QUASI-DEFORMATIONS OF THE SPECIAL LINEAR LIE ALGEBRA OF DEGREE 2 USING TWISTED DERIVATIONS

By

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Declaration

I, **Dominique Sampa**, hereby declare that this dissertation represents my own work (except where otherwise indicated), and that it has not previously been submitted for a degree, diploma or other qualification at this or another university.

Signature: .... Date: ....

Certificate of Approval

This dissertation of Dominique Sampa has been approved as fulfilling the requirements or partial fulfilment of the requirements for the award of the degree of Master of Science in Mathematics by the University of Zambia.

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Abstract

In this dissertation we obtain quasi-deformations of the special linear Lie algebra of degree 2 over a field $\mathbb{F}$, abbreviated $\mathfrak{sl}_2(\mathbb{F})$. We take $\mathbb{F}$ to be a field of characteristic zero. Our approach involves obtaining and deforming a linear operator representation of $\mathfrak{sl}_2(\mathbb{F})$ using σ-twisted derivations (with σ an endomorphism on the base algebra). One of the main points of this deformation method is that the deformed algebra comes endowed with a canonical twisted Jacobi identity. Furthermore, we show that by choosing parameters suitably we can deform $\mathfrak{sl}_2(\mathbb{F})$ into the Heisenberg Lie algebra and some other 3- and 4-dimensional Lie algebras in addition to more exotic types of algebras. This is in stark contrast to the usual classical deformation schemes where $\mathfrak{sl}_2(\mathbb{F})$ is rigid.

To the cherished memory of my esteemed late parents,
Peter Mutale Sampa and Febby Mulenga Sampa.

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Dominique Sampa

Contents

List of Symbols

Below is a list of some symbols used throughout this dissertation and their meanings.

$\mathfrak{sl}_2(\mathbb{F})$	Special linear Lie algebra of degree 2 over a field $\mathbb{F}$
$\mathcal{U}(\mathfrak{g})$	Universal enveloping algebra of a Lie algebra $\mathfrak{g}$
$\circlearrowleft_{x,y,z}$	Cyclic summation with respect to x, y and z
σ	Algebra endomorphism
∂_σ	σ-derivation
$\mathcal{L}_{\mathbb{F}}(V)$	Set of $\mathbb{F}$-linear maps on the $\mathbb{F}$-vector space V
$\mathfrak{D}_\sigma(\mathcal{A})$	Set of σ-derivations on an algebra $\mathcal{A}$
$\mathcal{A} \cdot \partial_\sigma$	$\mathcal{A}$-submodule of $\mathfrak{D}_\sigma(\mathcal{A})$ generated by ∂_σ
$\mathbb{F}[t]$	Algebra of polynomials over a field $\mathbb{F}$ in the variable t
$\mathbb{F}[t]/(t^3)$	Algebra of polynomials of degree less than 3 over a field $\mathbb{F}$ in the variable t
$\mathbb{F}[t]/(t^4)$	Algebra of polynomials of degree less than 4 over a field $\mathbb{F}$ in the variable t

Chapter 1

Introduction

Deformation is the process of continuously changing the structure or properties of a mathematical object in a controlled way. In this dissertation, the basic undeformed object is the special linear Lie algebra of degree 2 over a field $\mathbb{F}$, abbreviated $\mathfrak{sl}_2(\mathbb{F})$. It is defined as the Lie algebra of 2×2 matrices with entries from $\mathbb{F}$ and trace zero. By a Lie algebra, we mean a vector space V over a field $\mathbb{F}$ equipped with a binary operation $[\cdot,\cdot] : V \times V \to V$ called the Lie bracket that satisfies the following conditions for all $X, Y, Z \in V$ and $a, b \in \mathbb{F}$:

(i) Bilinearity: $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ and $[X, aY + bZ] = a[X, Y] + b[X, Z]$

(ii) Alternativity: $[X, X] = 0$

(iii) Jacobi identity: $\big[X, [Y, Z]\big] + \big[Y, [Z, X]\big] + \big[Z, [X, Y]\big] = 0$

The first two conditions imply skew-symmetry, that is, $[X, Y] = -[Y, X]$ while the third condition shows that the Lie bracket is in general non-associative. In the case of $\mathfrak{sl}_2(\mathbb{F})$, the Lie bracket of two elements X and Y is given by the commutator

$$[X, Y] = XY - YX.$$

It is easy to see that the commutator defined in this manner satisfies all the three axioms of a Lie bracket given above, so $\mathfrak{sl}_2(\mathbb{F})$ is indeed a Lie Algebra. As a vector space, $\mathfrak{sl}_2(\mathbb{F})$ is generated by three matrices H, X and Y given by

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Using the definition of the commutator, we see that these matrices satisfy the relations

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H. \tag{1.1}$$

These are the defining relations of $\mathfrak{sl}_2(\mathbb{F})$, and since they contain the commutator, they will be referred to as the commutation relations defining $\mathfrak{sl}_2(\mathbb{F})$. A linear operator representation of these commutation relations is what will be deformed using twisted derivations. By a twisted derivation on an algebra $\mathcal{A}$ we mean a linear map $\partial_\sigma : \mathcal{A} \to \mathcal{A}$ satisfying the rule

$$\partial_\sigma(ab) = \partial_\sigma(a)b + \sigma(a)\partial_\sigma(b)$$

for an endomorphism σ on $\mathcal{A}$. This rule is a σ-deformed rule of the usual product rule of differentiation, $\partial(ab) = \partial(a)b + a\partial(b)$. For Lie algebras, the deformed objects seldom, if ever, belong to the original class of Lie algebras. However, one retains the undeformed objects in the appropriate limit. The deformations we consider in this dissertation do not necessarily have this important property. We thus refer to them as quasi-deformations thereby emphasizing this crucial difference explicitly. For a broader view of this active area of research, see, for example, [1, 2, 7–15, 25, 37, 39] and the references therein.

1.1 Statement of the Problem

Many studies have been done on deformations of $\mathfrak{sl}_2(\mathbb{C})$ using twisted derivations but nothing much has been done to generalise the results to a quasi-deformed $\mathfrak{sl}_2(\mathbb{F})$, where $\mathbb{F}$ is a field of characteristic zero (a field where $n \cdot 1 = 0$ implies $n = 0$). In this dissertation, we do the generalization and provide an elaborated example of a quasi-deformed $\mathfrak{sl}_2(\mathbb{F})$.

1.2 Aim of the Study

To obtain quasi-deformations of $\mathfrak{sl}_2(\mathbb{F})$, where $\mathbb{F}$ is a field of characteristic zero, using twisted derivations

1.3 Research Objectives

(i) To obtain and deform an operator representation of $\mathfrak{sl}_2(\mathbb{F})$.

(ii) To determine the algebraic structure represented by the deformed operators.

(iii) To provide an elaborated example of a quasi-deformed $\mathfrak{sl}_2(\mathbb{F})$.

1.4 Research Questions

(i) How can we obtain and deform an operator representation of $\mathfrak{sl}_2(\mathbb{F})$?

(ii) How can we determine the algebraic structure represented by the deformed operators?

(iii) How can we provide an elaborated example of a quasi-deformed $\mathfrak{sl}_2(\mathbb{F})$?

1.5 Significance of the Study

The study provides a method of obtaining new algebraic structures from various undeformed structures, thus adding to the body of knowledge in mathematics and possibly physics.

1.6 Literature Review

The idea to deform algebraic structures within the appropriate category is obviously not new. Quantum deformations (or q-deformations) associated with Lie algebras have been studied for some time now [7–15, 20, 23, 34]. This area underwent a period of rapid expansion in the mid-1980s when Drinfel [16] and Jimbo [21] independently studied deformations of $\mathcal{U}(\mathfrak{g})$, the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. This was motivated by their applications to the Yang-Baxter equation and quantum inverse scattering methods [24]. After that, several other versions of q-deformed Lie algebras appeared, especially in physical contexts such as string theory, vertex models and quantum field theory. The main objects for these deformations were infinite-dimensional algebras, primarily the Heisenberg type algebras, oscillator algebras, and the Virasoro algebra [2–5, 7, 27, 41].

In Hartwig [19], it was shown that for a σ-derivation ∂_σ defined on a commutative associative algebra $\mathcal{A}$ with unity, the module $\mathcal{A} \cdot \partial_\sigma$ admits a $\mathbb{C}$-algebra with a σ-deformed commutator

$$[a \cdot \partial_\sigma, b \cdot \partial_\sigma] = \sigma(a) \cdot \partial_\sigma(b \cdot \partial_\sigma) - \sigma(b) \cdot \partial_\sigma(a \cdot \partial_\sigma)$$

as a multiplication bracket, satisfying a twisted 6-term Jacobi identity. This study builds on Hartwig's work and provides an elaborated example of a quasi-deformed $\mathfrak{sl}_2(\mathbb{F})$, where $\mathbb{F}$ is a field of characteristic zero. The result becomes a natural quasi-deformation of $\mathfrak{sl}_2(\mathbb{F})$ and we have shown that it is a quasi-hom-Lie algebra in general. For more information on quasi-hom-Lie algebras, one can see Larsson and Silvestrov [26].

1.7 Research Methodology

Suppose that $\mathfrak{g}$ is the Lie algebra we want to deform and suppose that $\mathfrak{g} \xrightarrow{\rho} \mathfrak{gl}(\mathcal{A})$ is a representation of $\mathfrak{g}$ on a commutative, associative algebra $\mathcal{A}$ with unity, where $\mathfrak{gl}(\mathcal{A})$ is the Lie algebra (under the commutator bracket) of linear operators on the underlying vector space of $\mathcal{A}$. Then our deformation scheme can be diagrammatically depicted as follows:

$$\begin{tikzcd}
\mathfrak{g} \arrow[rr, "\rho"] & & \mathfrak{gl}(\mathcal{A}) \arrow[d, rightsquigarrow] \\
& \tilde{\mathfrak{g}} \arrow[ul, dotted, "\text{``limit''}"] \arrow[r, dotted, "\tilde{\rho}"'] & \widetilde{\mathfrak{gl}}(\mathcal{A}) \arrow[l, bend right]
\end{tikzcd}$$

The "squiggly" line is the "deformation" procedure of substituting the original operators with the deformed (σ-twisted) versions. The dotted arrows indicate the fact that we do not necessarily come back to the object we started with when going back the way the arrows point. This is the very reason for calling it "quasi-deformation". The algebra $\tilde{\mathfrak{g}}$ is then to be considered as the "deformation" of $\mathfrak{g}$. So what we actually change, or deform, is the given representation of $\mathfrak{g}$ and the multiplication bracket in $\mathfrak{gl}(\mathcal{A})$. This explains why the tilde appears above $\mathfrak{gl}(\mathcal{A})$ in the bottom row of the scheme diagram.

In this dissertation, the basic undeformed object ($\mathfrak{g}$ in the scheme diagram) is $\mathfrak{sl}_2(\mathbb{F})$. Recall that $\mathfrak{sl}_2(\mathbb{F})$ is generated as a vector space by three elements H, X and Y satisfying

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H.$$

The starting point for us is the following representation of $\mathfrak{sl}_2(\mathbb{F})$ in terms of first order differential operators acting on a vector space of functions in the variable t:

$$H \mapsto -2t\partial, \qquad X \mapsto \partial, \qquad Y \mapsto -t^2\partial.$$

This is what we will generalize to first order operators acting on an algebra $\mathcal{A}$, where ∂ is replaced by ∂_σ, a twisted derivation on $\mathcal{A}$.

1.8 Organisation of the Dissertation

The rest of the dissertation is organised as follows:

Chapter 2. In this chapter, we present some definitions and concepts needed to understand the work in this dissertation. Notions about fields are presented in Section 2.1 while the basic concepts of twisted derivations are presented in Section 2.2. The notion of a quasi-Lie algebra is introduced in Section 2.3 and in Section 2.4 we formally introduce $\mathfrak{sl}_2(\mathbb{F})$. In the last section we state the definition of an operator representation and give some examples.

Chapter 3. In this chapter we obtain quasi-deformations of $\mathfrak{sl}_2(\mathbb{F})$, where $\mathbb{F}$ is a field of characteristic 0 (a field where $n{\cdot}1 = 0$ implies $n = 0$). In Section 3.1, we take the base algebra $\mathcal{A}$ to be as general as it can be in this situation and deduce some necessary conditions for everything to make sense. In Section 3.2, we take the base algebra $\mathcal{A}$ to be the polynomial algebra $\mathbb{F}[t]$ and calculate some properties of the corresponding quasi-deformations. An elaborated example of a quasi-deformed $\mathfrak{sl}_2(\mathbb{F})$ is provided in Section 3.3.

Chapter 4. In this chapter we obtain quasi-deformations of $\mathfrak{sl}_2(\mathbb{F})$, where $\mathbb{F}$ is a field of characteristic 0 that includes all the third roots of unity (the solutions to the equation $z^3 = 1$). We take the base algebra $\mathcal{A}$ to be $\mathbb{F}[t]/(t^3)$, the algebra of polynomials of degree less than 3 over $\mathbb{F}$ in the variable t. In Section 4.1, we take the base algebra $\mathcal{A} = \mathbb{F}[t]/(t^3)$ to be as general as it can be in this situation and deduce some necessary conditions for everything to make sense. One special case of this deformation is considered in Section 4.2 while another special case is considered in Section 4.3.

Chapter 5. In this chapter we obtain quasi-deformations of $\mathfrak{sl}_2(\mathbb{F})$, where $\mathbb{F}$ is a field of characteristic 0 that includes all the fourth roots of unity (the solutions to the equation $z^4 = 1$). We take the base algebra $\mathcal{A}$ to be $\mathbb{F}[t]/(t^4)$, the algebra of polynomials of degree less than 4 over $\mathbb{F}$ in the variable t. In Section 5.1, we take the base algebra $\mathcal{A} = \mathbb{F}[t]/(t^4)$ to be as general as it can be in this situation and deduce some necessary conditions for everything to make sense. One special case of this deformation is considered in Section 5.2 while another special case is considered in Section 5.3.

Chapter 6. In this chapter, we delve into the results and implications of quasi-deformations of $\mathfrak{sl}_2(\mathbb{F})$. We reflect on the insights garnered from our exploration of this fascinating field of deformation theory. The aim and objectives of this dissertation are restated and discussed in Section 6.1 while the employed methodology to achieve these goals is discussed in Section 6.2. In Section 6.3 we discuss quasi-deformations of $\mathfrak{sl}_2(\mathbb{F})$, where $\mathbb{F}$ is a field of characteristic 0. Here the base algebra is first taken to be as general as it can be in this situation, and thereafter, taken to be the polynomial algebra $\mathbb{F}[t]$. In Section 6.4 we discuss quasi-deformations of $\mathfrak{sl}_2(\mathbb{F})$, where $\mathbb{F}$ is a field of characteristic 0 that includes all the third roots of unity. Here the base algebra is taken to be $\mathbb{F}[t]/(t^3)$, the algebra of polynomials of degree less than 3 over $\mathbb{F}$ in the variable t. In Section 6.5 we discuss quasi-deformations of $\mathfrak{sl}_2(\mathbb{F})$, where $\mathbb{F}$ is a field of characteristic 0 that includes all the fourth roots of unity. The base algebra is taken to be $\mathbb{F}[t]/(t^4)$, the algebra of polynomials of degree less than 4 over $\mathbb{F}$.

Chapter 7. In this final chapter, we summarize the main findings of our research and underscore their implications. We offer concrete recommendations based on our insights to guide future actions and initiatives. Ultimately, this chapter serves as a blueprint for leveraging our findings to enact positive change in the relevant domain.

Chapter 2

Preliminaries

In this chapter, we present some definitions and concepts needed to understand the work in this dissertation. Notions about fields are presented in Section 2.1 while the basic concepts of twisted derivations are presented in Section 2.2. The notion of a quasi-Lie algebra is introduced in Section 2.3 and in Section 2.4 we formally introduce $\mathfrak{sl}_2(\mathbb{F})$. In the last section we state the definition of an operator representation and give some examples.

2.1 Some Key Definitions Related to Fields

A field is a fundamental mathematical structure that generalizes the properties of familiar number systems such as the rational numbers, real numbers, and complex numbers. Here are some key definitions and properties related to fields:

Definition 2.1.1. A field $\mathbb{F}$ is a set equipped with two binary operations, usually denoted as addition $(+)$ and multiplication $(\cdot)$, such that the following properties hold for all elements a, b, c in the field:

(i) **Closure:** For any a, b in $\mathbb{F}$, both $a + b$ and $a \cdot b$ are also in $\mathbb{F}$.

(ii) **Associativity:** Addition and multiplication are associative operations, meaning $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all a, b, c in $\mathbb{F}$.

(iii) **Commutativity:** Addition and multiplication are commutative, meaning $a+b = b+a$ and $a \cdot b = b \cdot a$ for all a, b in $\mathbb{F}$.

(iv) **Additive Identity:** There exists an element 0 (called the additive identity) such that $a + 0 = a$ for all a in $\mathbb{F}$.

(v) **Multiplicative Identity:** There exists an element 1 (called the multiplicative identity) such that $a \cdot 1 = a$ for all a in $\mathbb{F}$, where $1 \neq 0$.

(vi) **Additive Inverse:** For every element a in $\mathbb{F}$, there exists an element $-a$ (called the additive inverse) such that $a + (-a) = 0$.

(vii) **Multiplicative Inverse:** For every nonzero element a in $\mathbb{F}$, there exists an element a^{-1} (called the multiplicative inverse) such that $a \cdot a^{-1} = 1$.

Definition 2.1.2. The characteristic of a field $\mathbb{F}$ is the smallest positive integer n such that $n \cdot 1 = 0$, or 0 if no such n exists. That is, the order of the multiplicative identity element.

Definition 2.1.3. Let $\mathbb{F}$ be a field.

(a) **Subfield:** A subfield of $\mathbb{F}$ is a subset of $\mathbb{F}$ that forms a field under the same operations of addition and multiplication.

(b) **Finite Field:** A field with a finite number of elements is called a finite field or Galois field. The order of a finite field is the number of elements it contains.

(c) **Prime Field:** The smallest field is called the prime field, denoted by $\mathbb{F}_p$, where p is a prime number. This field contains p elements.

(d) **Algebraic Closure:** A field $\mathbb{F}$ is algebraically closed if every non-constant polynomial with coefficients in $\mathbb{F}$ has a root in $\mathbb{F}$.

(e) **Field Extension:** If $\mathbb{K}$ is a field containing $\mathbb{F}$, then $\mathbb{K}$ is said to be an extension field of $\mathbb{F}$. The degree of the extension is the dimension of $\mathbb{K}$ as a vector space over $\mathbb{F}$.

Understanding the properties and structures of fields is crucial in various areas of mathematics, including algebra, analysis and number theory. Fields serve as foundational structures for studying arithmetic operations and solving equations in various contexts.

2.2 σ-Derivations

We let $\mathbb{F}$ denote a field of characteristic zero and $\mathcal{A}$ be a commutative, associative $\mathbb{F}$-algebra with unity 1. Furthermore, we let σ denote an endomorphism on $\mathcal{A}$.

Definition 2.2.1. [27] A twisted derivation or σ-derivation on $\mathcal{A}$ is an $\mathbb{F}$-linear map $\partial_\sigma : \mathcal{A} \to \mathcal{A}$ satisfying the σ-twisted product (Leibniz) rule

$$\partial_\sigma(ab) = \partial_\sigma(a)b + \sigma(a)\partial_\sigma(b) \tag{2.1}$$

If $\sigma = \mathrm{id}_{\mathcal{A}}$, the identity map on $\mathcal{A}$, then $\partial_\sigma = \partial$, the ordinary derivative satisfying the ordinary product rule

$$\partial(ab) = \partial(a)b + a\partial(b).$$

We denote the set of all σ-derivations on an algebra $\mathcal{A}$ by $\mathfrak{D}_\sigma(\mathcal{A})$. When $\sigma = \mathrm{id}_{\mathcal{A}}$, we simply write $\mathfrak{D}(\mathcal{A})$ for this set.

In the following examples of σ-derivations, we always consider $\mathcal{A} = \mathbb{C}[t]$, the algebra of polynomials over the complex field $\mathbb{C}$ in the variable t.

Example 2.2.2. [27] If S is the shift endomorphism on the polynomial algebra $\mathbb{C}[t]$ given by

$$S(a)(t) = a(t+1),$$

then the shifted difference operator Δ acting on $\mathbb{C}[t]$ and defined by

$$\Delta(a)(t) = a(t+1) - a(t)$$

is an S-derivation on $\mathbb{C}[t]$. This follows since for all $a, b \in \mathbb{C}[t]$, we have

$$\begin{aligned}
\Delta(ab)(t) &= (ab)(t+1) - (ab)(t) \\
&= a(t+1)b(t+1) - a(t)b(t) \\
&= a(t+1)b(t+1) - a(t+1)b(t) + a(t+1)b(t) - a(t)b(t) \\
&= a(t+1)[b(t+1) - b(t)] + [a(t+1) - a(t)]b(t) \\
&= S(a)(t)\Delta(b)(t) + \Delta(a)(t)b(t) \\
&= \Delta(a)(t)b(t) + S(a)(t)\Delta(b)(t).
\end{aligned}$$

Suppressing the variable t, we obtain

$$\Delta(ab) = \Delta(a)b + S(a)\Delta(b),$$

which shows that Δ is indeed an S-derivation on $\mathbb{C}[t]$. Comparing with equation (2.1) in Definition 2.2.1, we observe that $\sigma = S$ in this case.

Example 2.2.3. [27] If α_q is the rescaling endomorphism on $\mathbb{C}[t]$ defined by

$$\alpha_q(a)(t) = a(qt)$$

for some $q \in \mathbb{C} \setminus \{0\}$, then the Jackson derivative D_q acting on $\mathbb{C}[t]$ and defined by

$$D_q(a)(t) = \frac{a(qt) - a(t)}{qt - t}$$

for some $q \in \mathbb{C} \setminus \{1\}$ is an α_q-derivation on $\mathbb{C}[t]$. This follows since for all $a, b \in \mathbb{C}[t]$,

$$\begin{aligned}
D_q(ab)(t) &= \frac{(ab)(qt) - (ab)(t)}{qt - t} \\
&= \frac{a(qt)b(qt) - a(t)b(t)}{qt - t} \\
&= \frac{a(qt)b(qt) - a(qt)b(t) + a(qt)b(t) - a(t)b(t)}{qt - t} \\
&= \frac{a(qt)[b(qt) - b(t)] + [a(qt) - a(t)]b(t)}{qt - t} \\
&= a(qt)(t)\frac{b(qt)) - b(t)}{qt - t} + \frac{a(qt)) - a(t)}{qt - t}b(t) \\
&= \alpha_q(a)(t)D_q(b)(t) + D_q(a)(t)b(t) \\
&= D_q(a)(t)b(t) + \alpha_q(a)(t)D_q(b)(t).
\end{aligned}$$

Suppressing the variable t, we obtain

$$D_q(ab) = D_q(a)b + \alpha_q(a)D_q(b),$$

which shows that D_q is indeed an α_q-derivation on $\mathbb{C}[t]$. Comparing with equation (2.1) in Definition 2.2.1, we observe that $\sigma = \alpha_q$ in this case.

Example 2.2.4. [37] If α_q is again the rescaling endomorphism on $\mathbb{C}[t]$ defined by

$$\alpha_q(a)(t) = a(qt)$$

for some $q \in \mathbb{C} \setminus \{0\}$, then another Jackson derivative $M_{t^2}D_q$ acting on $\mathbb{C}[t]$ and defined by

$$M_{t^2}D_q(a)(t) = \frac{ta(qt) - ta(t)}{q-1}$$

for some $q \in \mathbb{C} \setminus \{1\}$ is an α_q-derivation on $\mathbb{C}[t]$. This follows since for all $a, b \in \mathbb{C}[t]$,

$$\begin{aligned}
M_{t^2}D_q(ab)(t) &= \frac{t(ab)(qt) - t(ab)(t)}{q-1} \\
&= \frac{ta(qt)b(qt) - ta(t)b(t)}{q-1} \\
&= \frac{ta(qt)b(qt) - ta(qt)b(t) + ta(qt)b(t) - ta(t)b(t)}{q-1} \\
&= \frac{a(qt)[tb(qt) - tb(t)] + [ta(qt) - ta(t)]b(t)}{q-1} \\
&= a(qt)\frac{tb(qt) - tb(t)}{q-1} + \frac{ta(qt) - ta(t)}{q-1}b(t) \\
&= \alpha_q(a)(t)M_{t^2}D_q(b)(t) + M_{t^2}D_q(a)(t)b(t) \\
&= M_{t^2}D_q(a)(t)b(t) + \alpha_q(a)(t)M_{t^2}D_q(b)(t).
\end{aligned}$$

Suppressing the variable t, we obtain

$$M_{t^2}D_q(ab) = M_{t^2}D_q(a)b + \alpha_q(a)M_{t^2}D_q(b),$$

which shows that $M_{t^2}D_q$ is indeed an α_q-derivation on $\mathbb{C}[t]$. Comparing with equation (2.1) in Definition 2.2.1, we observe that $\sigma = \alpha_q$ in this case.

Example 2.2.5. [37] In general, if α_f is the endomorphism on $\mathbb{C}[t]$ defined by

$$\alpha_f(a)(t) = a(f(t))$$

for some $f \in \mathbb{C}[t] \setminus \{0\}$, then the operator D_f acting on $\mathbb{C}[t]$ and defined by

$$D_f(a)(t) = \frac{a(f(t)) - a(t)}{f(t) - t}, \quad f(t) \neq t$$

is an α_f-derivation on $\mathbb{C}[t]$. This follows since for all $a, b \in \mathbb{C}[t]$, we have

$$\begin{aligned}
D_f(ab)(t) &= \frac{(ab)(f(t)) - (ab)(t)}{f(t) - t} \\
&= \frac{a\big(f(t)\big)b\big(f(t)\big) - a(t)b(t)}{f(t) - t} \\
&= \frac{a\big(f(t)\big)b\big(f(t)\big) - a\big(f(t)\big)b(t) + a\big(f(t)\big)b(t) - a(t)b(t)}{f(t) - t} \\
&= a\big(f(t)\big)\frac{b\big(f(t)\big) - b(t)}{f(t) - t} + \frac{a\big(f(t)\big) - a(t)}{f(t) - t}b(t) \\
&= \alpha_f(a)(t)D_f(b)(t) + D_f(a)(t)b(t) \\
&= D_f(a)(t)b(t) + \alpha_f(a)(t)D_f(b)(t).
\end{aligned}$$

Suppressing the variable t, we obtain

$$D_f(ab) = D_f(a)b + \alpha_f(a)D_f(b),$$

which shows that D_f is indeed an α_f-derivation on $\mathbb{C}[t]$. Comparing with equation (2.1) in Definition 2.2.1, we observe that $\sigma = \alpha_f$ in this case.

2.3 Quasi-Lie Algebras

The concept of quasi-Lie algebras emanates from the work of Hartwig et al. [19] who introduced the notion of a hom-Lie algebra as a deformed version of a Lie algebras, motivated by some of the examples of deformations of the Witt and Virasoro algebras constructed using σ-derivations.

Before coming to some important definitions, we introduce a convenient notation. If $f : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{A} \cdot \partial_\sigma$ is a function we will write

$$\circlearrowleft_{a,b,c} f(a, b, c)$$

for the cyclic sum

$$f(a, b, c) + f(b, c, a) + f(c, a, b).$$

We note the following properties of the cyclic sum:

$$\circlearrowleft_{a,b,c} \big(x \cdot f(a,b,c) + y \cdot g(a,b,c)\big) = x\cdot \circlearrowleft_{a,b,c} f(a,b,c) + y\cdot \circlearrowleft_{a,b,c} g(a,b,c),$$
$$\circlearrowleft_{a,b,c} f(a,b,c) = \circlearrowleft_{a,b,c} f(b,c,a) = \circlearrowleft_{a,b,c} f(c,a,b),$$

where $f, g : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{A} \cdot \partial_\sigma$ are two functions and $x, y \in \mathcal{A}$. Combining the two identities we obtain

$$\begin{aligned}\circlearrowleft_{a,b,c} \big(f(a,b,c) + g(a,b,c)\big) &= \circlearrowleft_{a,b,c} \big(f(a,b,c) + g(b,c,a)\big)\\ &= \circlearrowleft_{a,b,c} \big(f(a,b,c) + g(c,a,b)\big).\end{aligned}$$

Definition 2.3.1. A Lie algebra is a pair $\big(V, [\cdot,\cdot]\big)$ where

- V is a vector space over a field $\mathbb{F}$,
- $[\cdot,\cdot] : V \times V \to V$ a binary operation called the Lie bracket (a multiplication in V),

such that the following conditions hold for all $x, y, z \in V$ and $a, b \in \mathbb{F}$:

(i) Bilinearity: $[ax + by, z] = a[x, z] + b[y, z]$ and $[x, ay + bz] = a[x, y] + b[x, z]$,

(ii) Alternativity: $[x, x] = 0$,

(iii) Jacobi identity: $\big[x, [y, z]\big] + \big[y, [z, x]\big] + \big[z, [x, y]\big] = 0$.

Using cyclic summation notation, the Jacobi identity can be written as $\circlearrowleft_{x,y,z} \big[x, [y, z]\big] = 0$.

The first two conditions imply skew-symmetry, that is, $[x, y] = -[y, x]$ while the third condition shows that the Lie bracket is in general non-associative. To see that the first two conditions imply skew-symmetry, we first observe that

$$\begin{aligned}0 &= [x + y, x + y], \quad \text{by alternativity}\\ &= [x, x] + [x, y] + [y, x] + [y, y], \quad \text{by bilineality}\\ &= 0 + [x, y] + [y, x] + 0, \quad \text{by alternativity}\\ &= [x, y] + [y, x]\end{aligned}$$

That is, $[x, y] + [y, x] = 0$, which implies $[x, y] = -[y, x]$. Therefore, bilinearity and alternativity indeed imply skew symmetry, and in view of this, we present another definition of a Lie algebra which is equivalent to Definition 2.3.1.

Definition 2.3.2. [27] A Lie algebra is a pair $\big(V, [\cdot,\cdot]\big)$ where

- V is a vector space over a field $\mathbb{F}$,
- $[\cdot,\cdot] : V \times V \to V$ is a bilinear map,

such that the following conditions hold:

(i) Skew-symmetry: $[x,y] = -[y,x]$ for all $x, y \in V$,

(ii) Jacobi identity: $\circlearrowleft_{x,y,z} \big[x,[y,z]\big] = 0$ for all $x, y, z \in V$,

where $\circlearrowleft_{x,y,z}$ as before denotes cyclic summation with respect to x, y, z.

Definition 2.3.3. [27] A hom-Lie algebra is a triple $\big(V, [\cdot,\cdot], \alpha\big)$ where

- V is a vector space over a field $\mathbb{F}$,
- $[\cdot,\cdot] : V \times V \to V$ is a bilinear map,
- $\alpha : V \to V$ is a linear map,

such that the following conditions hold:

(i) Skew-symmetry: $[x,y] = -[y,x]$ for all $x, y \in V$,

(ii) Hom-Jacobi identity: $\circlearrowleft_{x,y,z} \big[\alpha(x),[y,z]\big] = 0$ for all $x, y, z \in V$.

Note that in this definition if $\alpha = \mathrm{id}_V$ (the identity linear map on V), then we retain Definition 2.3.2 of a Lie algebra. In the following definition, $\mathcal{L}_{\mathbb{F}}(V)$ denotes the vector space of linear maps on a vector space V over a field $\mathbb{F}$.

Definition 2.3.4. [41] A quasi-Lie algebra is a tuple $\big(V, [\cdot,\cdot], \alpha, \beta, \theta, \omega\big)$ where

- V is a vector space over a field $\mathbb{F}$,
- $[\cdot,\cdot] : V \times V \to V$ is a bilinear map,
- $\alpha, \beta : V \to V$ are linear maps,
- $\theta, \omega : D_\theta, D_\omega \to \mathcal{L}_{\mathbb{F}}(V)$ are maps with domains of definition $D_\theta, D_\omega \subseteq V \times V$,

such that the following conditions hold:

(i) ω-symmetry: $[x,y] = \omega(x,y)[y,x]$ for all $(x,y) \in D_\omega$,

(ii) Quasi-Jacobi identity: $\circlearrowleft_{x,y,z} \Big\{\theta(z,x)\Big(\big[\alpha(x),[y,z]\big] + \beta\big[x,[y,z]\big]\Big)\Big\} = 0$ for all (z,y), (y,z), $(x,y) \in D_\theta$.

Note that in this definition if $\beta = \mathrm{id}$ and $\omega = \theta = -\mathrm{id}$, then we retain Definition 2.3.3 of a hom-Lie algebra. To see this, we first observe that under these conditions, skew symmetry follows from ω-symmetry since for all $(x,y) \in D_\omega$, we have

$$
\begin{aligned}
&[x,y] - \omega(x,y)[y,x] = 0, \quad \text{by } \omega\text{-symmetry}\\
&[x,y] - \omega(x,y)\omega(y,x)[x,y] = 0, \quad \text{again by } \omega\text{-symmetry}\\
&[x,y](\mathrm{id} - \omega(x,y)\omega(y,x)) = 0,
\end{aligned}
$$

which implies $\omega(x,y)\omega(y,x) - \mathrm{id} = 0$ so that $\omega(x,y) = -1$ whenever $\omega = -\mathrm{id}$. Secondly, under these same conditions, the hom-Jacobi identity follows from the quasi-Jacobi identity since for all (z,y), (y,z), $(x,y) \in D_\theta$, we have

$$\begin{aligned}
&\circlearrowleft_{x,y,z} \left\{\theta(z,x)\Big(\big[\alpha(x),[y,z]\big] + \beta\big[x,[y,z]\big]\Big)\right\} = 0, \\
&\circlearrowleft_{x,y,z} \Big(\big[\alpha(x),[y,z]\big] + \big[x,[y,z]\big]\Big) = 0, \quad \text{applying the conditions } \beta = \mathrm{id},\ \omega = \theta = -\mathrm{id} \\
&\circlearrowleft_{x,y,z} \big[\alpha(x),[y,z]\big] + \circlearrowleft_{x,y,z} \big[x,[y,z]\big] = 0, \quad \text{distributing the cyclic sum} \\
&\circlearrowleft_{x,y,z} \big[\alpha(x),[y,z]\big] = 0, \quad \text{since } \circlearrowleft_{x,y,z} \big[x,[y,z]\big] = 0, \text{ the Jacobi identity.}
\end{aligned}$$

So indeed if $\beta = \mathrm{id}$ and $\omega = \theta = -\mathrm{id}$, then we retain Definition 2.3.3 of a hom-Lie algebra.

In the following definition we specialise in the case where $\theta = \omega$, and additionally, we impose the β-twisting condition, $\big[\alpha(x), \alpha(y)\big] = \beta \circ \alpha[x,y]$, where $\circ$ denotes composition of maps. This leads us to another algebraic structure called the quasi-hom-Lie algebra.

Definition 2.3.5. [25] A quasi-hom-Lie algebra is a tuple $\big(V, [\cdot,\cdot], \alpha, \beta, \omega\big)$ where

- V is a vector space over a field $\mathbb{F}$,
- $[\cdot,\cdot] : V \times V \to V$ is a bilinear map,
- $\alpha, \beta : V \to V$ are linear maps,
- $\omega : D_\omega \to \mathcal{L}_{\mathbb{F}}(V)$ is a map with domain of definition $D_\omega \subseteq V \times V$,

such that the following conditions hold:

(i) β-twisting: $\big[\alpha(x), \alpha(y)\big] = \beta \circ \alpha[x,y]$ for all $x, y \in V$, where $\circ$ is composition of maps,

(ii) ω-symmetry: $[x,y] = \omega(x,y)[y,x]$ for all $(x,y) \in D_\omega$,

(iii) Quasi-hom-Jacobi identity: $\circlearrowleft_{x,y,z} \left\{\omega(z,x)\Big(\big[\alpha(x),[y,z]\big] + \beta\big[x,[y,z]\big]\Big)\right\} = 0$ for all $(z,x), (x,y), (y,z) \in D_\omega$.

We end this section by presenting an important result from Hartwig et al. [19]. This result is of fundamental importance in this dissertation as it introduces an $\mathbb{F}$-algebra structure on a given submodule and turns this submodule into a quasi-hom-Lie algebra. To present this result, we first introduce some terminology and notation. We denote as before the set of all σ-derivations on an algebra $\mathcal{A}$ by $\mathfrak{D}_\sigma(\mathcal{A})$, and we let

$$\mathrm{Ann}(\partial_\sigma) = \big\{a \in \mathcal{A} : a \cdot \partial_\sigma = 0\big\},$$

the annihilator of the σ-derivation ∂_σ. Furthermore, we fix an endomorphism $\sigma : \mathcal{A} \to \mathcal{A}$, an element $\partial_\sigma \in \mathfrak{D}_\sigma(\mathcal{A})$, and an element $\delta \in \mathcal{A}$ and assume that the following conditions are satisfied:

$$\sigma\big(\mathrm{Ann}(\partial_\sigma)\big) \subseteq \mathrm{Ann}(\partial_\sigma) \tag{2.2}$$

$$\partial_\sigma\big(\sigma(a)\big) = \delta\sigma\big(\partial_\sigma(a)\big) \quad \text{for } a \in \mathcal{A}. \tag{2.3}$$

Furthermore, we let

$$\mathcal{A} \cdot \partial_\sigma = \big\{a \cdot \partial_\sigma : a \in \mathcal{A}\big\},$$

the cyclic left $\mathcal{A}$-submodule of $\mathfrak{D}_\sigma(\mathcal{A})$ generated by ∂_σ, and we extend the endomorphism σ to the submodule $\mathcal{A} \cdot \partial_\sigma$ by

$$\sigma(a \cdot \partial_\sigma) = \sigma(a) \cdot \partial_\sigma. \tag{2.4}$$

Finally, we let "$\circ$" denote composition of maps. The following proposition introduces an $\mathbb{F}$-algebra structure on the submodule $\mathcal{A} \cdot \partial_\sigma$ and turns it into a quasi-hom-Lie algebra.

Proposition 2.3.6. *[19] If condition* (2.2) *holds, then the map* $[\cdot,\cdot]_\sigma$ *defined for all* $a, b \in \mathcal{A}$ *by*

$$[a \cdot \partial_\sigma, b \cdot \partial_\sigma]_\sigma = \big(\sigma(a) \cdot \partial_\sigma\big) \circ (b \cdot \partial_\sigma) - \big(\sigma(b) \cdot \partial_\sigma\big) \circ (a \cdot \partial_\sigma) \tag{2.5}$$

is a well-defined $\mathbb{F}$*-algebra product on the* $\mathbb{F}$*-linear space* $\mathcal{A} \cdot \partial_\sigma$*. This product satisfies the following identities for all* $a, b, c \in \mathcal{A}$*:*

$$[a \cdot \partial_\sigma, b \cdot \partial_\sigma]_\sigma = \Big(\sigma(a)\partial_\sigma(b) - \sigma(b)\partial_\sigma(a)\Big)\partial_\sigma, \tag{2.6}$$

$$[a \cdot \partial_\sigma, b \cdot \partial_\sigma]_\sigma = -[b \cdot \partial_\sigma, a \cdot \partial_\sigma]_\sigma, \tag{2.7}$$

and if in addition condition (2.3) *holds, then we have the* 6*-term deformed Jacobi identity:*

$$\circlearrowleft_{a,b,c} \Big(\big[\sigma(a) \cdot \partial_\sigma, [b \cdot \partial_\sigma, c \cdot \partial_\sigma]_\sigma\big]_\sigma + \delta \cdot \big[a \cdot \partial_\sigma, [b \cdot \partial_\sigma, c \cdot \partial_\sigma]_\sigma\big]_\sigma\Big) = 0. \tag{2.8}$$

Remark 2.3.7. The algebra $\mathcal{A} \cdot \partial_\sigma$ in this proposition is then a quasi-hom-Lie algebra with $\alpha = \sigma$, $\beta = \delta \mathrm{id}_{\mathcal{A}\cdot\partial_\sigma}$ and $\omega = -\mathrm{id}_{\mathcal{A}\cdot\partial_\sigma}$ (see Corollary 2.3.8). If moreover $\delta \in \mathbb{F}$, then after pairing the terms, the 6-term twisted Jacobi identity is reduced to a 3-term twisted Jacobi identity, and hence in this case $\mathcal{A} \cdot \partial_\sigma$ is a hom-Lie algebra. Such class of σ-derivations ∂_σ has been studied also in ring theoretical works in connection to iterated skew polynomial extension rings, actions of hopf algebras, quantum algebras and quantum groups see, for example, [5, 7, 9, 13, 18, 28, 34]. In this sense, our results and constructions can be also seen as a contribution into these directions.

We now prove Proposition 2.3.6

Proof of proposition 2.3.6. We first prove that the bracket $[\cdot,\cdot]_\sigma$ is well defined. That is, for $a_1, a_2, b \in \mathcal{A}$, the condition $a_1 \cdot \partial_\sigma = a_2 \cdot \partial_\sigma$ implies the following equations:

$$[a_1 \cdot \partial_\sigma, b \cdot \partial_\sigma]_\sigma = [a_2 \cdot \partial_\sigma, b \cdot \partial_\sigma]_\sigma, \tag{2.9}$$

$$[b \cdot \partial_\sigma, a_1 \cdot \partial_\sigma]_\sigma = [b \cdot \partial_\sigma, a_2 \cdot \partial_\sigma]_\sigma. \tag{2.10}$$

To prove this, we first note that $a_1 \cdot \partial_\sigma = a_2 \cdot \partial_\sigma$ is equivalent to $(a_1 - a_2) \cdot \partial_\sigma = 0$ which implies that $a_1 - a_2 \in \mathrm{Ann}(\partial_\sigma)$. Furthermore, $\sigma(a_1 - a_2) \in \mathrm{Ann}(\partial_\sigma)$ by condition (2.2). Therefore, using the given definition of the bracket $[\cdot,\cdot]_\sigma$, we have

$$\begin{aligned}
[a_1 \cdot \partial_\sigma, b \cdot \partial_\sigma]_\sigma - [a_2 \cdot \partial_\sigma, b \cdot \partial_\sigma]_\sigma &= \big(\sigma(a_1) \cdot \partial_\sigma\big) \circ (b \cdot \partial_\sigma) - \big(\sigma(b) \cdot \partial_\sigma\big) \circ (a_1 \cdot \partial_\sigma) \\
&\quad - \big(\sigma(a_2) \cdot \partial_\sigma\big) \circ (b \cdot \partial_\sigma) + \big(\sigma(b) \cdot \partial_\sigma\big) \circ (a_2 \cdot \partial_\sigma) \\
&= \big(\sigma(a_1) \cdot \partial_\sigma\big) \circ (b \cdot \partial_\sigma) - \big(\sigma(a_2) \cdot \partial_\sigma\big) \circ (b \cdot \partial_\sigma) \\
&\quad - \big(\sigma(b) \cdot \partial_\sigma\big) \circ (a_1 \cdot \partial_\sigma) + \big(\sigma(b) \cdot \partial_\sigma\big) \circ (a_2 \cdot \partial_\sigma) \\
&= \Big(\sigma(a_1 - a_2) \cdot \partial_\sigma\Big) \circ (b \cdot \partial_\sigma) - \big(\sigma(b) \cdot \partial_\sigma\big) \circ \Big((a_1 - a_2) \cdot \partial_\sigma\Big) \\
&= 0, \quad \text{since } a_1 - a_2 \in \mathrm{Ann}(\partial_\sigma) \text{ and } \sigma(a_1 - a_2) \in \mathrm{Ann}(\partial_\sigma),
\end{aligned}$$

which proves equation (2.9). Similarly, we have

$$\begin{aligned}
[b \cdot \partial_\sigma, a_1 \cdot \partial_\sigma]_\sigma - [b \cdot \partial_\sigma, a_2 \cdot \partial_\sigma]_\sigma &= \big(\sigma(b) \cdot \partial_\sigma\big) \circ (a_1 \cdot \partial_\sigma) - \big(\sigma(a_1) \cdot \partial_\sigma\big) \circ (b \cdot \partial_\sigma) \\
&\quad - \big(\sigma(b) \cdot \partial_\sigma\big) \circ (a_2 \cdot \partial_\sigma) + \big(\sigma(a_2) \cdot \partial_\sigma\big) \circ (b \cdot \partial_\sigma) \\
&= \big(\sigma(b) \cdot \partial_\sigma\big) \circ (a_1 \cdot \partial_\sigma) - \big(\sigma(b) \cdot \partial_\sigma\big) \circ (a_2 \cdot \partial_\sigma) \\
&\quad - \big(\sigma(a_1) \cdot \partial_\sigma\big) \circ (b \cdot \partial_\sigma) + \big(\sigma(a_2) \cdot \partial_\sigma\big) \circ (b \cdot \partial_\sigma) \\
&= \big(\sigma(b) \cdot \partial_\sigma\big) \circ \Big((a_1 - a_2) \cdot \partial_\sigma\Big) - \Big(\sigma(a_1 - a_2) \cdot \partial_\sigma\Big) \circ (b \cdot \partial_\sigma) \\
&= 0, \quad \text{since } a_1 - a_2 \in \mathrm{Ann}(\partial_\sigma) \text{ and } \sigma(a_1 - a_2) \in \mathrm{Ann}(\partial_\sigma),
\end{aligned}$$

which proves equation (2.10). Next, we prove identity (2.6) using the definitions of the bracket $[\cdot,\cdot]_\sigma$ and the σ-derivation ∂_σ. For $a, b \in \mathcal{A}$, we have

$$\begin{aligned}
[a \cdot \partial_\sigma, b \cdot \partial_\sigma]_\sigma &= \sigma(a) \cdot \partial_\sigma \circ (b \cdot \partial_\sigma) - \sigma(b) \cdot \partial_\sigma \circ (a \cdot \partial_\sigma) \\
&= \sigma(a) \cdot \Big(\partial_\sigma(b)\partial_\sigma + \sigma(b)\partial_\sigma(\partial_\sigma)\Big) - \sigma(b) \cdot \Big(\partial_\sigma(a)\partial_\sigma + \sigma(a)\partial_\sigma(\partial_\sigma)\Big) \\
&= \sigma(a)\partial_\sigma(b)\partial_\sigma + \sigma(a)\sigma(b)\partial_\sigma(\partial_\sigma) - \sigma(b)\partial_\sigma(a)\partial_\sigma - \sigma(b)\sigma(a)\partial_\sigma(\partial_\sigma) \\
&= \sigma(a)\partial_\sigma(b)\partial_\sigma - \sigma(b)\partial_\sigma(a)\partial_\sigma + \sigma(a)\sigma(b)\partial_\sigma(\partial_\sigma) - \sigma(b)\sigma(a)\partial_\sigma(\partial_\sigma) \\
&= \Big(\sigma(a)\partial_\sigma(b) - \sigma(b)\partial_\sigma(a)\Big)\partial_\sigma + \Big(\sigma(a)\sigma(b) - \sigma(b)\sigma(a)\Big)\partial_\sigma(\partial_\sigma)
\end{aligned}$$

Since the algebra $\mathcal{A}$ is commutatitve, the last term is zero and this proves identity 2.6. We can now use identity (2.6) to prove identity (2.7). For $a, b \in \mathcal{A}$, we have

$$\begin{aligned}
[a \cdot \partial_\sigma, b \cdot \partial_\sigma]_\sigma &= \Big(\sigma(a)\partial_\sigma(b) - \sigma(b)\partial_\sigma(a)\Big)\partial_\sigma, \quad \text{by identity (2.6)} \\
&= \Big(- \sigma(b)\partial_\sigma(a) + \sigma(a)\partial_\sigma(b)\Big)\partial_\sigma \\
&= -\Big(\sigma(b)\partial_\sigma(a) - \sigma(a)\partial_\sigma(b)\Big)\partial_\sigma \\
&= -[b \cdot \partial_\sigma, a \cdot \partial_\sigma]_\sigma, \quad \text{by identity (2.6)},
\end{aligned}$$

as required. Finally, we prove the Jacobi identity (2.8). For all $a, b, c \in \mathcal{A}$, we write it as

$$\circlearrowleft_{a,b,c} \big[\sigma(a) \cdot \partial_\sigma, [b \cdot \partial_\sigma, c \cdot \partial_\sigma]_\sigma\big]_\sigma + \circlearrowleft_{x,y,z} \delta \cdot \big[a \cdot \partial_\sigma, [b \cdot \partial_\sigma, c \cdot \partial_\sigma]_\sigma\big]_\sigma = 0$$

and prove it in parts. The first part can be expanded using identity (2.6) as follows

$$\begin{aligned}
&\circlearrowleft_{a,b,c} \big[\sigma(a) \cdot \partial_\sigma, [b \cdot \partial_\sigma, c \cdot \partial_\sigma]_\sigma\big]_\sigma \\
&= \circlearrowleft_{a,b,c} \left[\sigma(a) \cdot \partial_\sigma, \Big(\sigma(b)\partial_\sigma(c) - \sigma(c)\partial_\sigma(b)\Big)\partial_\sigma\right]_\sigma \\
&= \circlearrowleft_{a,b,c} \left(\sigma^2(a)\partial_\sigma\Big(\sigma(b)\partial_\sigma(c) - \sigma(c)\partial_\sigma(b)\Big) - \sigma\Big(\sigma(b)\partial_\sigma(c) - \sigma(c)\partial_\sigma(b)\Big)\partial_\sigma(\sigma(a))\right)\partial_\sigma \\
&= \circlearrowleft_{a,b,c} \Bigg(\sigma^2(a)\partial_\sigma\Big(\sigma(b) \cdot \partial_\sigma(c)\Big) - \sigma^2(a)\partial_\sigma\Big(\sigma(c) \cdot \partial_\sigma(b)\Big) - \sigma^2(b)\sigma(\partial_\sigma(c))\partial_\sigma(\sigma(a)) \\
&\qquad + \sigma^2(c)\sigma(\partial_\sigma(b))\partial_\sigma\big(\sigma(a)\big)\Bigg)\partial_\sigma \\
&= \circlearrowleft_{a,b,c} \Bigg(\sigma^2(a)\partial_\sigma\big(\sigma(b)\big)\sigma\big(\partial_\sigma(c)\big) + \sigma^2(a)\partial_\sigma^2(c)\sigma^2(b) - \sigma^2(a)\partial_\sigma(\sigma(c))\sigma\big(\partial_\sigma(b)\big) - \sigma^2(a)\partial_\sigma^2(b)\sigma^2(c) \\
&\qquad - \sigma^2(b)\sigma\big(\partial_\sigma(c)\big)\partial_\sigma(\sigma(a)) + \sigma^2(c)\sigma\big(\partial_\sigma(b)\big)\partial_\sigma\big(\sigma(a)\big)\Bigg)\partial_\sigma \\
&= \circlearrowleft_{a,b,c} \Bigg(\sigma^2(a)\partial_\sigma(\sigma(b))\sigma\big(\partial_\sigma(c)\big) \cdot \partial_\sigma + \sigma^2(a)\partial_\sigma^2(c)\sigma^2(b) \cdot \partial_\sigma - \sigma^2(a)\partial_\sigma\big(\sigma(c)\big)\sigma\big(\partial_\sigma(b)\big) \cdot \partial_\sigma \\
&\qquad - \sigma^2(a)\partial_\sigma^2(b)\sigma^2(c) \cdot \partial_\sigma - \sigma^2(b)\sigma\big(\partial_\sigma(c)\big)\partial_\sigma(\partial_\sigma(a)) \cdot \partial_\sigma + \sigma^2(c)\sigma\big(\partial_\sigma(b)\big)\partial_\sigma\big(\sigma(a)\big) \cdot \partial_\sigma\Bigg) \\
&= \circlearrowleft_{a,b,c} \Big(\sigma^2(a)\partial_\sigma(\sigma(b))\sigma\big(\partial_\sigma(c)\big) \cdot \partial_\sigma - \sigma^2(a)\partial_\sigma\big(\sigma(c)\big)\sigma\big(\partial_\sigma(b)\big) \cdot \partial_\sigma\Big) \\
&\qquad + \circlearrowleft_{a,b,c} \Big(\sigma^2(a)\partial_\sigma^2(c)\sigma^2(b) \cdot \partial_\sigma - \sigma^2(a)\partial_\sigma^2(b)\sigma^2(c) \cdot \partial_\sigma\Big) \\
&\qquad + \circlearrowleft_{a,b,c} \Big(- \sigma^2(b)\sigma(\partial_\sigma(c))\partial_\sigma(\sigma(a)) \cdot \partial_\sigma + \sigma^2(c)\sigma\big(\partial_\sigma(b)\big)\partial_\sigma\big(\sigma(a)\big) \cdot \partial_\sigma\Big)
\end{aligned}$$

From here, the second and third term vanish when adding up cyclically. Therefore, we remain with

$$= \circlearrowleft_{a,b,c} \Big(\sigma^2(a)\partial_\sigma(\sigma(b))\sigma(\partial_\sigma(c)) \cdot \partial_\sigma - \sigma^2(a)\partial_\sigma(\sigma(c))\sigma(\partial_\sigma(b)) \cdot \partial_\sigma\Big)$$

$$= \circlearrowleft_{a,b,c} \Big(\sigma^2(a)\partial_\sigma(\sigma(b))\partial_\sigma(c) \cdot \partial_\sigma - \sigma^2(a)\partial_\sigma(\sigma(c))\partial_\sigma(b) \cdot \partial_\sigma\Big) \tag{2.11}$$

Computing the second part of the Jacobi identity, we obtain

$$\circlearrowleft_{a,b,c} \delta \cdot \big[a \cdot \partial_\sigma, [b \cdot \partial_\sigma, c \cdot \partial_\sigma]_\sigma\big]_\sigma$$

$$= \circlearrowleft_{a,b,c} \delta \cdot \Big[a \cdot \partial_\sigma, \Big(\sigma(b)\partial_\sigma(c) - \sigma(c)\partial_\sigma(b)\Big)\partial_\sigma\Big]_\sigma$$

$$\begin{aligned}
&= \circlearrowleft_{x,y,z} \delta \cdot \Bigg(\sigma(a)\partial_\sigma\Big(\sigma(b)\partial_\sigma(c) - \sigma(c)\partial_\sigma(b)\Big) - \sigma\Big(\sigma(b)\partial_\sigma(c) - \sigma(c)\partial_\sigma(b)\Big)\partial_\sigma(a)\Bigg)\partial_\sigma \\
&= \circlearrowleft_{a,b,c} \delta\Bigg(\sigma(a)\partial_\sigma\Big(\sigma(b)\partial_\sigma(c)\Big) - \sigma(a)\partial_\sigma\Big(\sigma(c) \cdot \partial_\sigma(b)\Big) - \sigma^2(b)\sigma\big(\partial_\sigma(c)\big)\partial_\sigma(a) \\
&\qquad - \sigma^2(c)\sigma\big(\partial_\sigma(b)\big)\partial_\sigma(a)\Bigg)\partial_\sigma \\
&= \circlearrowleft_{a,b,c} \delta \cdot \Bigg(\sigma(a)\Big(\partial_\sigma(\sigma(b))\sigma\big(\partial_\sigma(c)\big) + \partial_\sigma^2(c)\sigma^2(b)\Big) - \sigma(a)\partial_\sigma\big(\sigma(c)\big)\sigma\big(\partial_\sigma(b)\big) - \sigma(a)\partial_\sigma^2(b)\sigma^2(c) \\
&\qquad - \Big(\sigma^2(b)\sigma\big(\partial_\sigma(c)\big) - \sigma^2(c)\sigma\big(\partial_\sigma(b)\big)\Big)\partial_\sigma(a)\Bigg)\partial_\sigma \\
&= \circlearrowleft_{a,b,c} \Bigg(\delta \cdot \sigma(a)\partial_\sigma\big(\sigma(b)\big)\sigma\big(\partial_\sigma(c)\big) \cdot \partial_\sigma + \delta \cdot \sigma(a)\partial_\sigma^2(c)\sigma(b) \cdot \partial_\sigma - \delta \cdot \sigma(a)\partial_\sigma\big(\sigma(c)\big)\sigma\big(\partial_\sigma(b)\big) \cdot \partial_\sigma \\
&\qquad - \delta \cdot \sigma(a)\partial_\sigma^2(b)\sigma^2(c) \cdot \partial_\sigma - \delta \cdot \sigma^2(b)\sigma\big(\partial_\sigma(c)\big)\partial_\sigma(a) \cdot \partial_\sigma + \delta \cdot \sigma^2(c)\sigma\big(\partial_\sigma(b)\big)\partial_\sigma(a) \cdot \partial_\sigma\Bigg) \\
&= \circlearrowleft_{a,b,c} \Bigg(\sigma(a)\partial_\sigma\big(\sigma(b)\big)\partial_\sigma\big(\sigma(c)\big) \cdot \partial_\sigma + \delta \cdot \sigma(a)\partial_\sigma^2(c)\sigma(b) \cdot \partial_\sigma - \sigma(a)\partial_\sigma(\sigma(c))\partial_\sigma\big(\sigma(b)\big) \cdot \partial_\sigma \\
&\qquad - \delta \cdot \sigma(a)\partial_\sigma^2(b)\sigma(c) \cdot \partial_\sigma - \sigma^2(b)\partial_\sigma\big(\sigma(c)\big)\partial_\sigma(a) \cdot \partial_\sigma \\
&\qquad + \sigma^2(c)\partial_\sigma\big(\sigma(b)\big)\partial_\sigma(a) \cdot \partial_\sigma\Bigg), \qquad \text{by condition 2.3} \\
&= \circlearrowleft_{a,b,c} \Big(\delta \cdot \sigma(a)\partial_\sigma^2(c)\sigma(b) \cdot \partial_\sigma - \delta \cdot \sigma(a)\partial_\sigma^2(b)\sigma(c) \cdot \partial_\sigma \\
&\qquad - \sigma^2(b)\partial_\sigma\big(\sigma(c)\big)\partial_\sigma(a) \cdot \partial_\sigma + \sigma^2(c)\partial_\sigma\big(\sigma(b)\big)\partial_\sigma(a) \cdot \partial_\sigma\Big) \\
&= \circlearrowleft_{a,b,c} \big(\delta \cdot \sigma(a)\partial_\sigma^2(c)\sigma(b) \cdot \partial_\sigma - \delta \cdot \sigma(a)\partial_\sigma^2(b)\sigma(c) \cdot \partial_\sigma\big) \\
&\qquad + \circlearrowleft_{a,b,c} \Big(-\sigma^2(b)\partial_\sigma\big(\sigma(c)\big)\partial_\sigma(a) \cdot \partial_\sigma + \sigma^2(c)\partial_\sigma\big(\sigma(b)\big)\partial_\sigma(a) \cdot \partial_\sigma\Big).
\end{aligned}$$

By applying the cyclic summation and taking note that $\mathcal{A}$ is commutative, the first term vanish and we remain with

$$\circlearrowleft_{a,b,c} \Big(-\sigma^2(b)\partial_\sigma\big(\sigma(c)\big)\partial_\sigma(a) \cdot \partial_\sigma + \sigma^2(c)\partial_\sigma\big(\sigma(b)\big)\partial_\sigma(a) \cdot \partial_\sigma\Big). \tag{2.12}$$

Finally, when we combine (2.11) and (2.12), we obtain

$$\begin{aligned}
&\circlearrowleft_{a,b,c} \Big(\sigma^2(a)\partial_\sigma(\sigma(b))\partial_\sigma(c)\cdot\partial_\sigma - \sigma^2(a)\partial_\sigma(\sigma(c))\partial_\sigma(b)\cdot\partial_\sigma\Big)\\
&+ \circlearrowleft_{a,b,c} \Big(-\sigma^2(b)\partial_\sigma\big(\sigma(c)\big)\partial_\sigma(a)\cdot\partial_\sigma + \sigma^2(c)\partial_\sigma\big(\sigma(b)\big)\partial_\sigma(a)\cdot\partial_\sigma\Big)\\
&= \sigma^2(a)\partial_\sigma(\sigma(b))\partial_\sigma(c)\cdot\partial_\sigma + \sigma^2(b)\partial_\sigma(\sigma(c))\partial_\sigma(a)\cdot\partial_\sigma + \sigma^2(c)\partial_\sigma(\sigma(a))\partial_\sigma(b)\cdot\partial_\sigma\\
&- \sigma^2(a)\partial_\sigma(\sigma(c))\partial_\sigma(b)\cdot\partial_\sigma - \sigma^2(c)\partial_\sigma(\sigma(b))\partial_\sigma(a)\cdot\partial_\sigma - \sigma^2(b)\partial_\sigma(\sigma(a))\partial_\sigma(c)\cdot\partial_\sigma\\
&- \sigma^2(b)\partial_\sigma\big(\sigma(c)\big)\partial_\sigma(a)\cdot\partial_\sigma - \sigma^2(c)\partial_\sigma\big(\sigma(a)\big)\partial_\sigma(b)\cdot\partial_\sigma - \sigma^2(a)\partial_\sigma\big(\sigma(b)\big)\partial_\sigma(c)\cdot\partial_\sigma\\
&+ \sigma^2(c)\partial_\sigma\big(\sigma(b)\big)\partial_\sigma(a)\cdot\partial_\sigma + \sigma^2(b)\partial_\sigma\big(\sigma(a)\big)\partial_\sigma(c)\cdot\partial_\sigma + \sigma^2(a)\partial_\sigma\big(\sigma(c)\big)\partial_\sigma(b)\cdot\partial_\sigma\\
&= 0.
\end{aligned}$$

This gives the required proof of the Jacobi identity (2.8). ■

Corollary 2.3.8. *[27] The algebra $\mathcal{A}\cdot\partial_\sigma$ in Proposition 2.3.6 is a quasi-hom-Lie algebra with $\alpha = \sigma,\ \beta = \delta id_{\mathcal{A}\cdot\partial_\sigma}$ and $\omega = -id_{\mathcal{A}\cdot\partial_\sigma}$*

Proof. We show that $\mathcal{A}\cdot\partial_\sigma$ satisfies all the three conditions for a quasi-hom-Lie algebra. The β-twisting condition follows since for all $a, b \in \mathcal{A}$, we have

$$\begin{aligned}
\big[\sigma(a)\cdot\partial_\sigma, \sigma(b)\cdot\partial_\sigma\big]_\sigma &= \Big(\sigma^2(a)\partial_\sigma(\sigma(b)) - \sigma^2(b)\partial_\sigma(\sigma(a))\Big)\partial_\sigma, \quad \text{by condition 2.6}\\
&= \Big(\sigma^2(a)\delta\cdot\sigma(\partial_\sigma(\sigma(b))) - \sigma^2(b)\delta\cdot\sigma(\partial_\sigma(\sigma(a)))\Big)\partial_\sigma, \quad \text{by condition 2.3}\\
&= \delta\cdot\sigma\Big(\big(\sigma(a)\cdot\partial_\sigma(\sigma(b)) - \sigma(b)\cdot\partial_\sigma(\sigma(a))\big)\partial_\sigma\Big), \quad \text{by condition 2.4}\\
&= \delta\cdot\sigma[a\cdot\partial_\sigma, b\cdot\partial_\sigma]_\sigma.
\end{aligned}$$

Skew symmetry follows since $\omega(a\cdot\partial_\sigma, a\cdot\partial_\sigma) = -1$ whenever we choose $\omega = -\mathrm{id}_{\mathcal{A}\cdot\partial_\sigma}$ (see pages 10–11). Thus the ω-symmetry in Definition 2.3.5 is equivalent to

$$[a\cdot\partial_\sigma, b\cdot\partial_\sigma]_\sigma = -[b\cdot\partial_\sigma, a\cdot\partial_\sigma]_\sigma$$

in Proposition 2.3.6 whenever we choose $\omega = -\mathrm{id}_{\mathcal{A}\cdot\partial_\sigma}$. Finally, the Jacobi identity follows since by condition 2.4, we have

$$\begin{aligned}
&\circlearrowleft_{a,b,c} \Big(\big[\sigma(a)\cdot\partial_\sigma, [b\cdot\partial_\sigma, c\cdot\partial_\sigma]_\sigma\big]_\sigma + \delta\cdot\big[a\cdot\partial_\sigma, [b\cdot\partial_\sigma, c\cdot\partial_\sigma]_\sigma\big]_\sigma\Big) = 0\\
=&\circlearrowleft_{a,b,c} \Big(\big[\sigma(a\cdot\partial_\sigma), [b\cdot\partial_\sigma, c\cdot\partial_\sigma]_\sigma\big]_\sigma + \delta\cdot\big[a\cdot\partial_\sigma, [b\cdot\partial_\sigma, c\cdot\partial_\sigma]_\sigma\big]_\sigma\Big) = 0,
\end{aligned}$$

which is equivalent to the Jacobi identity of Definition 2.3.5 with $\alpha = \sigma,\ \beta = \delta\mathrm{id}_{\mathcal{A}\cdot\partial_\sigma}$ and $\omega = -\mathrm{id}_{\mathcal{A}\cdot\partial_\sigma}$. ■

2.4 The Lie Algebra $\mathfrak{sl}_2(\mathbb{F})$

In this dissertation, the basic undeformed object is the special linear Lie algebra of degree 2 over a field $\mathbb{F}$, abbreviated $\mathfrak{sl}_2(\mathbb{F})$. It is defined as the Lie algebra of 2×2 matrices with entries from $\mathbb{F}$ and trace zero. As defined in Section 2.2, a Lie algebra is a vector space V over a field $\mathbb{F}$ equipped with a binary operation $[\cdot,\cdot] : V \times V \to V$ called the Lie bracket that satisfies the following axioms for all $x, y, z \in V$ and $a, b \in \mathbb{F}$:

(i) Bilinearity: $[ax + by, z] = a[x, z] + b[y, z]$ and $[x, ay + bz] = a[x, y] + b[x, z]$

(ii) Alternativity: $[x, x] = 0$

(iii) Jacobi identity: $\big[x, [y, z]\big] + \big[y, [z, x]\big] + \big[z, [x, y]\big] = 0$

The first two axioms imply skew-symmetry, that is, $[x, y] = -[y, x]$ while the third axiom shows that the Lie bracket is in general non-associative. In the case of $\mathfrak{sl}_2(\mathbb{F})$, the Lie bracket of two elements x and y is given by the commutator

$$[x, y] = xy - yx.$$

It is easy to see that the commutator defined in this manner satisfies all the three axioms of a Lie bracket given above, so $\mathfrak{sl}_2(\mathbb{F})$ is indeed a Lie Algebra. To check that the commutator satisfies the bilineality condition, we observe that for all $x, y, z \in \mathfrak{sl}_2(\mathbb{F})$ and $a, b \in \mathbb{F}$,

$$\begin{aligned}
[ax + by, z] &= (ax + by)z - z(ax + by) \\
&= axz + byz - azx - bzy \\
&= axz - azx + byz - bzy \\
&= a(xz - zx) + b(yz - zy) \\
&= a[x, z] + b[y, z],
\end{aligned}$$

which shows that the commutator is linear in the first argument. Furthermore, we have

$$\begin{aligned}
[x, ay + bz] &= x(ay + bz) - (ay + bz)x \\
&= axy + bxz - ayx - bzx \\
&= axy - ayx + bxz - bzx \\
&= a(xy - yx) + b(xz - zx) \\
&= a[x, y] + b[x, z],
\end{aligned}$$

which shows that the commutator is linear in the second argument. The alternativity condition follows since for all $x \in \mathfrak{sl}_2(\mathbb{F})$, we have

$$[x, x] = xx - xx = 0.$$

Finally, the commutator satisfies the Jacobi identity since for all $x, y, z \in \mathfrak{sl}_2(\mathbb{F})$, we have

$$\begin{aligned}
\big[x, [y, z]\big] + \big[y, [z, x]\big] + \big[z, [x, y]\big] &= [x, yz - zy] + [y, zx - xz] + [z, xy - yx] \\
&= x(yz - zy) - (yz - zy)x + y(zx - xz) \\
&\qquad - (zx - xz)y + z(xy - yx) - (xy - yx)z \\
&= xyz - xzy - yzx + zyx + yzx - yxz \\
&\qquad - zxy + xzy + zxy - zyx - xyz + yxz \\
&= (xyz - xyz) + (yzx - yzx) + (zyx - zyx) + (yxz - yxz) \\
&\qquad + (zxy - zxy) + (xzy - xzy) \\
&= 0 + 0 + 0 + 0 + 0 + 0 \\
&= 0.
\end{aligned}$$

As a vector space, $\mathfrak{sl}_2(\mathbb{F})$ is generated by three matrices h, x and y given by

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

These matrices satisfy the following relations:

$$[h, x] = 2x, \qquad [h, y] = -2y, \qquad [x, y] = h.$$

These are the defining relations of $\mathfrak{sl}_2(\mathbb{F})$, and since they contain the commutator, they will be referred to as the commutation relations defining $\mathfrak{sl}_2(\mathbb{F})$. To verify these commutation relations we use the definition of the commutator. For the first relation, we have

$$\begin{aligned}
[h, x] = hx - xh &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \\
&= 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&= 2x.
\end{aligned}$$

For the second relation, we have

$$\begin{aligned}
[h, y] = hy - yh &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
&= -2\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
&= -2y,
\end{aligned}$$

and for the third relation, we have

$$\begin{aligned}
[x, y] = xy - yx &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= h.
\end{aligned}$$

In view of the preceding narrations and the commutation relations, the following definition of $\mathfrak{sl}_2(\mathbb{F})$ is considered in this dissertation.

Definition 2.4.1. The special linear Lie algebra of degree 2 over a field $\mathbb{F}$, abbreviated $\mathfrak{sl}_2(\mathbb{F})$, is a Lie algebra generated as a vector space by three elements h, x and y satisfying the commutation relations

$$[h, x] = 2x, \qquad [h, y] = -2y, \qquad [x, y] = h. \tag{2.13}$$

In this dissertation, the starting point of our deformation process is the following representation of $\mathfrak{sl}_2(\mathbb{F})$ in terms of first order differential operators acting on a vector space of functions in the variable t:

$$h \mapsto -2t\partial, \qquad x \mapsto \partial, \qquad y \mapsto -t^2\partial.$$

This is what we will generalize to first order operators acting on an algebra $\mathcal{A}$, where ∂ is replaced by ∂_σ, a twisted derivation on $\mathcal{A}$. In the following section, we formally introduce representations of algebras by linear operators.

2.5 Operator Representations

Consider the commutation relations (2.13) in Definition 2.4.1. An operator representation of these commutation relations (or of the Lie Algebra $\mathfrak{sl}_2(\mathbb{F})$) is any set of linear operators satisfying these commutation relations. To indicate that generator A is represented by operator A', we write $A \mapsto A'$.

Example 2.5.1. [37] The classical Heisenberg–Lie algebra is generated by two elements S and Q satisfying the commutation relation

$$SQ - QS = S.$$

A concrete representation of this commutation relation (or of the Heisenberg–Lie algebra) is given by the association $S \mapsto \alpha_1$ and $Q \mapsto Q_x$, where α_1 and Q_x are the linear operators

$$\alpha_1(f)(x) = f(x+1) \quad \text{and} \quad Q_x(f)(x) = xf(x)$$

acting on polynomials or other suitable functions. To see this, we first observe that

$$\begin{aligned} \alpha_1 Q_x(f)(x) &= \alpha_1(\mathrm{id} \cdot f)(x) \\ &= (x+1)f(x+1) \\ &= (x+1)\alpha_1(f)(x), \end{aligned} \tag{2.14}$$

where id denotes the identity function: $\mathrm{id}(x) = x$. Interpreting the multiplication with the variable x as an application of the multiplication operator Q_x and suppressing the operand f, equation (2.14) can be written as the commutation relation

$$\alpha_1 Q_x - Q_x \alpha_1 = \alpha_1.$$

Example 2.5.2. [37] Consider a deformed Heisenberg–Lie algebra generated by two elements S and Q satisfying the commutation relation

$$SQ - qQS = cS$$

for some scalars q and c. A concrete representation of this commutation relation (or of the deformed Heisenberg–Lie algebra) is given by the association $S \mapsto \alpha_2$ and $Q \mapsto Q_x$, where α_2 and Q_x are the linear operators

$$\alpha_2(f)(x) = f(qx+c) \quad \text{and} \quad Q_x(f)(x) = xf(x)$$

acting on polynomials or some other suitable functions. To see this, we first observe that

$$\begin{aligned}\alpha_2 Q_x(f)(x) &= \alpha_2(\mathrm{id} \cdot f)(x) \\ &= (qx+c)f(qx+c) \\ &= (qx+c)\alpha_2(f)(x),\end{aligned} \tag{2.15}$$

where id denotes the identity function: $\mathrm{id}(x) = x$. Interpreting the multiplication with the variable x as an application of the multiplication operator Q_x and suppressing the operand f, equation (2.15) can be written as the commutation relation

$$\alpha_2 Q_x - qQ_x\alpha_2 = c\alpha_2.$$

Example 2.5.3. [37] Generalizing the previous two examples, we consider the algebra generated by two elements S and Q satisfying the commutation relation

$$SQ = \sigma(Q)S,$$

where σ is a polynomial. A concrete representation of this relation is given by the association $S \mapsto \alpha_\sigma$ and $Q \mapsto Q_x$, where α_σ and Q_x are the linear operators

$$\alpha_\sigma(f)(x) = f(\sigma(x)) \quad \text{and} \quad Q_x(f)(x) = xf(x)$$

acting on polynomials or some other suitable functions. To see this, we first observe that

$$\begin{aligned}\alpha_\sigma Q_x(f)(x) &= \alpha_\sigma(\mathrm{id} \cdot f)(x) \\ &= \sigma(x)f(\sigma(x)) \\ &= \sigma(x)\alpha_\sigma(f)(x),\end{aligned} \tag{2.16}$$

where id denotes the identity function: $\mathrm{id}(x) = x$. Interpreting the multiplication with the variable x as an application of the multiplication operator Q_x and suppressing the operand f, equation (2.16) can be written as the commutation relation

$$\alpha_\sigma Q_x = \sigma(Q_x)\alpha_\sigma.$$

Example 2.5.4. [37] We consider the 3-dimensional Lie-type algebra generated by three elements R, J and Q satisfying the commutation relations

$$\begin{aligned} RQ - QR &= -J, \\ JQ - QJ &= R, \\ RJ - JR &= 0. \end{aligned}$$

A concrete representation of these commutation relations is given by the association $R \mapsto R_i$, $J \mapsto J_i$ and $Q \mapsto Q_x$, where R_i, J_i and Q_x are the linear operators

$$\begin{aligned} R_i(f)(x) &= \frac{f(x+i) + f(x-i)}{2} \\ J_i(f)(x) &= \frac{f(x+i) - f(x-i)}{2i} \\ Q_x(f)(x) &= xf(x) \end{aligned}$$

acting on polynomials or some other suitable functions. Three systems of orthogonal polynomials belonging to the class of Meixner–Pollaczek polynomials that are connected by these operators were presented in [35, 36, 38]. Boundedness properties of the operators R^{-1} and JR^{-1} in the function spaces related to the three systems of orthogonal polynomials were investigated in [22, 36, 38]. To see this, we observe that

$$\begin{aligned} R_iQ_x(f)(x) &= R_i(\mathrm{id} \cdot f)(x) \\ &= \frac{(x+i)f(x+i) + (x-i)f(x-i)}{2} \\ &= \frac{xf(x+i) + if(x+i) + xf(x-i) - if(x-i)}{2} \\ &= \frac{xf(x+i) + xf(x-i) + if(x+i) - if(x-i)}{2} \\ &= \frac{x[f(x+i) + f(x-i)]}{2} + \frac{i[f(x+i) - f(x-i)]}{2} \\ &= \frac{x[f(x+i) + f(x-i)]}{2} - \frac{[f(x+i) - f(x-i)]}{2i} \\ &= xR_i(f)(x) - J_i(f)(x), \end{aligned} \tag{2.17}$$

where id denotes the identity function: $\mathrm{id}(x) = x$. Interpreting the multiplication with the variable x as an application of the multiplication operator Q_x and suppressing the operand f, equation (2.17) can be written as the commutation relation

$$R_iQ_x - Q_xR_i = -J_i.$$

Similarly, for the commutation relation $JQ - QJ = R$, we have

$$
\begin{aligned}
J_iQ_x(f)(x) &= J_i(\mathrm{id}\cdot f)(x) \\
&= \frac{(x+i)f(x+i) - (x-i)f(x-i)}{2i} \\
&= \frac{xf(x+i) + if(x+i) - xf(x-i) + if(x-i)}{2i} \\
&= \frac{xf(x+i) - xf(x-i) + if(x+i) + if(x-i)}{2i} \\
&= \frac{x[f(x+i) - f(x-i)]}{2i} + \frac{i[f(x+i) + f(x-i)]}{2i} \\
&= \frac{x[f(x+i) - f(x-i)]}{2i} + \frac{[f(x+i) + f(x-i)]}{2} \\
&= xJ_i(f)(x) + R_i(f)(x).
\end{aligned}
\tag{2.18}
$$

Interpreting the multiplication with the variable x as an application of the multiplication operator Q_x and suppressing the operand f, equation (2.18) can be written as the commutation relation

$$J_iQ_x - Q_xJ_i = R_i.$$

Finally, for the commutation relation $RJ - JR = 0$, we have

$$
\begin{aligned}
&R_i(f)(x)J_i(f)(x) - J_i(f)(x)R_i(f)(x) \\
&= \left(\frac{f(x+i) + f(x-i)}{2}\right)\left(\frac{f(x+i) - f(x-i)}{2i}\right) \\
&\quad - \left(\frac{f(x+i) - f(x-i)}{2i}\right)\left(\frac{f(x+i) + f(x-i)}{2}\right) \\
&= \frac{f^2(x+i) + f(x+i)f(x-i) - f(x-i)f(x+i) - f^2(x+i)}{4i} \\
&\quad - \left(\frac{f^2(x+i) + f(x+i)f(x-i) - f(x-i)f(x+i) - f^2(x-i)}{4i}\right) \\
&= \frac{f^2(x+i) - f^2(x+i) + f(x+i)f(x-i) - f(x+i)f(x-i)}{4i} \\
&\quad \frac{-f(x-i)f(x+i) + f(x-i)f(x+i) - f^2(x-i) + f^2(x-i)}{4i} \\
&= \frac{0}{4i} + \frac{0}{4i} \\
&= 0.
\end{aligned}
$$

And suppressing the operand f, the equation can be written as the commutation relation

$$R_iJ_i - J_iR_i = 0.$$

Example 2.5.5. [27] The Lie algebra $\mathfrak{sl}_2(\mathbb{F})$ is generated as a vector space by three elements H, X and Y satisfying the commutation relations

$$[H, X] = 2X, \qquad [H, Y] = -2Y, \qquad [X, Y] = H. \tag{2.19}$$

A concrete representation of these commutation relations (or of $\mathfrak{sl}_2(\mathbb{F})$) is given by

$$H \mapsto -2t\partial, \qquad X \mapsto \partial, \qquad Y \mapsto -t^2\partial,$$

where ∂ is the first order derivative acting on polynomials or some other suitable functions in the variable t. To verify this, we use the definition of the commutator. For the first commutation relation, we have

$$\begin{aligned}
[H, X] &= [-2t\partial, \partial] \\
&= -2t\partial(\partial) - \partial(-2t\partial) \\
&= -2t\partial^2 - (-2\partial + (-2t)\partial^2) \\
&= -2t\partial^2 + 2\partial + 2t\partial^2 \\
&= 2\partial \\
&= 2X.
\end{aligned}$$

For the second commutation relation, we have

$$\begin{aligned}
[H, Y] &= \left[-2t\partial, -t^2\partial\right] \\
&= -2t\partial(-t^2\partial) - (-t^2\partial)(-2t\partial) \\
&= -2t(-2t\partial + (-t^2\partial^2)) - (-t^2(-2\partial + (-2t)\partial^2)) \\
&= 4t^2\partial + 2t^3\partial^2 - 2t^2\partial - 2t^3\partial^2 \\
&= 2t^2\partial \\
&= -2Y.
\end{aligned}$$

For the third commutation relation, we have

$$\begin{aligned}
[X, Y] &= \left[\partial, -t^2\partial\right] \\
&= \partial(-t^2\partial) - (-t^2\partial)(\partial) \\
&= -2t\partial - t^2\partial^2 + t^2\partial^2 \\
&= -2t\partial \\
&= H.
\end{aligned}$$

So indeed, the association $H \mapsto -2t\partial$, $X \mapsto \partial$, $Y \mapsto -t^2\partial$ is a representation of $\mathfrak{sl}_2(\mathbb{F})$, and this result is of fundamental importance in this dissertation.

Chapter 3

Quasi-Deformations of $\mathfrak{sl}_2(\mathbb{F})$

In this chapter we obtain quasi-deformations of $\mathfrak{sl}_2(\mathbb{F})$, where $\mathbb{F}$ is a field of characteristic 0 (a field where $n \cdot 1 = 0$ implies $n = 0$). In Section 3.1, we take the base algebra $\mathcal{A}$ to be as general as it can be in this situation and deduce some necessary conditions for everything to make sense. In Section 3.2, we take the base algebra $\mathcal{A}$ to be the polynomial algebra $\mathbb{F}[t]$ and calculate some properties of the corresponding quasi-deformations. An elaborated example of a quasi-deformed $\mathfrak{sl}_2(\mathbb{F})$ is provided in Section 3.3.

3.1 Quasi-Deformations with General Base Algebra $\mathcal{A}$ and the Necessary Conditions

Let $\mathcal{A}$ be a commutative, associative algebra with unity 1 over a field $\mathbb{F}$ of characteristic 0, and let σ denote an endomorphism on $\mathcal{A}$. We fix an element $t \in \mathcal{A}$, choose a σ-derivation (twisted derivation) ∂_σ on $\mathcal{A}$, and consider the $\mathcal{A}$-module

$$\mathcal{A} \cdot \partial_\sigma = \big\{ a \cdot \partial_\sigma : a \in \mathcal{A} \big\}.$$

To simplify notation, we will usually write $a\partial_\sigma$ to mean $a \cdot \partial_\sigma$. By Proposition 2.3.6, there exists a skew-symmetric algebra product on this $\mathcal{A}$-module given by

$$[a \cdot \partial_\sigma, b \cdot \partial_\sigma]_\sigma = \big(\sigma(a) \cdot \partial_\sigma\big) \circ (b \cdot \partial_\sigma) - \big(\sigma(b) \cdot \partial_\sigma\big) \circ (a \cdot \partial_\sigma). \tag{3.1}$$

Furthermore, the three elements

$$H = -2t\partial_\sigma, \quad X = \partial_\sigma, \quad Y = -t^2\partial_\sigma$$

span an $\mathbb{F}$-linear subspace of the $\mathcal{A}$-module $\mathcal{A} \cdot \partial_\sigma$, which we denote and write as

$$\mathcal{S} = \mathrm{LinSpan}_{\mathbb{F}}\big\{ -2t\partial_\sigma,\, \partial_\sigma, -t^2\partial_\sigma \big\} = \mathrm{LinSpan}_{\mathbb{F}}\{H, X, Y\}.$$

We restrict the product (3.1) to $\mathcal{S}$ without, at this point, assuming closure. In addition, we make the general ansatz that

$$\partial_\sigma(1) = d_0 + d_1 t + \cdots + d_k t^k \qquad \text{and} \qquad \sigma(1) = s_0 + s_1 t + \cdots + s_n t^n.$$

If all non-negative integer powers of t are linearly independent over the field $\mathbb{F}$, then $\sigma(1) = 1$ or $\sigma(1) = 0$ since $\sigma(1) = \sigma(1 \cdot 1){=}\sigma(1)^2$, and thus, s_0 is either 1 or 0 and $s_1 = \cdots = s_n = 0$. In addition to this, in this case,

$$\begin{aligned} d_0 + d_1 t + \cdots + d_k t^k &= \partial_\sigma(1) \\ &= \partial_\sigma(1 \cdot 1) \\ &= \partial_\sigma(1)1 + \sigma(1)\partial_\sigma(1), \quad \text{since } \partial_\sigma \text{ is a } \sigma\text{-derivation} \\ &= \big(1 + \sigma(1)\big)\partial_\sigma(1) \\ &= (1 + s_0)\Big(d_0 + d_1 t + \cdots + d_k t^k\Big) \end{aligned} \tag{3.2}$$

leading to $d_0 = d_1 = \cdots = d_k = 0$ if $s_0 = 1$, and arbitrary $d_0, \ldots, d_k$ if $s_0 = 0$. But if $s_0 = 0$, that is $\sigma(1) = 0$, then $\sigma(t^w) = 0$ for all $w \in \mathbb{N}$ since $\sigma(t^w) = \sigma(1 \cdot t^w) = \sigma(1)\sigma(t^w) = 0$, and all our computations involving σ-derivations will collapse. Therefore, $\sigma(1) \neq 0$ and we thus assume that $\sigma(1) = 1$ which implies that $\partial_\sigma(1) = 0$ by equation (3.2). In this case,

$$\partial_\sigma(t^2) = \partial_\sigma(t \cdot t) = \partial_\sigma(t)t + \sigma(t)\partial_\sigma(t) = \big(t + \sigma(t)\big)\partial_\sigma(t). \tag{3.3}$$

We will also make use of the following identity from Proposition 2.3.6:

$$[a \cdot \partial_\sigma, b \cdot \partial_\sigma]_\sigma = \Big(\sigma(a)\partial_\sigma(b) - \sigma(b)\partial_\sigma(a)\Big)\partial_\sigma. \tag{3.4}$$

We can now state the main result of this section.

Theorem 3.1.1. [27] Let $\mathcal{A}$ be a commutative, associative algebra with unity 1 over a field $\mathbb{F}$ of characteristic 0 and fix an element $t \in \mathcal{A}$. Let σ be an endomorphism on $\mathcal{A}$ and ∂_σ a σ-derivation (twisted derivation) on $\mathcal{A}$. Furthermore, assume that $\sigma(1) = 1$ and $\partial_\sigma(1) = 0$. Then the $\mathcal{A}$-module elements $H = -2t\partial_\sigma$, $\;X = \partial_\sigma$ and $Y = -t^2\partial_\sigma$ satisfy the following:

$$[H, X]_\sigma = 2\partial_\sigma(t)\partial_\sigma, \tag{3.5}$$

$$[H, Y]_\sigma = 2\sigma(t)\partial_\sigma(t)t\partial_\sigma, \tag{3.6}$$

$$[X, Y]_\sigma = -\big(t + \sigma(t)\big)\partial_\sigma(t)\partial_\sigma. \tag{3.7}$$

Remark 3.1.2. Note that when $\sigma = \mathrm{id}_\mathcal{A}$ and $\partial_\sigma(t) = 1$, we retain the usual $\mathfrak{sl}_2(\mathbb{F})$ with commutation relations:

$$\begin{aligned} [H, X] &= 2X, \\ [H, Y] &= -2Y, \\ [X, Y] &= H. \end{aligned}$$

Proof of Theorem 3.1.1. For the first relation, we have

$$\begin{aligned}
[H,X]_\sigma &= [-2t\partial_\sigma, \partial_\sigma]_\sigma \\
&= -2[t\partial_\sigma, \partial_\sigma]_\sigma, \quad \text{since the bracket } [\cdot,\cdot]_\sigma \text{ is bilinear} \\
&= -2\Big(\sigma(t)\partial_\sigma(1) - \sigma(1)\partial_\sigma(t)\Big)\partial_\sigma, \quad \text{by identity (3.4)} \\
&= -2\big(0 - \partial_\sigma(t)\big)\partial_\sigma, \quad \text{since } \sigma(1) = 1 \text{ and } \partial_\sigma(1) = 0 \\
&= 2\partial_\sigma(t)\partial_\sigma,
\end{aligned}$$

as required. For the second relation, we have

$$\begin{aligned}
[H,Y]_\sigma &= \big[-2t\partial_\sigma, -t^2\partial_\sigma\big]_\sigma \\
&= 2\big[t\partial_\sigma, t^2\partial_\sigma\big]_\sigma, \quad \text{since the bracket } [\cdot,\cdot]_\sigma \text{ is bilinear} \\
&= 2\Big(\sigma(t)\partial_\sigma(t^2) - \sigma(t^2)\partial_\sigma(t)\Big)\partial_\sigma, \quad \text{by identity (3.4)} \\
&= 2\Big(\sigma(t)\Big(\big(t+\sigma(t)\big)\partial_\sigma(t)\Big) - \sigma(t^2)\partial_\sigma(t)\Big)\partial_\sigma, \quad \text{by (3.3)} \\
&= 2\Big(\sigma(t)t\partial_\sigma(t) + \sigma^2(t)\partial_\sigma(t) - \sigma(t^2)\partial_\sigma(t)\Big)\partial_\sigma \\
&= 2\Big(\sigma(t)t\partial_\sigma(t) + 0\Big)\partial_\sigma \\
&= 2\sigma(t)\partial_\sigma(t)t\partial_\sigma,
\end{aligned}$$

as required. For the third relation, we have

$$\begin{aligned}
[X,Y]_\sigma &= \big[\partial_\sigma, -t^2\partial_\sigma\big]_\sigma \\
&= -\big[\partial_\sigma, t^2\partial_\sigma\big]_\sigma, \quad \text{since the bracket } [\cdot,\cdot]_\sigma \text{ is bilinear} \\
&= -\Big(\sigma(1)\partial_\sigma(t^2) - \sigma(t^2)\partial_\sigma(1)\Big)\partial_\sigma, \quad \text{by identity (3.4)} \\
&= -\Big(\partial_\sigma(t^2) - 0\Big)\partial_\sigma, \quad \text{since } \sigma(1) = 1 \text{ and } \partial_\sigma(1) = 0 \\
&= -\Big(\big(t+\sigma(t)\big)\partial_\sigma(t)\Big)\partial_\sigma, \quad \text{by (3.3)} \\
&= -\big(t+\sigma(t)\big)\partial_\sigma(t)\partial_\sigma,
\end{aligned}$$

as required. ■

3.2 Quasi-Deformations with Base Algebra $\mathcal{A}=\mathbb{F}[t]$

In this section, we take the base algebra $\mathcal{A}$ to be the polynomial algebra $\mathbb{F}[t]$, and as before, we assume that $\sigma(1) = 1$ and $\partial_\sigma(1) = 0$. In the polynomial algebra $\mathbb{F}[t]$, all non-negative integer powers of t are linearly independent over $\mathbb{F}$, so this assumption is valid. Suppose that $\sigma(t) = q(t)$ and $\partial_\sigma(t) = p(t)$, where $p(t), q(t) \in \mathbb{F}[t]$. To have closure of relations (3.5)–(3.7) over the field $\mathbb{F}$, the polynomials $p(t)$ and $q(t)$ are far from arbitrary. Relation (3.6) implies that, to have this closure, we need at least

$$\deg\big(\partial_\sigma(t)\sigma(t)\big) = \deg\big(p(t)q(t)\big) \leq 1,$$

where by $\deg\big(f(t)\big)$ we mean the degree of the polynomial $f(t)$. Thus three cases arise:

$$\begin{aligned}
&\text{Case 1}: \ \sigma(t) = q(t) = q_0 + q_1 t, \qquad q_1 \neq 0, \qquad p(t) = p_0.\\
&\text{Case 2}: \ \sigma(t) = q(t) = q_0, \qquad q_0 \neq 0, \qquad p(t) = p_0 + p_1 t.\\
&\text{Case 3}: \ \sigma(t) = q(t) = 0, \qquad p(t) = p_0 + p_1 t + \cdots + p_n t^n.
\end{aligned}$$

In all three cases we assume $p(t) \neq 0$. Note that if $p(t) = 0$ then $\partial_\sigma = 0$ and so the original operator representation collapses.

Remark 3.2.1. If we allow $\sigma(t) = q(t)$ and $\partial_\sigma(t) = p(t)$ where p and q are arbitrary polynomials in t then we get a deformation of $\mathfrak{sl}_2(\mathbb{F})$ which does not preserve dimension; that is, brackets of the basis elements H, X and Y are not simply linear combinations in these elements but include more new “basis” elements. This phenomena could possibly be interesting to study further.

3.2.1 Case 1

For this case we assume $q(t) = q_0 + q_1 t$, implying that $p(t) = p_0$ for some constant $p_0 \neq 0$. This leads us to the following result.

Theorem 3.2.2. [27] Let $\mathcal{A}$ be a commutative, associative algebra with unity 1 over a field $\mathbb{F}$ of characteristic 0 and fix an element $t \in \mathcal{A}$. Let σ be an endomorphism on $\mathcal{A}$ and ∂_σ a σ-derivation (twisted derivation) on $\mathcal{A}$ where $\sigma(t) = q_0 + q_1 t$ and $\partial_\sigma(t) = p_0$. Furthermore, assume that $\sigma(1) = 1$ and $\partial_\sigma(1) = 0$. Then the $\mathcal{A}$-module elements

$$H = -2t\partial_\sigma, \quad X = \partial_\sigma, \quad Y = -t^2\partial_\sigma$$

satisfy the following relations:

$$[H, X]_\sigma : -2q_0X^2 + q_1HX - XH = 2p_0X, \tag{3.8}$$

$$[H, Y]_\sigma : -2q_0XY + q_1HY + q_0^2XH - q_0q_1H^2 - q_1^2YH = -q_0p_0H - 2q_1p_0Y, \tag{3.9}$$

$$[X, Y]_\sigma : XY + q_0^2X^2 - 2q_0q_1HX - q_1^2YX = -q_0p_0X + \frac{(q_1 + 1)}{2}p_0H. \tag{3.10}$$

Remark 3.2.3. Note that when $q_0 = 0$, $q_1 = 1$ and $p_0 = 1$, we retain the usual $\mathfrak{sl}_2(\mathbb{F})$ with

$$[H, X] : HX - XH = 2X, \qquad [H, Y] : HY - YH = -2Y, \qquad [X, Y] : XY - YX = H.$$

Also note that changing the role of H and Y in relation (3.9) does not correspond to changing H and Y in identity (3.4). This means that, in a sense, the skew-symmetry of identity (3.4) is "hidden" in relation (3.9). When $[Y, H]$ is calculated from (3.4), one sees that indeed

$$\begin{aligned}
[Y, H]_\sigma &= \big[-t^2\partial_\sigma, -2t\partial_\sigma\big]_\sigma \\
&= 2\big[t^2\partial_\sigma, t\partial_\sigma\big]_\sigma \\
&= 2\Big(\sigma(t^2)\partial_\sigma(t) - \sigma(t)\partial_\sigma(t^2)\Big)\partial_\sigma, \quad \text{by definition of } [\cdot, \cdot]_\sigma \\
&= 2\Big(\sigma(t^2)\partial_\sigma(t) - \sigma(t)(\partial_\sigma(t)t + \sigma(t)\partial_\sigma(t))\Big)\partial_\sigma, \quad \text{by definition of } [\cdot, \cdot]_\sigma \\
&= -2\sigma(t)\partial_\sigma(t)t\partial_\sigma \\
&= -[H, Y]
\end{aligned}$$

as one would expect, and one gets exactly minus the left-hand side of relation (3.9).

Proof of Theorem 3.2.2. We start by proving relation (3.8). For the left-hand side,

$$\begin{aligned}
[H, X]_\sigma &= [-2t\partial_\sigma, \partial_\sigma]_\sigma \\
&= -2[t\partial_\sigma, \partial_\sigma]_\sigma, \quad \text{since the bracket } [\cdot, \cdot]_\sigma \text{ is bilinear} \\
&= -2\Big(\sigma(t)\partial_\sigma(\partial_\sigma) - \sigma(1)\partial_\sigma(t\partial_\sigma)\Big), \quad \text{by definition of } [\cdot, \cdot]_\sigma \\
&= -2\Big((q_0 + q_1 t)\partial_\sigma(\partial_\sigma) - \partial_\sigma(t\partial_\sigma)\Big), \quad \text{since } \sigma(t) = q_0 + q_1 t \\
&= -2q_0\partial_\sigma^2 - 2q_1 t\partial_\sigma\partial_\sigma - \partial_\sigma(-2t\partial_\sigma), \quad \text{since } \partial_\sigma \text{ is } \mathbb{F}\text{-linear} \\
&= -2q_0 X^2 + q_1 HX - XH, \quad \text{since } X = \partial_\sigma \text{ and } H = -2t\partial_\sigma,
\end{aligned}$$

as required. And for the right-hand side, we have

$$\begin{aligned}
[H, X]_\sigma &= 2\partial_\sigma(t)\partial_\sigma, \quad \text{by relation (3.5) of Theorem (3.1.1)} \\
&= 2p_0 X, \quad \text{since } \partial_\sigma(t) = p_0 \text{ and } \partial_\sigma = X.
\end{aligned}$$

as required. Next, we prove relation (3.9) in a similar way. For the left-hand side, we have

$$\begin{aligned}
[H, Y]_\sigma &= \big[-2t\partial_\sigma, -t^2\partial_\sigma\big]_\sigma \\
&= 2\big[t\partial_\sigma, t^2\partial_\sigma\big]_\sigma \\
&= 2\Big(\sigma(t)\partial_\sigma(t^2\partial_\sigma) - \sigma(t^2)\partial_\sigma(t\partial_\sigma)\Big), \quad \text{by definition of } [\cdot, \cdot]_\sigma
\end{aligned}$$

$$
\begin{aligned}
&= 2\left((q_0 + q_1 t)\partial_\sigma(-Y) - (q_0 + q_1 t)^2 \partial_\sigma\left(\frac{-H}{2}\right)\right), \quad \text{since } \sigma(t) = q_0 + q_1 t \\
&= -2q_0 XY + q_1 HY + q_0^2 XH - q_0 q_1 H^2 - q_1^2 YH,
\end{aligned}
$$

as required. And for the right-hand side, we have

$$
\begin{aligned}
[H, Y]_\sigma &= 2\sigma(t)\partial_\sigma(t)t\partial_\sigma, \quad \text{by relation (3.6) of Theorem (3.1.1)} \\
&= 2(q_0 + q_1 t)(p_0)(t\partial_\sigma) \\
&= 2q_0 p_0 t\partial_\sigma + 2q_1 p_0 t^2 \partial_\sigma \\
&= -q_0 p_0 H - 2q_1 p_0 Y,
\end{aligned}
$$

as required. Finally, we prove relation (3.10). For the left-hand side, we have

$$
\begin{aligned}
[X, Y]_\sigma &= \big[\partial_\sigma, -t^2\partial_\sigma\big]_\sigma \\
&= -\big[\partial_\sigma, t^2\partial_\sigma\big]_\sigma \\
&= -\Big(\sigma(1)\partial_\sigma(t^2\partial_\sigma) - \sigma(t^2)\partial_\sigma(\partial_\sigma)\Big), \quad \text{by definition of } [\cdot,\cdot]_\sigma \\
&= -\Big(\partial_\sigma(t^2\partial_\sigma) - (q_0 + q_1 t)^2 \partial_\sigma(\partial_\sigma)\Big), \quad \text{since } \sigma(t) = q_0 + q_1 t \\
&= XY + q_0^2 X^2 - 2q_0 q_1 HX - q_1^2 YX,
\end{aligned}
$$

as required. And for the right-hand side, we have

$$
\begin{aligned}
[X, Y]_\sigma &= -\big(t + \sigma(t)\big)\partial_\sigma(t)\partial_\sigma, \quad \text{by relation (3.7) of Theorem (3.1.1)} \\
&= -(q_0 + q_1 t + t)p_0 \partial_\sigma \\
&= -q_0 p_0 X + \frac{q_1 p_0 H}{2} + \frac{p_0 H}{2} \\
&= -q_0 p_0 X + \frac{(q_1 + 1)}{2} p_0 H,
\end{aligned}
$$

as required. ∎

For the present case, the associative algebra with three abstract generators X, H and Y and the defining relations (3.9), (3.8) and (3.10) can be seen as multiparameter quasi-Lie deformations of $U(\mathfrak{sl}_2(\mathbb{F}))$, the universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{F})$.

Twisted Jacobi Identity for Case 1

We now determine the Jacobi identity applicable for the present case. We will make use of the following result concerning condition (2.3).

Lemma 3.2.4. *[27] The following identity holds for all $a \in \mathcal{A} = \mathbb{F}[t]$:*

$$\partial_\sigma(\sigma(a)) = \delta \cdot \sigma(\partial_\sigma(a)) \tag{3.11}$$

Proof. It suffices to prove this assertion for the monomials $a = t^k$, where k is a non-negative integer. We make use of the assumptions $\sigma(1) = 1$ and $\partial_\sigma(1) = 0$. We proceed by induction on k. When $k = 0$, we have $a = t^0 = 1$ and thus

$$\begin{aligned}
\text{L.H.S} &= \partial_\sigma(\sigma(1)) = \partial_\sigma(1) = 0 \\
\text{R.H.S} &= \delta \cdot \sigma(\partial_\sigma(1)) = \delta \cdot \sigma(0) = \delta \cdot 0 = 0
\end{aligned}$$

which proves (3.11) for $k = 0$.

Suppose now that (3.11) holds for some integer $k > 0$, then

$$\begin{aligned}
\text{R.H.S} = \delta \cdot \sigma\Big(\partial_\sigma(t^{k+1})\Big) &= \delta \cdot \sigma(\partial_\sigma(tt^k)) \\
&= \delta \cdot \sigma\Big(\partial_\sigma(t)t^k + \sigma(t)\partial_\sigma(t^k)\Big), \quad \text{since } \partial_\sigma \text{ is } \sigma\text{-derivation on } \mathbb{F}[t] \\
&= \delta \cdot \sigma\big(\partial_\sigma(t)\big)\sigma(t^k) + \delta \cdot \sigma^2(t)\sigma\Big(\partial_\sigma(t^k)\Big) \\
&= \delta \cdot \sigma(\partial_\sigma(t))\sigma(t^k) + \sigma^2(t) \cdot \delta \cdot \sigma(\partial_\sigma(t^k)) \\
&= \partial_\sigma(\sigma(t))\sigma(t^k) + \sigma^2(t)\partial_\sigma(\sigma(t^k)) \\
&= \partial_\sigma(\sigma(t^{k+1})) = \text{L.H.S.}
\end{aligned}$$

which proves (3.11) for $k+1$. Hence (3.11) holds for all non-negative integer values of k. ■

Now, by Proposition 2.3.6, the bracket $[\cdot,\cdot]_\sigma$ on the $\mathcal{A}$-module $\mathcal{A} \cdot \partial_\sigma = \mathbb{F}[t] \cdot \partial_\sigma$ satisfies the 6-term deformed Jacobi identity

$$\circlearrowleft_{x,y,z} \left(\big[\sigma(x), [y,z]_\sigma\big]_\sigma + \delta\big[x, [y,z]_\sigma\big]_\sigma \right) = 0 \tag{3.12}$$

We need to find the possible $\delta \in \mathcal{A} = \mathbb{F}[t]$ in this deformed Jacobi identity. By Lemma (3.2.4), equation (3.11) holds for all monomials $a = t^k$, where k is a non-negative integer. Now for $k = 1$, we have $a = t^1 = t$ and thus

$$\begin{aligned}
\text{L.H.S} &= \partial_\sigma(\sigma(t)) = \partial_\sigma(q_0 + q_1 t) = \partial_\sigma(q_0) + q_1(\partial_\sigma(t)) = q_1 p_0, \\
\text{R.H.S} &= \delta \cdot \sigma(\partial_\sigma(t)) = \delta \cdot \sigma(p_0) = \delta p_0.
\end{aligned}$$

Since equation (3.11) holds for $k = 1$, we have L.H.S = R.H.S, which implies that $\delta = q_1$.

We thus have the following Jacobi identity on $\mathcal{A} \cdot \partial_\sigma = \mathbb{F}[t] \cdot \partial_\sigma$ for the present case:

$$\circlearrowleft_{x,y,z} \left(\left[\sigma(x), [y, z]_\sigma\right]_\sigma + q_1 \left[x, [y, z]_\sigma\right]_\sigma \right) = 0. \tag{3.13}$$

Corollary 3.2.5. *[27] By defining $\alpha(x) = q_1^{-1}\sigma(x)$, the Jacobi-identity* (3.13) *can be rewritten as the Jacobi-like relation for the hom-Lie algebra given by*

$$\circlearrowleft_{x,y,z} \left[(\alpha + id)(x), [y, z]_\sigma\right]_\sigma = 0. \tag{3.14}$$

Proof. This follows immediately from the following observation:

$$\begin{aligned}
\left[(\alpha + \mathrm{id})(x), [y, z]_\sigma\right]_\sigma &= \left[\alpha(x) + x, [y, z]_\sigma\right]_\sigma \\
&= \left[\alpha(x), [y, z]_\sigma\right]_\sigma + \left[x, [y, z]_\sigma\right]_\sigma, \quad \text{since } [\cdot, \cdot]_\sigma \text{ is bilinear} \\
&= \left[q^{-1}\sigma(x), [y, z]_\sigma\right]_\sigma + q_1 q_1^{-1} \left[x, [y, z]_\sigma\right]_\sigma
\end{aligned}$$

■

We take note that the deformed Jacobi-identity (3.13) is directly generated from the general Proposition 2.3.6. This shows that we do not need to go through lengthy trial-and-error computation with various arbitrary assumptions on the form of the identity.

We conclude that the deformed $\mathfrak{sl}_2(\mathbb{F})$ is a quasi-hom-Lie algebra with Jacobi identity

$$\circlearrowleft_{x,y,z} \left(\left[\sigma(x), [y, z]_\sigma\right]_\sigma + q_1 \left[x, [y, z]_\sigma\right]_\sigma \right) = 0.$$

Alternatively, the deformed $\mathfrak{sl}_2(\mathbb{F})$ is a hom-Lie algebra with Jacobi identity

$$\circlearrowleft_{x,y,z} \left[(\alpha + \mathrm{id})(x), [y, z]_\sigma\right]_\sigma = 0,$$

where $\alpha(x) = q_1^{-1}\sigma(x)$.

3.2.2 Case 2

For this case we assume that $\sigma(t) = q(t) = q_0 \neq 0$, implying that $\partial_\sigma(t) = p(t) = p_0 + p_1 t$ for some constant $p_1 \neq 0$. This leads us to the following result.

Theorem 3.2.6. [27] Let $\mathcal{A}$ be a commutative, associative algebra with unity 1 over a field $\mathbb{F}$ of characteristic 0 and fix an element $t \in \mathcal{A}$. Let σ be an endomorphism on $\mathcal{A}$ and ∂_σ a σ-derivation (twisted derivation) on $\mathcal{A}$ where $\sigma(t) = q_0$ and $\partial_\sigma(t) = p_0 + p_1 t$. Furthermore, assume that $\sigma(1) = 1$ and $\partial_\sigma(1) = 0$. Then the $\mathcal{A}$-module elements

$$H = -2t\partial_\sigma, \quad X = \partial_\sigma, \quad Y = -t^2\partial_\sigma$$

satisfy the following relations:

$$[H,X]_\sigma : -2q_0X^2 - XH = 2p_0X - p_1H, \tag{3.15}$$
$$[H,Y]_\sigma : -2q_0XY + q_0^2XH = -q_0p_0H - 2q_0p_1Y, \tag{3.16}$$
$$[X,Y]_\sigma : XY + q_0^2X^2 = -q_0p_0X + \frac{q_0p_1 + p_0}{2}H + p_1Y. \tag{3.17}$$

Proof. We start by proving relation (3.15). For the left-hand side, we have

$$\begin{aligned}
[H,X]_\sigma &= [-2t\partial_\sigma, \partial_\sigma]_\sigma \\
&= -2[t\partial_\sigma, \partial_\sigma]_\sigma, \quad \text{since the bracket } [\cdot,\cdot]_\sigma \text{ is bilinear} \\
&= -2\Big(\sigma(t)\partial_\sigma(\partial_\sigma) - \sigma(1)\partial_\sigma(t\partial_\sigma)\Big), \quad \text{by definition of } [\cdot,\cdot]_\sigma \\
&= -2\Big(q_0\partial_\sigma(\partial_\sigma) - \partial_\sigma(t\partial_\sigma)\Big), \quad \text{since } \sigma(t) = q_0 \\
&= -2q_0\partial_\sigma^2 - \partial_\sigma(-2t\partial_\sigma), \quad \text{since } \partial_\sigma \text{ is } \mathbb{F}\text{-linear} \\
&= -2q_0X^2 - XH, \quad \text{since } X = \partial_\sigma \text{ and } H = -2t\partial_\sigma,
\end{aligned}$$

as required. And for the right-hand side, we have

$$\begin{aligned}
[H,X]_\sigma &= 2\partial_\sigma(t)\partial_\sigma, \quad \text{by relation (3.5) of Theorem (3.1.1)} \\
&= 2(p_0 + p_1t)\partial_\sigma, \quad \text{since } \partial_\sigma(t) = p_0 + p_1t \\
&= 2p_0X - p_1H, \quad \text{since } X = \partial_\sigma \text{ and } H = -2t\partial_\sigma,
\end{aligned}$$

as required. Next, we prove relation (3.16) in a similar way. For the left-hand side, we have

$$\begin{aligned}
[H,Y]_\sigma &= \big[-2t\partial_\sigma, -t^2\partial_\sigma\big]_\sigma \\
&= 2\big[t\partial_\sigma, t^2\partial_\sigma\big]_\sigma \\
&= 2\Big(\sigma(t)\partial_\sigma(t^2\partial_\sigma) - \sigma(t^2)\partial_\sigma(t\partial_\sigma)\Big), \quad \text{by definition of } [\cdot,\cdot]_\sigma \\
&= 2q_0\partial_\sigma(t^2\partial_\sigma) + q_0^2\partial_\sigma(-2t\partial_\sigma), \quad \text{since } \sigma(t) = q_0 \\
&= -2q_0XY + q_0^2XH,
\end{aligned}$$

as required. And for the right-hand side, we have

$$\begin{aligned}
[H,Y]_\sigma &= 2\sigma(t)\partial_\sigma(t)t\partial_\sigma, \quad \text{by relation (3.6) of Theorem (3.1.1)} \\
&= 2q_0(p_0 + p_1t)t\partial_\sigma \\
&= -q_0p_0H - 2q_0p_1Y,
\end{aligned}$$

as required.

Finally, we prove relation (3.17). For the left-hand side, we have

$$\begin{aligned}
[X, Y]_\sigma &= -\big[\partial_\sigma, t^2\partial_\sigma\big]_\sigma \\
&= -\Big(\sigma(1)\partial_\sigma(t^2\partial_\sigma) - \sigma(t^2)\partial_\sigma(\partial_\sigma)\Big), \quad \text{by definition of } [\cdot,\cdot]_\sigma \\
&= -\Big(\partial_\sigma(t^2\partial_\sigma) - q_0^2\partial_\sigma(\partial_\sigma)\Big), \quad \text{since } \sigma(t) = q_0 \\
&= XY + q_0^2X^2,
\end{aligned}$$

as required. And for the right-hand side, we have

$$\begin{aligned}
[X, Y]_\sigma &= -\big(t + \sigma(t)\big)\partial_\sigma(t)\partial_\sigma, \quad \text{by relation (3.7) of Theorem (3.1.1)} \\
&= -(q_0 + t)(p_0 + p_1t)\partial_\sigma \\
&= -q_0p_0X + \frac{q_0p_1H}{2} + \frac{p_0H}{2} + p_1Y \\
&= -q_0p_0X + \frac{q_0p_1 + p_0}{2}H + p_1Y,
\end{aligned}$$

as required. The proof is now complete. ■

Since $H = -2t\partial_\sigma$, $X = \partial_\sigma$ and $Y = -t^2\partial_\sigma$, we have

$$\sigma(H) = -2\sigma(t)\partial_\sigma = -2q_0X, \tag{3.18}$$

$$\sigma(X) = \sigma(\partial_\sigma) = \partial_\sigma = X, \tag{3.19}$$

$$\sigma(Y) = -\sigma(t)^2\partial_\sigma = -q_0^2X. \tag{3.20}$$

Therefore, we cannot recover the usual $\mathfrak{sl}_2(\mathbb{F})$ from the present deformation by specifying suitable parameters since, in a sense, the commutators "collapsed" due to the choice of σ. Indeed, when we substitute (3.18)–(3.20) into relations (3.15)–(3.17), we obtain

$$\sigma(H)X - \sigma(X)H = -\sigma(H) - p_1H \tag{3.21}$$

$$-2\sigma(H)Y - \sigma(Y) = -q_0p_0H - 2q_0p_1Y \tag{3.22}$$

$$\sigma(X)Y - \sigma(Y)X = -q_0p_0\sigma(X) + \frac{q_0p_1 + p_0}{2}H + p_1Y. \tag{3.23}$$

So it is impossible to retain the undeformed relation $HX - XH = -2X$ from relation (3.21), since this will require $\sigma = \mathrm{id}$, $p_0 = -1$ and $p_1 = 0$, but $p_0 \neq 0$ by assumption. Similarly, relations $HY - YH = -2Y$ and $XY - YX = H$ can not be retained from (3.22) and (3.23), respectively. Therefore, we have a **quasi-deformation** for the present case.

Twisted Jacobi Identity for Case 2

We now determine the Jacobi identity applicable for the present case. We need to find the possible $\delta \in \mathcal{A} = \mathbb{F}[t]$ in the Jacobi identity (3.12). By Lemma (3.2.4), equation (3.11) holds for all monomials $a = t^k$. For $k = 1$, we have $a = t^1 = t$ and thus

$$\begin{aligned}
&\text{L.H.S} = \partial_\sigma(\sigma(t)) = \partial_\sigma(q_0) = 0,\\
&\text{R.H.S} = \delta \cdot \sigma(\partial_\sigma(t)) = \delta \cdot \sigma(p_0 + p_1 t) = \delta \cdot (p_0 + q_0 p_1)
\end{aligned}$$

Since equation (3.11) holds for $k = 1$, we have L.H.S = R.H.S, which implies that $\delta = 0$. We thus have the following Jacobi identity on $\mathcal{A} \cdot \partial_\sigma = \mathbb{F}[t] \cdot \partial_\sigma$ for the present case:

$$\circlearrowleft_{x,y,z} \left(\left[\sigma(x), [y,z]_\sigma\right]_\sigma + 0 \cdot \left[x, [y,z]_\sigma\right]_\sigma \right) = 0,$$

which simplifies to the hom-Jacobi identity

$$\circlearrowleft_{x,y,z} \left[\sigma(x), [y,z]_\sigma\right]_\sigma = 0. \tag{3.24}$$

We conclude that the deformed $\mathfrak{sl}_2(\mathbb{F})$ in this case is a hom-Lie algebra.

3.2.3 Case 3

For this case we assume $\sigma(t) = q(t) = 0$, implying that $\partial_\sigma(t) = p(t) = p_0 + p_1 t + \cdots + p_n t^n$. However, substituting $\sigma(t) = 0$ into relation (3.7) implies that $\deg(p(t)) \leq 1$ (since we need relations (3.5)–(3.7) to be closed over $\mathbb{F}$). This leads us to the following result.

Theorem 3.2.7. [27] Let $\mathcal{A}$ be a commutative, associative algebra with unity 1 over a field $\mathbb{F}$ of characteristic 0 and fix an element $t \in \mathcal{A}$. Let σ be an endomorphism on $\mathcal{A}$ and ∂_σ a σ-derivation (twisted derivation) on $\mathcal{A}$ where $\sigma(t) = 0$ and $\partial_\sigma(t) = p_0$. Furthermore, assume that $\sigma(1) = 1$ and $\partial_\sigma(t) = p(t) = p_0 + p_1 t + \cdots + p_n t^n$. Then the $\mathcal{A}$-module elements

$$H = -2t\partial_\sigma, \quad X = \partial_\sigma, \quad Y = -t^2\partial_\sigma$$

satisfy the following relations:

$$[H,X]_\sigma : -XH = 2p_0 X - p_1 H, \tag{3.25}$$

$$[H,Y]_\sigma : 0 = 0, \tag{3.26}$$

$$[X,Y]_\sigma : XY = \frac{p_0 H}{2} + p_1 Y. \tag{3.27}$$

Proof. We start by proving relation (3.25). For the left-hand side, we have

$$
\begin{aligned}
[H, X]_\sigma &= [-2t\partial_\sigma, \partial_\sigma]_\sigma \\
&= -2[t\partial_\sigma, \partial_\sigma]_\sigma, \quad \text{since the bracket } [\cdot,\cdot]_\sigma \text{ is bilinear} \\
&= -2\Big(\sigma(t)\partial_\sigma(\partial_\sigma) - \sigma(1)\partial_\sigma(t\partial_\sigma)\Big), \quad \text{by definition of } [\cdot,\cdot]_\sigma \\
&= -2\Big(0 - \partial_\sigma(t\partial_\sigma)\Big), \quad \text{since } \sigma(t) = 0 \\
&= -\partial_\sigma(-2t\partial_\sigma), \quad \text{since } \partial_\sigma \text{ is } \mathbb{F}\text{-linear} \\
&= -XH, \quad \text{since } X = \partial_\sigma \text{ and } H = -2t\partial_\sigma,
\end{aligned}
$$

as required. And for the right-hand side, we have

$$
\begin{aligned}
[H, X]_\sigma &= 2\partial_\sigma(t)\partial_\sigma, \quad \text{by relation (3.5) of Theorem (3.1.1)} \\
&= 2(p_0 + p_1 t)\partial_\sigma, \quad \text{since } \partial_\sigma(t) = p(t) = p_0 + p_1 t + \cdots + p_n t^n \text{ where deg(p(t))} \leq 1 \\
&= 2p_0 X - p_1 H, \quad \text{since } X = \partial_\sigma \text{ and } H = -2t\partial_\sigma,
\end{aligned}
$$

as required. Next, we prove relation (3.26) in a similar way. For the left-hand side, we have

$$
\begin{aligned}
[H, Y]_\sigma &= 2\big[t\partial_\sigma, t^2\partial_\sigma\big]_\sigma \\
&= 2\Big(\sigma(t)\partial_\sigma(t^2\partial_\sigma) - \sigma(t^2)\partial_\sigma(t\partial_\sigma)\Big), \quad \text{by definition of } [\cdot,\cdot]_\sigma \\
&= 2\Big(q(t)\partial_\sigma(t^2\partial_\sigma) - q^2(t)\partial_\sigma(t\partial_\sigma)\Big) = 0, \quad \text{since } \sigma(t) = q(t) = 0,
\end{aligned}
$$

as required. And for the right-hand side, we have

$$
\begin{aligned}
[H, Y]_\sigma &= 2\sigma(t)\partial_\sigma(t)t\partial_\sigma, \quad \text{by relation (3.6) of Theorem (3.1.1)} \\
&= 2q(t)(p_0 + p_1 t)\partial_\sigma(t)t\partial_\sigma, \quad \text{since } \sigma(t) = q(t) \\
&= 0, \quad \text{since } q(t) = 0,
\end{aligned}
$$

as required. Finally, we prove relation (3.27). For the left-hand side, we have

$$
\begin{aligned}
[X, Y]_\sigma &= -\big[\partial_\sigma, t^2\partial_\sigma\big]_\sigma \\
&= -\Big(\sigma(1)\partial_\sigma(t^2\partial_\sigma) - \sigma(t^2)\partial_\sigma(\partial_\sigma)\Big), \quad \text{by definition of } [\cdot,\cdot]_\sigma \\
&= -\Big(\partial_\sigma(t^2\partial_\sigma) - \sigma^2(t)\partial_\sigma(\partial_\sigma)\Big), \quad \text{since } \sigma(1) = 1 \text{ and } \sigma(t^2) = \sigma(tt) = \sigma(t)\sigma(t) \\
&= -\Big(\partial_\sigma(t^2\partial_\sigma) - q^2(t)\partial_\sigma(\partial_\sigma)\Big), \quad \text{since } \sigma(t) = q(t) \\
&= -\partial_\sigma(t^2\partial_\sigma) = \partial_\sigma(-t^2\partial_\sigma) = XY,
\end{aligned}
$$

as required.

And for the right-hand side, we have

$$\begin{aligned}
[X,Y]_\sigma &= -\big(t+\sigma(t)\big)\partial_\sigma(t)\partial_\sigma, \quad \text{by relation (3.7) of Theorem (3.1.1)}\\
&= -t(p_0+p_1t)\partial_\sigma, \quad \text{since } \sigma(t)=q(t)=0\\
&= \frac{p_0H}{2}+p_1Y,
\end{aligned}$$

as required. The proof is now complete. ■

Twisted Jacobi Identity for Case 3

We now determine the Jacobi identity applicable for the present case. We need to find the possible $\delta \in \mathcal{A} = \mathbb{F}[t]$ in the Jacobi identity (3.12). By Lemma (3.2.4), equation (3.11) holds for all monomials $a = t^k$. For $k = 1$, we have $a = t^1 = t$ and thus

$$\begin{aligned}
\text{L.H.S} &= \partial_\sigma(\sigma(t)) = \partial_\sigma(0) = 0,\\
\text{R.H.S} &= \delta \cdot \sigma(\partial_\sigma(t^k)) = \delta \cdot \sigma(t^k) = \delta \cdot 0.
\end{aligned}$$

Hence δ can be chosen completely arbitrary in this case, for example, $\delta = 0$, giving as in Case 2 the following hom Jacobi identity on $\mathcal{A} \cdot \partial_\sigma = \mathbb{F}[t] \cdot \partial_\sigma$:

$$\circlearrowleft_{x,y,z} \Big(\big[\sigma(x),[y,z]_\sigma\big]_\sigma + 0\cdot\big[x,[y,z]_\sigma\big]_\sigma\Big) = 0 \quad \text{or} \quad \circlearrowleft_{x,y,z} \big[\sigma(x),[y,z]_\sigma\big]_\sigma = 0.$$

We conclude that the deformed $\mathfrak{sl}_2(\mathbb{F})$ in this case is a hom-Lie algebra.

3.3 An Elaborated Example

Consider relations (3.8)–(3.10) in Case 1. By taking $q_0 = 0$ and $q_1 = q \neq 0$, we obtain a deformation of $\mathfrak{sl}_2(\mathbb{F})$ corresponding to replacing the ordinary derivation operator ∂ with the Jackson q-derivative D_q acting on $\mathbb{C}[t]$ (see Section 2.2):

$$D_q(a)(t) = \frac{a(qt)-a(t)}{qt-t}, \qquad q \in \mathbb{C}\setminus\{1\}.$$

Due to the intriguing nature of this deformation, we explicitly record it here.

Relations (3.8)–(3.10) in Case 1 are given by:

$$\begin{aligned}
-2q_0X^2 + q_1HX - XH &= 2p_0X,\\
-2q_0XY + q_1HY + q_0^2XH - q_0q_1H^2 - q_1^2YH &= -q_0p_0H - 2q_1p_0Y,\\
XY + q_0^2X^2 - 2q_0q_1HX - q_1^2YX &= -q_0p_0X + \frac{(q_1+1)}{2}p_0H.
\end{aligned}$$

Taking $q_0 = 0$ and $q_1 = q \neq 0$ in these relations, we obtain

$$\begin{aligned} qHX - XH &= 2p_0 X, \\ qHY - q^2 YH &= -2qp_0 Y, \\ XY - q^2 YX &= \frac{(q+1)}{2} p_0 H. \end{aligned}$$

Dividing the first two relations through by q, we obtain

$$HX - q^{-1}XH = 2q^{-1}p_0 X, \tag{3.28}$$

$$HY - qYH = -2p_0 Y, \tag{3.29}$$

$$XY - q^2 YX = \frac{(q+1)}{2} p_0 H. \tag{3.30}$$

Note that, by taking $q = 1$ and $p_0 = 1$ in relations (3.28)–(3.30), we obtain the usual commutation relations for $\mathfrak{sl}_2(\mathbb{F})$, corresponding to the ordinary derivation operator ∂:

$$\begin{aligned} HX - XH &= 2X, \\ HY - YH &= -2Y, \\ XY - YX &= H. \end{aligned}$$

By taking $q = 1$ and $p_0 = 0$ in relations (3.28)–(3.30) we get the "abelianized" versions:

$$\begin{aligned} HX - XH &= 0, \\ HY - YH &= 0, \\ XY - YX &= 0. \end{aligned}$$

However, the original operator representation

$$H = -2t\partial_\sigma, \quad X = \partial_\sigma, \quad Y = -t^2\partial_\sigma$$

collapses in this case since $\partial_\sigma = 0$.

Consider now the "lifting" of the right-hand-side of relations (3.8)–(3.10):

$$\begin{aligned} [H, X]_\sigma &= 2p_0 X, \\ [H, Y]_\sigma &= -q_0 p_0 H - 2q_1 p_0 Y, \\ [X, Y]_\sigma &= -q_0 p_0 X + \frac{(q_1+1)}{2} p_0 H. \end{aligned}$$

If $p_0 = 1$, $q_0 = 0$, $q_1 = q \neq 0$, we obtain an abstract skew-symmetric algebra with products:

$$[H,X]_\sigma = 2X, \qquad [H,Y]_\sigma = -2qY, \qquad [X,Y]_\sigma = \frac{(q+1)}{2}H.$$

We denote this algebra by ${}_q\mathfrak{sl}_2(\mathbb{F})$ and informally call it the "Jackson $\mathfrak{sl}_2(\mathbb{F})$". It will be shown in Example 3.3.2 that ${}_q\mathfrak{sl}_2(\mathbb{F})$ is a (quasi)-hom-Lie algebra. When $p_0 \neq 1$ we get a natural 1-parameter deformation of ${}_q\mathfrak{sl}_2(\mathbb{F})$.

Proposition 3.3.1. *[27] For $q \neq 1$, we can rewrite relations (3.28)–(3.30) using transformations*

$$H \mapsto 2p_0q^{-1}y + \frac{2p_0}{q-1}\mathbf{1}, \qquad X \mapsto z, \qquad Y \mapsto x,$$

where $\mathbf{1}$ denotes the identity element, in the form

$$yz - q^{-1}zy = 0, \qquad xy - q^{-1}yx = 0, \qquad zx - q^2xz = p_0^2(1+q^{-1})y + p_0^2\frac{q+1}{q-1}\mathbf{1}. \quad (3.31)$$

Proof. We rewrite relation (3.28) as follows:

$$\begin{aligned}
HX - q^{-1}XH &= 2q^{-1}p_0X \\
\left(2p_0q^{-1}y + \frac{2p_0}{q-1}\mathbf{1}\right)z - q^{-1}z(2p_0q^{-1}y + \frac{2p_0}{q-1}\mathbf{1}) &= 2q^{-1}p_0z \\
2p_0q^{-1}yz + \frac{2p_0z}{q-1} - 2q^{-2}p_0zy - \frac{2q^{-1}p_0z}{q-1} &= 2q^{-1}p_0z \\
2p_0q^{-1}yz - 2q^{-2}p_0zy &= 2q^{-1}p_0z - \frac{2p_0z}{q-1} + \frac{2q^{-1}p_0z}{q-1} \\
2p_0q^{-1}yz - 2q^{-2}p_0zy &= \frac{(q-1)2q^{-1}p_0z - 2p_0z + 2q^{-1}p_0z}{q-1} \\
2p_0q^{-1}yz - 2q^{-2}p_0zy &= \frac{2p_0z - 2q^{-1}p_0z - 2p_0z + 2q^{-1}p_0z}{q-1} \\
2p_0q^{-1}yz - 2q^{-2}p_0zy &= \frac{2p_0(z - q^{-1}z - z + q^{-1}z)}{q-1} \\
yz - q^{-1}zy &= \frac{2p_0(z - q^{-1}z - z + q^{-1}z)}{2p_0q^{-1}(q-1)} \\
yz - q^{-1}zy &= 0,
\end{aligned}$$

as required. Similarly, we rewrite relation (3.29) as follows:

$$\begin{aligned}
HY - qYH &= -2p_0Y \\
\left(2p_0q^{-1}y + \frac{2p_0}{q-1}\mathbf{1}\right)x - qx(2p_0q^{-1}y + \frac{2p_0}{q-1}\mathbf{1}) &= -2p_0x
\end{aligned}$$

$$
\begin{aligned}
2p_0q^{-1}yx + \frac{2p_0x}{q-1} - 2p_0xy - \frac{2p_0qx}{q-1} &= -2p_0x \\
2p_0q^{-1}yx - 2p_0xy &= -2p_0x - \frac{2p_0x}{q-1} + \frac{2p_0qx}{q-1} \\
2p_0q^{-1}yx - 2p_0xy &= \frac{-2p_0x(q-1) - 2p_0x + 2p_0qx}{q-1} \\
2p_0q^{-1}yx - 2p_0xy &= \frac{-2p_0qx + 2p_0x - 2p_0x + 2p_0qx}{q-1} \\
q^{-1}yx - xy &= \frac{-2p_0(qx - x + x - qx)}{2p_0(q-1)} \\
xy - q^{-1}yx &= \frac{0}{q^{-1}(q-1)} \\
xy - q^{-1}yx &= 0,
\end{aligned}
$$

as required. Finally, we rewrite relation (3.30) as follows:

$$
\begin{aligned}
XY - q^2YX &= \frac{(q+1)}{2}p_0H \\
zx - q^2xz &= \frac{q+1}{2}p_0(2p_0q^{-1}y + \frac{2p_0}{q-1}\mathbf{1}) \\
zx - q^2xz &= p_0^2(1+q^{-1})y + p_0^2\frac{q+1}{q-1}\mathbf{1},
\end{aligned}
$$

as required. ■

Example 3.3.2. [27] On page 39 we defined ${}_q\mathfrak{sl}_2(\mathbb{F})$ as the algebra generated by three elements H, X and Y satisfying the relations

$$
[H,X]_\sigma = 2X, \qquad [H,Y]_\sigma = -2qY, \qquad [X,Y]_\sigma = \frac{(q+1)}{2}H. \tag{3.32}
$$

The algebra ${}_q\mathfrak{sl}_2(\mathbb{F})$ is a hom-Lie algebra with the Jacobi identity as given by (3.14):

$$
\circlearrowleft_{x,y,z} \big[(\alpha + \mathrm{id})(x), [y,z]_\sigma\big]_\sigma = 0,
$$

where $\alpha = q^{-1}\sigma$. This is not clear a priori since ${}_q\mathfrak{sl}_2(\mathbb{F})$ is not a subalgebra of $\mathcal{A}\cdot\partial_\sigma$. When ${}_q\mathfrak{sl}_2(\mathbb{F})$ is represented by its "nonlifted" operators

$$
H \mapsto -2t\partial_\sigma, \quad X \mapsto \partial_\sigma, \quad Y \mapsto -t^2\partial_\sigma,
$$

this representation becomes a subalgebra of $\mathcal{A}\cdot\partial_\sigma$, and hence, a hom-Lie algebra. However, the algebra ${}_q\mathfrak{sl}_2(\mathbb{F})$ is defined abstractly without reference to a particular representation. It is initially possible that the twisted Jacobi identity is a result of some extra relations (for example, by the σ-Leibniz rule) available in that representation as σ-derivations.[27]

To prove that ${}_q\mathfrak{sl}_2(\mathbb{F})$ is a hom-Lie algebra, we proceed as follows. First we define α on the basis vectors H, X and Y respectively as

$$\alpha(H) = H, \qquad \alpha(X) = q^{-1}X, \qquad \alpha(Y) = qY. \tag{3.33}$$

It is enough to consider the case when $x = X,\ y = Y$ and $z = H$. That is,

$$\begin{aligned}
&\circlearrowleft_{x,y,z} \big[(\alpha + \mathrm{id})(x), [y, z]_\sigma\big]_\sigma \\
&\quad = \circlearrowleft_{X,Y,H} \big[(\alpha + \mathrm{id})(X), [Y, H]_\sigma\big]_\sigma \\
&\quad = \big[(\alpha + \mathrm{id})(X), [Y, H]_\sigma\big]_\sigma + \big[(\alpha + \mathrm{id})(Y), [H, X]_\sigma\big]_\sigma + \big[(\alpha + \mathrm{id})(H), [X, Y]_\sigma\big]_\sigma \\
&\quad = \big[\alpha(X) + \mathrm{id}(X), [Y, H]_\sigma\big]_\sigma + \big[\alpha(Y) + \mathrm{id}(Y), [H, X]_\sigma\big]_\sigma + \big[\alpha(H) + \mathrm{id}(H), [X, Y]_\sigma\big]_\sigma \\
&\quad = \big[q^{-1}X + X, [Y, H]_\sigma\big]_\sigma + \big[qY + Y, [H, X]_\sigma\big]_\sigma + \big[H + H, [X, Y]_\sigma\big]_\sigma, \quad \text{by equations (3.33)} \\
&\quad = \Big[\big(q^{-1} + 1\big)X, 2qY\Big]_\sigma + \big[(q + 1)Y, 2X\big]_\sigma + \left[2H, \frac{(q + 1)}{2}H\right]_\sigma, \quad \text{by relations (3.32)} \\
&\quad = 2q(q^{-1} + 1)[X, Y]_\sigma + 2(q + 1)[Y, X]_\sigma + (q + 1)[H, H]_\sigma, \quad \text{since the bracket } [\cdot, \cdot]_\sigma \text{ is bilinear} \\
&\quad = 2q(q^{-1} + 1)[X, Y]_\sigma - 2(q + 1)[X, Y]_\sigma + 0, \quad \text{since } [Y, X]_\sigma = -[X, Y]_\sigma \text{ and } [H, H]_\sigma = 0 \\
&\quad = 2(1 + q)[X, Y]_\sigma - 2(q + 1)[X, Y]_\sigma \\
&\quad = 0,
\end{aligned}$$

as required.

Remark 3.3.3. Unlike in the case of the undeformed universal enveloping algebra $U(\mathfrak{sl}_2(\mathbb{F}))$, the 1-parameter deformation of $\mathcal{U}_q$ (with $q \neq 1$) has nontrivial 1-dimensional representations. Indeed, taking $H = 2p_0/(q - 1)$, the relations (3.28)–(3.30) become

$$XY = \frac{-p^2}{(q - 1)^2}. \tag{3.34}$$

So X and Y can be chosen as any numbers satisfying (3.34). When $p_0 = 1$ and $q \to 1$, we "approach $\mathfrak{sl}_2(\mathbb{F})$" in the relations (3.28)–(3.30), but the representation (3.34) goes to infinity and thus does not converge to a representation of $\mathfrak{sl}_2(\mathbb{F})$.

Remark 3.3.4. It is important to note that Proposition (2.3.6) gives us two alternative ways to view our algebras:

(i) Either one uses Proposition 2.3.6 to construct a bracket on the vector space $\mathcal{A} \cdot \partial_\sigma$ with the help of equation (2.5) viewing this bracket as a product; or

(ii) One uses equation (2.5) in adjunction with the Definition 2.6 of the bracket to obtain quadratic relations and "moding" out these from the tensor algebra, thereby giving us an associative quadratic algebra which loosely can be thought of as a deformed "enveloping algebra" of the corresponding algebra from viewpoint (i).

Chapter 4

Quasi-Deformations of $\mathfrak{sl}_2(\mathbb{F})$ with Base $\mathbb{F}[t]/(t^3)$

In this chapter we obtain quasi-deformations of $\mathfrak{sl}_2(\mathbb{F})$, where $\mathbb{F}$ is a field of characteristic 0 that includes all the third roots of unity (the solutions to the equation $z^3 = 1$). We take the base algebra $\mathcal{A}$ to be $\mathbb{F}[t]/(t^3)$, the algebra of polynomials of degree less than 3 over $\mathbb{F}$ in the variable t. In Section 4.1, we take the base algebra $\mathcal{A} = \mathbb{F}[t]/(t^3)$ to be as general as it can be in this situation and deduce some necessary conditions for everything to make sense. One special case of this deformation is considered in Section 4.2 while another special case is considered in Section 4.3.

4.1 General Case and the Necessary Conditions

We begin this section by recalling that $\mathfrak{sl}_2(\mathbb{F})$ is generated by three elements H, X and Y satisfying the commutation relations

$$HX - XH = 2X, \qquad HY - YH = -2Y, \qquad XY - YX = H.$$

We now let $\mathbb{F}$ be a field of characteristic 0 that includes all the third roots of unity and take the base algebra $\mathcal{A}$ to be $\mathbb{F}[t]/(t^3)$. This is obviously a 3-dimensional $\mathbb{F}$-vector space and a finitely generated $\mathbb{F}[t]$-module with basis $\{1, t, t^2\}$. We also take

$$H = -2t\partial_\sigma, \quad X = \partial_\sigma, \quad Y = -t^2\partial_\sigma.$$

We observe from this representation that

$$-2t \cdot X = H, \qquad t \cdot H = 2Y, \qquad t \cdot Y = 0.$$

Since we are considering the base algebra $\mathbb{F}[t]/(t^3)$, we take

$$\partial_\sigma(t) = p_0 + p_1 t + p_2 t^2 \quad \text{and} \quad \sigma(t) = q_0 + q_1 t + q_2 t^2. \tag{4.1}$$

We assume that $\sigma(1) = 1$ and $\partial_\sigma(1) = 0$. Consequently, the following relations given in Theorem 3.1.1 still hold:

$$[H, X]_\sigma = 2\partial_\sigma(t)\partial_\sigma, \tag{4.2}$$

$$[H, Y]_\sigma = 2\sigma(t)\partial_\sigma(t)t\partial_\sigma, \tag{4.3}$$

$$[X, Y]_\sigma = -\big(t + \sigma(t)\big)\partial_\sigma(t)\partial_\sigma. \tag{4.4}$$

Now equations (4.1) have to be compatible with our assumption of $t^3 = 0$. This means that

$$\begin{aligned}
0 &= \sigma(0) \\
&= \sigma(t^3), \quad \text{since } t^3 = 0 \\
&= \sigma(t)\sigma(t)\sigma(t), \quad \text{since } \sigma \text{ is an endormophism} \\
&= \sigma^3(t) \\
&= (q_0 + q_1 t + q_2 t^2)^3 \\
&= (q_0 + q_1 t)^3 + 3(q_0 + q_1 t)^2 (q_2 t^2) + 3(q_0 + q_1 t)(q_2 t^2)^2 + (q_3 t^2)^3 \\
&= q_0^3 + 3q_0^2 q_1 t + (3q_0^2 q_2 + 3q_0 q_1^2)t^2, \quad \text{since terms with higher powers of } t \text{ vanish,}
\end{aligned}$$

implying that $q_0 = 0$, and thus, $\sigma(t)$ becomes $\sigma(t) = q_1 t + q_2 t^2$. Furthermore, we have

$$\begin{aligned}
\partial_\sigma(t^2) &= \partial_\sigma(tt) \\
&= \partial_\sigma(t)t + \sigma(t)\partial_\sigma(t) \\
&= (p_0 + p_1 t + p_2 t^2)t + (q_1 t + q_2 t^2)(p_0 + p_1 t + p_2 t^2), \quad \text{substituting for } \partial_\sigma(t) \text{ and } \sigma(t) \\
&= p_0 t + p_1 t^2 + p_0 q_1 t + p_1 q_1 t^2 + p_0 q_2 t^2, \quad \text{since terms with higher powers of } t \text{ vanish} \\
&= (p_0 + p_0 q_1)t + (p_1 + p_1 q_1 + p_0 q_2)t^2. \qquad (4.5)
\end{aligned}$$

Finally, we observe that

$$\begin{aligned}
0 &= \partial_\sigma(0) \\
&= \partial_\sigma(t^3), \quad \text{since } t^3 = 0 \\
&= \partial_\sigma(tt^2) \\
&= \partial_\sigma(t)t^2 + \sigma(t)\partial_\sigma(t^2), \quad \text{since } \partial_\sigma \text{ is a } \sigma\text{-derivation} \\
&= (p_0 + p_1 t + p_2 t^2)t^2 + (q_1 t + q_2 t^2)\Big((p_0 + p_0 q_1)t + (p_1 + p_1 q_1 + p_0 q_2)t^2\Big) \\
&= p_0 t^2 + p_0 q_1 t^2 + p_0 q_1^2 t^2, \quad \text{since terms with higher powers of } t \text{ vanish} \\
&= p_0\big(1 + q_1 + q_1^2\big)t^2. \qquad (4.6)
\end{aligned}$$

We now have the following result.

Proposition 4.1.1. *[27] The action of the module $\mathcal{A} \cdot \partial_\sigma$ on $\mathcal{A} = \mathbb{F}[t]/(t^3)$ has the following matrix representation with respect to the basis $\{1, t, t^2\}$:*

$$H \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2p_0 & 0 \\ 0 & -2p_1 & -2p_0(1+q_1) \end{pmatrix}, \quad X \mapsto \begin{pmatrix} 0 & p_0 & 0 \\ 0 & p_1 & p_0(1+q_1) \\ 0 & p_2 & p_1(1+q_1) + p_0 q_2 \end{pmatrix}, \quad Y \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -p_0 & 0 \end{pmatrix}.$$

Remark 4.1.2. Note that when $p_0 = 1, \quad p_1 = p_2 = 0, \quad q_0 = q_2 = 0$ and $q_1 = 1$, which seemingly would correspond to the case of $\mathfrak{sl}_2(\mathbb{F})$, we surprisingly obtain the matrices

$$H \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{pmatrix}, \qquad X \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Y \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix},$$

satisfying the relations: $HX - XH = 2X,\ \ HY - YH = -2Y$ and $XY + 2YX = H$. The reason for this anomaly is that the values of the parameter so chosen are not compatible with the σ-Leibniz rule on $\mathcal{A} = \mathbb{F}[t]/(t^3)$ since (4.6) becomes $0 = 3t^2$, the equality which is not possible in $\mathcal{A}$ over a field of characteristic zero.

Proof of Proposition 4.1.1. Let H_1, H_2 and H_3 be column vectors of the matrix H. Since $\mathbb{F}[t]/(t^3)$ is a finitely generated $\mathbb{F}[t]$-module with basis $\{1, t, t^2\}$, we have

$$\begin{aligned}
&\begin{pmatrix} 1 & t & t^2 \end{pmatrix} H_1 = -2t\partial_\sigma(1) = 0, \\
&\begin{pmatrix} 1 & t & t^2 \end{pmatrix} H_2 = -2t\partial_\sigma(t) = -2t(p_0 + p_1 t + p_2 t^2) = -2p_0 t - 2p_1 t^2, \\
&\begin{pmatrix} 1 & t & t^2 \end{pmatrix} H_3 = -2t\partial_\sigma(t^2) = -2t\Big((p_0 + p_0 q_1)t + (p_1 + p_1 q_1 + p_0 q_2)t^2\Big) = -2p_0(1+q_1)t^2.
\end{aligned}$$

If a_{ij} is the (i,j)th entry of the 3×3 matrix H, then

$$\begin{aligned}
\begin{pmatrix} 1 & t & t^2 \end{pmatrix} H_1 &= a_{11} + a_{21}t + a_{31}t^2, \\
\begin{pmatrix} 1 & t & t^2 \end{pmatrix} H_2 &= a_{12} + a_{22}t + a_{32}t^2, \\
\begin{pmatrix} 1 & t & t^2 \end{pmatrix} H_3 &= a_{13} + a_{23}t + a_{33}t^2.
\end{aligned}$$

Equating the right-hand sides gives

$$\begin{aligned}
a_{11} + a_{21}t + a_{31}t^2 &= 0, \\
a_{12} + a_{22}t + a_{32}t^2 &= -2p_0 t - 2p_1 t^2, \\
a_{13} + a_{23}t + a_{33}t^2 &= -2p_0(1+q_1)t^2.
\end{aligned}$$

Equating terms with the same powers of t gives

$$\begin{aligned}
a_{11} &= 0, & a_{21} &= 0, & a_{31} &= 0, \\
a_{12} &= 0, & a_{22} &= -2p_0, & a_{32} &= -2p_1, \\
a_{13} &= 0, & a_{23} &= 0, & a_{33} &= -2p_0(1+q_1),
\end{aligned}$$

as required.

Similarly, if X_1, X_2 and X_3 are the column vectors of the matrix X, then

$$\begin{aligned}
\begin{pmatrix}1 & t & t^2\end{pmatrix} X_1 &= \partial_\sigma(1) = 0,\\
\begin{pmatrix}1 & t & t^2\end{pmatrix} X_2 &= \partial_\sigma(t) = p_0 + p_1 t + p_2 t^2,\\
\begin{pmatrix}1 & t & t^2\end{pmatrix} X_3 &= \partial_\sigma(t^2) = (p_0 + p_0 q_1)t + (p_1 + p_1 q_1 + p_0 q_2)t^2.
\end{aligned}$$

If a_{ij} is the (i,j)th entry of the 3×3 matrix X, then

$$\begin{aligned}
\begin{pmatrix}1 & t & t^2\end{pmatrix} X_1 &= a_{11} + a_{21} t + a_{31} t^2,\\
\begin{pmatrix}1 & t & t^2\end{pmatrix} X_2 &= a_{12} + a_{22} t + a_{32} t^2,\\
\begin{pmatrix}1 & t & t^2\end{pmatrix} X_3 &= a_{13} + a_{23} t + a_{33} t^2.
\end{aligned}$$

Equating the right-hand sides gives

$$\begin{aligned}
a_{11} + a_{21} t + a_{31} t^2 &= 0,\\
a_{12} + a_{22} t + a_{32} t^2 &= p_0 + p_1 t + p_2 t^2,\\
a_{13} + a_{23} t + a_{33} t^2 &= (p_0 + p_0 q_1)t + (p_1 + p_1 q_1 + p_0 q_2)t^2.
\end{aligned}$$

Equating terms with the same powers of t gives

$$\begin{aligned}
a_{11} &= 0, & a_{21} &= 0, & a_{31} &= 0,\\
a_{12} &= p_0, & a_{22} &= p_1, & a_{32} &= p_2,\\
a_{13} &= 0, & a_{23} &= p_0(1 + q_1), & a_{33} &= p_1(1 + q_1) + p_0 q_2,
\end{aligned}$$

as required. Finally, if Y_1, Y_2 and Y_3 are the column vectors of the matrix Y, then

$$\begin{aligned}
\begin{pmatrix}1 & t & t^2\end{pmatrix} Y_1 &= -t^2 \partial_\sigma(1) = 0,\\
\begin{pmatrix}1 & t & t^2\end{pmatrix} Y_2 &= -t^2 \partial_\sigma(t) = -t^2(p_0 + p_1 t + p_2 t^2) = -p_0 t^2,\\
\begin{pmatrix}1 & t & t^2\end{pmatrix} Y_3 &= -t^2 \partial_\sigma(t^2) = -t^2\big((p_0 + p_0 q_1)t + (p_1 + p_1 q_1 + p_0 q_2)t^2\big) = 0.
\end{aligned}$$

If a_{ij} is the (i,j)th entry of the 3×3 matrix Y, then

$$\begin{aligned}
\begin{pmatrix}1 & t & t^2\end{pmatrix} Y_1 &= a_{11} + a_{21} t + a_{31} t^2,\\
\begin{pmatrix}1 & t & t^2\end{pmatrix} Y_2 &= a_{12} + a_{22} t + a_{32} t^2,\\
\begin{pmatrix}1 & t & t^2\end{pmatrix} Y_3 &= a_{13} + a_{23} t + a_{33} t^2.
\end{aligned}$$

Equating the right-hand sides gives

$$\begin{aligned}
a_{11} + a_{21}t + a_{31}t^2 &= 0, \\
a_{12} + a_{22}t + a_{32}t^2 &= -p_0 t^2, \\
a_{13} + a_{23}t + a_{33}t^2 &= 0.
\end{aligned}$$

Equating terms with the same powers of t gives

$$\begin{aligned}
a_{11} &= 0, & a_{21} &= 0, & a_{31} &= 0, \\
a_{12} &= 0, & a_{22} &= 0, & a_{32} &= -p_0, \\
a_{13} &= 0, & a_{23} &= 0, & a_{33} &= 0,
\end{aligned}$$

as required. ■

We now compute the bracket $[\cdot,\cdot]_\sigma$ explicitly under the present conditions. The result is presented in the following theorem and the required computations in the subsequent proof.

Theorem 4.1.3. [27] Let $\mathcal{A}$ be a commutative, associative algebra with unity 1 over a field $\mathbb{F}$ of characteristic 0 and fix an element $t \in \mathcal{A}$. If conditions 4.1 holds where we let σ be an endomorphism on $\mathcal{A}$ and ∂_σ a σ-derivation (twisted derivation) on $\mathcal{A}$. Furthermore, assume that $\sigma(1) = 1$ and $\partial_\sigma(1) = 0$. Then the $\mathcal{A}$-module elements

$$H = -2t\partial_\sigma, \quad X = \partial_\sigma, \quad Y = -t^2\partial_\sigma$$

satisfy the following relations:

$$[H, X]_\sigma : q_1 HX + 2q_2 YX - XH = 2p_0 X - p_1 H - 2p_2 Y, \tag{4.7}$$

$$[H, Y]_\sigma : q_1 HY + 2q_2 Y^2 - q_1^2 YH = -2q_1 p_0 Y, \tag{4.8}$$

$$[X, Y]_\sigma : XY - q_1^2 YX = (q_1 p_1 + q_2 p_0 + p_1)Y + \frac{1}{2}p_0(q_1 + 1)H. \tag{4.9}$$

Proof. We start by proving relation (4.7). For the left-hand side,

$$\begin{aligned}
[H, X]_\sigma &= [-2t\partial_\sigma, \partial_\sigma]_\sigma \\
&= -2[t\partial_\sigma, \partial_\sigma]_\sigma, \quad \text{since the bracket } [\cdot,\cdot]_\sigma \text{ is bilinear} \\
&= -2\Big(\sigma(t)\partial_\sigma(\partial_\sigma) - \sigma(1)\partial_\sigma(t\partial_\sigma)\Big), \quad \text{by (3.1)} \\
&= -2\Big((q_1 t + q_2 t^2)\partial_\sigma(\partial_\sigma) - \partial_\sigma(t\partial_\sigma)\Big), \quad \text{since } \sigma(t) = q_1 t + q_2 t^2 \\
&= -2q_1 t\partial_\sigma(\partial_\sigma) - 2q_2 t^2\partial_\sigma(\partial_\sigma) - \partial_\sigma(-2t\partial_\sigma), \quad \text{since } \partial_\sigma \text{ is } \mathbb{F}\text{-linear} \\
&= q_1 HX + 2q_2 YX - XH, \quad \text{since } H = -2t\partial_\sigma,\ X = \partial_\sigma \text{ and } Y = -t^2\partial_\sigma
\end{aligned}$$

as required. And for the right-hand side, we have

$$
\begin{aligned}
[H, X]_\sigma &= 2\partial_\sigma(t)\partial_\sigma, \quad \text{by relation (4.2)} \\
&= 2(p_0 + p_1 t + p_2 t^2)\partial_\sigma, \quad \text{since } \partial_\sigma(t) = p_0 + p_1 t + p_2 t^2 \\
&= 2p_0 X - p_1 H - 2p_2 Y, \quad \text{since } H = -2t\partial_\sigma,\ X = \partial_\sigma \text{ and } Y = -t^2\partial_\sigma,
\end{aligned}
$$

as required. Next, we prove relation (4.8) in a similar way. For the left-hand side, we have

$$
\begin{aligned}
[H, Y]_\sigma &= \big[-2t\partial_\sigma, -t^2\partial_\sigma\big]_\sigma = 2\big[t\partial_\sigma, t^2\partial_\sigma\big]_\sigma \\
&= 2\Big(\sigma(t)\partial_\sigma(t^2\partial_\sigma) - \sigma(t^2)\partial_\sigma(t\partial_\sigma)\Big), \quad \text{by (3.1)} \\
&= 2\Big((q_1 t + q_2 t^2)\partial_\sigma(t^2\partial_\sigma) - (q_1 t + q_2 t^2)^2\partial_\sigma(t\partial_\sigma)\Big), \quad \text{since } \sigma(t) = q_1 t + q_2 t^2 \\
&= 2q_1 t\partial_\sigma(t^2\partial_\sigma) + 2q_2 t^2\partial_\sigma(t^2\partial_\sigma) - 2q_1^2 t^2\partial_\sigma(t\partial_\sigma), \quad \text{since } t^3 = 0 \\
&= q_1 HY + 2q_2 Y^2 - q_1^2 YH,
\end{aligned}
$$

as required. And for the right-hand side, we have

$$
\begin{aligned}
[H, Y]_\sigma &= 2\sigma(t)\partial_\sigma(t)t\partial_\sigma, \quad \text{by relation (4.3)} \\
&= 2(q_1 t + q_2 t^2)(p_0 + p_1 t + p_2 t^2)t\partial_\sigma \\
&= 2q_1 p_0 t^2\partial_\sigma = -2q_1 p_0 Y,
\end{aligned}
$$

as required. Finally, we prove relation (4.9). For the left-hand side, we have

$$
\begin{aligned}
[X, Y]_\sigma = -\big[\partial_\sigma, t^2\partial_\sigma\big]_\sigma &= -\Big(\sigma(1)\partial_\sigma(t^2\partial_\sigma) - \sigma(t^2)\partial_\sigma(\partial_\sigma)\Big), \quad \text{by (3.1)} \\
&= -\Big(\partial_\sigma(t^2\partial_\sigma) - (q_1 t + q_q t^2)^2\partial_\sigma(\partial_\sigma)\Big), \quad \text{since } \sigma(t) = q_1 t + q_2 t^2 \\
&= -\partial_\sigma(t^2\partial_\sigma) + q_1^2 t^2\partial_\sigma(\partial_\sigma), \quad \text{since } t^3 = 0 \\
&= XY - q_1^2 YX,
\end{aligned}
$$

as required. And for the right-hand side, we have

$$
\begin{aligned}
[X, Y]_\sigma &= -\big(t + \sigma(t)\big)\partial_\sigma(t)\partial_\sigma, \quad \text{by relation (4.4)} \\
&= -(t + q_1 t + q_2 t^2)(p_0 + p_1 t + p_2 t^2)\partial_\sigma \\
&= -(p_0 t + q_1 p_0 t + p_1 t^2 + q_1 p_1 t^2 + q_2 p_0 t^2)\partial_\sigma \\
&= p_0\frac{H}{2} + q_1 p_0\frac{H}{2} + p_1 Y + q_1 p_1 Y + q_2 p_0 Y \\
&= (q_1 p_1 + q_2 p_0 + p_1)Y + \frac{1}{2}p_0(q_1 + 1)H,
\end{aligned}
$$

as required. ∎

Let us now consider equation (4.6) which is given as

$$p_0\big(q_1^2+q_1+1\big)t^2=0.$$

This equation shows that if $p_0=0$, then q_1 is a true free deformation parameter, and if $p_0\neq 0$, we generate deformations at the imaginary third roots of unity. This means that we have to subdivide our presentation from this point into two special cases.

4.2 Case 1: $p_0 = 0$

For this case we assume that $p_0=0$, and this leads us to the following result.

Corollary 4.2.1. *[27] Let $\mathcal{A}$ be a commutative, associative algebra with unity 1 over a field $\mathbb{F}$ of characteristic 0 and fix an element $t\in\mathcal{A}$. If conditions 4.1 holds where $p_0=0$, and we let σ be an endomorphism on $\mathcal{A}$ and ∂_σ a σ-derivation (twisted derivation) on $\mathcal{A}$. Furthermore, assume that $\sigma(1)=1$ and $\partial_\sigma(1)=0$. Then the $\mathcal{A}$-module elements*

$$H=-2t\partial_\sigma,\quad X=\partial_\sigma,\quad Y=-t^2\partial_\sigma$$

satisfy the following relations:

$$[H,X]_\sigma : q_1HX+2q_2YX-XH=-p_1H-2p_2Y, \tag{4.10}$$
$$[H,Y]_\sigma : q_1HY+2q_2Y^2-q_1^2YH=0, \tag{4.11}$$
$$[X,Y]_\sigma : XY-q_1^2YX=p_1(q_1+1)Y. \tag{4.12}$$

Proof. Take $p_0=0$ in relations (4.7)–(4.9) of Theorem 4.1.3. ■

Note that we cannot retain $\mathfrak{sl}_2(\mathbb{F})$ from this deformation by choosing suitable parameter values. This is because $\mathfrak{sl}_2(\mathbb{F})$ is generated by relations

$$[H,X]=2X,\qquad [H,Y]=-2Y,\qquad [X,Y]=H,$$

but in commutation relations (4.10)–(4.12), the commutator $[X,Y]$ can never be H, the commutator $[H,X]$ can never be $2X$, and the commutator $[H,Y]$ is identically 0.

If $p_1=p_2=0$, $q_1=1$, and $\epsilon=-2q_2$, then relation (4.11) becomes

$$HY-YH=\epsilon Y^2,$$

which is the defining relation for the Jordanian quantum plane or the ϵ-deformed quantum plane (see Cho et al.[12]). After the re-scaling $H\mapsto\epsilon H$, the quadratic algebra with this defining relation becomes one of the two possible Artin–Schelter regular algebras (graded in degree 1) of global dimension 2, while the other is the ordinary quantum plane

$$HY - qYH = 0.$$

For a broader view, see Artin and Schelter [1].

Suppose for this paragraph that $\mathbb{F} = \mathbb{C}$. When $q_1 = 1, q_2 = -\frac{1}{2}$, and $p_1 = p_2 = 0$, then relations (4.10)–(4.12) become

$$HY - YH = Y^2, \qquad HX - XH = YX, \qquad XY - YX = 0,$$

which can be viewed as a kind of "noncommutative deformation in Y-direction" of the algebra of polynomials in two commuting variables indeed recovered for $Y = 0$. Furthermore, if $p_1 = 1, \;\; p_2 = \frac{a}{2}, \;\; q_1 = 1$ and $q_2 = 0$, then relations (4.10)–(4.12) become

$$HY - YH = 0, \qquad HX - XH = -H - aY, \qquad XY - YX = 2Y.$$

These are the defining relations of a 3-dimensional Lie algebra $\mathfrak{g}$. This algebra is solvable for every $a \in \mathbb{C}$. Furthermore, if $p_1 = 0$, $p_2 = -1/2$, $q_1 = 1$ and $q_2 = 0$, then

$$HY - YH = 0, \qquad HX - XH = Y, \qquad XY - YX = 0.$$

These are the defining relations of the Heisenberg Lie algebra. If instead $q_1 = 1$, $q_2 = 0$ and $p_1 = p_2 = 0$ we obtain the 3-dimensional polynomial algebra in three commuting variables:

$$HY - YH = 0, \qquad HX - XH = 0, \qquad XY - YX = 0.$$

In view of the above observations, we conclude that the class of algebras with three generators and defining relations of Corollary 4.2.1, being obtained via a special twisting of $\mathfrak{sl}_2(\mathbb{F})$, is a multiparameter deformation of the polynomial algebra in three commuting variables, of the Heisenberg Lie algebra, and of the solvable Lie algebra.

Twisted Jacobi Identity for Case 1

We now determine the Jacobi identity applicable for the present case. By Proposition 2.3.6, the bracket $[\cdot, \cdot]_\sigma$ on the module $\mathbb{F}[t] \cdot \partial_\sigma$ satisfies the 6-term deformed Jacobi identity

$$\circlearrowleft_{x,y,z} \Big(\big[\sigma(x), [y, z]_\sigma\big]_\sigma + \delta\big[x, [y, z]_\sigma\big]_\sigma \Big) = 0. \tag{4.13}$$

We need to find the possible values of $\delta \in \mathcal{A} = \mathbb{F}[t]/(t^3)$ in this deformed Jacobi identity. By Lemma 3.2.4, the identity

$$\partial_\sigma\big(\sigma(a)\big) = \delta \cdot \sigma\big(\partial_\sigma(a)\big)$$

holds for all monomials $a = t^k$, where k is a non-negative integer. In particular, for $a = t$,

$$\begin{aligned}
\text{L.H.S} &= \partial_\sigma(\sigma(t)) = \partial_\sigma\big(q_1 t + q_2 t^2\big) \\
&= q_1\partial_\sigma(t) + q_2\partial_\sigma(t^2) \\
&= q_1\partial_\sigma(t) + q_2\big(\partial_\sigma(t)t + \sigma(t)\partial_\sigma(t)\big) \\
&= q_1(p_1 t + p_2 t^2) + q_2\big((q_1 t + q_2 t^2)(p_1 t + p_2 t^2) + t(p_1 t + p_2 t^2)\big) \\
&= p_1 q_1 t + p_2 q_1 t^2 + p_1 q_1 q_2 t^2 + p_1 q_2 t^2 \\
&= p_1 q_1 t + (p_2 q_1 + p_1 q_1 q_2 + p_1 q_2)t^2,
\end{aligned}$$

$$\begin{aligned}
\text{R.H.S} &= \delta \cdot \sigma(\partial_\sigma(t)) = \delta \cdot \sigma(p_1 t + p_2 t^2) \\
&= \delta \cdot \big(p_1\sigma(t) + p_2\sigma(t^2)\big) \\
&= \delta \cdot \big(p_1(q_1 t + q_2 t^2) + p_2(q_1 t + q_2 t^2)^2\big) \\
&= \delta \cdot \Big(p_1 q_1 t + \big(p_1 q_2 + p_2 q_1^2\big)t^2\Big),
\end{aligned}$$

which implies that

$$\delta \cdot \Big(p_1 q_1 t + \big(p_1 q_2 + p_2 q_1^2\big)t^2\Big) = p_1 q_1 t + (p_2 q_1 + p_1 q_1 q_2 + p_1 q_2)t^2. \tag{4.14}$$

Since $\delta \in \mathcal{A} = \mathbb{F}[t]/(t^3)$, we can write $\delta = \delta_0 + \delta_1 t + \delta_2 t^2$ for some $\delta_0, \delta_1, \delta_2 \in \mathbb{F}$. Furthermore, we let ξ_0, ξ_1 and ξ_2 denote three free parameters. Equation (4.14) then becomes

$$\big(\delta_0 + \delta_1 t + \delta_2 t^2\big)\big(p_1 q_1 t + (p_1 q_2 + p_2 q_1^2)t^2\big) = p_1 q_1 t + (p_2 q_1 + p_1 q_1 q_2 + p_1 q_2)t^2.$$

Since $t^3 = 0$, this equation can be simplified to the equation

$$p_1 q_1 \delta_0 t + \big((p_1 q_2 + p_2 q_1^2)\delta_0 + p_1 q_1 \delta_1\big)t^2 = p_1 q_1 t + (p_2 q_1 + p_1 q_1 q_2 + p_1 q_2)t^2.$$

Equating the coefficients of the same powers of t and rearranging gives the following linear system of equations for δ_0, δ_1 and δ_2:

$$\begin{cases}
p_1 q_1 \delta_0 - p_1 q_1 = 0, \\
(p_1 q_2 + p_2 q_1^2)\delta_0 + p_1 q_1 \delta_1 - p_2 q_1 - p_1 q_1 q_2 - p_1 q_2 = 0.
\end{cases} \tag{4.15}$$

Since δ_2 is not involved at all, it can be chosen arbitrarily, say $\delta_2 = \xi_2$. We now have several cases to consider:

(1) If $p_1 q_1 \neq 0$, then the first equation of system (4.15) implies that

$$\delta_0 - 1 = 0,$$

giving $\delta_0 = 1$. Substituting $\delta_0 = 1$ in the second equation of system (4.15) gives

$$\begin{aligned}
&p_1 q_2 + p_2 q_1^2 + p_1 q_1 \delta_1 - p_2 q_1 - p_1 q_1 q_2 - p_1 q_2 = 0 \\
&p_2 q_1^2 + p_1 q_1 \delta_1 - p_2 q_1 - p_1 q_1 q_2 = 0 \\
&p_1 q_1 \delta_1 = p_1 q_1 q_2 + p_2 q_1 - p_2 q_1^2 \\
&\delta_1 = \frac{p_2 + q_2 p_1 - p_2 q_1}{p_1}.
\end{aligned}$$

With these values of δ_0, δ_1 and δ_2, we have

$$\begin{aligned}
\delta &= \delta_0 + \delta_1 t + \delta_2 t^2 \\
&= 1 + \frac{p_2 + q_2 p_1 - p_2 q_1}{p_1} t + \xi_2 t^2.
\end{aligned}$$

Substituting this δ in equation (4.19) gives the Jacobi identity

$$\circlearrowleft_{x,y,z} \left(\big[\sigma(x), [y, z]\big] + \left(1 + \frac{p_2 + p_1 q_2 - p_2 q_1}{p_1} t + \xi_2 t^2\right) \big[x, [y, z]\big] \right) = 0.$$

Therefore, for the case $p_1 q_1 \neq 0$, the deformed $\mathfrak{sl}_2(\mathbb{F})$ is a quasi-hom-Lie algebra.

(2) If $p_1 \neq 0$, $q_1 = 0$ and $q_2 \neq 0$, then the first equation of system (4.15) vanishes while the second equation becomes

$$p_1 q_2 \delta_0 - p_1 q_2 = 0,$$

which implies that $\delta_0 = 1$. Since δ_1 and δ_2 are not involved at all, they can be chosen arbitrarily, say $\delta_1 = \xi_1$ and $\delta_2 = \xi_2$. With these values of δ_0, δ_1 and δ_2, we have

$$\begin{aligned}
\delta &= \delta_0 + \delta_1 t + \delta_2 t^2 \\
&= 1 + \xi_1 t + \xi_2 t^2.
\end{aligned}$$

Substituting this δ in equation (4.19) gives the Jacobi identity

$$\circlearrowleft_{x,y,z} \left(\big[\sigma(x), [y, z]\big] + \left(1 + \xi_1 t + \xi_2 t^2\right) \big[x, [y, z]\big] \right) = 0.$$

Now ξ_1 and ξ_2 can be chosen completely arbitrarily in this case, for example, $\xi_1 = 0$ and $\xi_2 = 0$, giving the hom-Jacobi identity

$$\circlearrowleft_{x,y,z} \big[(\sigma + \mathrm{id})(x), [y, z]\big] = 0.$$

Therefore, for the case $p_1 \neq 0$, $q_1 = 0$, $q_2 \neq 0$, the deformed $\mathfrak{sl}_2(\mathbb{F})$ is a hom-Lie algebra.

(3) If $p_1 \neq 0$ and $q_1 = q_2 = 0$, then both first and second equations of system (4.15) vanish. Since δ_1, δ_1 and δ_2 are not involved at all, they can be chosen arbitrarily, say $\delta_0 = \xi_0$, $\delta_1 = \xi_1$ and $\delta_2 = \xi_2$. With these values of δ_0, δ_1 and δ_2, we have

$$\begin{aligned} \delta &= \delta_0 + \delta_1 t + \delta_2 t^2 \\ &= \xi_0 + \xi_1 t + \xi_2 t^2. \end{aligned}$$

Substituting this δ in equation (4.19) gives the Jacobi identity

$$\circlearrowleft_{x,y,z} \Big(\big[\sigma(x), [y, z]\big] + \big(\xi_0 + \xi_1 t + \xi_2 t^2\big)\big[x, [y, z]\big]\Big) = 0$$

Now ξ_0, ξ_1 and ξ_2 can be chosen completely arbitrarily in this case, for example, $\xi_0 = 0$, $\xi_1 = 0$ and $\xi_2 = 0$, giving the hom-Jacobi identity

$$\circlearrowleft_{x,y,z} \big[(\sigma(x), [y, z]\big] = 0.$$

Therefore, for the case $p_1 \neq 0$, $q_1 = q_2 = 0$, the deformed $\mathfrak{sl}_2(\mathbb{F})$ is a hom-Lie algebra.

(4) If $p_1 = 0$, $p_2 \neq 0$ and $q_1 \neq 0$, then the first equation of system (4.15) vanishes while the second equation becomes

$$p_2 q_1^2 \delta_0 - p_2 q_1 = 0,$$

which implies that $\delta_0 = q_1^{-1}$. Since δ_1 and δ_2 are not involved at all, they can be chosen arbitrarily, say $\delta_1 = \xi_1$ and $\delta_2 = \xi_2$. With these values of δ_0, δ_1 and δ_2, we have

$$\begin{aligned} \delta &= \delta_0 + \delta_1 t + \delta_2 t^2 \\ &= q_1^{-1} + \xi_1 t + \xi_2 t^2. \end{aligned}$$

Substituting this δ in equation (4.19) gives the Jacobi identity

$$\circlearrowleft_{x,y,z} \Big(\big[\sigma(x), [y, z]\big] + \big(q_1^{-1} + \xi_1 t + \xi_2 t^2\big)\big[x, [y, z]\big]\Big) = 0.$$

Now ξ_1 and ξ_2 can be chosen completely arbitrarily in this case, for example, $\xi_1 = 0$

and $\xi_2 = 0$, giving the hom-Jacobi identity

$$\circlearrowleft_{x,y,z} \Big(\big[\sigma(x), [y,z]\big] + q_1^{-1} \big[x, [y,z]\big] \Big) = 0.$$

Therefore, for the case $p_1 = 0$, $p_2 \neq 0$, $q_1 \neq 0$, the deformed $\mathfrak{sl}_2(\mathbb{F})$ is a quasi-hom-Lie algebra. However, Corollary 3.2.5 implies that we also have the Jacobi identity

$$\circlearrowleft_{x,y,z} \big[(\alpha + \mathrm{id})(x), [y,z]_\sigma\big]_\sigma = 0,$$

where $\alpha(x) = q_1\sigma(x)$. So the deformed $\mathfrak{sl}_2(\mathbb{F})$ can also be viewed as a hom-Lie algebra.

(5) If $p_1 = p_2 = 0$ or $p_1 = q_1 = 0$, then both first and second equations of system (4.15) vanish. Since δ_1, δ_1 and δ_2 are not involved at all, they can be chosen arbitrarily, say $\delta_0 = \xi_0$, $\delta_1 = \xi_1$ and $\delta_2 = \xi_2$. With these values of δ_0, δ_1 and δ_2, we have

$$\delta = \xi_0 + \xi_1 t + \xi_2 t^2.$$

Substituting this δ in equation (4.19) gives the Jacobi identity

$$\circlearrowleft_{x,y,z} \Big(\big[\sigma(x), [y,z]\big] + \big(\xi_0 + \xi_1 t + \xi_2 t^2\big) \big[x, [y,z]\big] \Big) = 0$$

Now ξ_0, ξ_1 and ξ_2 can be chosen completely arbitrarily in this case, for example, $\xi_0 = 0$, $\xi_1 = 0$ and $\xi_2 = 0$, giving the hom-Jacobi identity

$$\circlearrowleft_{x,y,z} \big[(\sigma(x), [y,z]\big] = 0.$$

Therefore, when $p_1 = p_2 = 0$ or $p_1 = q_1 = 0$, the deformed $\mathfrak{sl}_2(\mathbb{F})$ is a hom-Lie algebra.

4.3 Case 2: $p_0 \neq 0$

For this case we assume that $p_0 \neq 0$. Equation (4.6), given as

$$p_0\big(q_1^2 + q_1 + 1\big)t^2 = 0,$$

then implies that $q_1^2 + q_1 + 1 = 0$, giving the roots $q_1 = \omega^k \neq 1$, where $\omega = e^{\frac{2\pi}{3}i}$ is a third root of unity. This leads us to the following result.

Corollary 4.3.1. *[27] Let $\mathcal{A}$ be a commutative, associative algebra with unity 1 over a field $\mathbb{F}$ of characteristic 0 and fix an element $t \in \mathcal{A}$. If conditions 4.1 holds where $p_0 \neq 0$, and we let σ be an endomorphism on $\mathcal{A}$ and ∂_σ a σ-derivation (twisted derivation) on $\mathcal{A}$.*

Furthermore, assume that $\sigma(1) = 1$ *and* $\partial_\sigma(1) = 0$. *Then the* $\mathcal{A}$*-module elements*

$$H = -2t\partial_\sigma, \quad X = \partial_\sigma, \quad Y = -t^2\partial_\sigma$$

satisfy the following relations:

$$[H, X]_\sigma : \omega^k HX + 2q_2 YX - XH = 2p_0 X - p_1 H - 2p_2 Y, \tag{4.16}$$

$$[H, Y]_\sigma : \omega^k HY + 2q_2 Y^2 - \omega^{2k} YH = -2\omega^k p_0 Y, \tag{4.17}$$

$$[X, Y]_\sigma : XY - \omega^{2k} YX = \Big((1 + \omega^k)p_1 + q_2 p_0\Big)Y + \frac{1}{2}p_0(\omega^k + 1)H. \tag{4.18}$$

Proof of Corollary 4.3.1. Put $q_1 = \omega^k$ in relations (4.7)–(4.9) of Theorem 4.1.3. ■

Remark 4.3.2. When $p_1 = p_2 = q_2 = 0$ and $\omega^k = q$, then we retain relations (3.28)–(3.30):

$$\begin{aligned} HX - q^{-1}XH &= 2q^{-1}p_0 X, \\ HY - qYH &= -2p_0 Y, \\ XY - q^2 YX &= \frac{(q+1)}{2} p_0 H. \end{aligned}$$

These are the defining relations for the Jackson $\mathfrak{sl}_2(\mathbb{F})$ (see page 37 of Chapter 3).

Twisted Jacobi identity for case 2

We now determine the Jacobi identity applicable for the present case. By Proposition 2.3.6, the bracket $[\cdot, \cdot]_\sigma$ on the module $\mathbb{F}[t] \cdot \partial_\sigma$ satisfies the 6-term deformed Jacobi identity

$$\circlearrowleft_{x,y,z} \Big(\big[\sigma(x), [y, z]_\sigma\big]_\sigma + \delta\big[x, [y, z]_\sigma\big]_\sigma \Big) = 0. \tag{4.19}$$

We need to find the possible values of $\delta \in \mathcal{A} = \mathbb{F}[t]/(t^3)$ in this deformed Jacobi identity. By Lemma 3.2.4, the identity

$$\partial_\sigma\big(\sigma(a)\big) = \delta \cdot \sigma\big(\partial_\sigma(a)\big)$$

holds for all monomials $a = t^k$, where k is a non-negative integer. In particular, for $a = t$,

$$\begin{aligned} \text{L.H.S} &= \partial_\sigma(\sigma(t)) = \partial_\sigma(\omega^k t + q_2 t^2) \\ &= \omega^k \partial_\sigma(t) + q_2 \partial_\sigma(t^2) \\ &= \omega^k \partial_\sigma(t) + q_2 \partial_\sigma(t)t + q_2 \sigma(t)\partial_\sigma(t) \\ &= \omega^k \big(p_0 + p_1 t + p_2 t^2\big) + q_2 \Big((\omega^k t + q_2 t^2)\big(p_0 + p_1 t + p_2 t^2\big) \Big) \\ &\quad + t q_2 \big(p_0 + p_1 t + p_2 t^2\big) \end{aligned}$$

$$= \omega^k p_0 + \Big(\omega^k p_1 + q_2 p_0 \omega^k + p_0 q_2\Big)t + \Big(\omega^k p_2 + p_1 q_2 \omega^k + p_0 q_2^2 + p_1 q_2\Big)t^2,$$

$$\begin{aligned}
\text{R.H.S} &= \delta \cdot \sigma(\partial_\sigma(t)) = \delta \cdot \sigma\big(p_0 + p_1 t + p_2 t^2\big) \\
&= \delta \cdot \big(p_0 + p_1\sigma(t) + p_2\sigma(t)^2\big) \\
&= \delta \cdot (p_0 + p_1)\Big(\omega^k t + q_2 t^2\Big) + p_2\Big(\omega^k t + q_2 t^2\Big)^2 \\
&= \delta \cdot \Big(p_0 + p_1\omega^k t + p_1 q_2 t^2 + p_2 \omega^{2k} t^2\Big),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\delta \cdot \Big(p_0 + p_1\omega^k t + p_1 q_2 t^2 + p_2 \omega^{2k} t^2\Big) \\
&\quad = \omega^k p_0 + \Big(\omega^k p_1 + q_2 p_0 \omega^k + p_0 q_2\Big)t + \Big(\omega^k p_2 + p_0 q_2 \omega^k + p_0 q_2^2 + p_1 q_2\Big)t^2. \qquad (4.20)
\end{aligned}$$

Since $\delta \in \mathcal{A} = \mathbb{F}[t]/(t^3)$, we can write $\delta = \delta_0 + \delta_1 t + \delta_2 t^2$ for some $\delta_0, \delta_1, \delta_2 \in \mathbb{F}$. Furthermore, we let ξ_0, ξ_1 and ξ_2 denote three free parameters. Equation (4.20) then becomes

$$\begin{aligned}
&\big(\delta_0 + \delta_1 t + \delta_2 t^2\big)\Big(p_0 + p_1\omega^k t + p_1 q_2 t^2 + p_2\omega^{2k} t^2\Big) \\
&\quad = \omega^k p_0 + \Big(\omega^k p_1 + q_2 p_0 \omega^k + p_0 q_2\Big)t + \Big(\omega^k p_2 + p_0 q_2 \omega^k + p_0 q_2^2 + p_1 q_2\Big)t^2.
\end{aligned}$$

Since $t^3 = 0$, this equation can be simplified as follows:

$$\begin{aligned}
&\delta_0 p_0 + \Big(\delta_0 p_1 \omega^k + \delta_1 p_0\Big)t + \Big(\delta_0 p_1 q_2 + \delta_0 p_2 \omega^{2k} + \delta_1 p_1 \omega^k + \delta_2 p_0\Big)t^2 \\
&\quad = \omega^k p_0 + \Big(\omega^k p_1 + q_2 p_0 \omega^k + p_0 q_2\Big)t + \Big(\omega^k p_2 + p_1 q_2 \omega^k + p_0 q_2^2 + p_1 q_2\Big)t^2.
\end{aligned}$$

Equating the coefficients of the same powers of t and rearranging gives the following linear system of equations for δ_0, δ_1 and δ_2:

$$\begin{cases}
p_0\delta_0 = \omega^k p_0 \\
\delta_0 p_1 \omega^k + \delta_1 p_0 = p_0 q_2 + p_0 q_2 \omega^k + \omega^k p_1 \\
\delta_0 p_1 q_2 + \delta_0 p_2 \omega^{2k} + \delta_1 p_1 \omega^k + \delta_2 p_0 = p_0 q_2^2 + p_1 q_2 + \omega^k p_2 + p_1 q_2 \omega^k.
\end{cases} \qquad (4.21)$$

Since $p \neq 0$, the first equation of system (4.21) implies that $\delta_0 = \omega^k$. Substituting $\delta_0 = \omega^k$ in the second equation of system (4.21) gives

$$\begin{aligned}
p_1\omega^{2k} + \delta_1 p_0 &= p_0 q_2 + p_0 q_2 \omega^k + \omega^k p_1 \\
\delta_1 p_0 &= p_0 q_2 + p_0 q_2 \omega^k + \omega^k p_1 - p_1 \omega^{2k} \\
\delta_1 &= \frac{p_0 q_2 + (p_0 q_2 + p_1)\omega^k - p_1\omega^{2k}}{p_0}.
\end{aligned}$$

Substituting these obtained values of δ_0 and δ_1 in the third equation of system (4.21), and remembering that $t^3 = 0$ and $\omega^3 = 1$, gives

$$\begin{aligned} p_0q_2^2 + p_1q_2 + \omega^k p_2 + p_1q_2\omega^k &= \omega^k p_1q_2 + \omega^k p_2\omega^{2k} + \Big(\frac{p_0q_2 + (p_0q_2 + p_1)\omega^k - p_1\omega^{2k}}{p_0}\Big)p_1\omega^k + \delta_2 p_0 \\ &= \omega^k p_1q_2 + p_2\omega^{3k} + \frac{p_0p_1q_2\omega^k + (p_0p_1q_2 + p_1^2)\omega^{2k} - p_1^2\omega^{3k}}{p_0} + \delta_2 p_0 \\ &= \omega^k p_1q_2 + p_2 + \frac{p_0p_1q_2\omega^k + (p_0p_1q_2 + p_1^2)\omega^{2k} - p_1^2}{p_0} + \delta_2 p_0. \end{aligned}$$

Solving for δ_2, we have

$$\begin{aligned} \delta_2 p_0 &= \omega^k p_2 + p_1q_2\omega^k + p_0q_2^2 + p_1q_2 - p_1q_2\omega^k - p_2 - \left(\frac{p_0p_1q_2\omega^k + (p_0p_1q_2 + p_1^2)\omega^{2k} - p_1^2}{p_0}\right) \\ \delta_2 p_0^2 &= \omega^k p_0p_2 + p_0p_1q_2\omega^k + p_0^2q_2^2 + p_0p_1q_2 - p_0p_1q_2\omega^k - p_0p_2 - p_0p_1q_2\omega^k - (p_0p_1q_2 + p_1^2)\omega^{2k} + p_1^2 \\ \delta_2 p_0^2 &= p_0p_1q_2 - p_0p_2 + p_0^2q_2^2 + p_1^2 + (p_0p_2 - p_0p_1q_2)\omega^k - (p_0p_1q_2 + p_1^2)\omega^{2k} \\ \delta_2 &= \frac{p_0p_1q_2 - p_0p_2 + p_0^2q_2^2 + p_1^2 + (p_0p_2 - p_0p_1q_2)\omega^k - (p_0p_1q_2 + p_1^2)\omega^{2k}}{p_0^2}. \end{aligned}$$

In summary, the values of the parameters δ_0, δ_1 and δ_2 that we have obtained are:

$$\begin{pmatrix} \delta_0 \\ \delta_1 \\ \delta_2 \end{pmatrix} = \begin{pmatrix} \omega^k \\ \frac{p_0q_2+(p_0q_2+p_1)\omega^k - p_1\omega^{2k}}{p_0} \\ \frac{p_0p_1q_2 - p_0p_2 + p_0^2q_2^2 + p_1^2 + (p_0p_2 - p_0p_1q_2)\omega^k - (p_0p_1q_2+p_1^2)\omega^{2k}}{p_0^2} \end{pmatrix} = \begin{pmatrix} \omega^k \\ \mathbf{w_1} \\ \mathbf{w_2} \end{pmatrix}$$

With these values of δ_0, δ_1 and δ_2, we have

$$\begin{aligned} \delta &= \delta_0 + \delta_1 t + \delta_2 t^2 \\ &= \omega^k + \mathbf{w_1} t + \mathbf{w_2} t^2 \end{aligned}$$

Substituting this δ in equation (4.19) gives the Jacobi identity

$$\circlearrowleft_{x,y,z} \left(\big[\sigma(x), [y,z]\big] + \Big(\omega^k + \mathbf{w_1} t + \mathbf{w_2} t^2\Big)\big[x, [y,z]\big] \right) = 0$$

which corresponds to a quasi-hom-Lie algebra. This becomes a hom-Lie algebra when $\mathbf{w_1} = \mathbf{w_2} = 0$, that is, for some special choices of the parameters defining $\sigma(t)$ and $\partial_\sigma(t)$.

Chapter 5

Deformations of $\mathfrak{sl}_2(\mathbb{F})$ with Base $\mathbb{F}[t]/(t^4)$

In this chapter we obtain quasi-deformations of $\mathfrak{sl}_2(\mathbb{F})$, where $\mathbb{F}$ is a field of characteristic 0 that includes all the fourth roots of unity (the solutions to the equation $z^4 = 1$). We take the base algebra $\mathcal{A}$ to be $\mathbb{F}[t]/(t^4)$, the algebra of polynomials of degree less than 4 over $\mathbb{F}$ in the variable t. In Section 5.1, we take the base algebra $\mathcal{A} = \mathbb{F}[t]/(t^4)$ to be as general as it can be in this situation and deduce some necessary conditions for everything to make sense. One special case of this deformation is considered in Section 5.2 while another special case is considered in Section 5.3.

5.1 General Case and the Necessary Conditions

Let $\mathbb{F}$ be a field of characteristic 0 that includes all the fourth roots of unity and take the base algebra $\mathcal{A}$ to be the algebra $\mathbb{F}[t]/(t^4)$. This is obviously a 4-dimensional $\mathbb{F}$-vector space and a finitely generated $\mathbb{F}[t]$-module with basis $\{1, t, t^2, t^3\}$. As before, we take

$$H = -2t\partial_\sigma, \quad X = \partial_\sigma, \quad Y = -t^2\partial_\sigma.$$

In addition, we introduce the fourth basis element $R = 2t^3\partial_\sigma$. We observe from this that

$$-2t \cdot X = H, \qquad t \cdot H = 2Y, \qquad -2t \cdot Y = R.$$

Since we are considering the base algebra $\mathbb{F}[t]/(t^4)$, we take

$$\partial_\sigma(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3 \quad \text{and} \quad \sigma(t) = q_0 + q_1 t + q_2 t^2 + q_3 t^3. \tag{5.1}$$

As in the previous chapters, we assume that $\sigma(1) = 1$ and $\partial_\sigma(1) = 0$. Consequently, the following relations given in Theorem 3.1.1 still hold:

$$[H, X]_\sigma = 2\partial_\sigma(t)\partial_\sigma, \tag{5.2}$$

$$[H, Y]_\sigma = 2\sigma(t)\partial_\sigma(t)t\partial_\sigma, \tag{5.3}$$

$$[X, Y]_\sigma = -\big(t + \sigma(t)\big)\partial_\sigma(t)\partial_\sigma. \tag{5.4}$$

We will also make use of the following identity from Proposition 2.3.6:

$$[a \cdot \partial_\sigma, b \cdot \partial_\sigma]_\sigma = \Big(\sigma(a)\partial_\sigma(b) - \sigma(b)\partial_\sigma(a)\Big)\partial_\sigma. \tag{5.5}$$

Now equations (5.1) have to be compatible with our assumption of $t^4 = 0$. This means that

$$\begin{aligned}
0 = \sigma(0) = \sigma(t^4), &\quad \text{since } t^4 = 0 \\
&= \sigma(t)\sigma(t)\sigma(t)\sigma(t), \quad \text{since } \sigma \text{ is an endormophism} \\
&= \sigma^4(t) \\
&= \big(q_0 + q_1 t + q_2 t^2 + q_3 t^3\big)^4 \\
&= q_0^4 + 4q_0^3 q_1 t + \big(6q_0^2 q_1^2 + 4q_0^3 q_2\big)t^2 + \big(4q_0 q_1^2 + 12q_0^2 q_1 q_2 + 4q_0^3 q_3\big)t^3, \qquad (5.6)
\end{aligned}$$

implying that $q_0 = 0$, and thus, $\sigma(t)$ becomes $\sigma(t) = q_1 t + q_2 t^2 + q_3 t^3$. Furthermore, we have

$$\begin{aligned}
\partial_\sigma(t^2) = \partial_\sigma(tt) &= \partial_\sigma(t)t + \sigma(t)\partial_\sigma(t) \\
&= (p_0 + p_1 t + p_2 t^2 + p_3 t^3)t + (q_1 t + q_2 t^2 + q_3 t^3)(p_0 + p_1 t + p_2 t^2 + p_3 t^3) \\
&= p_0 t + p_1 t^2 + p_2 t^3 + p_0 q_1 t + p_1 q_1 t^2 + p_2 q_1 t^3 + p_0 q_2 t^2 + p_1 q_2 t^3 + p_0 q_3 t^3 \\
&= (p_0 + p_0 q_1)t + (p_1 + p_1 q_1 + p_0 q_2)t^2 + (p_2 + p_2 q_1 + p_1 q_2 + p_0 q_3)t^3 \qquad (5.7)
\end{aligned}$$

and

$$\begin{aligned}
\partial_\sigma(t^3) &= \partial_\sigma(tt^2) \\
&= \partial_\sigma(t)t^2 + \sigma(t)\partial_\sigma(t^2), \quad \text{since } \partial_\sigma \text{ is a } \sigma\text{-derivation} \\
&= (p_0 + p_1 t + p_2 t^2 + p_3 t^3)t^2 + (q_1 t + q_2 t^2 + q_3 t^3)\Big((p_0 + p_0 q_1)t + (p_1 + p_1 q_1 + p_0 q_2)t^2 \\
&\qquad + (p_2 + p_2 q_1 + p_1 q_2 + p_0 q_3)t^3\Big) \\
&= p_0 t^2 + p_1 t^3 + p_0 q_1(1 + q_1)t^2 + q_1(p_1 + p_1 q_1 + p_0 q_2)t^3 + p_0 q_2(1 + q_1)t^3 \\
&= p_0 t^2 + p_0 q_1 t^2 + p_0 q_1^2 t^2 + p_1 t^3 + p_1 q_1 t^3 + p_1 q_1^2 t^3 + p_0 q_2 q_1 t^3 + p_0 q_2 t^3 + p_0 q_1 q_2 t^3 \\
&= p_0\big(1 + q_1 + q_1^2\big)t^2 + \Big(p_1(1 + q_1 + q_1^2) + p_0(q_2 + 2q_1 q_2)\Big)t^3. \qquad (5.8)
\end{aligned}$$

Finally, we observe that

$$\begin{aligned}
0 = \partial_\sigma(0) = \partial_\sigma(t^4), &\quad \text{since } t^4 = 0 \\
&= \partial_\sigma(tt^3) \\
&= \partial_\sigma(t)t^3 + \sigma(t)\partial_\sigma(t^3), \quad \text{since } \partial_\sigma \text{ is a } \sigma\text{-derivation} \\
&= (p_0 + p_1 t + p_2 t^2 + p_3 t^3)t^3 + (q_1 t + q_2 t^2 + q_3 t^3)\Big(p_0\big(1 + q_1 + q_1^2\big)t^2 \\
&\qquad + \Big(p_1(1 + q_1 + q_1^2) + p_0(q_1 q_2 + q_2 + q_1 q_2)\Big)t^3\Big) \\
&= p_0 t^3 + p_0 q_1(1 + q_1 + q_1^2)t^3 \\
&= p_0(1 + q_1 + q_1^2 + q_1^3)t^3. \qquad (5.9)
\end{aligned}$$

Let us now consider this equation (5.9) which is given as

$$p_0(1 + q_1 + q_1^2 + q_1^3)t^3 = 0.$$

This equation shows that if $p_0 = 0$, then q_1 is a true free deformation parameter, and if $p_0 \neq 0$, we generate deformations at the zeros of the polynomial $1 + q_1 + q_1^2 + q_1^3$.

Moreover, the introduction of the new generator R means that we obtain three additional relations using identity (5.5). This leads us to the following result.

Theorem 5.1.1. In addition to satisfying relations (5.2)–(5.4), the $\mathcal{A}$-module elements

$$H = -2t\partial_\sigma, \quad X = \partial_\sigma, \quad Y = -t^2\partial_\sigma, \quad R = 2t^3\partial_\sigma$$

satisfy the following relations:

$$[H, R]_\sigma = -4\big(\sigma(t) + t\big)t\sigma(t)\partial_\sigma(t)\partial_\sigma, \tag{5.10}$$

$$[X, R]_\sigma = 2(t^2 + \sigma(t)t + \sigma(t)^2)\partial_\sigma(t)\partial_\sigma, \tag{5.11}$$

$$[Y, R]_\sigma = -2t^2\sigma(t)^2\partial_\sigma(t)\partial_\sigma. \tag{5.12}$$

Proof. For the first relation, we have

$$\begin{aligned}
[H, R]_\sigma &= \big[-2t\partial_\sigma, 2t^3\partial_\sigma\big]_\sigma \\
&= -4\big[t\partial_\sigma, t^3\partial_\sigma\big]_\sigma, \quad \text{since the bracket } [\cdot,\cdot]_\sigma \text{ is bilinear} \\
&= -4\Big(\sigma(t)\partial_\sigma(t^3) - \sigma(t^3)\partial_\sigma(t)\Big)\partial_\sigma, \quad \text{by identity (3.4)} \\
&= -4\Big(\sigma(t)\big(\sigma(t^2) + t\sigma(t) + t^2\big)\partial_\sigma(t)) - \sigma(t^3)\partial_\sigma(t)\Big)\partial_\sigma \\
&= -4\Big(\sigma(t^3)\partial_\sigma(t) + t\sigma^2(t)\partial_\sigma(t) + \sigma(t)t^2\partial_\sigma(t) - \sigma(t^3)\partial_\sigma(t)\Big)\partial_\sigma \\
&= -4\Big(t\sigma^2(t)\partial_\sigma(t) + \sigma(t)t^2\partial_\sigma(t)\Big)\partial_\sigma \\
&= -4\big(\sigma(t) + t\big)t\sigma(t)\partial_\sigma(t)\partial_\sigma,
\end{aligned}$$

as required. For the second relation, we have

$$\begin{aligned}
[X, R]_\sigma &= \big[\partial_\sigma, 2t^3\partial_\sigma\big]_\sigma \\
&= 2\big[\partial_\sigma, t^3\partial_\sigma\big]_\sigma, \quad \text{since the bracket } [\cdot,\cdot]_\sigma \text{ is bilinear} \\
&= 2\Big(\sigma(1)\partial_\sigma(t^3) - \sigma(t^3)\partial_\sigma(1)\Big)\partial_\sigma, \quad \text{by identity (3.4)} \\
&= 2\partial_\sigma(t^3)\partial_\sigma, \quad \text{since } \sigma(1) = 1 \text{ and } \partial_\sigma(1) = 0 \\
&= 2\Big(\partial_\sigma(t)t^2 + \sigma(t)\big(\partial_\sigma(t)t + \sigma(t)\partial_\sigma(t)\big)\Big)\partial_\sigma \\
&= 2\big(t^2 + t\sigma(t) + \sigma^2(t)\big)\partial_\sigma(t)\partial_\sigma,
\end{aligned}$$

as required. For the third relation, we have

$$\begin{aligned}
[Y,R]_\sigma &= \big[-t^2\partial_\sigma, 2t^3\partial_\sigma\big]_\sigma \\
&= -2\big[t^2\partial_\sigma, t^3\partial_\sigma\big]_\sigma, \quad \text{since the bracket } [\cdot,\cdot]_\sigma \text{ is bilinear} \\
&= -2\Big(\sigma(t^2)\partial_\sigma(t^3) - \sigma(t^3)\partial_\sigma(t^2)\Big)\partial_\sigma, \quad \text{by identity (3.4)} \\
&= -2\Big(\sigma(t^2)(\sigma(t^2) + t\sigma(t) + t^2)\partial_\sigma(t)) - \sigma(t^3)\big(\partial_\sigma(t)t + \sigma(t)\partial_\sigma(t)\big)\Big)\partial_\sigma \\
&= -2\Big(\sigma^4(t)\partial_\sigma(t) + t\sigma(t^3)\partial_\sigma(t) + \sigma^2(t)t^2)\partial_\sigma(t) - \sigma(t^3)\partial_\sigma(t)t - \sigma^4(t)\partial_\sigma(t)\Big)\partial_\sigma \\
&= -2t^2\sigma(t)^2\partial_\sigma(t)\partial_\sigma,
\end{aligned}$$

as required. ∎

We also have the following result.

Proposition 5.1.2. *The action of the module $\mathcal{A}\cdot\partial_\sigma$ on $\mathcal{A} = \mathbb{F}[t]/(t^4)$ has the following matrix representation with respect to the basis $\{1, t, t^2, t^3\}$:*

$$H \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2p_0 & 0 & 0 \\ 0 & -2p_1 & -2p_0(1+q_1) & 0 \\ 0 & -2p_2 & -2(p_1(q_1+1)+p_0q_2) & -2p_0(1+q_1+q_1^2) \end{pmatrix},$$

$$X \mapsto \begin{pmatrix} 0 & p_0 & 0 & 0 \\ 0 & p_1 & p_0(1+q_1) & 0 \\ 0 & p_2 & p_1(1+q_1)+p_0q_2 & p_0(1+q_1+q_1^2) \\ 0 & p_3 & p_2(1+q_1)+p_1q_2+p_0q_3 & p_1(1+q_1+q_1^2)+p_0q_2(1+2q_1) \end{pmatrix},$$

$$Y \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -p_0 & 0 & 0 \\ 0 & -p_1 & -p_0(1+q_1) & 0 \end{pmatrix},$$

$$R \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2p_0 & 0, & 0 \end{pmatrix}.$$

Proof. Let H_1, H_2, H_3 and H_4 be column vectors of the matrix H. Since $\mathbb{F}[t]/(t^4)$ is a finitely generated $\mathbb{F}[t]$-module with basis $\{1, t, t^2, t^3\}$, we have

$$\begin{aligned}
\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix} H_1 &= -2t\partial_\sigma(1) = 0,\\
\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix} H_2 &= -2t\partial_\sigma(t) = -2t(p_0 + p_1 t + p_2 t^2 + p_3 t^3) = -2p_0 t - 2p_1 t^2 - 2p_2 t^3,\\
\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix} H_3 &= -2t\partial_\sigma(t^2) = -2t\Big((p_0 + p_0 q_1)t + (p_1 + p_1 q_1 + p_0 q_2)t^2\\
&\qquad + (p_2 + p_2 q_1 + p_1 q_2 + p_0 q_3)t^3\Big) = -2p_0(1+q_1)t^2 - 2\big(p_1(1+q_1) + p_0 q_2\big)t^3,\\
\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix} H_4 &= -2t\partial_\sigma(t^3) = -2t\Big(p_0\big(1 + q_1 + q_1^2\big)t^2 + \Big(p_1(1 + q_1 + q_1^2) + p_0(q_2 + 2q_1 q_2)\Big)t^3\Big)\\
&= -2p_0(1 + q_1 + q_1^2)t^3.
\end{aligned}$$

If a_{ij} is the (i,j)th entry of the 4×4 matrix H, then

$$\begin{aligned}
\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix} H_1 &= a_{11} + a_{21}t + a_{31}t^2 + a_{41}t^3,\\
\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix} H_2 &= a_{12} + a_{22}t + a_{32}t^2 + a_{42}t^3,\\
\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix} H_3 &= a_{13} + a_{23}t + a_{33}t^2 + a_{43}t^3,\\
\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix} H_4 &= a_{14} + a_{24}t + a_{34}t^2 + a_{44}t^3.
\end{aligned}$$

Equating the right-hand sides gives

$$\begin{aligned}
a_{11} + a_{21}t + a_{31}t^2 + a_{41}t^3 &= 0,\\
a_{12} + a_{22}t + a_{32}t^2 + a_{42}t^3 &= -2p_0 t - 2p_1 t^2 - 2p_2 t^3,\\
a_{13} + a_{23}t + a_{33}t^2 + a_{43}t^3 &= -2p_0(1+q_1)t^2 - 2\big(p_1(1+q_1) + p_0 q_2\big)t^3,\\
a_{14} + a_{24}t + a_{34}t^2 + a_{44}t^3 &= -2p_0(1 + q_1 + q_1^2)t^3.
\end{aligned}$$

Equating coefficients of the same powers of t gives

$$\begin{array}{llll}
a_{11} = 0, & a_{21} = 0, & a_{31} = 0, & a_{41} = 0,\\
a_{12} = 0, & a_{22} = -2p_0, & a_{32} = -2p_1, & a_{42} = -2p_2,\\
a_{13} = 0, & a_{23} = 0, & a_{33} = -2p_0(1+q_1), & a_{43} = -2\big(p_1(1+q_1) + p_0 q_2\big),\\
a_{14} = 0, & a_{24} = 0, & a_{34} = 0, & a_{44} = -2p_0(1 + q_1 + q_1^2),
\end{array}$$

as required. Similarly, if X_1, X_2, X_3 and X_4 are the column vectors of the matrix X, then

$$\begin{aligned}
\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix} X_1 &= \partial_\sigma(1) = 0,\\
\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix} X_2 &= \partial_\sigma(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3,
\end{aligned}$$

$$\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix}X_3 = \partial_\sigma(t^2) = (p_0 + p_0q_1)t + (p_1 + p_1q_1 + p_0q_2)t^2 + (p_2 + p_2q_1 + p_1q_2 + p_0q_3)t^3,$$
$$\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix}X_4 = \partial_\sigma(t^3) = p_0\big(1 + q_1 + q_1^2\big)t^2 + \Big(p_1(1 + q_1 + q_1^2) + p_0(q_2 + 2q_1q_2)\Big)t^3.$$

If a_{ij} is the (i, j)th entry of the 4×4 matrix X, then

$$\begin{aligned}
\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix}X_1 &= a_{11} + a_{21}t + a_{31}t^2 + a_{41}t^3,\\
\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix}X_2 &= a_{12} + a_{22}t + a_{32}t^2 + a_{42}t^3,\\
\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix}X_3 &= a_{13} + a_{23}t + a_{33}t^2 + a_{43}t^3,\\
\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix}X_4 &= a_{14} + a_{24}t + a_{34}t^2 + a_{44}t^3.
\end{aligned}$$

Equating the right-hand sides gives

$$\begin{aligned}
&a_{11} + a_{21}t + a_{31}t^2 + a_{41}t^3 = 0,\\
&a_{12} + a_{22}t + a_{32}t^2 + a_{42}t^3 = p_0 + p_1t + p_2t^2 + p_3t^3,\\
&a_{13} + a_{23}t + a_{33}t^2 + a_{43}t^3 = (p_0 + p_0q_1)t + (p_1 + p_1q_1 + p_0q_2)t^2 + (p_2 + p_2q_1 + p_1q_2 + p_0q_3)t^3,\\
&a_{14} + a_{24}t + a_{34}t^2 + a_{44}t^3 = p_0\big(1 + q_1 + q_1^2\big)t^2 + \Big(p_1(1 + q_1 + q_1^2) + p_0(q_1q_2 + q_2 + q_1q_2)\Big)t^3.
\end{aligned}$$

Equating coefficients of the same powers of t gives

$$\begin{aligned}
a_{11} &= 0, & a_{21} &= 0, & a_{31} &= 0, & a_{41} &= 0,\\
a_{12} &= P_0, & a_{22} &= p_1, & a_{32} &= p_2, & a_{42} &= p_3,\\
a_{13} &= 0, & a_{23} &= p_0(1 + q_1), & a_{33} &= \big(p_1(1 + q_1) + p_0q_2\big), & a_{43} &= \big(p_2(1 + q_1) + p_1q_2 + p_0q_3\big),\\
a_{14} &= 0, & a_{24} &= 0, & a_{34} &= p_0(1 + q_1 + q_1^2), & a_{44} &= p_1(1 + q_1 + q_1^2) + p_0q_2(2q_1 + 1),
\end{aligned}$$

as required. Similarly, if Y_1, Y_2, Y_3 and Y_4 are the column vectors of the matrix Y, then

$$\begin{aligned}
\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix}Y_1 &= -t^2\partial_\sigma(1) = 0,\\
\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix}Y_2 &= -t^2\partial_\sigma(t) = -t^2(p_0 + p_1t + p_2t^2 + p_3t^3) = -p_0t^2 - p_1t^3,\\
\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix}Y_3 &= -t^2\partial_\sigma(t^2) = -t^2\Big((p_0 + p_0q_1)t + (p_1 + p_1q_1 + p_0q_2)t^2\\
&\qquad + (p_2 + p_2q_1 + p_1q_2 + p_0q_3)t^3\Big) = -p_0(1 + q_1)t^3,\\
\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix}Y_4 &= -t^2\bigg(p_0\big(1 + q_1 + q_1^2\big)t^2 + \Big(p_1(1 + q_1 + q_1^2) + p_0(q_2 + 2q_1q_2)\Big)t^3\bigg) = 0.
\end{aligned}$$

If a_{ij} is the (i, j)th entry of the 4×4 matrix Y, then

$$\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix}Y_1 = a_{11} + a_{21}t + a_{31}t^2 + a_{41}t^3,$$

$$\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix} Y_2 = a_{12} + a_{22}t + a_{32}t^2 + a_{42}t^3,$$
$$\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix} Y_3 = a_{13} + a_{23}t + a_{33}t^2 + a_{43}t^3,$$
$$\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix} Y_4 = a_{14} + a_{24}t + a_{34}t^2 + a_{44}t^3.$$

Equating the right-hand sides gives

$$\begin{aligned}
a_{11} + a_{21}t + a_{31}t^2 + a_{41}t^3 &= 0,\\
a_{12} + a_{22}t + a_{32}t^2 + a_{42}t^3 &= -p_0t^2 - p_1t^3,\\
a_{13} + a_{23}t + a_{33}t^2 + a_{43}t^3 &= -p_0(1+q_q)t^3\\
a_{14} + a_{24}t + a_{34}t^2 + a_{44}t^3 &= 0.
\end{aligned}$$

Equating coefficients of the same powers of t gives

$$\begin{array}{llll}
a_{11} = 0, & a_{21} = 0, & a_{31} = 0, & a_{41} = 0,\\
a_{12} = 0, & a_{22} = 0, & a_{32} = -p_0, & a_{42} = -p_1,\\
a_{13} = 0, & a_{23} = 0, & a_{33} = 0, & a_{43} = -p_0(1+q_1),\\
a_{14} = 0, & a_{24} = 0, & a_{34} = 0, & a_{44} = 0,
\end{array}$$

as required. Finally, if R_1, R_2, R_3 and R_4 are the column vectors of the matrix R, then

$$\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix} R_1 = 2t^3\partial_\sigma(1) = 0,$$
$$\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix} R_2 = 2t^3\partial_\sigma(t) = 2t^3(p_0 + p_1t + p_2t^2 + p_3t^3) = 2p_0t^3,$$
$$\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix} R_3 = 2t^3\Big((p_0 + p_0q_1)t + (p_1 + p_1q_1 + p_0q_2)t^2 + (p_2 + p_2q_1 + p_1q_2 + p_0q_3)t^3\Big) = 0,$$
$$\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix} R_4 = 2t^3\Big(p_0\big(1 + q_1 + q_1^2\big)t^2 + \Big(p_1(1 + q_1 + q_1^2) + p_0(q_2 + 2q_1q_2)\Big)t^3\Big) = 0.$$

If a_{ij} is the (i,j)th entry of the 4×4 matrix R, then

$$\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix} R_1 = a_{11} + a_{21}t + a_{31}t^2 + a_{41}t^3,$$
$$\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix} R_2 = a_{12} + a_{22}t + a_{32}t^2 + a_{42}t^3,$$
$$\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix} R_3 = a_{13} + a_{23}t + a_{33}t^2 + a_{43}t^3,$$
$$\begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix} R_4 = a_{14} + a_{24}t + a_{34}t^2 + a_{44}t^3.$$

Equating the right-hand sides gives

$$a_{11} + a_{21}t + a_{31}t^2 + a_{41}t^3 = 0,$$

$$
\begin{aligned}
a_{12} + a_{22}t + a_{32}t^2 + a_{42}t^3 &= 2p_0t^3, \\
a_{13} + a_{23}t + a_{33}t^2 + a_{43}t^3 &= 0, \\
a_{14} + a_{24}t + a_{34}t^2 + a_{44}t^3 &= 0.
\end{aligned}
$$

Equating coefficients of the same powers of t gives

$$
\begin{array}{llll}
a_{11} = 0, & a_{21} = 0, & a_{31} = 0, & a_{41} = 0, \\
a_{12} = 0, & a_{22} = 0, & a_{32} = 0, & a_{42} = 2p_0, \\
a_{13} = 0, & a_{23} = 0, & a_{33} = 0, & a_{43} = 0, \\
a_{14} = 0, & a_{24} = 0, & a_{34} = 0, & a_{44} = 0.
\end{array}
$$

as required. ■

We now compute the bracket $[\cdot,\cdot]_\sigma$ explicitly under the present conditions. The result is presented in the following theorem and the required computations in the subsequent proof.

Theorem 5.1.3. Let $\mathcal{A}$ be a commutative, associative algebra with unity 1 over a field $\mathbb{F}$ of characteristic 0 and fix an element $t \in \mathcal{A}$. If conditions 5.1 holds where we let σ be an endomorphism on $\mathcal{A}$ and ∂_σ a σ-derivation (twisted derivation) on $\mathcal{A}$. Furthermore, assume that $\sigma(1) = 1$ and $\partial_\sigma(1) = 0$. Then the $\mathcal{A}$-module elements

$$
H = -2t\partial_\sigma, \quad X = \partial_\sigma, \quad Y = -t^2\partial_\sigma, \quad R = 2t^3\partial_\sigma
$$

satisfy the following relations:

$$
[H,X]_\sigma : q_1HX + 2q_2YX - q_3RX - XH = 2p_0X - p_1H - 2p_2Y + p_3R, \tag{5.13}
$$

$$
[H,Y]_\sigma : q_1HY + 2q_2Y^2 - q_3RY - q_1^2YH + q_1q_2RH = -2q_1p_0Y + q_1p_1R + q_2p_0R, \tag{5.14}
$$

$$
\begin{aligned}
[X,Y]_\sigma : XY - q_1^2YX + q_1q_2RX &= \frac{1}{2}p_0H + P_1Y - \frac{1}{2}p_2R + \frac{1}{2}p_0q_1H \\
&+ q_1p_1Y - \frac{1}{2}q_1p_2R + p_0q_2Y - \frac{1}{2}q_2p_2R - \frac{1}{2}p_0q_3R,
\end{aligned} \tag{5.15}
$$

$$
[H,R]_\sigma : q_1HR + 2q_2YR - q_3R^2 - q_1^2RH = -2q_1^2p_0R - 2q_1p_0R, \tag{5.16}
$$

$$
\begin{aligned}
[X,R]_\sigma : XR - q_1^3RX &= -2q_1^2p_0Y + q_1^2p_1R + 2q_1q_2p_0R - 2q_1p_0Y \\
&+ q_1p_1R + q_2p_0R - 2p_0Y + p_1R,
\end{aligned} \tag{5.17}
$$

$$
[Y,R]_\sigma : q_1^2YR - q_1q_2R^2 - q_1^3RY = 0. \tag{5.18}
$$

Proof. We start by proving relation (5.13). For the left-hand side,

$$\begin{aligned}
[H,X]_\sigma &= [-2t\partial_\sigma, \partial_\sigma]_\sigma = -2[t\partial_\sigma, \partial_\sigma]_\sigma, \quad \text{since the bracket } [\cdot,\cdot]_\sigma \text{ is bilinear} \\
&= -2\Big(\sigma(t)\partial_\sigma(\partial_\sigma) - \sigma(1)\partial_\sigma(t\partial_\sigma)\Big), \quad \text{by definition of } [\cdot,\cdot]_\sigma \\
&= -2\Big(q_1t + q_2t^2 + q_3t^3)\partial_\sigma(\partial_\sigma) - \partial_\sigma(t\partial_\sigma)\Big), \quad \text{since } \sigma(t) = q_1t + q_2t^2 + q_3t^3 \\
&= -2q_1t\partial_\sigma(\partial_\sigma) - 2q_2t^2\partial_\sigma(\partial_\sigma) - 2q_3t^3\partial_\sigma(\partial_\sigma) + \partial_\sigma(2t\partial_\sigma), \quad \text{since } \partial_\sigma \text{ is } \mathbb{F}\text{-linear} \\
&= q_1HX + 2q_2YX - q_3RX - XH,
\end{aligned}$$

as required. And for the right-hand side, we have

$$\begin{aligned}
[H,X]_\sigma &= 2\partial_\sigma(t)\partial_\sigma, \quad \text{by relation (5.2)} \\
&= -2\big(p_0 + p_1t + p_2t^2 + p_3t^3\big)\partial_\sigma \\
&= -2p_0\partial_\sigma - 2p_1t\partial_\sigma - p_2t^2\partial_\sigma - 2p_3t^3\partial_\sigma \\
&= 2p_0X - p_1H - 2p_2Y + p_3R,
\end{aligned}$$

as required. Next, we prove relation (5.14) in a similar way. For the left-hand side, we have

$$\begin{aligned}
[H,Y]_\sigma &= \big[-2t\partial_\sigma, -t^2\partial_\sigma\big]_\sigma = 2\big[t\partial_\sigma, t^2\partial_\sigma\big]_\sigma \\
&= 2\Big(\sigma(t)\partial_\sigma(t^2\partial_\sigma) - \sigma(t^2)\partial_\sigma(t\partial_\sigma)\Big), \quad \text{by definition of } [\cdot,\cdot]_\sigma \\
&= 2\Big((q_1t + q_2t^2 + q_3t^3)\partial_\sigma(t^2\partial_\sigma) - (q_1t + q_2t^2 + q_3t^3)^2\partial_\sigma(t\partial_\sigma)\Big) \\
&= 2q_1t\partial_\sigma(t^2\partial_\sigma) + 2q_2t^2\partial_\sigma(t^2\partial_\sigma) + 2q_3t^3\partial_\sigma(t^2\partial_\sigma) - q_1^2t^2\partial_\sigma(2t\partial_\sigma) + 4q_1q_2t^3\partial_\sigma(t\partial_\sigma) \\
&= q_1HY + 2q_2Y^2 - q_3RY - q_1^2YH + q_1q_2RH,
\end{aligned}$$

as required. And for the right-hand side, we have

$$\begin{aligned}
[H,Y]_\sigma &= 2t\sigma(t)\partial_\sigma(t)\partial_\sigma, \quad \text{by relation (5.3)} \\
&= 2t\big(q_1t + q_2t^2 + q_3t^3\big)\big(p_0 + p_1t + p_2t^2 + p_3t^3\big)\partial_\sigma \\
&= 2q_1p_0t^2\partial_\sigma + 2q_1p_1t^3\partial_\sigma + 2q_2p_0t^3\partial_\sigma \\
&= -2q_1p_0Y + q_1p_1R + q_2p_0R,
\end{aligned}$$

as required. Next, we prove relation (5.15) in a similar way. For the left-hand side, we have

$$\begin{aligned}
[X,Y]_\sigma &= -\big[\partial_\sigma, t^2\partial_\sigma\big]_\sigma = -\Big(\sigma(1)\partial_\sigma(t^2\partial_\sigma) - \sigma(t^2)\partial_\sigma(\partial_\sigma)\Big), \quad \text{by definition of } [\cdot,\cdot]_\sigma \\
&= -\Big(\partial_\sigma(t^2\partial_\sigma) - (q_1t + q_2t^2 + q_3t^3)^2\partial_\sigma(\partial_\sigma)\Big) \\
&= -\partial_\sigma(t^2\partial_\sigma) + q_1^2t^2\partial_\sigma(\partial_\sigma) + 2q_1q_2t^3\partial_\sigma(\partial_\sigma) \\
&= XY - q_1^2YX + q_1q_2RX,
\end{aligned}$$

as required. And for the right-hand side, we have

$$
\begin{aligned}
[X,Y]_\sigma &= -\Big(t+\sigma(t)\Big)\partial_\sigma(t)\partial_\sigma, \quad \text{by relation (5.4)}\\
&= -\big(t+q_1t+q_2t^2+q_3t^3\big)\big(p_0+p_1t+p_2t^2+p_3t^3\big)\partial_\sigma\\
&= \big(-p_0t-p_1t^2-p_2t^3-q_1p_0t-q_1p_1t^2-q_1p_2t^3-q_2p_0t^2-q_1p_1t^3-p_0q_3t^3\big)\partial_\sigma\\
&= \frac{1}{2}p_0H+p_1Y-\frac{1}{2}p_2R+\frac{1}{2}p_0q_1H+q_1p_1Y-\frac{1}{2}q_1p_2R+p_0q_2Y-\frac{1}{2}q_2p_2R-\frac{1}{2}p_0q_3R,
\end{aligned}
$$

as required. Next, we prove relation (5.16) in a similar way. For the left-hand side, we have

$$
\begin{aligned}
[H,R]_\sigma &= -4\big[t\partial_\sigma, t^3\partial_\sigma\big]_\sigma\\
&= -4(\sigma(t)\partial_\sigma)(t^3\partial_\sigma)-4(\sigma(t^3)\partial_\sigma)t\partial_\sigma\\
&= -4\big((q_1t+q_2t^2+q_3t^3)\partial_\sigma(t^3\partial_\sigma)-(q_1t+q_2t^2+q_3t^3)^3\partial_\sigma(t\partial_\sigma)\big)\\
&= q_1HR+2q_2YR-q_3R^2-q_1^2RH,
\end{aligned}
$$

as required. And for the right-hand side, we have

$$
\begin{aligned}
[H,R]_\sigma &= -4(\sigma(t)+t)t\sigma(t)\partial_\sigma(t)\partial_\sigma, \quad \text{by relation (5.10)}\\
&= -4\big(q_1t+q_2t^2+q_3t^3+t\big)\big(q_1p_0t^2+q_1p_1t^3+q_2p_0t^3\big)\partial_\sigma\\
&= -4(q_1^2p_0t^3+q_1p_0t^3)\partial_\sigma\\
&= -2q_1^2p_0R-2q_1p_0R,
\end{aligned}
$$

as required. Next, we prove relation (5.17) in a similar way. For the left-hand side, we have

$$
\begin{aligned}
[X,R]_\sigma &= 2\big[\partial_\sigma, t^3\partial_\sigma\big]_\sigma\\
&= 2\Big(\sigma(1)\partial_\sigma(t^3\partial_\sigma)-\sigma(t^3)\partial_\sigma(\partial_\sigma)\Big)\\
&= XR-2\big(q_1t+q_2t^2+q_3t^3\big)^3\partial_\sigma)\partial_\sigma\\
&= XR-q_1^3RX,
\end{aligned}
\tag{5.19}
$$

as required. And for the right-hand side, we have

$$
\begin{aligned}
[X,R]_\sigma &= 2\Big(\sigma(t)^2+t\sigma(t)+t^2\Big)\partial_\sigma(t)\partial_\sigma, \quad \text{by relation (5.11)}\\
&= 2\big(q_1^2t^2+2q_1q_2t^3+q_1t^2+q_2t^3+t^2\big)\big(p_0+p_1t+p_2t^2+p_3t^3\big)\partial_\sigma\\
&= \big(2q_1^2p_0t^2+2q_1^2p_1t^3+4q_1q_2p_0t^3+q_1p_0t^2+2q_1p_1t^3+q_2p_0t^3+p_0t^2+p_1`t^3\big)\partial_\sigma\\
&= -2q_1^2p_0Y+q_1^2p_1R+2q_1q_2p_0R-2q_1p_0Y\\
&\qquad +q_1p_1R+q_2p_0R-2p_0Y+p_1R,
\end{aligned}
$$

as required.

Next, we prove relation (5.18) in a similar way. For the left-hand side, we have

$$
\begin{aligned}
[Y, R]_\sigma &= -2\big[t^2\partial_\sigma, t^3\partial_\sigma\big]_\sigma \\
&= -2(\sigma(t^2)\partial_\sigma(t^3\partial_\sigma) - (\sigma(t^3)\partial_\sigma)(t^2\partial_\sigma) \\
&= -2(q_1t + q_2t^2 + q_3t^3)^2\partial_\sigma(t^3\partial_\sigma) - (q_1t + q_2t^2 + q_3t^3)^3\partial_\sigma(t^2\partial_\sigma) \\
&= q_1^2 YR - q_1q_2R^2 - q_1^3 RY, \qquad (5.20)
\end{aligned}
$$

as required. And for the right-hand side, we have

$$
\begin{aligned}
[Y, R]_\sigma &= 2t^2\sigma(t^2)\partial_\sigma(t)\partial_\sigma, \quad \text{by relation (5.12)} \\
&= 2\Big(t^2(q_1t + q_2t^2 + q_3t^3)^2\big(p_0 + p_1t + p_2t^2 + p_3t^3\big)\Big)\partial_\sigma \\
&= 0,
\end{aligned}
$$

as required. ■

We now subdivide our presentation into two cases, $q_1 = 1$ and $q_1 = -1$, where $q_2 = q_3 = 0$.

5.2 Case 1: $q_1 = 1$ and $q_2 = q_3 = 0$

For this case we assume that $q_1 = 1$ and $q_2 = q_3 = 0$. Equation (5.9), given as

$$p_0(1 + q_1 + q_1^2 + q_1^3)t^3 = 0,$$

then implies that $p_0 = 0$. This leads us to the following result.

Corollary 5.2.1. *Let $\mathcal{A}$ be a commutative, associative algebra with unity 1 over a field $\mathbb{F}$ of characteristic 0 and fix an element $t \in \mathcal{A}$. If conditions 5.1 where $p_0 = 0$ holds where we let σ be an endomorphism on $\mathcal{A}$ and ∂_σ a σ-derivation (twisted derivation) on $\mathcal{A}$. Furthermore, assume that $\sigma(1) = 1$ and $\partial_\sigma(1) = 0$. Then the $\mathcal{A}$-module elements*

$$H = -2t\partial_\sigma, \quad X = \partial_\sigma, \quad Y = -t^2\partial_\sigma, \quad R = 2t^3\partial_\sigma$$

satisfy the following relations:

$$
\begin{aligned}
[H, X]_\sigma &: HX - XH = -p_1H - 2p_2Y + p_3R, && (5.21) \\
[H, Y]_\sigma &: HY - YH = p_1R, && (5.22) \\
[X, Y]_\sigma &: XY - YX = 2p_1Y - p_2R, && (5.23) \\
[H, R]_\sigma &: HR - RH = 0, && (5.24) \\
[X, R]_\sigma &: XR - RX = 3p_1R, && (5.25) \\
[Y, R]_\sigma &: YR - RY = 0. && (5.26)
\end{aligned}
$$

Proof. Take $p_0 = 0$, $q_1 = 1$ and $q_2 = q_3 = 0$ in relations (5.13)–(5.18) of Theorem 5.1.3. $\blacksquare$

Corollary 5.2.2. *For $p_0 = 0$, $q_1 = 1$ and $q_2 = q_3 = 0$, the action of the module $\mathcal{A} \cdot \partial_\sigma$ on $\mathcal{A} = \mathbb{F}[t]/(t^4)$ has the following matrix representation with respect to the basis $\{1, t, t^2, t^3\}$:*

$$H \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2p_1 & 0 & 0 \\ 0 & -2p_2 & -4p_1 & 0 \end{pmatrix}, \qquad X \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 \\ 0 & p_2 & 2p_1 & 0 \\ 0 & p_3 & 2p_2 & 3p_1 \end{pmatrix},$$

$$Y \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -p_1 & 0 & 0 \end{pmatrix}, \qquad R \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Proof. Put $p_0 = 0$, $q_1 = 1$ and $q_2 = q_3 = 0$ in the representation given in Theorem 5.1.2. $\blacksquare$

Twisted Jacobi Identity

We now determine the Jacobi identity applicable for the present case. By Proposition 2.3.6, the bracket $[\cdot, \cdot]_\sigma$ on the module $\mathbb{F}[t] \cdot \partial_\sigma$ satisfies the 6-term deformed Jacobi identity

$$\circlearrowleft_{x,y,z} \Big(\big[\sigma(x), [y, z]_\sigma\big]_\sigma + \delta\big[x, [y, z]_\sigma\big]_\sigma \Big) = 0. \tag{5.27}$$

We need to find the possible values of $\delta \in \mathcal{A} = \mathbb{F}[t]/(t^4)$ in this deformed Jacobi identity. By Lemma 3.2.4, the identity

$$\partial_\sigma\big(\sigma(a)\big) = \delta \cdot \sigma\big(\partial_\sigma(a)\big)$$

holds for all monomials $a = t^k$, where k is a non-negative integer. In particular, for $a = t$,

$$\begin{aligned} \text{L.H.S} = \partial_\sigma(\sigma(t)) &= \partial_\sigma\big(q_1 t + q_2 t^2 + q_3 t^3\big) \\ &= \partial_\sigma(t), \quad \text{since for this case } q_1 = 1 \text{ and } q_2 = q_2 = 0 \\ &= p_0 + p_1 t + p_2 t^2 + p_3 t^3 \\ &= p_1 t + p_2 t^2 + p_3 t^3, \quad \text{since for this case } p_0 = 0 \end{aligned}$$

$$
\begin{aligned}
\text{R.H.S} = \delta\cdot\sigma(\partial_\sigma(t)) &= \delta\cdot\sigma\big(p_0+p_1t+p_2t^2+p_3t^3\big)\\
&= \delta\cdot\big(p_1\sigma(t)+p_2\sigma(t^2)+p_3\sigma(t^3)\big), \quad \text{since for this case } p_0=0\\
&= \delta\cdot\big(p_1(q_1t+q_2t^2+q_2t^3)+p_2(q_1t+q_2t^2+q_3t^3)^2+p_3(q_1t+q_2t^2+q_3t^3)^3\big)\\
&= \delta\cdot\big(p_1t+p_2t^2+p_3t^3\big), \quad \text{since } q_1=1 \text{ and } q_2=q_2=0,
\end{aligned}
$$

which implies that

$$
\delta\cdot\big(p_1t+p_2t^2+p_3t^3\big)=p_1t+p_2t^2+p_3t^3. \tag{5.28}
$$

Since $\delta\in\mathcal{A}=\mathbb{F}[t]/(t^4)$, we can write $\delta=\delta_0+\delta_1t+\delta_2t^2+\delta_3t^3$ for some $\delta_0,\delta_1,\delta_2,\delta_3\in\mathbb{F}$. Furthermore, let ξ_0, ξ_1, ξ_2 and ξ_3 denote four free parameters. Equation (5.28) then becomes

$$
\big(\delta_0+\delta_1t+\delta_2t^2+\delta_3t^3\big)\big(p_1t+p_2t^2+p_3t^3\big)=p_1t+p_2t^2+p_3t^3.
$$

Since $t^4=0$, this equation can be simplified as follows:

$$
\delta_0p_1t+(\delta_0p_2+\delta_1p_1)t^2+(\delta_0p_3+\delta_1p_2+\delta_2p_1)t^3=p_1t+p_2t^2+p_3t^3.
$$

Equating the coefficients of the same powers of t gives the following linear system of equations for δ_0, δ_1, δ_2, and δ_3:

$$
\begin{cases}
\delta_0p_1=p_1\\
\delta_0p_2+\delta_1p_1=p_2\\
\delta_0p_3+\delta_1p_2+\delta_2p_1=p_3
\end{cases} \tag{5.29}
$$

Since $p_1\neq 0$, the first equation of system (5.29) implies that $\delta_0=1$. Substituting $\delta_0=1$ in the second equation of system (5.29) gives

$$
\begin{aligned}
p_2+\delta_1p_1&=p_2\\
\delta_1&=0.
\end{aligned}
$$

Substituting these obtained values of δ_0 and δ_1 in the third equation of system (5.29) gives

$$
\begin{aligned}
p_3+\delta_2p_1&=p_3\\
\delta_2&=0.
\end{aligned}
$$

Since δ_3 is not involved at all, it can be chosen arbitrarily, say $\delta_3=\xi_3$. With these values of δ_0, δ_1, δ_2 and δ_3, we have

$$
\begin{aligned}
\delta&=\delta_0+\delta_1t+\delta_2t^2+\delta_3t^3\\
&=1+\xi_3t^3.
\end{aligned}
$$

Substituting this δ in equation (5.27) gives the Jacobi identity

$$\circlearrowleft_{x,y,z} \Big(\big[\sigma(x), [y,z]\big] + \big(1+\xi_3 t^3\big) \big[x, [y,z]\big] \Big) = 0,$$

Now ξ_3 can be chosen completely arbitrarily in this case, for example, $\xi_3 = 0$, giving the hom-Jacobi identity

$$\circlearrowleft_{x,y,z} \big[(\sigma + \mathrm{id})(x), [y,z]\big] = 0.$$

Therefore, for the case $p_0 = 0$, $q_1 = 1$, $q_2 = q_3 = 0$, the deformed $\mathfrak{sl}_2(\mathbb{F})$ is a hom-Lie algebra.

5.3 Case 2: $q_1 = -1$ and $q_2 = q_3 = 0$

For this case we assume that $q_1 = -1$ and $q_2 = q_3 = 0$. Equation (5.9), given as

$$p_0(1 + q_1 + q_1^2 + q_1^3)t^3 = 0,$$

then gives $p_0 \cdot 0 = 0$, implying that p_0 can be chosen completely arbitrarily in this case.

Corollary 5.3.1. *Let $\mathcal{A}$ be a commutative, associative algebra with unity 1 over a field $\mathbb{F}$ of characteristic 0 and fix an element $t \in \mathcal{A}$. If conditions 5.1 where $p_0 \neq 0$ holds where we let σ be an endomorphism on $\mathcal{A}$ and ∂_σ a σ-derivation (twisted derivation) on $\mathcal{A}$. Furthermore, assume that $\sigma(1) = 1$ and $\partial_\sigma(1) = 0$. Then the $\mathcal{A}$-module elements*

$$H = -2t\partial_\sigma, \quad X = \partial_\sigma, \quad Y = -t^2\partial_\sigma, \quad R = 2t^3\partial_\sigma$$

satisfy the following relations:

$$[H,X]_\sigma : HX + XH = -p_0X + p_1H + 2p_2Y - p_3R, \tag{5.30}$$
$$[H,Y]_\sigma : HY + YH = -2p_0Y + p_1R, \tag{5.31}$$
$$[X,Y]_\sigma : XY - YX = 0, \tag{5.32}$$
$$[H,R]_\sigma : HR - RH = 0, \tag{5.33}$$
$$[X,R]_\sigma : XR + RX = -2p_0Y + p_1R, \tag{5.34}$$
$$[Y,R]_\sigma : YR + RY = 0. \tag{5.35}$$

Proof. Put $q_1 = -1$, $q_2 = q_3 = 0$ in relations (5.13)–(5.15) of Theorem 5.1.3. ∎

Corollary 5.3.2. *For $q_1 = -1$ and $q_2 = q_3 = 0$, the action of the module $\mathcal{A} \cdot \partial_\sigma$ on $\mathcal{A} = \mathbb{F}[t]/(t^4)$ has the following matrix representation with respect to the basis $\{1, t, t^2, t^3\}$:*

$$H \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2p_0 & 0 & 0 \\ 0 & -2p_1 & 0 & 0 \\ 0 & -2p_2 & 0 & 2p_0 \end{pmatrix}, \qquad X \mapsto \begin{pmatrix} 0 & p_0 & 0 & 0 \\ 0 & p_1 & 0 & 0 \\ 0 & p_2 & 0 & p_0 \\ 0 & p_3 & 0 & p_1 \end{pmatrix},$$

$$Y \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -p_0 & 0 & 0 \\ 0 & -p_1 & 0 & 0 \end{pmatrix}, \qquad R \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2p_0 & 0 & 0 \end{pmatrix}.$$

Proof. Put $q_1 = -1$ and $q_2 = q_3 = 0$ in the representation given in Theorem 5.1.2. ■

Twisted Jacobi Identity

We now determine the Jacobi identity applicable for the present case. By Proposition 2.3.6, the bracket $[\cdot,\cdot]_\sigma$ on the module $\mathbb{F}[t] \cdot \partial_\sigma$ satisfies the 6-term deformed Jacobi identity

$$\circlearrowleft_{x,y,z} \Big(\big[\sigma(x), [y,z]_\sigma\big]_\sigma + \delta\big[x, [y,z]_\sigma\big]_\sigma \Big) = 0. \tag{5.36}$$

We need to find the possible values of $\delta \in \mathcal{A} = \mathbb{F}[t]/(t^4)$ in this deformed Jacobi identity. By Lemma 3.2.4, the identity $\partial_\sigma\big(\sigma(a)\big) = \delta \cdot \sigma\big(\partial_\sigma(a)\big)$ holds for all monomials $a = t^k$, where k is a non-negative integer. In particular, for $a = t$,

$$\begin{aligned} \text{L.H.S} = \partial_\sigma(\sigma(t)) &= \partial_\sigma\big(q_1 t + q_2 t^2 + q_3 t^3\big) \\ &= \partial_\sigma(q_1 t), \quad \text{since for this case } q_2 = q_3 = 0 \\ &= q_1 \partial_\sigma(t) \\ &= -1(p_0 + p_1 t + p_2 t^2 + p_3 t^3), \quad \text{since for this case } q_1 = -1 \\ &= -p_0 - p_1 t - p_2 t^2 - p_3 t^3 \end{aligned}$$

$$\begin{aligned} \text{R.H.S} = \delta\sigma(\partial_\sigma(t)) &= \delta\sigma\big(p_0 + p_1 t + p_2 t^2 + p_3 t^3\big) \\ &= \delta\big(p_0 + p_1\sigma(t) + p_2\sigma(t^2) + p_3\sigma(t^3)\big) \\ &= \delta\big(p_0 + p_1(q_1 t + q_2 t^2 + q_2 t^3) + p_2(q_1 t + q_2 t^2 + q_3 t^3)^2 + p_3(q_1 t + q_2 t^2 + q_3 t^3)\big) \\ &= \delta\big(p_0 - p_1 t + p_2 t^2 - p_3 t^3\big), \end{aligned}$$

which implies that

$$\delta \cdot \left(p_0 - p_1 t + p_2 t^2 - p_3 t^3\right) = -p_0 - p_1 t - p_2 t^2 - p_3 t^3. \tag{5.37}$$

Since $\delta \in \mathcal{A} = \mathbb{F}[t]/(t^4)$, we can write $\delta = \delta_0 + \delta_1 t + \delta_2 t^2 + \delta_3 t^3$ for some $\delta_0, \delta_1, \delta_2, \delta_3 \in \mathbb{F}$. Furthermore, let ξ_0, ξ_1, ξ_2 and ξ_3 denote four free parameters. Equation (5.37) then becomes

$$\left(\delta_0 + \delta_1 t + \delta_2 t^2 + \delta_3 t^3\right)\left(p_0 - p_1 t + p_2 t^2 - p_3 t^3\right) = -p_0 - p_1 t - p_2 t^2 - p_3 t^3.$$

Since $t^4 = 0$, this equation can be simplified as follows:

$$\begin{aligned} \delta_0 p_0 + (-\delta_1 p_1 + \delta_1 p_0)t + (\delta_0 p_2 - \delta_1 p_1 + \delta_2 p_0)t^2 + (-\delta_0 p_3 + \delta_1 p_2 - \delta_2 p_1 + \delta_3 p_0)t^3 \\ = -p_0 - p_1 t - p_2 t^2 - p_3 t^3. \end{aligned}$$

Equating the coefficients of the same powers of t gives the following linear system of equations for δ_0, δ_1, δ_2, and δ_3:

$$\begin{cases} \delta_0 p_0 = -p_0 \\ -\delta_0 p_1 + \delta_1 p_0 = -p_1 \\ \delta_0 p_2 - \delta_1 p_1 + \delta_2 p_0 = -p_2 \\ -\delta_0 p_3 + \delta_1 p_2 - \delta_2 p_1 + \delta_3 p_0 = -p_3. \end{cases} \tag{5.38}$$

Since $p_0 \neq 0$, the first equation of system (5.38) implies that $\delta_0 = -1$. Substituting $\delta_0 = -1$ in the second equation of system (5.38) gives

$$\begin{aligned} p_1 + \delta_1 p_0 &= -p_1 \\ \delta_1 &= \frac{-2p_1}{p_0}. \end{aligned}$$

Substituting these obtained values of δ_0 and δ_1 in the third equation of system (5.38) gives

$$\begin{aligned} &\delta_0 p_2 - \delta_1 p_1 + \delta_2 p_0 = -p_2 \\ &- p_2 + \frac{2p_1^2}{p_0} + \delta_2 p_0 = -p_2 \\ &\delta_2 = \frac{-2p_1^2}{p_0^2}. \end{aligned}$$

Substituting these values of δ_0, δ_1 and δ_2 in the fourth equation of system (5.38) gives

$$\begin{aligned}
&-\delta_0 p_3 + \delta_1 p_2 - \delta_2 p_1 + \delta_3 p_0 = -p_3. \\
&p_3 - \frac{2p_1}{p_0} p_2 + \frac{2p_1^2}{p_0^2} p_1 + \delta_3 p_0 = -p_3 \\
&p_0^2 p_3 - 2p_0 p_1 p_2 + 2p_1^3 + \delta_3 p_0^3 = -p_0^2 p_3 \\
&\delta_3 = \frac{2p_0 p_1 p_2 - 2p_0^2 p_3 - 2p_1^3}{p_0^3}.
\end{aligned}$$

With these values of δ_0, δ_1 , δ_2 and δ_2, we have

$$\begin{aligned}
\delta &= \delta_0 + \delta_1 t + \delta_2 t^2 + \delta_3 t^3 \\
&= -1 - \frac{2p_1}{p_0} t - \frac{2p_1^2}{p_0^2} t^2 + \frac{2p_0 p_1 p_2 - 2p_0^2 p_3 - 2p_1^3}{p_0^3} t^3.
\end{aligned}$$

Substituting this δ in equation (5.27) gives the Jacobi identity

$$\circlearrowleft_{x,y,z} \left(\big[\sigma(x), [y,z]\big] + \left(-1 - \frac{2p_1}{p_0} t - \frac{2p_1^2}{p_0^2} t^2 + \frac{2p_0 p_1 p_2 - 2p_0^2 p_3 - 2p_1^3}{p_0^3} t^3 \right) \big[x, [y,z]\big] \right) = 0$$

Therefore, for the case $q_1 = -1$, $q_2 = q_3 = 0$, the deformed $\mathfrak{sl}_2(\mathbb{F})$ is a quasi-hom-Lie algebra.

Remark 5.3.3. Note that when $p_1 = p_3 = 0$, this Jacobi identity reduces to

$$\circlearrowleft_{x,y,z} \Big(\big[\sigma(x), [y,z]\big] - \big[x, [y,z]\big] \Big) = 0,$$

which by Corollary 3.2.5 can be written as the hom-Jacobi identity

$$\circlearrowleft_{x,y,z} \big[(\alpha + \mathrm{id})(x), [y,z]_\sigma\big]_\sigma = 0,$$

where $\alpha(x) = -\sigma(x)$. So we have a hom-Lie algebra for this choice of the parameters.

Chapter 6

Discussion

In this chapter, we delve into the results and implications of quasi-deformations of $\mathfrak{sl}_2(\mathbb{F})$. We reflect on the insights garnered from our exploration of this fascinating field of deformation theory. The aim and objectives of this dissertation are restated and discussed in Section 6.1 while the employed methodology to achieve these goals is discussed in Section 6.2. In Section 6.3 we discuss quasi-deformations of $\mathfrak{sl}_2(\mathbb{F})$, where $\mathbb{F}$ is a field of characteristic 0. Here the base algebra is first taken to be as general as it can be in this situation, and thereafter, taken to be the polynomial algebra $\mathbb{F}[t]$. In Section 6.4 we discuss quasi-deformations of $\mathfrak{sl}_2(\mathbb{F})$, where $\mathbb{F}$ is a field of characteristic 0 that includes all the third roots of unity. Here the base algebra is taken to be $\mathbb{F}[t]/(t^3)$, the algebra of polynomials of degree less than 3 over $\mathbb{F}$ in the variable t. In Section 6.5 we discuss quasi-deformations of $\mathfrak{sl}_2(\mathbb{F})$, where $\mathbb{F}$ is a field of characteristic 0 that includes all the fourth roots of unity. Here the base algebra is taken to be $\mathbb{F}[t]/(t^4)$, the algebra of polynomials of degree less than 4 over $\mathbb{F}$.

6.1 On the Aim and Objectives

The primary aim of this dissertation was to explore and obtain quasi-deformations of $\mathfrak{sl}_2(\mathbb{F})$, where $\mathbb{F}$ denotes a field of characteristic zero, using twisted derivations. To achieve this aim, the following research objectives were formulated:

(a) **To obtain and deform an operator representation of $\mathfrak{sl}_2(\mathbb{F})$:** The first objective involved obtaining an operator representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{F})$ and deforming it using twisted derivations.

(b) **To determine the algebraic structure represented by the deformed operators:** The second objective focused on analysing the algebraic structure represented by the deformed operators. This involved studying the commutation relations and the corresponding Jacobi identities, and identifying any deviations from the usual $\mathfrak{sl}_2(\mathbb{F})$.

(c) **To provide an elaborated example of a quasi-deformed $\mathfrak{sl}_2(\mathbb{F})$:** The final objective was to illustrate the theoretical findings through an elaborated example. This example serves to demonstrate the practical implications of the quasi-deformations and provides insight into their behavior in specific scenarios.

6.2 On the Methodology

In this dissertation the basic undeformed object is $\mathfrak{sl}_2(\mathbb{F})$, which is defined as the Lie algebra generated by three elements H, X and Y satisfying the commutation relations

$$HX - XH = 2X, \quad HY - YH = -2Y, \quad XY - YX = H.$$

In order to achieve the objectives outlined in Section 6.1, our approach involved obtaining the following representation of $\mathfrak{sl}_2(\mathbb{F})$ in terms of first order differential operators acting on a vector space of functions in the variable t:

$$H \mapsto -2t\partial, \quad X \mapsto \partial, \quad Y \mapsto -t^2\partial.$$

This was then generalised to first order operators acting on an algebra $\mathcal{A}$ by replacing ∂ with ∂_σ, a σ-derivation on $\mathcal{A}$ (where σ is an endomorphism on $\mathcal{A}$). By choosing different base algebras $\mathcal{A}$, we got different algebraic structures on the module $\mathcal{A} \cdot \partial_\sigma$. So in a way we have two different deformation "parameters", namely, σ and $\mathcal{A}$. However, changing the algebra $\mathcal{A}$ and not changing the endomorphism σ is mathematically absurd, since σ is dependent on $\mathcal{A}$, but it is nice to loosely think about $\mathcal{A}$ as an independent deformation parameter.

6.3 On Quasi-Deformations of $\mathfrak{sl}_2(\mathbb{F})$

One of the main points of Chapter 2 is Proposition 2.3.6 which shows that for a σ-derivation ∂_σ defined on a commutative associative algebra $\mathcal{A}$ with unity, the module $\mathcal{A} \cdot \partial_\sigma$ admits a $\mathbb{C}$-algebra with a σ-deformed commutator

$$[a \cdot \partial_\sigma, b \cdot \partial_\sigma] = \sigma(a) \cdot \partial_\sigma(b \cdot \partial_\sigma) - \sigma(b) \cdot \partial_\sigma(a \cdot \partial_\sigma),$$

as a multiplication bracket, satisfying a twisted 6-term Jacobi identity. Chapter 3 builds on this and provides an elaborated example of a quasi-deformed $\mathfrak{sl}_2(\mathbb{F})$, where $\mathbb{F}$ is a field of characteristic zero. The result becomes a natural quasi-deformation of $\mathfrak{sl}_2(\mathbb{F})$ and we have shown that it is a quasi-hom-Lie algebra in general.

It is also important to note that Proposition 2.3.6 gives us two alternative ways to view our algebras. On the one hand, one can use Proposition 2.3.6 to construct a bracket on the vector space $\mathcal{A} \cdot \partial_\sigma$ with the help of relation (2.5) viewing this bracket as a product. On the other hand, one can use relation (2.5) in adjunction with relation (2.6) of the bracket to obtain quadratic relations and "moding" out these from the tensor algebra, thereby giving us an associative quadratic algebra which loosely can be thought of as a deformed "enveloping algebra" of the corresponding algebra from the viewpoint of the preceding sentence.

In Section 3.2 the base algebra $\mathcal{A}$ is taken to be the polynomial algebra $\mathbb{F}[t]$ and it is assumed that $\sigma(t) = q(t)$ and $\partial_\sigma(t) = p(t)$, where $p(t), q(t) \in \mathbb{F}[t]$. To have closure of the corresponding deformed commutation relations, the polynomials $p(t)$ and $q(t)$ are far from arbitrary. Note that if we allow $\sigma(t) = q(t)$ and $\partial_\sigma(t) = p(t)$ where p and q are arbitrary polynomials in t then we get a deformation of $\mathfrak{sl}_2(\mathbb{F})$ which does not preserve dimension; that is, brackets of the basis elements H, X and Y are not simply linear combinations in these elements but include more new "basis" elements. This phenomena could possibly be interesting to study further.

One intriguing deformation provided in the elaborated example in Section 3.3 is given by

$$\begin{aligned} HX - q^{-1}XH &= 2q^{-1}p_0X, \\ HY - qYH &= -2p_0Y, \\ XY - q^2YX &= \frac{(q+1)}{2}p_0H. \end{aligned}$$

The algebra generated by these relations is denoted by $\mathcal{U}_q$. Using the transformations

$$H \mapsto 2p_0q^{-1}y + \frac{2p_0}{q-1}\mathbf{1}, \qquad X \mapsto z, \qquad Y \mapsto x,$$

where $\mathbf{1}$ denotes the identity element, we obtained the relations

$$yz - q^{-1}zy = 0, \qquad xy - q^{-1}yx = 0, \qquad zx - q^2xz = p_0^2(1+q^{-1})y + p_0^2\frac{q+1}{q-1}\mathbf{1}.$$

Unlike in the case of the undeformed universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{F})$, the 1-parameter deformation of $\mathcal{U}_q$ (with $q \neq 1$) has nontrivial 1-dimensional representations. Indeed, taking $H = 2p_0/(q-1)$, then the defining relations of $\mathcal{U}_q$ become

$$XY = \frac{-p^2}{(q-1)^2}. \tag{6.1}$$

So X and Y can be chosen as any numbers satisfying (6.1). When $p_0 = 1$ and $q \to 1$, we "approach $\mathfrak{sl}_2(\mathbb{F})$" in the defining relations of $\mathcal{U}_q$, but the representation (6.1) explodes to infinity and thus does not converge to a representation of $\mathfrak{sl}_2(\mathbb{F})$.

6.4 On Deformations of $\mathfrak{sl}_2(\mathbb{F})$ with Base $\mathbb{F}[t]/(t^3)$

In Chapter 4 we obtained quasi-deformations of $\mathfrak{sl}_2(\mathbb{F})$, where $\mathbb{F}$ is a field of characteristic 0 that includes all the third roots of unity (the solutions to the equation $z^3 = 1$). Here the base algebra $\mathcal{A}$ is taken to be $\mathbb{F}[t]/(t^3)$, the algebra of polynomials of degree less than 3 over

$\mathbb{F}$ in the variable t. With this base algebra, we took

$$\partial_\sigma(t) = p_0 + p_1 t + p_2 t^2 \quad \text{and} \quad \sigma(t) = q_0 + q_1 t + q_2 t^2.$$

One of the main points of Chapter 4 is that the action of the module $\mathcal{A} \cdot \partial_\sigma$ on $\mathcal{A} = \mathbb{F}[t]/(t^3)$ has the following matrix representation with respect to the basis $\{1, t, t^2\}$:

$$H \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2p_0 & 0 \\ 0 & -2p_1 & -2p_0(1+q_1) \end{pmatrix}, \quad X \mapsto \begin{pmatrix} 0 & p_0 & 0 \\ 0 & p_1 & p_0(1+q_1) \\ 0 & p_2 & p_1(1+q_1)+p_0q_2 \end{pmatrix}, \quad Y \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -p_0 & 0 \end{pmatrix}.$$

However, when $p_0 = 1, \quad p_1 = p_2 = 0, \quad q_0 = q_2 = 0$ and $q_1 = 1$, which seemingly would correspond to the case of $\mathfrak{sl}_2(\mathbb{F})$, we surprisingly obtained the matrices

$$H \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{pmatrix}, \qquad X \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Y \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix},$$

satisfying the relations: $HX - XH = 2X, \quad HY - YH = -2Y$ and $XY + 2YX = H$. The reason for this anomaly is that the values of the parameter so chosen are not compatible with the σ-Leibniz rule on $\mathcal{A} = \mathbb{F}[t]/(t^3)$ since the necessary condition

$$p_0\big(1 + q_1 + q_1^2\big)t^2 = 0$$

becomes $3t^2 = 0$, the equality which is not possible in $\mathcal{A}$ over a field of characteristic zero.

Furthermore, the condition $p_0\big(q_1^2 + q_1 + 1\big)t^2 = 0$ shows that if $p_0 = 0$, then q_1 is a true free deformation parameter, and if $p_0 \neq 0$, we generate deformations at the imaginary third roots of unity. This meant that we had to subdivide our presentation into two special cases.

The case $p_0 = 0$ resulted in deformed commutation relations where we could not retain $\mathfrak{sl}_2(\mathbb{F})$ by choosing suitable values of the parameters. In fact, the commutator of the basis elements H and Y was identically zero. In general, the class of algebras for the case $p_0 = 0$ obtained via a twisting of $\mathfrak{sl}_2(\mathbb{F})$, is a multiparameter deformation of the polynomial algebra in three commuting variables, of the Heisenberg Lie algebra, and of the solvable Lie algebra.

For the case $p_0 \neq 0$, the condition $p_0\big(q_1^2 + q_1 + 1\big)t^2 = 0$ implies that we generate deformations at the zeros of the polynomial $q_1^2 + q_1 + 1$. The deformed $\mathfrak{sl}_2(\mathbb{F})$ for this case corresponds to a quasi-hom-Lie algebra. However, it can also be viewed as a hom-Lie algebra for some special choices of the parameters defining $\sigma(t)$ and $\partial_\sigma(t)$. Furthermore, in the special case $p_1 = p_2 = q_2 = 0$ and $\omega^k = q$ we obtained the defining relations for the Jackson $\mathfrak{sl}_2(\mathbb{F})$.

We remark that the construction to generate deformations at third roots of unity can be generalized to deformations at n^{th} roots of unity by using $\mathcal{A} = \mathbb{F}[t]/(t^n)$ (where $\mathbb{F}$ is a field including the relevant primitive roots of unity) in exactly the same way as in Chapter 4.

6.5 On Deformations of $\mathfrak{sl}_2(\mathbb{F})$ with Base $\mathbb{F}[t]/(t^4)$

In Chapter 5 we obtained quasi-deformations of $\mathfrak{sl}_2(\mathbb{F})$, where $\mathbb{F}$ is a field of characteristic 0 that includes all the fourth roots of unity (the solutions to the equation $z^4 = 1$). Here the base algebra $\mathcal{A}$ was taken to be $\mathbb{F}[t]/(t^4)$, the algebra of polynomials of degree less than 4 over $\mathbb{F}$ in the variable t. With this base algebra, we took

$$\partial_\sigma(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3 \quad \text{and} \quad \sigma(t) = q_0 + q_1 t + q_2 t^2 + q_3 t^3.$$

One necessary condition for the existence of the deformations under this case is that

$$p_0(1 + q_1 + q_1^2 + q_1^3)t^3 = 0.$$

This condition implies that if $p_0 = 0$, then q_1 is a true free deformation parameter, and if $p_0 \neq 0$, then we generate deformations at the zeros of the polynomial $1 + q_1 + q_1^2 + q_1^3$. This meant that we had to subdivide our presentation into two special cases.

The case $p_0 = 0$ is equivalent to the case $q_1 = 1$ since $p_0(1 + q_1 + q_1^2 + q_1^3)t^3 = 0$. Therefore, for this case we considered $q_1 = 1$, and for simplicity, we took $q_2 = q_3 = 0$. One of the main points under this case is that the action of the module $\mathcal{A}\cdot\partial_\sigma$ on $\mathcal{A} = \mathbb{F}[t]/(t^4)$ has the matrix representation with respect to the basis $\{1, t, t^2, t^3\}$ where the R element is represented by the zero matrix (see Corollary 5.2.2). This means that all commutation relations involving R collapse and we retain the usual three commutation relations of a deformed $\mathfrak{sl}_2(\mathbb{F})$. The deformed $\mathfrak{sl}_2(\mathbb{F})$ for this case turns out to be a hom-Lie algebra.

The case $p_0 \neq 0$ is equivalent to the case $q_1 = -1$ since $p_0(1+q_1+q_1^2+q_1^3)t^3 = 0$. Therefore, for this case we considered $q_1 = -1$, and for simplicity, we took $q_2 = q_3 = 0$. One of the main points under this case is that, for $p_0 = p_1 = p_2 = p_3 = 0$, the action of the module $\mathcal{A} \cdot \partial_\sigma$ on $\mathcal{A} = \mathbb{F}[t]/(t^4)$ has the matrix representation with respect to the basis $\{1, t, t^2, t^3\}$ where the R element is represented by a non-zero matrix (see Corollary 5.3.3). This means that we retain all the six commutation relations we started with.

Chapter 7

Conclusion and Recommendations

In this final chapter, we summarize the main findings of our research and underscore their implications. We offer recommendations based on our insights to guide future research. The summary of the main findings is given in Section 7.1 while the recommendations are given in Section 7.2.

7.1 Conclusion

The theoretical framework developed in this dissertation lays the foundation for understanding quasi-deformations in the context of Lie algebras. By establishing representations and studying commutation relations, we have elucidated the algebraic structures arising from these deformations. Our investigation has revealed the intricate dependence of deformations on parameters such as an endomorphism σ and the choice of base algebra $\mathcal{A}$. The interplay between these parameters influences the resulting algebraic structures, leading to diverse deformation phenomena. Special cases, such as deformations at roots of unity, have unveiled unexpected outcomes and mathematical anomalies. These anomalies provide valuable insights into the underlying mathematics and suggest avenues for further exploration. The elaborated example presented in this dissertation demonstrates the practical implications of quasi-deformations. By illustrating specific scenarios, we have showcased the relevance and applicability of these deformations in mathematical contexts and theoretical physics.

In conclusion, the exploration of quasi-deformations of $\mathfrak{sl}_2(\mathbb{F})$ presented in this dissertation opens up exciting possibilities for future research and underscores the richness and complexity of algebraic structures that emerge from deformation theory.

7.2 Recommendations

Building upon the insights gained, several avenues for future research emerge:

1. **Generalisation:** Further investigation into generalisations of quasi-deformations, including deformations at higher-order roots of unity and extensions to other Lie algebras, can deepen our understanding of deformation theory and uncover new algebraic structures.
2. **Generalisation to Other Fields:** Extend the analysis to fields beyond characteristic

zero, examining the effects of characteristic on the properties of quasi-deformations and exploring potential applications in algebraic geometry and mathematical physics.

3. **Study of Special Cases**: Investigate special cases and limiting behaviors of quasi-deformations, such as the behavior as parameters approach certain values or the study of specific subclasses of deformations for their unique properties.

4. **Further Exploration of Parameter Configurations:** Given the sensitivity of quasi-deformations to parameter configurations, further exploration is warranted to uncover additional intriguing algebraic structures. Investigating different parameter choices and their implications could lead to the discovery of novel algebraic phenomena.

   For example, in Chapter 3, if we allow $\sigma(t) = q(t)$ and $\partial_\sigma(t) = p(t)$ where p and q are arbitrary polynomials in t then we get a deformation of $\mathfrak{sl}_2(\mathbb{F})$ which does not preserve dimension; that is, brackets of the basis elements H, X and Y are not simply linear combinations in these elements but include more new "basis" elements. This phenomena could possibly be interesting to study further.

5. **Applications:** Investigating applications of quasi-deformations in related fields such as mathematical physics, representation theory, and algebraic geometry can uncover connections and potential practical uses of these deformations.

6. **Computational Approaches:** Utilizing computational tools and techniques could facilitate the study and analysis of quasi-deformations. Employing computer algebra systems for computations, simulations, and symbolic manipulation could aid in exploring complex parameter spaces and verifying theoretical results.

7. **Collaborative Research:** Collaborating with researchers from diverse mathematical backgrounds could enrich our understanding of quasi-deformations. Engaging in interdisciplinary collaborations could lead to innovative approaches, new perspectives, and fruitful exchanges of ideas.

By pursuing these avenues of research, we can further deepen our understanding of quasi-deformations and their significance in mathematics and theoretical physics, opening up new directions for exploration and discovery in this fascinating field.

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