}\lambda\theta_{12}\lambda^{-1}\dots\lambda\theta_{1t}\lambda^{-1}.$$

Now the $\lambda\theta_{1i}\lambda^{-1}$'s ($i=1,\dots,t$) are disjoint cycles of $\lambda\theta_1\lambda^{-1}$ since the application of λ to each disjoint cycle θ_{1i} of θ_1 moves every symbol in θ_{1i} into a unique symbol. By lemma 3.1.3, the length of θ_{1i} is equal to that of $\lambda\theta_{1i}\lambda^{-1}$.

$$\begin{aligned} \text{Let } \text{Typ}(\theta_1) &= (a_{pq}(\theta_1)) = \sum_{i=1}^t \text{Typ}(\theta_{1i}) \\ &= \sum_{i=1}^t (a_{pq}(\theta_{1i})). \end{aligned}$$

Now as a consequence of lemma 3.2.2, it is enough to show that $\text{Typ}(\theta_{1i}) = \text{Typ}(\lambda\theta_{1i}\lambda^{-1})$.

$\text{Typ}(\theta_{1i}) = (a_{pq}(\theta_{1i}))$ means that θ_{1i} is of length q and $f(\theta_{1i}) \equiv p \pmod{m}$ i.e. $f(\theta_{1i}) = km + p$, for some $k \in \mathbb{N}^*$. By corollary 3.2.3,

$$\begin{aligned} f(\lambda\theta_{1i}\lambda^{-1}) &= f(\lambda) + f(\theta_{1i}) + f(\lambda^{-1}) \\ &= f(\lambda) + f(\lambda^{-1}) + km + p \\ &= f(e) + km + p \\ &= lm + km + p, \text{ for some } l \in \mathbb{N}^* \\ &= k'm + p, \text{ for some } k' = l+k \in \mathbb{N}^*. \end{aligned}$$

so $f(\lambda\theta_{1i}\lambda^{-1}) \equiv p \pmod{m}$. It follows that

$$\text{Typ}(\theta_{1i}) = \text{Typ}(\lambda\theta_{1i}\lambda^{-1}) \text{ and therefore } \text{Typ}(\theta_1) = \text{Typ}(\theta_2).$$

Conversely, suppose $\text{Typ}(\theta_1) = \text{Typ}(\theta_2)$. Then $(a_{p,q}(\theta_1)) = (b_{p,q}(\theta_2))$, so that the number of θ_{1i} 's involved in θ_1 of length q such that $f(\theta_{1i}) \equiv p \pmod{m}$ is equal to that of the θ_{2i} 's of length q involved in θ_2 such that $f(\theta_{2i}) \equiv p \pmod{m}$. Hence equal numbers of disjoint cycles of equal length's q are involved in the decompositions of θ_1 and θ_2 into disjoint cycles. Thus appealing to the homomorphism ϕ given in lemma 3.1.1 we have $\phi(\theta_1)$ is conjugate to $\phi(\theta_2)$ in S_n .

Now let $f(\theta_{1i}) = km + p$ and $f(\theta_{2i}) = k'm + p, k, k' \in \mathbb{N}^*$. Without loss of generality, we may assume $k' > k$, say by setting $k' = k + 1, l \in \mathbb{N}^*$. Then $f(\theta_{2i}) = f(\theta_{1i}) + lm$; and from lemma 3.2.2 we have $\theta_{2i} = \theta_{1i} \mu$, for some $\mu \in B_n^m$ where θ_{1i} and μ are disjoint and $f(\mu) \equiv 0 \pmod{m}$. Hence μ represents a sign change in B_n^m , so that we consider μ to be of the form $\mu = \begin{pmatrix} j \\ \xi^{1m} j \end{pmatrix}$, where ξ is some primitive m th root of unity. Rewriting μ as a product of cycles $\rho_j \rho_j^{-1}$ as

$$\mu = \rho_1 \rho_1^{-1} \rho_2 \rho_2^{-1} \dots \rho_j \rho_j^{-1} \dots = \begin{pmatrix} j \\ \xi^{1m} j \end{pmatrix}$$

$$\text{where } \rho_j \rho_j^{-1} = \begin{pmatrix} 1 & 2 & \dots & k \\ \xi^{t_1} 2 & \xi^{t_2} 3 \dots \xi^{t_k} 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & \dots & k \\ \xi^{u_1} k & \xi^{u_2} 1 \dots \xi^{u_k} (k-1) \end{pmatrix}$$

$$\text{i.e. } \rho_j \rho_j^{-1} = \begin{pmatrix} j \\ \xi^{t_i + u_i} j \end{pmatrix}$$

and $t_i + u_i = lm$ for $i=j$, in only one cycle $\rho_j \rho_j^{-1}$ of μ , and $t_i + u_i = m$ in all the other cycles of μ . Now write μ in the form $(\rho_1 \rho_2 \dots)(\rho_1^{-1} \rho_2^{-1} \dots) = \rho \rho^{-1}$

where the ρ_j^{-1} 's are disjoint cycles of μ . We have $\theta_{21} = \theta_{11} \mu = \rho \theta_{11} \rho^{-1}$, $\rho \in B_n^m$. Generalising this expression for θ_{21} to θ_2 we get $\theta_2 = \rho \theta_1 \rho^{-1}$, that is θ_1 is conjugate to θ_2 .

From above we conclude that B_n^m contains as many conjugacy classes as there are distinct types $(a_{pq}(\theta))$. We need a definition before proving the main theorem of this section.

DEFN 3.2.5

An m-set of partitions of n, $(\theta(t_1), \dots, \theta(t_m))$ is a set of partitions $\theta(t_p)$ ($p=1, \dots, m$) such that $\theta(t_p) = (a_{p1}, a_{p2}, \dots, a_{pn})$ is a partition of t_p , where $t_p = \sum_{q=1}^m q a_{pq}$ and $\sum_{p=1}^m t_p = n$. Here we shall denote by $(\alpha)_m$ an m-set of partitions of n; and sometimes write $\theta(t_p)$ simply as t_p . A 2-set of partitions of n, $(\alpha)_2$ will also be called a double partition of n. We now prove the following theorem.

THEOREM 3.2.6

Let $(\alpha)_m = (\theta(t_1), \dots, \theta(t_m))$ be an m -set of partitions of n . Let $p(t_p)$ be the number of partitions of t_p , $t_p \in \mathbb{N}$ and $p(0) = 1$, and let $p_m(n)$ be the number of the m -sets of partitions of n . Then the number of conjugacy classes of B_n^m is given by

$$p_m(n) = \sum_{j=1}^m \left\{ P(t_1) P(t_2) \dots P(t_m) \right\}_j,$$

where the summation is taken over all the m -sets of partitions of n .

PROOF

To a conjugacy class $Ccl(\theta)$ of θ of type $(a_{pq}(\theta))$ associate an m -set of partitions of n defined by $(\alpha)_m = (\theta(t_1), \dots, \theta(t_m))$, where $\theta(t_p) = (a_{p1}, a_{p2}, \dots, a_{pn})$ and $t_p = \sum_{q=1}^n qa_{pq}$. This m -set of partitions of n is unique since by definition an m -set of partitions of n different from this $(\alpha)_m$ shall correspond to a different conjugacy class not containing θ .

Conversely, let $(\alpha)_m = (\theta(t_1), \dots, \theta(t_m))$ be an m -set of partitions of n , so that $\theta(t_p) = (a_{p1}, \dots, a_{pn})$. Then the set $\Omega = \{1, 2, \dots, n\}$ is decomposed into a disjoint union of subsets of the form

$$\theta(t_p) = (a_{p1}, \dots, a_{pn}) \text{ as follows;}$$

$$(\alpha)_m = \left(\theta(t_1), \dots, \theta(t_p), \dots, \theta(t_m) \right)$$

$$\left(\theta(t_1), \dots, \underbrace{(a_{p1}^1, a_{p2}^1, \dots, a_{pn}^1)}_{=\theta^1(t_p)}, \dots, \theta(t_m) \right)$$

$$\left(\theta(t_1), \dots, \underbrace{(a_{p1}^2, a_{p2}^2, \dots, a_{pn}^2)}_{=\theta^2(t_p)}, \dots, \theta(t_m) \right)$$

⋮

$$\left(\theta(t_1), \dots, \underbrace{(a_{p1}^1, a_{p2}^1, \dots, a_{pn}^1)}_{=\theta^1(t_p)}, \dots, \theta(t_m) \right)$$

(3.3)

where $\sum_{p=1}^m t_p = n$, $t_p = \sum_{q=1}^n q a_{pq}$ and $l = p(t_p)$.

This is done to each $\theta(t_p)$ with the other $\theta(t_p)$'s fixed. We see that there are $\sum_{p,q} a_{pq}$ subsets $\theta(t_p)$ such that exactly $\sum_{p=1}^m a_{pq}$ of them have q elements as seen in the diagram (3.3) above. To see this clearly, for each $\theta(t_p)$ and q fixed, q appears $p(t_p)$ times down (3.3) above. Now on each such a subset with q elements, we may define a cycle θ_s of length q by

$$\theta_s = \left(\begin{array}{cccc} 1 & 2 & \dots & q \\ a_1 & a_2 & \dots & \xi^k a_q \end{array} \right),$$

where $S = \{1, 2, \dots, q\}$ and we may take $k=p$ for the p th a_{pq} subsets of length q .

Let θ be a product of these cycles θ_s . Then $\text{Typ}(\theta) = (a_{pq}(\theta))$. Thus a given m -set of partitions of n $(\alpha)_m$ defines a conjugacy class $\text{Ccl}(\theta)$ of θ in B_n^m of Type $(a_{pq}(\theta))$.

Furthermore, on varying the q this m -set of partitions of n corresponds to other conjugacy classes different from $\text{Ccl}(\theta)$. Now since there are $p(t_p)$ possibilities for row t_p to be a row of a type the number of conjugacy classes of B_n^m associated with each ordinary partition (t_1, t_2, \dots, t_m) of n is $p(t_1)p(t_2)\dots p(t_m)$. Summing these numbers over all the m -sets of partitions of n gives the result.

CHAPTER IV

REPRESENTATION THEORY OF THE GENERALISED SYMMETRIC
GROUP, B_n^m .

All representations discussed here are considered over the field \mathbb{C} of complex numbers. Therefore the number of inequivalent i.o.r's of B_n^m is equal to the number of conjugacy classes of B_n^m . We aim to give a construction of this number of inequivalent i.o.r's of B_n^m .

Since there is a 1-1 correspondence between the number of conjugacy classes of B_n^m and the m -sets of partitions $(\alpha)_m$ of n , we obtain a complete set of inequivalent i.o.r's of B_n^m by associating each of the $(\alpha)_m$'s with an i.o.r of B_n^m accordingly as in theorem 3.2.6. To this end we first consider the o.r's of special subgroups of B_n^m which play a similar role to the representation theory of B_n^m as do the young subgroups to the representation theory of the symmetric group, S_n .

4.1 GENERALISED YOUNG SUBGROUPS AND BASIC REPRESENTATIONS

We have the following definition.

DEFN 4.1.1

Let $(\alpha) = (t_1, t_2, \dots, t_k)$, be a k -tuple such that $t_i \in \{0, 1, \dots, n\}$ and $\sum_{i=1}^k t_i = n$, $k \leq m$. We

call (α) a permissible k -tuple of n . Define $P_0 = 0$ and

$$P_i = \sum_{j=1}^i t_j \quad (i=1, \dots, k). \quad \text{If } t_i \neq 0, \text{ let } B_{t_i}^m \text{ be the}$$

corresponding generalised symmetric group on the t_i

symbols $P_i = \{P_{i-1} + 1, \dots, P_i\}$, where $1 \leq i \leq k$ and B_0^m

is the trivial generalised symmetric group. That

is $B_{t_i}^m$ is a subgroup of B_n^m . Then the direct

product $B_{t_1}^m \times B_{t_2}^m \times \dots \times B_{t_k}^m$ is called the

generalised young subgroup of B_n^m determined by the

k -tuple $(\alpha) = (t_1, \dots, t_k)$. Here we shall denote the

generalised young subgroup of B_n^m by $B_{(t_1, \dots, t_k)}^m$

or simply $B_{(\alpha)}^m$.

We shall denote by $|(\alpha)|$ the number of permissible m -tuples of n .

PROPOSITION 4.1.2

Let π be a map: $B_{t_1}^m \longrightarrow \mathbb{C}^*$ defined by
 $\pi(\theta_1) = \xi^{if(\theta_1)}$ for all $\theta_1 \in B_{t_1}^m$,

where f is as defined in definition 3.2.1 and ξ is a primitive m^{th} root of unity. Then

- (i) π is well defined.
(ii) π gives rise to an irreducible linear representation P_{t_1} of $B_{t_1}^m$.

PROOF

(i) π is well defined: let $\theta_1, \theta'_1 \in B_{t_1}^m$.

$$\text{Then } \pi(\theta_1) = \xi^{if(\theta_1)} \neq \pi(\theta'_1) = \xi^{if(\theta'_1)}$$

if $f(\theta_1) \neq f(\theta'_1)$ which implies $\theta_1 \neq \theta'_1$, by definition 3.2.1.

(ii) For any $\theta_1 \in B_{t_1}^m$, define a map

$$P_{t_1} : B_{t_1}^m \longrightarrow \mathbb{C}^* \text{ by } P_{t_1}(\theta_1) = \pi(\theta_1).$$

Then for any $\theta_1, \theta'_1 \in B_{t_1}^m$, we have

$$\begin{aligned} P_{t_1}(\theta_1 \theta'_1) &= \pi(\theta_1 \theta'_1) = \xi^{if(\theta_1 \theta'_1)} = \xi^{i(f(\theta_1) + f(\theta'_1))} \\ &= \xi^{if(\theta_1)} \xi^{if(\theta'_1)} = \pi(\theta_1) \pi(\theta'_1) \\ &= P_{t_1}(\theta) P_{t_1}(\theta'_1), \end{aligned}$$

so that P_{t_1} is a homomorphism from $B_{t_1}^m$ to \mathbb{C}^* . P_{t_1} is irreducible since it is linear.

Now extending the map π linearly to the whole group $B_{(\alpha)}^m$, we have for any $\theta = \theta_1 \theta_2 \dots \theta_k \in B_{(\alpha)}^m$,

$$\pi(\theta) = \xi^{\sum_{i=1}^k \text{if}(\theta_i)} \quad \bullet$$

we state the following result.

LEMMA 4.1.3

We keep the above notation. $P_{t_1} \# P_{t_2} \# \dots \# P_{t_k}$ is an irreducible linear representation of $B_{(\alpha)}^m$.

PROOF This follows from definition 1.2.2 since $P_{t_1} \# \dots \# P_{t_k}$ is an outer tensor product representation of $B_{(\alpha)}^m$.

In what follows the representation $P_{t_1} \# \dots \# P_{t_k}$ of $B_{(\alpha)}^m$ shall be denoted by $P_{(t_1, \dots, t_k)}$ or simply $P_{(\alpha)}$. The number of the representations $P_{(\alpha)}$ of $B_{(\alpha)}^m$ is equal to the number of k -sets of partitions of n , $(\alpha)_k$ since there is a unique map π defined on each $B_{t_i}^m$, and changing the t_i leads to definition of P_{t_i} on another generalised symmetric subgroup which is already taken care of by (α) being a permissible k -tuple. Thus the number of the $P_{(\alpha)}$ depends only on the number of permissible k -tuples (α) and is equal to the number of

k-sets of partitions of n . Also for $\lambda \in B_n^m$ and some $x \in \ker \phi$ of lemma 3.1.1 we shall write $P_{(\alpha)}^\lambda(x)$ for $P_{(\alpha)}(\lambda x \lambda^{-1})$.

THEOREM 4.1.4

We keep the above notation. Then

- (i) If (α) and (α') are any two distinct permissible k -tuples, $P_{(\alpha)}^\lambda(x) \neq P_{(\alpha')}^\lambda(x)$, for some $x \in \ker \phi$ and for all $\lambda \in B_n^m$.
- (ii) If $(\alpha) = (\alpha')$, then $P_{(\alpha)}^\lambda(x) \neq P_{(\alpha)}^\lambda(x)$ for some $x \in \ker \phi$ and for all $\lambda \in B_n^m \setminus B_{(\alpha)}^m$.

PROOF

- (i) Let $(\alpha) = (t_1, \dots, t_k)$ and $(\alpha') = (t'_1, t'_2, \dots, t'_k)$ be distinct k -tuples. Then let i be the least index such that $t_i \neq t'_i$. Then we may assume without loss of generality that $t_i < t'_i$. By definition. 4.1.1 $P_i \subset P'_i$ and from lemma 3.1.1 if $\lambda \in B_n^m$ either $\phi(\lambda)P_i = P_i$ or $\phi(\lambda)P_i \neq P_i$. If $\phi(\lambda)P_i = P_i$, let $j \in P'_i \setminus P_i$ so that $\phi(\lambda)(j) \in P_i$, some $l \neq i$. If $\phi(\lambda)P_i \neq P_i$, there exists some $j \in P_i \subset P'_i$ such that $\phi(\lambda)(j) \in P_l$, $1 \leq l \leq k$, for some $l \neq i$. In either case choose $x = w_j \in \ker \phi$ for the j chosen. We have

$$\begin{aligned} P_{(\alpha')}^\lambda(x) &= P_{(\alpha')}^\lambda \left(\begin{matrix} j \\ \xi j \end{matrix} \right) = P_{t'_j}^\lambda \left(\begin{matrix} j \\ \xi j \end{matrix} \right) = \xi^{j f(w_j)} \\ &= \xi^1 \text{ (} j=i \text{ for } P'_i \supset P_i \text{)}. \end{aligned}$$

$$\text{And } P_{(\alpha)}^\lambda(x) = P_{(\alpha)} \left(\lambda \left(\begin{matrix} j \\ \xi_j \end{matrix} \right) \lambda^{-1} \right) = P_{(\alpha)} \left(\Phi(\lambda) \left(\begin{matrix} j \\ \xi_j \end{matrix} \right) \Phi(\lambda)^{-1} \right).$$

By lemma 3.1.3 it follows that

$$P_{(\alpha)}^\lambda(x) = P_{(\alpha)} \left(\begin{matrix} \Phi(\lambda) & (j) \\ \xi \Phi(\lambda) & (j) \end{matrix} \right),$$

$$\text{that is } P_{(\alpha)}^\lambda(x) = P_{(\alpha)} \left(\begin{matrix} 1 \\ \xi^1 \end{matrix} \right) = \xi^{1f(w_1)} = \xi^1.$$

Now $l \neq i$, so $P_{(\alpha)}^\lambda(x) \neq P_{(\alpha')}^\lambda(x)$, proving (i).

(ii) If $(\alpha') = (\alpha)$ and $\lambda \in B_n^m \setminus B_{(\alpha)}^m$ there exists atleast one i , $1 \leq i \leq k$ and a symbol $j \in P_i$ such that

$\Phi(\lambda)(i) \in P_i$, $l \neq i$. Let $x = w_j \in \ker \Phi$ as above.

We have

$$P_{(\alpha)}^\lambda(x) = P_{t_i}^\lambda \left(\begin{matrix} j \\ \xi_j \end{matrix} \right) = \xi^1 \text{ as above. But } P_{(\alpha)}(x) = \xi^1$$

($j = \text{some } i \in P_i$) and $\xi^1 \neq \xi^i$ since $l \neq i$, proving (ii).

DEFN 4.1.5

Keeping the notation above, we shall call the representation $P_{(\alpha)}$ the basic ordinary representation of $B_{(\alpha)}^m$.

4.2 REPRESENTATIONS OF THE GENERALISED YOUNG SUBGROUPS

We obtain the i.o.r's of the generalised young subgroups $B_{(\alpha)}^m$ by appealing to definition 1.2.8 and the homomorphism Φ of lemma 3.1.1 restricted to the young subgroup $B_{(\alpha)}^m$. First we have the following result.

THEOREM 4.2.1

We keep the above notation. For $\theta \in B_{(\alpha)}^m$ define $P^*: B_{(\alpha)}^m \longrightarrow GL_n(\mathbb{C})$ by $P^*(\theta) = P((\ker \phi)\lambda) = P(\phi(\lambda)) \in GL_n(\mathbb{C})$, where λ represents θ in the transversal of $\ker \phi$ in $B_{(\alpha)}^m$. Then P^* is an i.o.r. of $B_{(\alpha)}^m$.

PROOF

This follows from definition 1.2.8, lemma 3.1.1 and theorem 2.3.1; and P^* is a representation of $B_{(\alpha)}^m$ lifted from the i.o.r's P of the young subgroup $S_{(\alpha)}$ of S_n .

In all that follows we shall denote by \mathbb{P} the representation $P^* = ([\alpha_1] \otimes \dots \otimes [\alpha_m])^*$ of $B_{(\alpha)}^m$.

We now use the representations $P_{(\alpha)}$ and \mathbb{P} of $B_{(\alpha)}^m$ to construct further representations of $B_{(\alpha)}^m$. We have the following result.

LEMMA 4.2.2

We keep the above notation. The tensor product $\mathbb{P} \otimes P_{(\alpha)} = ([\alpha_1] \otimes \dots \otimes [\alpha_m])^* \otimes P_{(t_1, \dots, t_m)}$ is an o.r. of $B_{(\alpha)}^m$

PROOF

This follows directly from definition 1.2.1.

4.3 REPRESENTATIONS OF THE GENERALISED SYMMETRIC GROUP

Our aim in this section is to obtain a full set of inequivalent i.o.r's of B_n^m . We do this by inducing the o.r's $P \otimes P_{(\alpha)}$ of the generalised young subgroups $B_{(\alpha)}^m$ to give a representation

$$(P \otimes P_{(\alpha)}) \uparrow B_n^m = (([\alpha_1] \otimes \dots \otimes [\alpha_m])^* \otimes P_{(t_1, \dots, t_m)}) \uparrow B_n^m \quad (4.3)$$

of B_n^m . It is remarkable that the full set of inequivalent i.o.r's of B_n^m is obtained in this way.

We now give our main result.

THEOREM 4.3.1

We keep the above notation. A full set of inequivalent i.o.r's of B_n^m is given by

$$\left\{ (P \otimes P_{(\alpha)}) \uparrow B_n^m = (([\alpha_1] \otimes \dots \otimes [\alpha_m])^* \otimes P_{(t_1, \dots, t_m)}) \uparrow B_n^m \right\}$$

PROOF

We show that the set above is complete by showing that its cardinality equals the number of the conjugacy classes of B_n^m . Consider the tensor product $P \otimes P_{(\alpha)}$. Let $(\alpha) = (t_1, t_2, \dots, t_m)$ be an m-tuple, then by the remarks following lemma 4.1.3 and Keeping P fixed, it is seen that the number of the resulting

o.r.'s $P_{(\alpha)}$ of $B_{(\alpha)}^m$ is equal to $P(t_1)P(t_2)\dots P(t_m)$. Thus this construction gives rise to a total $p_m(n)$ of $\sum_{j=1}^{p_m(n)} \{P(t_1)P(t_2)\dots P(t_m)\}$ representations of B_n^m , where $p_m(n)$ is the number of m -sets of partitions of n . This equals the number of conjugacy classes of B_n^m . Now we appeal to character theory to show that the representations above are inequivalent and irreducible. Let $(\alpha) = (t_1, \dots, t_m)$ and $(\alpha') = (t'_1, \dots, t'_m)$ be any two distinct permissible m -tuple. Let $P_{(\alpha)}$ and $P_{(\alpha')}$ be the basic i.o.r.'s of $B_{(\alpha)}^m$ and $B_{(\alpha')}^m$. Let P and P' be i.o.r of $B_{(\alpha)}^m$ and $B_{(\alpha')}^m$, lifted from i.o.r.'s P and P' of $S_{(\alpha)}$ and $S_{(\alpha')}$ respectively. Let $\chi_1, \chi'_1, \chi_2, \chi'_2, \chi_3$ and χ'_3 denote the characters of $P_{(\alpha)}, P_{(\alpha')}, P, P', P$ and P' respectively. We consider the following inner product of characters of B_n^m .

$$\left[(\chi_2 \chi_1) \uparrow B_n^m, (\chi'_2 \chi'_1) \uparrow B_n^m \right] \text{----- (I)}$$

Now (I) = 0 if $(\alpha) \neq (\alpha')$ by the orthogonality relations, that is the representations above are inequivalent. Using theorem 1.2.5 we obtain

$$(I) = \left[(\chi_2 \chi_1), ((\chi'_2 \chi'_1) \uparrow B_n^m) \downarrow B_{(\alpha)}^m \right] B_{(\alpha)}^m$$

which on using lemma 1.2.6 gives

$$(I) = \sum_{x \in B_{(\alpha)}^m \times B_{(\alpha')}^m} \left\{ (\chi_2 \chi_1), ((\chi'_2 \chi'_1)^x \downarrow Bx \uparrow B_{(\alpha)}^m) \right\}_{B_{(\alpha)}^m}.$$

where $Bx = B_{(\alpha)}^m \cap xB_{(\alpha')}^m x^{-1}$

We use theorem 1.2.5 again and obtain

$$(I) = \sum_{x \in B_{(\alpha)}^m \times B_{(\alpha')}^m} ((\chi_2 \chi_1) \downarrow Bx, (\chi'_2 \chi'_1)^x \downarrow Bx)_{Bx}$$

and $B_{(\alpha)}^m \times B_{(\alpha')}^m$ is a double coset of B_n^m . Now (I) equals zero if the representations above are inequivalent as shown. Suppose (I) is not zero, that is for some $x \in B_{(\alpha)}^m \times B_{(\alpha')}^m$

$$\left\{ (\chi_2 \chi_1) \downarrow B_{(\alpha)}^m \cap xB_{(\alpha')}^m x^{-1}, (\chi'_2 \chi'_1)^x \downarrow B_{(\alpha)}^m \cap xB_{(\alpha')}^m x^{-1} \right\} \neq 0,$$

$$\text{we have } \left\{ (\chi_3 \chi_1) \downarrow \ker \phi, (\chi'_3 \chi'_1)^x \downarrow \ker \phi \right\} \neq 0$$

since $\ker \phi \subseteq Bx$ and where χ_2 and χ'_2 become χ_3 and χ'_3 on $\ker \phi$ respectively. By theorem 1.1.21, this implies that the products $\left(\chi_3 \downarrow \ker \phi \right) \left(\chi_1 \downarrow \ker \phi \right)$ and $\left(\chi_3^x \downarrow \ker \phi \right) \left(\chi_1^x \downarrow \ker \phi \right)$ have at least one constituent in common. But $\chi_1 \downarrow \ker \phi$, $\chi_3 \downarrow \ker \phi$, $\chi_1^x \downarrow \ker \phi$, $\chi_3^x \downarrow \ker \phi$ are all irreducible, so that we must have $\chi_1 \downarrow \ker \phi = \chi_1^x \downarrow \ker \phi$ and $\chi_3 \downarrow \ker \phi = \chi_3^x \downarrow \ker \phi$. This implies that $P_{(\alpha)} = P_{(\alpha')}^x$ on $\ker \phi$. Now theorem 4.1.4 implies that $(\alpha) = (\alpha')$ and $x \in B_{(\alpha)}^m$.

Thus it suffices to work in $B_{(\alpha)}^m$, we therefore consider the inner product

$$\left(\chi_2 \chi_1, \chi'_2 \chi'_1 \right)_{B_{(\alpha)}^m} = \frac{1}{|B_{(\alpha)}^m|} \sum_{\theta \in B_{(\alpha)}^m} \chi_2 \chi_1(\theta) \overline{\chi'_2 \chi'_1(\theta)} \quad \text{--- (II)}$$

where $\chi'_1 = \chi_1$ since $(\alpha) = (\alpha')$, $x \in B_{(\alpha)}^m$ and

$\overline{\chi'_2 \chi_1} = (\chi'_2 \chi_1)^x$. We now have

$$\begin{aligned} \text{(II)} &= \frac{1}{|B_{(\alpha)}^m|} \sum_{\theta \in B_{(\alpha)}^m} \chi_2(\theta) \overline{\chi'_2(\theta)} \chi_1(\theta) \overline{\chi_1(\theta)} \\ &= \frac{1}{|B_{(\alpha)}^m|} \sum_{\theta \in B_{(\alpha)}^m} \chi_2(\theta) \overline{\chi'_2(\theta)}, \chi_1(\theta) \overline{\chi_1(\theta)}. \end{aligned}$$

And on applying theorem 1.2.5 we have

$$\text{(II)} = \frac{1}{|S_{(\alpha)}|} \sum_{\theta \in S_{(\alpha)}} \chi_3(\theta) \chi'_3(\theta),$$

where $\chi_2 = \chi_3$ and $\chi'_2 = \chi'_3$ when restricted to $S_{(\alpha)}$.

Thus $\text{(II)} = (\chi_3, \chi'_3)$. Now if $\text{(II)} \neq 0$, it must equal 1, that is $\chi_3 = \chi'_3$ since χ_3 and χ'_3 are irreducible. Thus the representations above are irreducible. This completes the proof of the main theorem.

REMARK 4.3.2

The i.o.r's of the generalised symmetric group B_n^m have been investigated by Osima [13], Puttaswamaiah [14] using a method similar to that used in obtaining the i.o.r's of S_n as given in theorem 2.2.6. That is by inducing linear representations of the generalised young subgroups. Kerber [10], Read [16] and Hughes [8] have obtained i.o.r's of B_n^m by using Clifford's theory. It is seen that the i.o.r's of B_n^m obtained in all these cases correspond to our results in theorem 4.3.1.

CHAPTER V

APPLICATIONS TO THE REPRESENTATION THEORY OF THE
HYPEROCTAHEDRAL GROUP, B_n^2

In this chapter, we apply the results obtained so far to the generalised symmetric group, B_n^2 which is isomorphic to the Weyl group of type B_n . The group B_n^2 is also called the hyperoctahedral group, and was first studied by A. Young in 1930. We shall use the notation above without further reference.

5.1 THE HYPEROCTAHEDRAL GROUP, B_n^2

The group B_n^2 is a semi-direct product $C_2^n : S_n$ of the normal subgroup C_2^n of order 2^n generated by w_1, w_2, \dots, w_n , with the symmetric group S_n generated by r_i ($i=1, \dots, n-1$). The action of r_i on w_j is given by $r_i w_j = w_j r_i$ if $j \neq i, i+1$ and $r_i w_i = w_{i+1} r_i$ (see (3.1)). Thus each element of B_n^2 has a canonical form wr as a product, where $w \in C_2^n$ and $r \in S_n$. Thus B_n^2 permutes the symbols of $\Omega = (1, \dots, n)$ as well as changing the sign of subsets of Ω .

The group B_n^2 has a faithful permutation representation of degree $2n$ obtained by mapping w_i to the transposition $(i, i+n)$ and r_i to the permutation $(i, i+1)(i+n, i+1+n)$. Now taking $s_i = r_i$ ($i=1, \dots, n-1$) and $s_n = w_n$, we have (see [14]),

$$\begin{aligned} s_i^2 &= 1 \quad (i=1, \dots, n) \\ (s_i s_{i+1})^3 &= 1 \quad (i=1, \dots, n-2) \\ (s_i s_j)^2 &= 1 \quad (i=1, \dots, n), \quad j \neq i, i+1 \\ (s_{n-1} s_n)^4 &= 1. \end{aligned} \quad (5.1)$$

Hence B_n^2 has order $2^n n!$. Since each member θ of B_n^2 is a permutation of $\Omega = \{1, \dots, n\}$, we can uniquely express θ as a product of disjoint cycles (see (3.3)).

DEFN 5.1.1

A k -cycle $\theta_i = \begin{pmatrix} 1 & 2 & \dots & k \\ \pm 2 & \pm 3 & \dots & \pm 1 \end{pmatrix}$ of θ is said to be positive if $\theta_i^k = 1$ and negative otherwise. The ordered lengths $(\alpha_i \geq \alpha_{i+1})$ of the cyclic factors in the cycle-notation of θ together with their signs give a set of positive and negative integers called the signed-cycle type of θ e.g. an element $\theta \in B_n^2$ which has a_1 positive 1-cycles, a_2 positive 2-cycles etc, and b_1 negative 1-cycles, b_2 negative 2-cycles etc, has signed-cycle type given by

$$\left[1^{a_1} \ 2^{a_2} \ \dots; \ 1^{-b_1} \ 2^{-b_2} \ \dots \right] \quad (5.2)$$

It is clear from (5.2) and the length of θ that the signed-cycle type of θ satisfies $|t_1| + |t_2| = n$, where $t_1 = \sum i a_i$ and $t_2 = \sum i |b_i|$. Thus (t_1, t_2) is a permissible 2-tuple. We have a lemma.

LEMMA 5.1.2

The elements of B_n^2 of the same signed cycle-type are conjugate to each other. Furthermore there is a 1-1 correspondence between the conjugacy classes of B_n^2 and double partitions of n .

PROOF

The signed-cycle type of θ is simply a special form of describing the Type, $(a_{pq}(\theta))$, of θ (see 3.2.1). The result then follows from lemma 3.2.4. Further since the signed-cycle types define double partitions of n as seen above, the other result follows (see also [3]).

We prove the following result which gives the number of conjugacy classes of B_n^2 .

LEMMA 5.1.3

Let $n \in \mathbb{N}^*$, then $|\alpha| = n+1$. Furthermore the number of conjugacy classes of B_n^2 is given by

$$\sum_{j=1}^{n+1} P(t_j) P(n-t_j)$$

where $t_j \leq n$, $t_j = j-1$, and j ranges over all permissible 2-tuples of n .

PROOF

Let $(\alpha) = (t_1, t_2)$ be any permissible 2-tuple of n , that is $t_1 + t_2 = n$. Thus $t_2 = n - t_1$, so that $(t_1, t_2) = (t_1, n - t_1)$. In general we can write $(t_j, t_k) = (t_j, n - t_j)$ for all $t_j, t_k \leq n$, $t_j + t_k = n$.

Thus $\{(0, n), (1, n-1), \dots, (n-1, 1), (n, 0)\}$ is the complete set of permissible 2-tuples of n , giving a total of $n+1$ of them. Hence $|(\alpha)| = n+1$.

By theorem 3.2.6, the number of conjugacy classes of B_n^2 will be given by

$$p_2^{(n)} = \sum_{j=1}^n (p(t_1)p(t_2))j$$

which is now equal to the following

$$\sum_{j=1}^{n+1} p(t_j)p(n-t_j)$$

where clearly from the set of permissible 2-tuples of n $t_j = j-1$, $j=1, 2, \dots, n+1$.

5.2 REPRESENTATIONS OF THE GROUP B_n^2

The irreducible ordinary representations of B_n^2 have been described by A. Young (see [21]). Here we obtain these representations by applying the methods developed in chapter IV. First we will establish some notation.

Let $\theta = \theta_1 \theta_2$ be a decomposition of $\theta \in B_{t_j}^2 \times B_{n-t_j}^2$ into disjoint cycles θ_1 and θ_2 where $\theta_1 \in B_{t_j}^2$, $\theta_2 \in B_{n-t_j}^2$ are given by $\theta_1 = \theta_{11} \theta_{12} \dots \theta_{1t}$ and $\theta_2 = \theta_{21} \theta_{22} \dots \theta_{2s}$. Let $\sigma(\theta)$ denote the number of symbols of Ω involved in θ_1 which are mapped onto negative symbols by θ . Define a map $P_{(t_j, n-t_j)}$ from $B_{(t_j, n-t_j)}^2$ onto Z_2 by

$$P_{(t_j, n-t_j)}(\theta) = \begin{cases} 1 & \text{if } \sigma(\theta) \text{ is even} \\ -1 & \text{if } \sigma(\theta) \text{ is odd.} \end{cases} \quad (5.3)$$

We now have the following result.

LEMMA 5.2.1

The map $P_{(t_j, n-t_j)}$ is an i.o.r of $B_{(t_j, n-t_j)}^2$, $t_j = j-1$, $j=1, \dots, n+1$.

PROOF

By lemma 5.1.3 $(t_j, n-t_j)$ is a permissible 2-tuple of n , so that $B_{(t_j, n-t_j)}^2$ is a generalised young subgroup of B_n^2 . Now taking $\xi = -1$ and $\sigma(\theta) = f(\theta_1)$ where f is as defined in definition 3.2.1, then lemma 4.1.3 gives the result.

If now ϕ is the homomorphism in lemma 3.1.1, then $\ker \phi = C_2^n$, so that

$$(B_{(t_j, n-t_j)}^2 / \ker \phi) \cong S_{(t_j, n-t_j)}.$$

Thus the i.o.r's of $S_{(t_j, n-t_j)}$ may be

lifted to give i.o.r's of $B_{(t_j, n-t_j)}^2$. In the table

below for each generalised young subgroup

$B_{(t_j, n-t_j)}^2$ we give the corresponding young subgroup

$S_{(t_j, n-t_j)}$ from which the representation \mathbb{P} of

$B_{(t_j, n-t_j)}^2$ may be lifted. We have

$$\mathbb{P} = \left(\left[\alpha_{t_j} \right] \otimes \left[\alpha_{n-t_j} \right] \right)^* \quad (5.4)$$

where $[\alpha_{t_j}]$ denotes the representation of S_{t_j}

corresponding to the partition (α_{t_j}) of t_j , $[\alpha_{n-t_j}]$

is that of S_{n-t_j} corresponding to the partition

(α_{n-t_j}) of $n-t_j$; and $*$ denotes the lifting of

representations.

<u>GENERALISED YOUNG SUBGROUP</u>	<u>CORRESPONDING YOUNG</u>
$B_{(t_j, n-t_j)}^2$	<u>SUBGROUP, $S_{(t_j, n-t_j)}$</u>
$B_{(0, n)}^2 \cong B_n^2$	$S_{(0, n)} \cong S_n$
$B_{(1, n-1)}^2$	$S_{(1, n-1)}$
⋮	⋮
$B_{(n-1, 1)}^2$	$S_{(n-1, 1)}$
$B_{(n, 0)}^2 \cong B_n^2$	$S_{(n, 0)} \cong S_n$

(5.5)

And we see that corresponding to each permissible 2-tuple $(t_j, n-t_j)$ of n , there are $P(t_j)P(n-t_j)$ o.r.'s $\mathbb{P}=\mathbb{P}^*$ of $B_{(t_j, n-t_j)}^2$ obtained in this way. Thus the total number of such o.r.'s \mathbb{P} equals $\sum_{j=1}^{n+1} P(t_j)P(n-t_j)$ where j ranges over all permissible 2-tuples of n . Hence the number of the representations \mathbb{P} of $B_{(t_j, n-t_j)}^2$ is equal to the number of i.o.r.'s of $S_{(t_j, n-t_j)}$ counted over all 2-tuples $(t_j, n-t_j)$, of n .

Now if $\mathbb{P}_{(t_j, n-t_j)}$ is the o.r. of $B_{(t_j, n-t_j)}^2$ obtained in lemma 5.2.1, for each o.r. \mathbb{P} of $B_{(t_j, n-t_j)}^2$ given by (5.4), we form the tensor products

$$\mathbb{P} \otimes \mathbb{P}_{(\alpha)} = \left(\left[\alpha_{t_j} \right] \otimes \left[\alpha_{n-t_j} \right] \right)^* \otimes \mathbb{P}_{(t_j, n-t_j)} \quad (5.6)$$

If $\left(\mathbb{P} \otimes \mathbb{P}_{(t_j, n-t_j)} \right) \uparrow B_n^2$ denotes an induced representation of B_n^2 , we have our main result.

THEOREM 5.2.2

A full set of inequivalent i.o.r's of B_n^2 is given by

$$\left\{ \left(P \otimes P_{(t_j, n-t_j)} \right) \uparrow B_n^2 \right\}$$

where $(\alpha) = (t_j, n-t_j)$ ranges over all permissible 2-tuples of n .

PROOF

From the above construction the total number of these o.r's of B_n^2 is $\sum_{j=1}^{n+1} P(t_j)P(n-t_j)$, where j ranges over all permissible 2-tuples of n .

This gives a full set of i.o.r's of B_n^2 by lemma 5.1.3 and theorem 4.3.1.

REMARK 5.2.3

The i.o.r's of the Weyl group of type B_n which is isomorphic to the hyperoctahedral group B_n^2 have been investigated by J.J. Mayer [12], Hughes [8] and Gessinger and Kinch [22]. It is seen that the results obtained there correspond to those given in theorem 5.2.2.

5.3 ORDINARY REPRESENTATIONS OF B_3^2 AND B_4^2

In this section we give an explicit construction of the i.o.r's of the hyperoctahedral groups B_3^2 and B_4^2 as a direct application of theorem 5.2.2.

5.3.1 REPRESENTATIONS OF B_3^2

The group B_3^2 has order 48. In the table below we give the conjugacy class representatives of B_3^2 together with their types and orders.

Type	Class Representative(s)	Order
$ccl_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	e	1
$ccl_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 12 \\ 21 \end{pmatrix}$ or $\begin{pmatrix} 12 \\ -2-1 \end{pmatrix}$	6
$ccl_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 123 \\ 231 \end{pmatrix}$ or $\begin{pmatrix} 1 & 2 & 3 \\ -2 & -3 & 1 \end{pmatrix}$	8
$ccl_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	3
$ccl_5 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$	6
$ccl_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ -2 & 3 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 2 & 3 \\ -2 & -3 & -1 \end{pmatrix}$	8
$ccl_7 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}$	3
$ccl_8 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \end{pmatrix}$	1
$ccl_9 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -3 \end{pmatrix}$ or $\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ -3 \end{pmatrix}$	6
$ccl_{10} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -3 \end{pmatrix}$ or $\begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ -3 \end{pmatrix}$	6

The permissible 2-tuples of $n=3$ are

$(t_j, 3-t_j) \in \{(0,3), (1,2), (2,1), (3,0)\}$ so that $|(\alpha)|=4$. By lemma 5.1.3 the number of conjugacy classes of B_3^2 is $\sum_{j=1}^4 P(t_j)P(3-t_j) = \underline{10}$.

Hence B_3^2 has 10 inequivalent i.o.r's, we construct these step by step.

REPRESENTATIONS OF THE YOUNG SUBGROUPS $S_{(t_j, 3-t_j)}$

The partition $(t_j, 3-t_j) = (2, 1)$

The young subgroup $S_{(2, 1)}$ has two i.o.r's given by $[2] \otimes [1]$ and $[1^2] \otimes [1]$.

The partition $(t_j, 3-t_j) = (3)$ or $(3, 0)$

The young subgroup $S_{(3)}$ which is isomorphic to S_3 , has three representations given by $[3]$, $[21]$, $[1^3]$.

Thus the full set of relevant i.o.r's P of the young subgroups of S_3 is

$$\{ [2] \otimes [1], [1^2] \otimes [1], [3], [21], [1^3] \}$$

REPRESENTATIONS OF THE GENERALISED YOUNG SUBGROUPS

$B_{(t_j, 3-t_j)}^2$

Keeping the above notation, the i.o.r's P of the young subgroups of S_3 are lifted to give the i.o.r's $P=P^*$ of the generalised young subgroups. The full sets of the i.o.r's P of $B_{(t_j, 3-t_j)}^2$ are given below for each subgroup:

$$\begin{array}{l}
 \text{For } B^2_{(2,1)} \\
 \hline
 \{ ([2] \otimes [1])^*, ([1^2] \otimes [1])^* \} \\
 \\
 \text{For } B^2_{(1,2)} \\
 \hline
 \{ ([1] \otimes [2])^*, ([1] \otimes [1^2])^* \} \\
 \\
 \text{For } B^2_{(0,3)} \cong B^2_{(3,0)} (\cong B^2_{(3)}) \\
 \hline
 \{ [3]^*, [21]^*, [1^3]^* \} \qquad (5.7)
 \end{array}$$

BASIC REPRESENTATIONS OF THE $B^2_{(t_j, 3-t_j)}$

For the permissible 2-tuples of $n=3$ $(t_j, 3-t_j)$ in $\{(0,3), (1,2), (2,1), (3,0)\}$, we have the basic o.r's $P_{(t_j, 3-t_j)}$ as in lemma 5.2.1 given by

$$\begin{array}{l}
 P_{(0,3)} = P_0 \otimes P_3 \\
 P_{(1,2)} = P_1 \otimes P_2 \\
 P_{(2,1)} = P_2 \otimes P_1 \\
 P_{(3,0)} = P_3 \otimes P_0. \qquad (5.8)
 \end{array}$$

As in (5.6) we tensor the o.r's $P=P^*$ obtained earlier with the representations $P_{(t_j, 3-t_j)}$ above to give the following o.r's $P \otimes P_{(t_j, 3-t_j)}$ of $B^2_{(t_j, 3-t_j)}$, $t_j = j-1$, $(j=1, 2, \dots, 4)$

$$\begin{array}{l}
 ([1] \otimes [2])^* \otimes P_{(1,2)} \quad [3]^* \otimes P_{(0,3)} \\
 ([1] \otimes [1^2])^* \otimes P_{(1,2)} \quad [21]^* \otimes P_{(0,3)} \\
 ([2] \otimes [1])^* \otimes P_{(2,1)} \quad [1^3]^* \otimes P_{(0,3)} \\
 \qquad \qquad \qquad [3]^* \otimes P_{(3,0)} \\
 ([1^2] \otimes [1])^* \otimes P_{(2,1)} \quad [21]^* \otimes P_{(3,0)} \\
 \qquad \qquad \qquad [1^3]^* \otimes P_{(3,0)} \qquad (5.9)
 \end{array}$$

REPRESENTATION OF B_3^2

The o.r's $P \otimes P_{(t_j, 3-t_j)}$ of $B_{(t_j, 3-t_j)}^2$ are now induced to B_3^2 as in theorem 5.2.2, to give the full set of i.or's of B_3^2 .

$$\begin{aligned}
 T_1 &= (([1] \otimes [2]) \otimes^* P_{(1,2)}) \uparrow B_3^2; & T_6 &= ([3] \otimes^* P_{(0,3)}) \uparrow B_3^2 \\
 T_2 &= (([1] \otimes [1^2]) \otimes^* P_{(1,2)}) \uparrow B_3^2; & T_7 &= ([21] \otimes^* P_{(0,3)}) \uparrow B_3^2 \\
 T_3 &= (([2] \otimes [1]) \otimes^* P_{(2,1)}) \uparrow B_3^2; & T_8 &= ([1^3] \otimes^* P_{(0,3)}) \uparrow B_3^2 \\
 T_4 &= (([1^2] \otimes [1]) \otimes^* P_{(2,1)}) \uparrow B_3^2; & T_9 &= ([3] \otimes^* P_{(3,0)}) \uparrow B_3^2 \\
 T_5 &= ([21] \otimes^* P_{(3,0)}) \uparrow B_3^2; & T_{10} &= ([1^3] \otimes^* P_{(3,0)}) \uparrow B_3^2
 \end{aligned}$$

(5.10)

The characters of B_3^2 are given in Table III of the appendix.

5.3.2 REPRESENTATIONS OF B_4^2

The group B_4^2 has order 384. The table below shows the representatives of the classes of B_4^2 together with their types and orders.

<u>Type</u>	<u>Class representatives</u>	<u>Order</u>
$ccl_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$	e	1
$ccl_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}$ or $\begin{pmatrix} 12 \\ 21 \end{pmatrix}$	12
$ccl_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ -2 & -3 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$	32
$ccl_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ -2 & -3 & 4 & -1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$	48
$ccl_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$	4
$ccl_6 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$	12
$ccl_7 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ -2 & 3 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 2 & 3 \\ -2 & -3 & -1 \end{pmatrix}$	32
$ccl_8 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ -2 & 3 & 4 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 2 & 3 & 4 \\ -2 & -3 & -4 & 1 \end{pmatrix}$	48
$ccl_9 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ -3 \end{pmatrix}$ or $\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -3 \end{pmatrix}$	24
$ccl_{10} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ -2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ -4 \end{pmatrix}$ or $\begin{pmatrix} 1 & 2 & 3 \\ -2 & -3 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ -4 \end{pmatrix}$	32
$ccl_{11} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -3 \end{pmatrix}$ or $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ -4 \end{pmatrix}$	24
$ccl_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ -4 \end{pmatrix}$ or $\begin{pmatrix} 1 & 2 & 3 \\ -2 & -3 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ -4 \end{pmatrix}$	32
$ccl_{13} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}$ or $\begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}$	24
$ccl_{14} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \end{pmatrix}$	6
$ccl_{15} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}$	12
$ccl_{16} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}$ or $\begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ -4 & -3 \end{pmatrix}$	12
$ccl_{17} = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -3 \end{pmatrix} \begin{pmatrix} 4 \\ -4 \end{pmatrix}$ or $\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ -3 \end{pmatrix} \begin{pmatrix} 4 \\ -4 \end{pmatrix}$	12
$ccl_{18} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 12 \\ 21 \end{pmatrix} \begin{pmatrix} 3 \\ -3 \end{pmatrix} \begin{pmatrix} 4 \\ -4 \end{pmatrix}$ or $\begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ -3 \end{pmatrix} \begin{pmatrix} 4 \\ -4 \end{pmatrix}$	12

$$\begin{aligned}
 ccl_{19} &= \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \end{pmatrix} \begin{pmatrix} 3 \\ -3 \end{pmatrix} & 4 \\
 ccl_{20} &= \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \end{pmatrix} \begin{pmatrix} 3 \\ -3 \end{pmatrix} \begin{pmatrix} 4 \\ -4 \end{pmatrix} & 1
 \end{aligned}
 \tag{5.11}$$

The permissible 2-tuples of $n=4$ are:

$(t_j, 4-t_j) \in \{(0,4), (1,3), (2,2), (3,1), (4,0)\}$ so that $|\alpha|=5$. By lemma 5.1.3 the number of conjugacy classes of B_4^2 is $\sum_{j=1}^5 P(t_j)P(4-t_j)$ which equals 20. Hence B_4^2 has 20 inequivalent i.o.r's.

We construct these as in (5.3.1).

REPRESENTATIONS OF THE YOUNG SUBGROUPS $S_{(t_j, 4-t_j)}$

The partition $(t_j, 4-t_j) = (2,2)$

The young subgroup $S_{(2,2)} = S_2 \times S_2$ has 4 i.o.r's given by $[2] \otimes [2]$, $[2] \otimes [1^2]$, $[1^2] \otimes [1^2]$, $[1^2] \otimes [2]$.

The partition $(t_j, 4-t_j) = (3,1)$

The Young subgroup $S_{(3,1)} = S_3 \times S_1$ is isomorphic to S_3 and has three i.o.r's given by $[3] \otimes [1]$, $[21] \otimes [1]$, $[1^3] \otimes [1]$.

The partition $(t_j, 4-t_j) = (4) = (4,0) = (0,4)$

The young subgroup $S_{(4)}$ is isomorphic to S_4 and has five i.o.r's given by $[4]$, $[31]$, $[21^2]$, $[2^2]$ and $[1^4]$. (See example 2.1).

Thus the full set of relevant i.o.r's P of the Young subgroups $S_{(t_j, 4-t_j)}$ of S_4 is

$$\left\{ \begin{array}{l} [2] \otimes [2], [2] \otimes [1^2], [1^2] \otimes [2], [1^2] \otimes [1^2], [3] \otimes [1] \\ [21] \otimes [1], [1^3] \otimes [1], [4], [31], [2^2], [21^2], [1^4] \end{array} \right\} \quad (5.12)$$

REPRESENTATIONS OF THE GENERALISED YOUNG SUBGROUPS

$$\underline{B_{(t_j, 4-t_j)}^2}$$

We keep the above notation. The i.o.r's P of the young subgroups $S_{(t_j, 4-t_j)}$ of S_4 are lifted to give i.o.r's $P = P^*$ of the generalised young subgroups $B_{(t_j, 4-t_j)}^2$ of B_4^2 . The full sets of the i.o.r's P of $B_{(t_j, 4-t_j)}^2$ are given below for each subgroup.

$$\underline{\text{For } B_{(2,2)}^2}$$

$$\left\{ ([2] \otimes [2])^*, ([2] \otimes [1^2])^*, ([1^2] \otimes [2])^*, ([1^2] \otimes [1^2])^* \right\}$$

$$\underline{\text{For } B_{(3,1)}^2}$$

$$\left\{ ([3] \otimes [1])^*, ([21] \otimes [1])^*, ([1^3] \otimes [1])^* \right\}$$

$$\underline{\text{For } B_{(1,3)}^2}$$

$$\left\{ ([1] \otimes [3])^*, ([1] \otimes [21])^*, ([1] \otimes [1^3])^* \right\}$$

$$\underline{\text{For } B_{(4)}^2 \text{ or } B_{(4,0)}^2 \text{ or } B_{(0,4)}^2}$$

$$\left\{ [4]^*, [31]^*, [21^2]^*, [2^2]^*, [1^4]^* \right\}$$

BASIC REPRESENTATIONS OF $B^2_{(t_j, 4-t_j)}$

For the permissible 2-tuples of $n=4$, $(t_j, 4-t_j)$ in $\{(0,4), (1,3), (2,2), (3,1), (4,0)\}$ we have the basic o.r's $P_{(t_j, 4-t_j)}$ as in lemma 5.2.1 given by

$$\begin{aligned}
 P_{(0,4)} &= P_0 \otimes P_4; & P_{(3,1)} &= P_3 \otimes P_1 \\
 P_{(1,3)} &= P_1 \otimes P_3; & P_{(4,0)} &= P_4 \otimes P_0. \\
 P_{(2,2)} &= P_2 \otimes P_2
 \end{aligned}
 \tag{5.13}$$

As in (5.6) we tensor the o.r's $P = P^*$ obtained earlier with the $P_{(t_j, 4-t_j)}$ above to give the following o.r's

$$P \otimes P_{(t_j, 4-t_j)}, \quad t_j = j-1, \quad j = 1, \dots, 5.$$

$$\begin{aligned}
 ([2] \otimes [2])^* \otimes P_{(2,2)}; & \quad ([1^2] \otimes [1^2])^* \otimes P_{(2,2)} \\
 ([2] \otimes [1^2])^* \otimes P_{(2,2)}; & \quad ([3] \otimes [1])^* \otimes P_{(3,1)} \\
 ([1^2] \otimes [2])^* \otimes P_{(2,2)}; & \quad ([21] \otimes [1])^* \otimes P_{(3,1)} \\
 ([1^3] \otimes [1])^* \otimes P_{(3,1)}; & \quad ([1] \otimes [21])^* \otimes P_{(1,3)} \\
 ([1] \otimes [3])^* \otimes P_{(1,3)}; & \quad ([1] \otimes [1^3])^* \otimes P_{(1,3)} \\
 [4]^* \otimes P_{(0,4)}; & \quad ([4]^* \otimes P_{(4,0)}) \\
 [31]^* \otimes P_{(0,4)}; & \quad [31]^* \otimes P_{(4,0)} \\
 [21^2]^* \otimes P_{(0,4)}; & \quad [21^2]^* \otimes P_{(4,0)} \\
 [2^2]^* \otimes P_{(0,4)}; & \quad [2^2]^* \otimes P_{(4,0)} \\
 [1^4]^* \otimes P_{(0,4)}; & \quad [1^4]^* \otimes P_{(4,0)}
 \end{aligned}$$

(5.14)

REPRESENTATIONS OF B_4^2

The o.r's $P \otimes P_{(t_j, 4-t_j)}$ of $B_{(t_j, 4-t_j)}^2$ are now induced to B_4^2 as in theorem 5.2.2 to give the full set of i.o.r's of B_4^2 :

$$\begin{aligned}
 T_1 &= (([2] \otimes [2])^* \otimes P_{(2,2)}) \uparrow B_4^2; & T_5 &= ((([3] \otimes [1])^* \otimes P_{(3,1)}) \uparrow B_4^2 \\
 T_2 &= ((([2] \otimes [1^2])^* \otimes P_{(2,2)}) \uparrow B_4^2; & T_6 &= ((([21] \otimes [1])^* \otimes P_{(3,1)}) \uparrow B_4^2 \\
 T_3 &= ((([1^2] \otimes [2])^* \otimes P_{(2,2)}) \uparrow B_4^2; & T_7 &= ((([1^3] \otimes [1])^* \otimes P_{(3,1)}) \uparrow B_4^2 \\
 T_4 &= ((([1^2] \otimes [1^2])^* \otimes P_{(2,2)}) \uparrow B_4^2; & T_8 &= ((([1] \otimes [3])^* \otimes P_{(1,3)}) \uparrow B_4^2 \\
 & & T_9 &= ((([1] \otimes [21])^* \otimes P_{(1,3)}) \uparrow B_4^2 \\
 T_{11} &= ([4]^* \otimes P_{(0,4)}) \uparrow B_4^2; & T_{10} &= ([1] \otimes [1^3])^* \otimes P_{(1,3)} \uparrow B_4^2 \\
 T_{12} &= ([31]^* \otimes P_{(0,4)}) \uparrow B_4^2; & T_{14} &= ([2^2]^* \otimes P_{(0,4)}) \uparrow B_4^2 \\
 T_{13} &= ([21^2]^* \otimes P_{(0,4)}) \uparrow B_4^2; & T_{15} &= ([1^4]^* \otimes P_{(0,4)}) \uparrow B_4^2 \\
 T_{16} &= ([4]^* \otimes P_{(4,0)}) \uparrow B_4^2; & T_{17} &= ([31]^* \otimes P_{(4,0)}) B_4^2 \\
 T_{18} &= ([21^2]^* \otimes P_{(4,0)}) \uparrow B_4^2; & T_{19} &= ([2^2]^* \otimes P_{(4,0)}) B_4^2 \\
 T_{20} &= ([1^4]^* \otimes P_{(4,0)}) \uparrow B_4^2;
 \end{aligned}$$

(5.15)

The characters of B_4^2 are given in table IV of the appendix.

APPENDIX

CLASS REPRESENTATIVES	e	(12)	(123)
CLASS ORDERS	1	3	2
$\chi_{[3]}$	1	1	1
$\chi_{[21]}$	2	0	-1
$\chi_{[1^3]}$	1	-1	1

TABLE I: CHARACTERS OF S_3

CLASS REPRESENTATIVES	e	(12)	(123)	(12)(34)	(1234)
CLASS ORDERS	1	6	8	3	6
$\chi_{[4]}$	1	1	1	1	1
$\chi_{[31]}$	3	1	0	-1	-1
$\chi_{[2^2]}$	2	0	-1	2	0
$\chi_{[21^2]}$	3	-1	0	-1	1
$\chi_{[1^4]}$	1	-1	1	1	-1

TABLE II: CHARACTERS OF S_4

CLASSES OF B_3^2	$cc1_1$	$cc1_2$	$cc1_3$	$cc1_4$	$cc1_5$	$cc1_6$	$cc1_7$	$cc1_8$	$cc1_9$	$cc1_{10}$
CLASS ORDERS	1	6	8	3	3	6	1	6	6	8
$\chi_{T_1}^2$	3	1	0	1	-1	1	-3	-1	-1	0
$\chi_{T_2}^2$	3	-1	0	1	-1	-1	-3	1	1	0
$\chi_{T_3}^2$	3	1	0	-1	-1	-1	3	-1	1	0
$\chi_{T_4}^2$	3	-1	0	-1	-1	1	3	1	-1	0
$\chi_{T_5}^2$	2	0	-1	2	2	0	2	0	0	-1
$\chi_{T_6}^2$	1	1	1	-1	1	-1	-1	1	-1	-1
$\chi_{T_7}^2$	2	0	-1	-2	2	0	-2	0	0	1
$\chi_{T_8}^2$	1	-1	1	-1	1	1	-1	-1	1	-1
$\chi_{T_9}^2$	1	1	1	1	1	1	1	1	1	1
$\chi_{T_{10}}^2$	1	1	1	1	1	-1	1	-1	1	1

TABLE III: CHARACTERS OF B_3^2

CLASSES OF B_4	$cc1_1$	$cc1_2$	$cc1_3$	$cc1_4$	$cc1_5$	$cc1_6$	$cc1_7$	$cc1_8$	$cc1_9$	$cc1_{10}$	$cc1_{11}$	$cc1_{12}$	$cc1_{13}$	$cc1_{14}$	$cc1_{15}$	$cc1_{16}$	$cc1_{17}$	$cc1_{18}$	$cc1_{19}$	$cc1_{20}$
ORDERS	1	12	32	48	4	12	32	48	24	32	24	32	24	6	12	12	12	12	4	1
χ_{T_1}	6	2	0	0	0	0	0	0	-2	0	0	0	0	-2	-2	2	0	2	0	6
χ_{T_2}	6	0	0	0	0	-2	0	0	0	0	2	0	0	-2	2	-2	-2	0	0	6
χ_{T_3}	6	0	0	0	0	2	0	0	0	0	-2	0	0	-2	2	-2	2	0	0	6
χ_{T_4}	6	-2	0	0	0	0	0	2	0	0	0	0	0	-2	2	0	0	-2	0	6
χ_{T_5}	4	2	1	0	-2	-2	-1	0	0	-1	0	1	0	0	0	0	2	-2	2	-4
χ_{T_6}	8	0	-1	0	-4	0	1	0	0	1	0	-1	0	0	0	0	0	0	4	-8
χ_{T_7}	4	-2	1	0	-2	2	-1	0	0	-1	0	1	0	0	0	0	-2	-2	2	-4
χ_{T_8}	4	2	1	0	2	2	1	0	0	-1	0	-1	0	0	0	0	-2	-2	-2	-4
χ_{T_9}	8	0	-1	0	4	0	-1	0	0	1	0	1	0	0	0	0	0	0	-4	-8
$\chi_{T_{10}}$	4	-2	1	0	2	-2	1	0	0	-1	0	-1	0	0	0	0	2	2	-2	-4
$\chi_{T_{11}}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\chi_{T_{12}}$	3	1	0	-1	3	1	0	-1	1	0	1	0	-1	3	-1	-1	1	1	3	3
$\chi_{T_{13}}$	3	-1	0	1	3	-1	0	1	-1	0	-1	0	-1	3	-1	-1	-1	-1	3	3
$\chi_{T_{14}}$	2	0	-1	0	2	0	-1	0	0	-1	0	-1	2	2	2	2	0	0	2	2
$\chi_{T_{15}}$	1	-1	1	-1	1	-1	-1	-1	-1	1	-1	1	1	-1	1	1	-1	-1	1	1
$\chi_{T_{16}}$	1	1	1	1	-1	-1	-1	-1	1	1	-1	-1	-1	1	1	1	-1	1	-1	1
$\chi_{T_{17}}$	3	1	0	-1	-3	-1	0	1	1	0	-1	0	1	3	-1	-1	-1	1	-3	3
$\chi_{T_{18}}$	3	-1	0	1	-3	1	0	-1	-1	0	1	0	1	3	-1	-1	1	-1	-3	3
$\chi_{T_{19}}$	2	0	-1	0	-2	0	1	0	0	-1	0	1	-2	2	2	2	0	0	-2	2
$\chi_{T_{20}}$	1	-1	1	-1	1	-1	-1	-1	-1	1	-1	1	-1	1	1	1	1	-1	-1	1

TABLE IV: CHARACTERS OF B_4^2

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