HOPSCOTCH - A FAST ALGORITHM FOR PARTIAL DIFFERENTIAL EQUATIONS

A dissertation submitted to the school of Natural Sciences of the University of Zambia in partial-fulfillment of the requirements for the degree of Master of Science

by

YAMBOTO MUMBULA

OCTOBER, 1983

Signature of Author ______________

Certified by: ______________

Dissertation Supervisor

Accepted by ______________

Chairman

This dissertation has been examined by a committee as follows

PROFESSOR A.M. JASHON

EXTERNAL EXAMINER

DR. E.M. LUNCH

DISSERTATION SUPERVISOR

PROF. E.F. BARTHOLOMEUSZ

CHAIRMAN

DECLARATION

I hereby declare that this dissertation is my own work and that it has not been previously submitted for degree purposes here or any other University.

A C K N O W L E D G E M E N T S

The author would like to thank Dr. E.M. Lungu who tirelessly supervised the writing of this work. Appreciations also go to the Directorate of Manpower Development and Training for their financial assistance.

Last but not least, the author would like to thank Miss Rose Mweendo who typed the work.

I N T R O D U C T I O N

The rapid development of high speed computers and the increasing realization by engineers and scientists that many of their most significant problems can be formulated into partial differential equations and solved satisfactorily with the help of numerical methods has led to a lot of effort being thrown into the development of these methods.

A number of numerical methods have been proposed. Notable among the existing works are what Gourlay (1971) classifies into three categories:

(i) Fast general purpose algorithms which conserve both man and machine time;

(ii) classical finite difference methods which require more programming effort;

(iii) methods which give guaranteed accuracy at the expense of considerable programming effort.

Broadly speaking, all the methods which are explicit in nature fall under category (i). The second category consists of implicit schemes such as the Crank-Nicolson, alternating direction implicit and over-relaxation algorithms. The Garlekin-type methods constitute the methods in category (iii).

The need for a fast general purpose algorithm which guarantees accuracy and conserves both man and machine time has motivated a number of numerical analysts to concentrate their researches in this area. Recent work in this direction by Gourlay (1970) tries to combine the already existing finite difference schemes inorder to achieve the desired results. In so doing he came up with an algorithm which he called "hopscotch", a name derived from the hopscotch game which the algorithm resembles.

This work is a review of the literature so far available on the subject. The first chapter looks at linear parabolic equations while chapter 2 develops the algorithm for linear elliptic equations. In both cases analyses for the stability and convergence criteria have been provided. Chapter 3 looks at nonlinear hyperbolic equations in the same spirit. Each of the three chapters ends with an illustrative example where the numerical solution has been compared with the analytic one. Concluding remarks have been given in the section for conclusion.

Finally, in the appendix an attempt has been made to try and summaris some of the important aspects of the work which, due to time constrients it was found unnecessary to discuss in detail.

C O N T E N T S

CHAPTER 1

Parabolic Equations

Physical phenomena such as diffusion of material, heat conduction, fluid flow through some porous media, flow of electricity in cables all give rise to parabolic partial differential equations. These are of the form

$$\frac{\partial u}{\partial t} = L\,u + g \qquad (1.0)$$

where L is a second order elliptic partial differential operator which may be linear or nonlinear; and g is a function of either the space and time variables only, or may include u as well. We start the development of the hopscotch algorithm by considering equations of the type (1.0).

1.1 Development of the Algorithm

In the analysis that follows, we restrict ourselves for the moment to the linear equation in two space dimensions, that is

$$\frac{\partial u}{\partial t} = L\,u + g(x,y,t) \qquad (1.1)$$

where L is a second-order linear, elliptic partial differential operator in th space variables x and y. We require the solution in the cuboid $R \times (0 \leq t \leq T)$, where R is a closed rectangular region in the $x - y$ plane with a continuous boundary ∂R. The initial and boundary conditions have been specified on $t = 0$ and $\partial R \times (0 \leq t \leq T)$ respectively.

Let $\Delta x = \Delta y = h$ and $\Delta t = k$ be the mesh spacings in space and time respectively, such that we have a set of points $(ih, jh) \in R_h$, where i, j are integers and $1 \leq i, j \leq n$, $n \in N$. The cross-section of the cuboid in which the solution is sought is shown in fig. 1.1

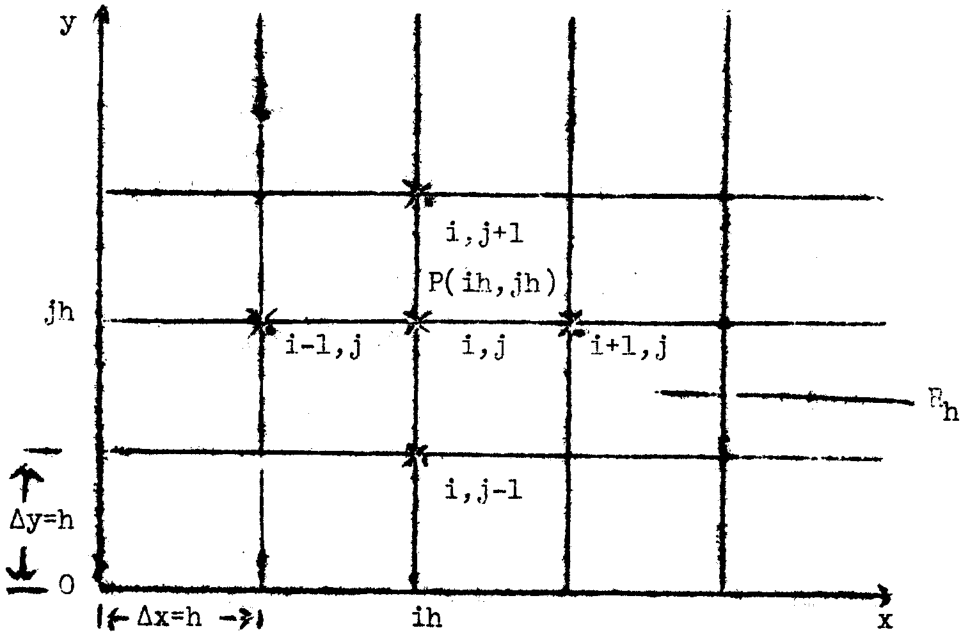

Fig. 1.1

Let $U^m_{i,j}$ denote the numerical solution of equation (1.1) at the point (ih, jh, mk) and $u(ih, jh, mk)$ denote the analytical solution at this same point. The solution of equation (1.1) is required at the grid points $R_h X\, t_m$, where $t_m = mk$ is a plane parallel to R and m is an integer.

The hopscotch process, as developed by Gourlay (1970) depends on Gordon's explicit and implicit schemes

$$U^{m+1}_{i,j} = U^m_{i,j} + k(L_h U^m_{i,j} + g^m_{i,j}) \qquad (1.2)$$

$$U^{m+1}_{i,j} = U^m_{i,j} + k(L_h U^{m+1}_{i,j} + g^{m+1}_{i,j}) \qquad (1.3)$$

where L_h is a finite difference replacement of the linear operator L.

In order to make the hopscotch process explicit, we require that $L_h U_{i,j}^{m+1}$ be a replacement which involves $U_{i,j}^{m+1}$ and its nearest neighbours (i.e $U_{i+1,j}^{m+1}$, $U_{i-1,j}^{m+1}$, $U_{i,j+1}^{m+1}$, $U_{i,j-1}^{m+1}$) along the grid line. We shall call an operator with this property an E-operator. Note that if L_h is not an E-operator the hopscotch process is implicitly defined and thus defeating the whole purpose for which the algorithm is intended.

Defining the odd-even function $\theta_{i,j}^{m}$ by

$$\theta_{i,j}^{m} = \begin{cases} 1 & \text{if } m+i+j \text{ is an odd integer} \\ 0 & \text{if } m+i+j \text{ is an even integer} \end{cases} \tag{1.4}$$

equations (1.2) and (1.3) can be combined into a single equation of the form

$$U_{i,j}^{m+1} - k\theta_{i,j}^{m+1} (L_h U_{i,j}^{m+1} + g_{i,j}^{m+1}) = U_{i,j}^{m} + k\theta_{i,j}^{m}(L_h U_{i,j}^{m} + g_{i,j}^{m}). \tag{1.5}$$

Equation (1.5) defines the hopscotch algorithm locally at all points in the domain R_h.

Note that increasing the time step in equation (1.5) by 1 and writing down the two successive equations we obtain

$$U_{i,j}^{m+1} - k\theta_{i,j}^{m+1} (L_h U_{i,j}^{m+1} + g_{i,j}^{m+1}) = U_{i,j}^{m} + k\theta_{i,j}^{m} (L_h U_{i,j}^{m} + g_{i,j}^{m})$$

$$U_{i,j}^{m+2} - k\theta_{i,j}^{m+2} (L_h U_{i,j}^{m+2} + g_{i,j}^{m+2}) = U_{i,j}^{m+1} + k\theta_{i,j}^{m+1} (L_h U_{i,j}^{m+1} + g_{i,j}^{m+1}) \quad (1.6)$$

$$= 2U_{i,j}^{m+1} - \{U_{i,j}^{m} + k\theta_{i,j}^{m} (L_h U_{i,j}^{m} + g_{i,j}^{m})\} \quad (1.7)$$

The second form (1.7) of equation (1.6) is obtained after replacing $k\theta_{i,j}^{m+1} (L_h U_{i,j}^{m+1} + g_{i,j}^{m+1})$ in (1.6) by its value from equation (1.5)

It is clear that when (m+i+j) is an even integer, equation (1.7) reduces to

$$U_{i,j}^{m+2} = 2U_{i,j}^{m+1} - U_{i,j}^{m} . \quad (1.8)$$

Consequently we can now define the two step hopscotch algorithm as follows: As we advance forward in time from time step $t = mk$ the steps **taken** are:

(1) For the interior mesh points with (m+i+j) an odd integer, equation (1.5) reduces to equation (1.2). Thus for all i, j satisfying (m+i+j) odd along the current time step, the values of $U_{i,j}^{m+1}$ are obtained using equation (1.2). It is important to observe that (1) can only be used either in the first time step with initial values at t=0 known, or if the print - out has been given in the previous time step. This step is also the **restart** procedure.

(2) When $(m+i+j)$ is an even integer, equation (1.5) reduces to equation (1.3). Therefore, if the values of $U^{m+1}_{i,j}$ are required for print-out, use equation (1.3) to evaluate all the remaining points along the current time step with the property $(m+i+j)$ even. We notice here that although equation (1.3) appears to be implicit in nature, it is actually explicit. This is evident from the fact that all the values of $U^{m+1}_{i\pm1,j}$, $U^{m+1}_{i,j\pm1}$ required in this formula have already been calculated in step (1).

After the print out, m is incremented and we return to step (1). Otherwise if the values of $U^{m+1}_{i,j}$ are not required for print-out, proceed to step (3).

(3) Since we have skipped step (2) when print-out is not required after step (1) has been performed, it means some points along the current time step still remain to be obtained. And these are the points with the property $(m+i+j)$ an even interger. As already seen in step (2) these points are obtained with the help of equation (1.3). We therefore use equation (1.3) for all those points satisfying the condition $(m+i+j)$ an even integer. For this fixed value of i,j with $(m+i+j)$ even, overwrite the value of $U^{m+1}_{i,j}$ with $U^{m+2}_{i,j}$ defined by equation (1.8).

$$\text{i.e} \quad U^{m+2}_{i,j} = 2U^{m+1}_{i,j} - U^{m}_{i,j}$$

Repeat this process for all allowable (i,j) with $(m+i+j)$ an even integer. Then increment m by 1. If the values of $U_{i,j}^{m+1}$ are required for print-out return to step (2), otherwise return to step (3) and repeat. .

Having outlined the hopscotch procedure we now turn to the examination of the stability and convergence criteria. Since the analysis involves use of matrices, it is perhaps gratifying to observe at this stage that matrix analysis pertaining to partial differential operators has been adquately covered by Varga (1965). Much of this analysis will therefore be based on Varga's analysis.

1.2 STABILITY

It is convenient to introduce vectors and matrices at this stage for the purpose of this analysis. Let u_{2m} denote the vector with components $U_{i,j}^{2m}$ for some ordering of internal points in R_h. The usual odering along rows for each fixed column will be adopted, (see example 1.1 below).

Define the matrix H by

$$H = -L_h. \tag{1.9}$$

Further define two diagonal matrices I_1 and I_2 of the same order as H, but whose entries are either + 1 or 0 thus:

I_1 is a diagonal matrix whose (i,i) element is + 1 if the i^{th} entry in the vector u_{2m} is a mesh function $U_{i,j}^{2m}$ with $(i+j)$ an odd integer and zeros elewhere.

$$I_2 = I - I_1 \tag{1.10}$$

where I is the identity matrix.

We note in particular the properties of the matrices I_1 and I_2 .

$$I_1 I_2 = I_2 I_1 = 0 \text{ (a null matrix)} \tag{1.11}$$

$$I_1^2 = I_1 \ , \ I_2^2 = I_2 \tag{1.12}$$

With this notation, the global form of the two step hopscotch process (1.5) and (1.6) becomes

$$(I + kI_2H)\, u_{2m+1} = (I - kI_1H)u_{2m} + k(I_2 g_{2m+1} + I_1 g_{2m}) \tag{1.13}$$

$$(I + kI_1H\,)u_{2m+2} = (I - kI_2H)u_{2m+1} + k(I_1 g_{2m+2} + I_2 g_{2m+1}) \tag{1.14}$$

We illustrate the foregoing with an example.

Example 1.1

Consider the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \ , \quad 0 \leq x \leq 1 \tag{1.15}$$

where $u = 0$ at $x = 0$ and 1, $t>0$, and u is known when $t = 0$.

The solution is required in the rectangular mesh shown in Fig. 1.2. The interval $0 \leq x \leq 1$ is subdivided into N mesh spacings such that $Nh = 1$.

$$U_0^m = U_N^m = 0.$$

We shall assume N an even integer, so that $(N-1)$ is odd. This assumption is important in that it enables us to predetermine the type of equation to be used for a mesh point lying on the $(N-1)^{th}$ mesh line at any given time level.

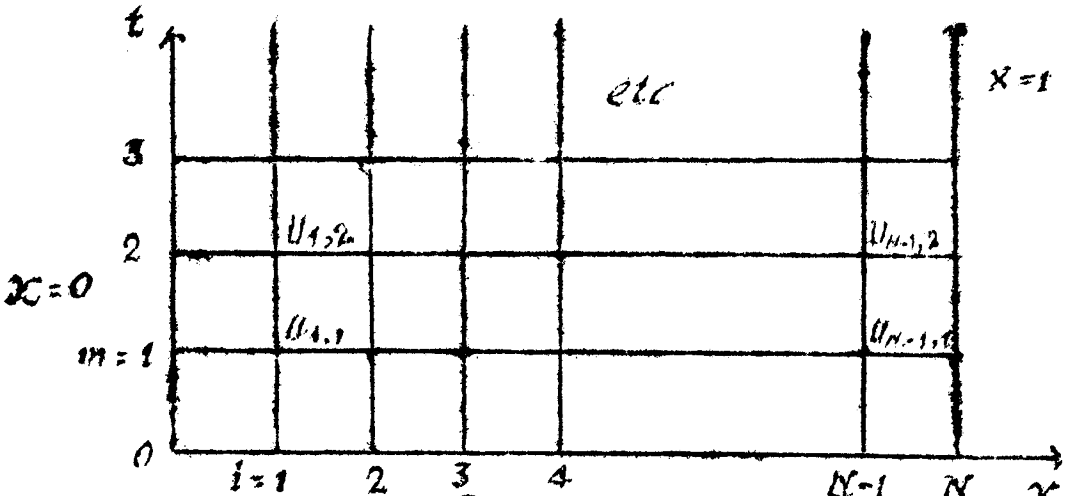

Fig. 1.2

The hopscotch process for equation (1.15) is given by

$$U_i^{m+1} - k\theta_i^{m+1}(L_h U_i^{m+1}) = U_i^m + k\theta_i^m (L_h U_i^m) \tag{1.16}$$

$$U_i^{m+2} - k\theta_i^{m+2}(L_h U_i^{m+2}) = U_i^{m+1} + k\theta_i^{m+1}(L_h U_i^{m+1}) \tag{1.17}$$

$$= 2U_i^{m+1} - \{U_i^m + k\theta_i^m(L_h U_i^m)\}$$

Equation (1.16) can be written as

$$U_i^{m+1} - k\theta_i^{m+1} \{\frac{1}{h^2}(U_{i+1}^{m+1} - 2U_i^{m+1} + U_{i-1}^{m+1})\} = U_i^m + k\theta_i^m\{\frac{1}{h^2}(U_{i+1}^m - 2U_i^m + U_{i-1}^m)\}$$

$$- \quad (1.18)$$

Application of equation (1.18) along odd time levels (i.e along $m = 1, 3, 5,$ ----) yields the following set of alternate equations:

$$U_{1,2m+1} = U_{1,2m} \quad k(\frac{1}{h^2}U_{2,2m+1} - {}^2/h^2\, U_{1,2m+1})$$

$$U_{2,2m+1} = U_{2,2m} + k(\frac{1}{h^2}U_{3,2m} - {}^2/h^2\, U_{2,2m} + \frac{1}{h^2}U_{1,2m}) \quad \Big\} \; (1.19)$$

$$U_{3,2m+1} = U_{3,2m} + k(\frac{1}{h^2}U_{4,2m+1} - {}^2/h^2\, U_{3,2m+1} + \frac{1}{h^2}U_{2,2m+1})$$

$$U_{4,2m+1} = U_{4,2m} + k(\frac{1}{h^2}U_{5,2m} - {}^2/h^2\, U_{4,2m} + \frac{1}{h^2}U_{3,2m})$$

$$U_{N-1,2m+1} = U_{N-1,2m} + k\left(-\frac{2}{h^2}U_{N-1,2m+1} + \frac{1}{h^2}U_{N-2,2m+1}\right)$$

In matrix form, the set of equations (1.19) is given by

$$\begin{bmatrix} U_{1,2m+1} \\ U_{2,2m+1} \\ U_{3,2m+1} \\ U_{4,2m+1} \\ - \\ - \\ - \\ - \\ - \\ U_{N-1,2m+1} \end{bmatrix} = \begin{bmatrix} U_{1,2m} \\ U_{2,2m} \\ U_{3,2m} \\ U_{4,2m} \\ - \\ - \\ - \\ - \\ - \\ U_{N-1,2m} \end{bmatrix} + \begin{bmatrix} -2/h^2 & \frac{1}{h^2} & & & 0 \\ 1/h^2 & -2/h^2 & \frac{1}{h^2} & & \\ 0 & \frac{1}{h^2} & -2/h^2 & \frac{1}{h^2} & \\ 0 & 0 & 1/h^2 & -2/h^2 & \frac{1}{h^2} \\ - & - & - & - & - \quad - \\ 0 & & & \frac{1}{h^2} & -2/h^2 \end{bmatrix} k$$

$$\times \begin{bmatrix} U_{1,s} \\ U_{2,s} \\ U_{3,s} \\ U_{4,s} \\ - \\ - \\ - \\ U_{N-1,s} \end{bmatrix} \quad -\ (1.20$$

Denoting the column vectors with components $U_{i,2m+1}$, $U_{i,2m}$ and $U_{i,s}$ by vector symbols U_{2m+1}, U_{2m} and U_s respectively and the matrix in equation (1.20) by L_h, equation (1.20) can be symbolically written as

$$U_{2m+1} = u_{2m} + kL_h u_s \tag{1.21}$$

We define s as, s = 2m whenever the column vector u_s multiplies an even row of the matrix L_h and s = 2m+1 whenever u_s multiplies the odd row.

The system (1.20) can be written as

$$\begin{bmatrix} 1 & 0 & 0 & - & - & - & - & 0 \\ 0 & 1 & 0 & - & - & - & - & 0 \\ 0 & 0 & 1 & - & - & - & - & 0 \\ - & - & - & & - & - & - & - \\ \\ 0 & 0 & 0 & - & - & - & - & 1 \end{bmatrix} \begin{bmatrix} U_{1,2m+1} \\ U_{2,2m+1} \\ U_{3,2m+1} \\ \vdots \\ \\ U_{N-1,2m+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & - & - & - & - & 0 \\ 0 & 1 & 0 & - & - & - & - & 0 \\ 0 & 0 & 1 & - & - & - & - & 0 \\ - & - & - & - & - & - & - & - \\ \\ 0 & 0 & 0 & - & - & - & - & 1 \end{bmatrix} \begin{bmatrix} U_{1,2m} \\ U_{2,2m} \\ U_{3,2m} \\ \vdots \\ \\ U_{N-1,2m} \end{bmatrix}$$

$$+k \begin{bmatrix} -2/h^2 & \frac{1}{h^2} & & & 0 \\ \frac{1}{h^2} & -2/h^2 & \frac{1}{h^2} & & \\ 0 & \frac{1}{h^2} & -2/h^2 & \frac{1}{h^2} & \\ - & - & - & - & \\ & 0 & & 1/h^2 & -2/h^2 \end{bmatrix} \begin{bmatrix} U_{1,s} \\ U_{2,s} \\ U_{3,s} \\ \vdots \\ U_{N-1,s} \end{bmatrix}$$

- (1.22)

or

$$Iu_{2m+1} = Iu_{2m} + k\, L_h us. \tag{1.23}$$

With the matrices I_1 and I_2 as defined above, we can write the product $L_h u_s$ as

$$L_h u_s = I_1 L_h\, u_{2m} + I_2 L_h u_{2m+1} \tag{1.24}$$

The matrices I_1 and I_2 in this case are given by

$$I_1 = \begin{bmatrix} 0 & 0 & 0 & - & - & - & 0 \\ 0 & 1 & 0 & - & - & - & 0 \\ - & - & - & - & - & - & - \\ 0 & 0 & 0 & - & - & - & 0 \end{bmatrix} \tag{1.25}$$

$$I_2 = \begin{bmatrix} 1 & 0 & 0 & - & - & - & 0 \\ 0 & 0 & 0 & - & - & - & 0 \\ - & - & - & - & - & - & - \\ 0 & 0 & 0 & - & - & - & 1 \end{bmatrix} \tag{1.26}$$

Thus equation (1.23) can be written as

$$I\, u_{2m+1} = I\, u_{2m} + k\, I_1 L_h\, u_{2m} + k\, I_2 L_h u_{2m+1} \tag{1.27}$$

With $H = - L_h$, the matrix equation (1.27) can be written as

$$I\, u_{2m+1} = I\, u_{2m} - kI_1 H\, u_{2m} - kI_2\, Hu_{2m+1}$$

$$\text{or} \qquad (I + kI_2 H)\, u_{2m+1} = (I - k\, I_1 H)\, u_{2m} \tag{1.28}$$

Equation (1.28) is similar in form to equation (1.12).

In the same way we can construct the global representation of equation (1.17) in the form of equation (1.14).

That is

$$(I + kI_1H)u_{2m+2} = (I - kI_2H)\, u_{2m+1}\ , \tag{1.29}$$

is the matrix equation for all the mesh points along the even time steps, m = 2,4,6, ------, in the rectangular mesh of fig. 1.2.

Now proceeding with the stability analysis, we first observe that the hopscotch process (1.13) and (1.14) is similar to the Peaceman - Rachford alternating-direction implicit (A.D.I) scheme discussed by Varga (1965). The splitting of the matrix H in this case, corresponding to that of the Peaceman - Rachford is given by

$$H = I_1H + I_2H \tag{1.30}$$

This formulation allows us to apply the alternating - direction implicit methods for stability and convergence analysis. It has been shown by Varga (1965) that the nxn matrix H arising from the general equation (1.1) with the operator L as defined, is an irreducible Stieltjes matrix. That is, H is a real symmetric and positive definite irreducible matrix with non-positive off-diagonal entries.

Combining the equations (1.13) and (1.14) into a single equation, we have

$$u_{2m+2} = Tu_{2m} + b_{2m} \tag{1.31}$$

where

$$\begin{aligned} T &= (I + kI_1H)^{-1}(I - kI_2H)(I+kI_2H)^{-1}(I-kI_1H) \\ &= (I + kI_1H)^{-1}\{(I-kI_2H)(I + kI_2H)^{-1}(I-kI_1H)^{-1}\}(I+kI_1H) \\ &= (I + kI_1H)^{-1}\,\hat{T}\,(I + kI_1H), \end{aligned}$$

u_{2m+2} is the unknown vector, u_{2m} is a known vector and b_{2m} is also a known vector.

Now since H is a positive definite square matrix, it has a positive definite square root $H^{\frac{1}{2}}$, so that

$$(I-kI_2H)(I + kI_2H)^{-1} = H^{-\frac{1}{2}}(I-kH^{\frac{1}{2}}I_2H^{\frac{1}{2}})(I + kH^{\frac{1}{2}}I_2H^{\frac{1}{2}})^{-1}H^{\frac{1}{2}}$$

$$= H^{-\frac{1}{2}}H_2H^{\frac{1}{2}} \qquad (1.32)$$

similarly

$$(I-kI_1H)(I + kI_1H)^{-1} = H^{-\frac{1}{2}}H_1H^{\frac{1}{2}} \qquad (1.33)$$

Hence $$T = \{H^{\frac{1}{2}}(I + kI_1H)\}^{-1} H_2H_1\{H^{\frac{1}{2}}(I+kI_1H)\} \qquad (1.34)$$

The two matrices $H^{-\frac{1}{2}}I_1H^{\frac{1}{2}}$ and $H^{-\frac{1}{2}}I_2H^{\frac{1}{2}}$ are non-negative definite.

The analysis by Gourlay (1970) requires the application of Kellog's lemmas whic are stated below. The reader is refered to Kellog (1964) for proofs.

Lemma 1 If $\rho > 0$ and $(B + B^*)$ is non-negative definite, then $(\rho I + B)$ has a bounded **inverse** and

$$||(\rho I + B)^{-1}|| \leq \frac{1}{\rho} \qquad (1.35)$$

Lemma 2 If $\rho > 0$ and $(B + B^*)$ is non-negative, then

$$||(\rho I - B)(\rho I + B)^{-1}|| \leq 1 \qquad (1.36)$$

Inorder to apply the two lemmas to our problem we must consider the form of the matrix H. It is clear from the matrix equation (1.20) that H can be written as

$$H = 1/_{h^2}\,\hat{H} \qquad (1.37)$$

Now let $r = {}^k/h^2$ be the mesh ratio assumed constant. In what follows we shall assume that H has been replaced by ${}^1/h^2\,\hat{H}$ but will maintain the notation H for $\hat{H}$.

Thus equation (1.34) can be written as

$$T = \{H^{\frac{1}{2}}(I + rI_1H)\}^{-1}H_2H_1\{H^{\frac{1}{2}}(I + rI_1H)\} \tag{1.38}$$

where

$$H_i = (I-rH^{\frac{1}{2}}I_iH^{\frac{1}{2}})\,(I + rH^{\frac{1}{2}}I_iH^{\frac{1}{2}})^{-1} \quad i = 1,2.$$

By Lemma 2 it follows that for $\gamma > 0$, constant,

$$H_i \leq 1, \quad i = 1,2.$$

For stability we require that powers of T be uniformly bounded. Raising the matrix T in equation (1.38) to the power m, we obtain

$$T^m = \{H^{\frac{1}{2}}(I + rI_1H\ )\}^{-1}(H_2H_1)^m\{H^{\frac{1}{2}}(I + rI_1H)\} \tag{1.39}$$

This result follows trivially from the fact that if we consider T to be of the form

$$T = A^{-1}\ B\ A$$

where $$A = H^{\frac{1}{2}}(I + rI_1H)$$

and $$B = H_2H_1,$$

then
$$\begin{aligned}
T^2 &= (A^{-1}\ B\ A)(A^{-1}\ B\ A) \\
&= A^{-1}\ B^2\ A \\
T^3 &= (A^{-1}B^2A)(A^{-1}B\ A) \\
&= A^{-1}\ B^3\ A \\
&- \quad - \quad - \\
&- \quad - \quad - \\
&- \quad - \quad - \\
&- \quad - \quad -- \\
T^m &= (A^{-1}B^{m-1}A)(A^{-1}B\ A) \\
&= A^{-1}B^mA.
\end{aligned}$$

Hence $$T^m = \{H^{\frac{1}{2}}(I + rI_1H)\}^{-1}(H_2H_1)^m\{H^{\frac{1}{2}}(I + rI_1H)\}$$

But $$||T^m|| \leq ||\{H^{\frac{1}{2}}(I + rI_1H)\}^{-1}||.||H^{\frac{1}{2}}(I+rI_1H)||$$

$$\leq C \text{ (a constant independent of } m)$$

proving that powers of T are uniformly bounded.

We can now sum up the foregoing in a theorem

Theorem 1

The hopscotch process is stable for the solution of equation (1.1) if the matrix H is positive definite.

Theorem 1 can be generalized by applying certain conditions on the matrices I_1H and I_2H. To prove the general case we start with the following lemma by Gourlay (1970).

Lemma 3

If the $n\times n$ non-singular matrix H has n independent left eigenvectors, then

(i) the matrices I_1H and I_2H also have n independent left eigenvectors

(ii) the null spaces of I_1H and I_2H are disjoint.

For the proof of this Lemma the reader is refered to Gourlay (1970).

The conditions which the matrices I_1H and I_2H have to satisfy for the hopscotch process to be stable are stated in the following theorem

Theorem 2

If the non-singular matrix H has a full set of independent eigenvectors, then the hopscotch process is stable if the eigenvalues of the two matrices I_1H, I_2H have non-negative real parts.

Proof

Since H is a real $n x n$ matrix arising from an elliptic operator, it has the following properties

$$h_{i,i} \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |h_{i,j}| \tag{1.40}$$

with $h_{i,j} \leq 0$, $i \neq j$

In short H is a Stieltjes matrix as already observed at the beginning of this section.

By Gerschgorin's theorem [Varga (1965)],

if $A = (a_{i,j})$ is an arbitrary $n x n$ complex matrix

and
$$S_i = \sum_{\substack{i=1 \\ j \neq i}}^{n} |a_{i,j}| \ , \quad 1 \leq i \leq n \tag{1.41}$$

then, all the eigenvalues λ of A lie in the union of the disks

$$|z - a_{i,i}| \leq S_i \ , \quad 1 \leq i \leq n. \tag{1.42}$$

Thus letting λ_i denote the n eigenvalues of the matrix H we have

$$|\lambda_i - h_{i,i}| \leq S_i \tag{1.43}$$

where $$S_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} |h_{i,j}| \ , \quad 1 \leq i \leq n \tag{1.44}$$

We notice here that the S_i's and $h_{i,i}$'s are all positive.

It is evident from (1.43) that

$$h_{i,i} - S_i \leq \lambda_i \leq S_i + h_{i,i} , \tag{1.45}$$

From the properties (1.40)

$$h_{i,i} \geq S_i .$$

Thus, both the left and right hand sides of (1.45) are positive, implying that all the λ_i's are positive. Since the matrices I_1H and I_2H are simply composed of rows from H and null rows, the proof follows.

Theorem 2 together with properties (1.40) now lead us to state the more general result in the following theorem

Theorem 3

The hopscotch process is stable if H is non-singular, has a full set of eigenvectors and satisfies the properties (1.40).

The proof for the case of problem (1.1) follows immediately from theorem 2.

Remark

With the above analysis, we can safely extend our results to problems in more than two space dimensions, so long as the matrix H arises from the replacement of an elliptic operator L.

The case where L is not an elliptic operator remains an open question for investigation.

1.3 CONVERGENCE

The global form of the two step hopscotch process for equation (1.1) is given by the matrix equation (1.31) viz;

$$u_{2m+2} = T\,u_{2m} + b_{2m}$$

where
$$T = (I + rI_1H)^{-1}(I - rI_2H)\,(I + rI_2H)^{-1}(I - rI_1H) \qquad (1.46)$$

in which $r = k/h^2$ is a constant mesh ratio.

If we denote the error vector associated with the vector iterate u_{2m} by

$$e_{2m} = u_{2m} - u \qquad (1.47)$$

then
$$e_{2m+2} = Te_{2m}$$

and
$$\begin{aligned} e_{2m} &= T\,e_{2m-2} \\ &= T\,(T\,e_{2m-4}) \\ &\;\;- \\ &\;\;- \\ &\;\;- \\ &\;\;- \\ &= T^m e_0 \end{aligned}$$

In general therefore, we have the error vector given by

$$e_{2m} = T^m e_o\ , \quad m \geq 1. \qquad (1.48)$$

To investigate the convergence of the power matrix T^m we recall that a matrix converges or diverges together with its powers, and that a matrix is convergent if its spectral radius is less than unity.

We now state an important result due to Collatz (1966) in the following lemma

Lemma 3

The matrix $\tilde{A} = XAX^{-1}$ which results from a transformation with the non-singular matrix X has the same eigenvalues as A.

Proof
$$\tilde{A} - \mu I = XAX^{-1} - \mu I$$
$$= X(A - \mu I)X^{-1}; \tag{1.49}$$

hence
$$\det(\tilde{A} - \mu I) = \det X \det (A - \mu I) \det X^{-1}$$
$$= \det (A - \mu I) \tag{1.50}$$

Now since matrix H is non-singular, the matrix I_1H is non-singular and so is $(I + rI_1H)$.

Using $(I + rI_1H)$ as the transformation matrix, the matrix T is transformed into
$$\tilde{T} = (I + rI_1H)T(I + rI_1H)^{-1} \tag{1.51}$$
$$= \{(I-rI_2H)(I + rI_2H)^{-1}\}.\{(I-rI_1H)(I + rI_1H)^{-1}\}$$

Using the notion that the spectral radius of an nxn matrix is less than or equal to its norm, we have
$$\rho(T) = \rho(\tilde{T}) \leq ||\tilde{T}|| , \quad \text{which implies}$$
$$\rho(T) \leq ||(I-rI_2H)(I+rI_2H)^{-1}|| \; . \; ||(I-rI_1H)(I + rI_1H)^{-1}|| \tag{1.52}$$

It has been proved in section (1.2) that the matrices I_1H and I_2H have positive eigenvalues.

Therefore, denoting the eigenvalues of I_2H by λ_j, $1 \leq j \leq n$, the eigenvalues of the matrix
$$(I-rI_2H)(I+rI_2H)^{-1}$$

are $\quad \left(\dfrac{1 - \lambda_j}{1 + \lambda_j}\right), \qquad 1 \leq j \leq n$

from which it is immediately seen that

$$\left(\frac{1 - \lambda_j}{1 + \lambda_j}\right) < 1, \text{ for all } j. \tag{1.53}$$

We introduce here another terminology common in matrix analysis. This is the Hermitian matrix. The reader is refered to Varga (1965) for the definition and results relating to this type of matrix. Here it only suffices to mention that all Stieltjes matrices are **Hermitian** . By definition of the norm of a **Hermitian** matrix we have that

$$||(I-rI_2H)(I + rI_2H)^{-1}|| = \max_{1\leq j\leq n} \left|\frac{1 - \lambda_j}{1 + \lambda_j}\right| < 1 \tag{1.54}$$

Applying the same argument to the matrix

$$(I-rI_1H)(I + rI_1H)^{-1}$$

we obtain $\quad ||(I-rI_1H)(I + rI_1H)|| < 1 \qquad (1.55)$

Thus the inequality (1.52) implies

$$\rho(T) < 1 \tag{1.56}$$

which completes the proof for convergence.

It is evident from the analysis of section (1.2) and (1.3) that the hopscotch process applied to linear parabolic equations is convergent and stable.

To develop a better mastery of the algorithm we illustrate the foregoing with an example.

Example 1.2

Find an approximation to the analytical solution of the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0<x<1 \tag{1.57}$$

satisfying the initial conditions

$$u = \sin \pi x, \quad \text{when } t = 0, \ 0 \leq x \leq 1 \tag{1.58}$$

and the boundary conditions

$$u = 0 \quad \text{at} \quad x = 0 \text{ and } 1, \quad \text{for all} \quad t > 0. \tag{1.59}$$

Solution

Comparison of equation (1.1) and (1.57) shows that

$$Lu = \frac{\partial^2 u}{\partial x^2} \quad \text{and} \quad g = 0$$

Thus $L_h U_i^m = {}^1/h^2(U_{i+1}^m - 2U_i^m + U_{i-1}^m), \ 1 \leq i \leq n,$

where $h = \Delta x.$

Equation (1.5) then gives

$$U_i^{m+1} - r\theta_i^{m+1}(U_i^{m+1} - 2U_i^{m+1} + U_{i-1}^{m+1}) = U_i^m + r\theta_i^m(U_{i+1}^m - 2U_i^m + U_{i-1}^{m+1}) \tag{1.60}$$

When $(m+i)$ is odd, (1.60) reduces to

$$U_i^{m+1} = U_i^m + r(U_{i+1}^m - 2U_i^m + U_{i-1}^m) \tag{1.61}$$

and when $(m+i)$ is even (1.60) becomes

$$U_i^{m+1} = U_i^m + r(U_{i+1}^{m+1} - 2U_i^{m+1} + U_{i-1}^{m+1}) \tag{1.62}$$

With $\Delta x = 0.1$, the interval $R_h = 0 \leq x \leq 1$ will consist of eleven mesh points. The solution is sought at nine internal mesh points since the values at $x = 0$ and 1 are already known. However, because the problem is symmetrical about the centre, we shall only require the solution at the first five mesh points (see Table 1.1).

Table 1.1

		i= 0	1	2	3	4	5
		x= 0	0.1	0.2	0.3	0.4	0.5
m=0	t=0	0	0.3090	0.5878	0.8090	0.9511	0.9902
m=1	0-001	0	x	x	x	x	x
2	0.002	0	x	x	x	x	x
⋮	⋮	⋮					

The values of U_i along the time row $t = 0$ are given by the initial condition (1.58).

So we have, for example

$$\text{at } x = 0.1, \quad u_1 = \sin \pi x$$
$$= \sin(0.1\pi) = 0.309$$

$$\text{at } x = 0.2 \quad u_2 = \sin(0.2\pi) = 0.5878$$

etc.

Given the values of U_i at $t = 0$, we are now ready to carry out the calculations along the first time step.

In the first time step, $(m+i)$ is odd at mesh points with $i=2,4$.

Therefore equation (1.61) gives

$$U_2^1 = U_2^0 + r(U_3^0 - 2U_2^0 + U_1^0)$$
$$U_4^1 = U_4^0 + r(U_5^0 - 2U_4^0 + U_3^0).$$

At mesh points with $i=1,3,5$ $(m+i)$ is even, and equation (1.62) gives

$$U_1^1 = U_1^0 + r(U_2^1 - 2U_1^1 + U_0^1)$$

$$= {}^1/_{1+2r}\ U_1^0 + {}^r/_{1+2r}\ (U_2^1 + U_0^1)$$

$$U_3^1 = U_3^0 + r(U_4^1 - 2U_3^1 + U_2^1)$$

$$= \frac{1}{1+2r} U_3^0 + {}^r/_{1+2r}\ (U_4^1 + U_2^1)$$

$$U_5^1 = U_5^0 + r(U_6^{11} - 2U_5^1 + U_4^1)$$

$$= U_5^0 + r(2U_4^1 - 2U_5^1)$$

$$= {}^1/_{1+2r}\ U_5^0 + \frac{2r}{1+2r} U_4^1 .$$

To proceed to the second time step we require the application of equation (1.7) which, in the context of the present example, is given by

$$U_i^{m+2} - r\theta_i^{m+2}(U_{i+1}^{m+2} - 2U_i^{m+2} + U_{i-1}^{m+2}) = 2U_i^{m+1} - \{U_i^m + r\theta_i^m(U_{i+1}^m - 2U_i^m + U_{i-1}^m)\} \qquad (1.63)$$

When $(m+i+1)$ is odd equation (1.63) reduces to simply

$$U_i^{m+2} = 2U_i^{m+1} - U_i^m \qquad (1.64)$$

Thus equation (1.64) is employed at all the mesh points with the property $(m+i+1)$ odd, along the second time step. These are the points with $i=1,3,5$;

so we have

$$U_1^2 = 2U_1^1 - U_0^1$$

$$U_3^2 = 2U_3^1 - U_3^0$$

$$U_5^2 = 2U_5^1 - U_5^0$$

For the remaining points along the second time step use equation (1.62). It will be noticed that the roles played by equation (1.61), (1.62) and (1.64) in the process as a whole as time increases are repeated after every other two steps. Therefore to calculate the values of U_i along the third and fourth time steps we employ the equations used in the first and second time steps respectively.

This problem was solved with $\Delta x = 0.1$ $\Delta t = 0.001$ and $r = 0.1$ and the results obtained were compared with the analytical solution. Table 1.2 shows the results at time $t = 0.002$.

Table 1.2

x	Analytical	t = 0.002	
		Numerical	Absolute Percent error
0.1	0.3030	0.3030	0.0
0.2	0.5763	0.5764	0.02
0.3	0.7932	0.7934	0.03
0.4	0.9325	0.9326	0.01
0.5	0.9805	0.9806	0.0

The analytical solution of the partial differential equation (1.57) satisfying the conditions (1.58) and (1.59) is given by

$$u = e^{-\pi^2 t} \sin \pi x.$$

Comparison of the two results in Table 1.2 shows that the hopscotch solution is quite accurate. The absolute percentage error is the absolute difference of the solutions expressed as a percentage of the analytical solution of the partial differential equation.

1.4. DERIVATIVE BOUNDARY CONDITIONS

In this section we shall apply the hopscotch method to a problem with derivative Boundary conditions.

Boundary Conditions expressed in terms of derivatives occur when, for example, the surface of a heat conducting material is thermally insulated. In this case there is no heat flow normal to the surface and the corresponding boundary condition is $\frac{\partial u}{\partial n} = 0$ at every point of the insulated surface. If the boundary walls are not insulated, a perfect conductivity condition rather than an insulating condition may be appropriate. Hence the relevant boundary condition in the case where there is heat flux across the boundary is generally given, in the case of one space dimensional problem, by

$$a \frac{\partial u}{\partial x} + bu = g(x) \tag{1.65}$$

where a,b are constants.

Using the central difference approximation for $\frac{\partial u}{\partial x}$ and taking the average of the neighbouring points of U_i, the finite difference representation of equation (1.65) becomes

$$\frac{a}{2h}(U_{i+1} - U_{i-1}) + \frac{b}{2}(U_{i+1}+U_{i-1}) = g_i. \tag{1.66}$$

when $i = 0$, equation (1.66) gives an expression for the left fictitious point U_{-1}, viz :

$$U_{-1} = \frac{2h}{h - a} g_o - \frac{bh + a}{bh - a} U_1, \tag{1.67}$$

while when $i = N$, we obtain an expression for the right fictitious point U_{N+1}, viz:

$$U_{N+1} = \frac{2h}{a + bh} g_N - \frac{bh - a}{bh + a} U_{N-1} \tag{1.68}$$

Example 1.3

Solve the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad , \quad 0<x<1 \tag{1.69}$$

satisfying the initial condition

$$u = 1 \quad \text{for} \quad 0 \le x \le 1 \quad \text{when} \quad t = 0 \tag{1.70}$$

and the boundary conditions

$$\frac{\partial u}{\partial x} = u \quad \text{at} \quad x = 0, \quad \text{for all } t \tag{1.71}$$

$$\frac{\partial u}{\partial x} = -u \quad \text{at} \quad x = 1, \quad \text{for all } t. \tag{1.72}$$

Solution

We first note that symmetry about $x = \frac{1}{2}$ allows us to compute only values in the interval $0 \le x \le \frac{1}{2}$

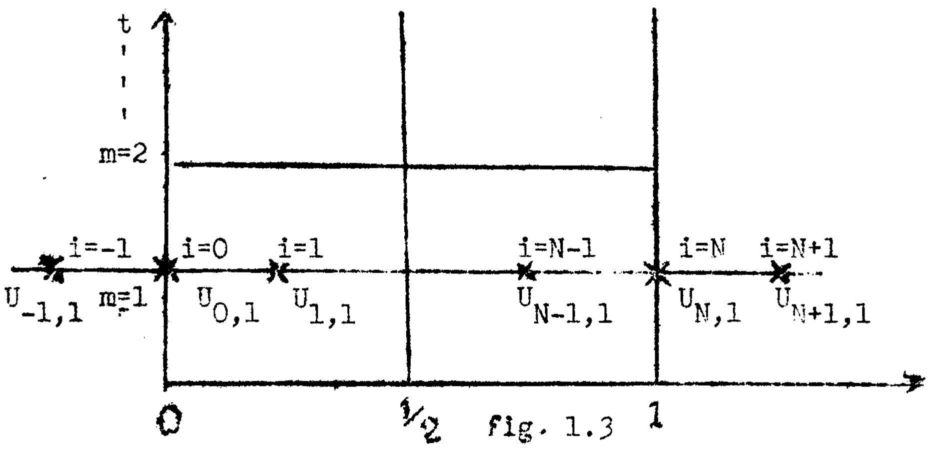

fig. 1.3

The calculations will then proceed as follows:

Along the first time step, the hopscotch process for equation (1.69) is given by

$$U_i^{m+1} = U_i^m + r(U_{i+1}^m - 2U_i^m + U_{i-1}^m), \quad \text{for} \quad (m+i) \text{ odd} \tag{1.73}$$

and

$$U_i^{m+1} = U_i^m + r(U_{i+1}^{m+1} - 2U_i^{m+1} + U_{i-1}^{m+1}), \quad \text{for} \quad (m+i) \text{ even} \tag{1.74}$$

The left boundary condition (1.71) gives

$$\frac{1}{2\Delta x}(U_{i+1}^{m}-U_{i-1}^{m}) = \tfrac{1}{2}(U_{i+1}^{m} + U_{i-1}^{m}), \text{ for } (m+i) \text{ odd} \quad (1.75)$$

or

$$\frac{1}{2\Delta x}(U_{i+1}^{m+1} - U_{i-1}^{m+1}) = \tfrac{1}{2}(U_{i+1}^{m+1} + U_{i-1}^{m+1}), \text{ for } (m+i) \text{ even} \quad (1.76)$$

At the point (0,1), that is the point on the left boundary along the first time step (see Fig. 1.3), the left fictitious point is given by equation (1.75) as

$$U_{-1}^{m} = \frac{1-\Delta x}{1+\Delta x} U_{1}^{m} \quad (1.77)$$

when $i = 0$, equation (1.73) gives

$$U_{o}^{m+1} = U_{o}^{m} + r(U_{1}^{m} - 2U_{o}^{m} + U_{-1}^{m}) \quad (1.78)$$

Upon replacing U_{-1}^{m} by $\frac{1-\Delta x}{1+\Delta x} U_{1}^{m}$ in equation (1.78) and simplifying, we obtain

$$U_{o}^{m+1} = (1-2r)U_{o}^{m} + r(1 + \frac{1-\Delta x}{1+\Delta x}) U_{1}^{m} \quad (1.79)$$

Therefore, along the first time step, equation (1.79) is used for the point on the boundary; equation (1.73) is applied at internal mesh points with (m+i) odd while equation (1.74) is employed at mesh points with (m+i) even.

Along the second time step, the hopscotch process for equation (1.69) is given by

$$U_{i}^{m+2} = 2U_{i}^{m+1} - U_{i}^{m}, \quad \text{for } (m+i+1) \text{ odd} \quad (1.80)$$

and

$$U_{i}^{m+2} = U_{i}^{m+1} + r(U_{i+1}^{m+2} - 2U_{i}^{m+2} + U_{i-1}^{m+2}), \text{ for} (m+i+1) \text{ even} \quad - (1.81)$$

At the point (0,2) that is the point on the left boundary along the second time step (see fig. 1.3), the left fictitious point is given by

$$U_{-1}^{m+2} = \frac{1-\Delta x}{1+\Delta x} U_{1}^{m+2} \quad (1.82)$$

when $i = 0$, equation (1.81) gives

$$U_0^{m+2} = U_0^{m+1} + r(U_1^{m+2} - 2U_0^{m+2} + U_{-1}^{m+2}) \tag{1.83}$$

Upon replacing U_{-1}^{m+2} by $\frac{1-\Delta x}{1+\Delta x} U_1^{m+2}$ in equation (1.83) we obtain, after some simplification, the equation

$$U_0^{m+2} = \frac{1}{1+2r} U_0^{m+1} + \frac{2r}{(1+2r)(1+\Delta x)} U_1^{m+2} \tag{1.84}$$

Therefore, along the second time step, equation (1.80) is applied at all those mesh points with $(m+i+1)$ odd; equation (1.84) is applied at the point on the boundary, while equation (1.81) is used at all those points with $(m+i+1)$ even.

The above process, starting from the first time step to the second time step, is repeated after every interval of two time steps.

Recall that the order of calculations in the hopscotch process should be strictly followed. That is, calculations along any given time step should begin with points with odd property ending with those points with even property.

The analytical solution of equation (1.69) satisfying the initial condition (1.70) and the Boundary conditions (1.71) and((1.72) is

$$u = 4 \sum_{n=1}^{\infty} \left\{\frac{\sec \alpha_n}{3+4\alpha_n^2}\right) e^{-4t\alpha_n^2} \cos 2\alpha_n(x-\tfrac{1}{2})\}, \quad 0<x<1 \tag{1.85}$$

where α_n are the positive roots of

$$\alpha \tan \alpha = \tfrac{1}{2}.$$

Table 1.3 shows the analytical values of u at time $t = 0.5$ together with hopscotch results for $r = 0.5$, $\Delta x = 0.1$ and $\Delta t = 0.05$.

Table 1.3

	$t = 0.5$		
x	Analytical	Hopscotch	Absolute Percent Error
0.0	0.3619	0.3618	0.03
0.1	0.3949	0.3946	0.08
0.2	0.4212	0.4208	0.09
0.3	0.4403	0.4398	0.11
0.4	0.4519	0.4513	0.13
0.5	0.4558	0.4552	0.13

Comparison of the results in Table 1.3 clearly shows a good agreement between the analytical and hopscotch solutions. It is therefore evident from this that the hopscotch method is also well suited for solving parabolic equations with derivative boundary conditions.

CHAPTER 2

Elliptic Equations

Elliptic partial differential equations usually arise from equilibrium or steady-state problems. The best known elliptic equations are Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = g(x,y) \qquad (2.1)$$

and Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \qquad (2.2)$$

Poisson's equation defines phenomena such as the slow motion of incompressi viscous fluid , and the inverse-square law theories of electricity, magnetis and gravitating matter at points where the charge density, pole strength or mass density respectively are non-zero. On the other hand Laplaces' equation arises in the theories associated with the steady flow of heat or electricity in homogeneous conductors. For further details and practical examples the reader is refered to, for instance Smith (1978).

2.1 DEVELOPMENT OF THE ALGORITHM

Consider the Poisson's equation (2.1). Applying the E-operator L_h of section (1.1) the finite difference replacement of equation (2.1) becomes

$$1/h^2(U_{i+1,j} - 2U_{i,j} + U_{i-1,j}) + 1/h^2(U_{i,j+1}-2U_{i,j}+U_{i,j-1})=g_{i,j} \qquad - (2.3)$$

where $h = \Delta x = \Delta y$.

Multiplying through by h^2 and simplifying, equation (2.3) gives

$$-4U_{i,j} + U_{i-1,j} + U_{i+1,j} + U_{i,j+1} + U_{i,j-1} = h^2 g_{i,j} \cdot \qquad (2.4)$$

Making $U_{i,j}$ the subject of the formula, equation (2.4) gives

$$U_{i,j} = \frac{1}{4}(U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1}) + \frac{1}{4} f_{i,j} \qquad (2.5)$$

where $f_{i,j} = -h^2 g_{i,j}$.

Equation (2.5) is Liebman's iterative scheme (see Curtis (1978)) for the numerical solution of equation (2.1).

Putting superscripts to show that a new value is computed from previous iterates gives

$$U^{m+1}_{i,j} = \frac{1}{4}(U^m_{i+1,j} + U^m_{i-1,j} + U^m_{i,j+1} + U^m_{i,j-1}) + \frac{1}{4} f^m_{i,j} \qquad (2.6)$$

Adding and subtracting $U^m_{i,j}$ on the right hand side of equation (2.6) yields

$$U^{m+1}_{i,j} = U^m_{i,j} + \frac{1}{4}(U^m_{i-1,j} + U^m_{i+1,j} + U^m_{i,j-1} + U^m_{i,j+1} - 4U^m_{i,j}) + \frac{1}{4} f_{i,j} \qquad (2.7)$$

For fast convergence to be achieved, equation (2.7) requires introduction of an over-relaxation factor w.

Thus equation (2.7) becomes

$$U^{m+1}_{i,j} = U^m_{i,j} + \frac{w}{4}(U^m_{i-1,j} + U^m_{i+1,j} + U^m_{i,j-1} + U^m_{i,j+1} - 4U^m_{i,j} + f^m_{i,j}) \qquad (2.8)$$

The above construction of equation (2.8) and the equation itself are precisely that of the successive over-relaxation (S.O.R) scheme as presented by Curtis (1978).

The hopscotch process requires us to have, in addition to equation (2.8), an implicit scheme

$$U^{m+1}_{i,j} = U^{m}_{i,j} + \frac{w}{4}\left(U^{m+1}_{i-1,j} + U^{m+1}_{i+1,j} + U^{m+1}_{i,j-1} + U^{m+1}_{i,j+1} - 4U^{m+1}_{i,j} + f^{m+1}_{i,j}\right) \quad -(2.9)$$

However, Gourlay (1971) proposes that, in order to further accelarate convergence, the factor $w/4$ in equation (2.9) should be replaced by $w/4-4w$, so that equation (2.9) gives

$$U^{m+1}_{i,j} = U^{m}_{i,j} + \frac{w}{4-4w}\left(U^{m+1}_{i-1,j} + U^{m+1}_{i+1,j} + U^{m+1}_{i,j-1} + U^{m+1}_{i,j+1} - 4U^{m+1}_{i,j} + f^{m+1}_{i,j}\right) \quad (2.10)$$

The next problem which arises is the estimation of the value of w. Gourlay (1970) suggests a way of estimating the value of this accelaration parameter. Here we shall use the method suggested for the S.O.R scheme by Curtis (1978) because application of the value of w obtained by using the latter method has produced results just as good as those obtained with the use of Gourlay's factor.

For the rectangular region having Dirichlet conditions (in other words constant boundary conditions), a reasonable estimate of the optimum value of w can be determined as the smaller root of the quadratic equation

$$(\cos \pi/p + \cos \pi/q)^2\, w^2 - 16w + 16 = 0 \quad (2.11)$$

where p and q are the number of mesh divisions on each side of the rectangular region.

For details regarding evaluation of w for non-rectangular regions without constant boundary conditions, the reader is refered to Forsythe and Wasow (1960), Fox(1962) and Varga (1965).

Solving equation (2.11) for w gives

$$w_{opt} = \frac{4}{2 + \sqrt{4 - c^2}} \quad (2.12)$$

where $c = \cos \pi/p + \cos \pi/q$.

We now introduce the odd-even function $\theta^m_{i,j}$ defined as

$$\theta^m_{i,j} = \begin{cases} 1 & \text{if } m+i+j \text{ is an odd integer} \\ 0 & \text{if } m+i+j \text{ is an even integer.} \end{cases} \tag{2.13}$$

Equation (2.13) enables us to write the two equations (2.8) and (2.9) as a single equation

$$U^{m+1}_{i,j} - \theta^{m+1}_{i,j} \frac{w}{4}(U^{m+1}_{i-1,j} + U^{m+1}_{i+1,j} + U^{m+1}_{i,j-1} + U^{m+1}_{i,j+1} - 4U^{m+1}_{i,j} + f^{m+1}_{i,j})$$

$$= U^m_{i,j} + \theta^m_{i,j} \frac{w}{4} (U^m_{i-1,j} + U^m_{i+1,j} + U^m_{i,j+1} - 4U^m_{i,j} + f^m_{i,j}) \tag{2.14}$$

Incrementing m by 1 in equation (2.24) we obtain

$$U^{m+2}_{i,j} - \theta^{m+2}_{i,j} \frac{w}{4} (U^{m+2}_{i-1,j} + U^{m+2}_{i+1,j} + U^{m+2}_{i,j-1} + U^{m+2}_{i,j+1} - 4U^{m+2}_{i,j} + f^{m+2}_{i,j})$$

$$= U^{m+1}_{i,j} + \theta^{m+1}_{i,j} \frac{w}{4}(U^{m+1}_{i-1,j} + U^{m+1}_{i+1,j} + U^{m+1}_{i,j-1} + U^{m+1}_{i,j+1} - 4U^{m+1}_{i,j} + f^{m+1}_{i,j})$$
$$= 2U^{m+1}_{i,j} - U^m_{i,j} - \theta^m_{i,j} \frac{w}{4}(U^m_{i-1,j} + U^m_{i,j-1} + U^m_{i,j+1} - 4U^m_{i,j} + f^m_{i,j}) \tag{2.15}$$

When $(m+i+j)$ is odd, equation (2.15) reduces to

$$U^{m+2}_{i,j} = 2U^{m+1}_{i,j} - U^m_{i,j}. \tag{2.16}$$

With equations (2.8), (2.10) and (2.16) we are now able to state the two level hopscotch process for the numerical solution of the elliptic equation (2.1).

In the first iteration, apply equation (2.8) at those mesh points with $(m+i+j)$ an odd integer, and use equation (2.10) at points with $(m+i+j)$ an even integer.

For the second iteration equation (2.16) is used at mesh points with (m+i+j+1) an odd integer where-as equation (2.10) is applied at mesh points with (m+i+j+1) an even integer. The whole process is repeated after every two iterations.

We **shall** now examine the convergence and stability of the process.

2.2. CONVERGENCE

Equations (2.8) and (2.9) can respectively be written as

$$\frac{4}{w} U^{m+1}_{i,j} = \frac{4}{w} U^{m}_{i,j} + (U^{m}_{i-1,j} + U^{m}_{i+1,j} + U^{m}_{i,j-1} + U^{m}_{i,j+1} - 4U^{m}_{i,j}) + f^{m}_{i,j} \quad (2.17)$$

and

$$\frac{4}{w} U^{m+1}_{i,j} = \frac{4}{w} U^{m}_{i,j} + (U^{m+1}_{i-1,j} + U^{m+1}_{i+1,j} + U^{m+1}_{i,j-1} + U^{m+1}_{i,j+1} - 4U^{m+1}_{i,j}) + f^{m+1}_{i,j} \quad (2.1$$

In view of equation (2.17) and (2.18) and in the spirit of section (1.2) of chapter 1, we can define the global form of the two level hopscotch process as

$$(wI + I_2 H)u_{2m+1} = (wI - I_1 H)u_{2m} + b \qquad (2.19)$$

$$(wI + I_1 H)u_{2m+2} = (wI - I_2 H)u_{2m+1} + b \qquad (2.20)$$

where the matrices I_1 , I_2 , H and the vectors u_{2m}, u_{2m+1} are as defined in section (1.2). b is a known vector.

Upon replacing $I_1 H$ and $I_2 H$ by H_1 and H_2 respectively, equations (2.19) and (2.20) respectively become

$$(wI + H_2)u_{2m+1} = (wI - H_1)u_{2m} + b \qquad (2.21)$$

$$(wI + H_1)u_{2m+2} = (wI - H_2)u_{2m} + b \qquad (2.22)$$

The system of equations (2.21) and (2.22) is precisely of the Peaceman-Rachford form discussed by Varga (1965). This analysis of the convergence criterion will therefore follow the analysis of Varga. It may be noted that we shall simply demonstrate convergence here. Analysis of the rate of convergence is beyond the scope of this work.

Combining the two equations (2.21) and (2.22) gives

$$u_{2m+2} = Tu_{2m} + b_{2m} \tag{2.23}$$

where the iterative matrix T is now given by

$$T = (wI + H_1)^{-1}(wI - H_2)(wI + H_2)^{-1}(wI - H_1) \tag{2.24}$$

However, Gourlay (1970) suggests that the replacement of the identity matrix I in (2.21) and (2.22) by a positive definite matrix D may accelarate convergence. This we suppose is in line with his proposed replacement of the factor $\frac{w}{4}$ by $\frac{w}{4 - 4w}$ in equation (2.9).

Thus replacing the identity matrix I by a positive definite matrix D in (2.21) and (2.22) yields the more general procedure

$$(wD + H_2)u_{2m+1} = (wD - H_1)u_{2m} + b \tag{2.25}$$

$$(wD + H_1)\, u_{2m+2} = (wD - H_2)u_{2m+1} + b \tag{2.26}$$

so that the combined form of equations (2.25) and (2.26)

$$u_{2m+2} = Tu_{2m} + b_{2m} \tag{2.27}$$

gives the iteration matrix T as

$$T = (wD + H_1)^{-1}(wD - H_2)(wD + H_2)^{-1}(wD - H_1). \tag{2.28}$$

Defining the vector v by

$$D^{\frac{1}{2}}u = v \tag{2.29}$$

the system of equations (2.25) and (2.26) takes the form

$$(wI + \hat{H}_2)v_{2m+1} = (wI - \hat{H}_1)v_{2m} + D^{-\frac{1}{2}}b \tag{2.30}$$

$$(wI + \hat{H}_1)v_{2m+2} = (wI - \hat{H}_2)v_{2m+1} + D^{-\frac{1}{2}}b \tag{2.31}$$

where $$\hat{H}_i = D^{-\frac{1}{2}}H_iD^{-\frac{1}{2}}, \quad i = 1,2.$$

Note that the original system of equations (2.25) and (2.26) may be recovered by replacing $\hat{H}_1$, $\hat{H}_2$ and V in (2.30) and (2.31) by $D^{-\frac{1}{2}}H_1D^{-\frac{1}{2}}$, $D^{-\frac{1}{2}}H_2D^{-\frac{1}{2}}$ and $D^{\frac{1}{2}}u$ respectively and then premultiply each equation by $D^{-\frac{1}{2}}$.

Now the iterative matrix T arising from the transformed equations (2.30) and (2.31) is given by

$$T = (wI + \hat{H}_1)^{-1}(wI - \hat{H}_2)(wI + \hat{H}_2)^{-1}(wI - \hat{H}_1) \tag{2.32}$$

Using lemma 3 of chapter 1 and the fact that H is a non-singular Stieltjes matrix H_1 is non-singular and so is $\hat{H}_1$. Therefore the matrix $(wI + \hat{H}_1)$ is also non-singular.

Using the matrix $(wI + \hat{H}_1)$ to transform T as in lemma 3 of chapter 1, we have

$$\hat{T} = (wI + \hat{H}_1)\, T\, (wI + \hat{H}_1)^{-1}$$

$$= \{(wI - \hat{H}_2)\,(wI + \hat{H}_2)^{-1}\}.\ \{(wI - \hat{H}_1)\,(wI + \hat{H}_1)^{-1}\} \tag{2.34}$$

Using the well known result that the norm of any square matrix is greater than or equal to its spectral radius, we have that

$$\rho(T) = \rho(\hat{T}) \leq ||\hat{T}|| \leq ||(wI-\hat{H}_2)(wI + \hat{H}_2)^{-1}||.||(wI-\hat{H}_1)\ (wI+\hat{H}_1)^{-1}|| \tag{2.35}$$

Since H is a Stieltjes matrix then H is a positive definite Hermitian matrix. This in turn implies that all the matrices arising from H in the above transformations are positive definite Hermitian matrices.

Therefore the product matrices of (2.35) are positive definite Hermitian matrices.

Denoting the eigenvalues of $\hat{H}_2$ and $\hat{H}_1$ by λ_j and u_j, $i \leq j \leq n$, respectively, the eigenvalues of the matrices $(wI - \hat{H}_2)(wI + \hat{H}_2)$ and $(wI - \hat{H}_1)(wI + \hat{H}_1)$ are respectively given by $(\frac{w - \lambda_j}{w + \lambda_j})$ and $(\frac{w - \mu_j}{w + \mu_j})$ where $1 \leq j \leq n$.

By definition, the spectral radius of an nxn matrix A with eigenvalue λ_j, $1 \leq j \leq n$, is given by

$$\rho(A) = \max_{1 \leq j \leq n} |\lambda_j| \tag{2.36}$$

Furthermore if A is a Hermitian matrix

$$||A|| = \rho(A) \cdot \tag{2.37}$$

Using results (2.36) and (2.37), we have that

$$||(wI - \hat{H}_2)(wI + \hat{H}_2)^{-1}|| = \max_{1 \leq j \leq n} \left| \frac{w - \lambda_j}{w + \lambda_j} \right| < 1 \tag{2.38}$$

and

$$||wI - \hat{H}_1)(wI + \hat{H}_1)^{-1}|| = \max_{1 \leq j \leq n} \left| \frac{w - \mu_j}{w + \mu_j} \right| < 1 \tag{2.39}$$

From results (2.38) and (2.39) we can conclude that the spectral radius of the matrix T as defined in (2.35) is less than unity, thus completin the proof.

2.3 STABILITY

As in convergence, the analysis for the stability creterion for elliptic equations follows exactly the same steps as those in the paraboli case.

We can therefore state that the hopscotch process (2.25), (2.26) is stable if the non-singular matrix H has a full set of eigenvectors and the two matrices H_1 and H_2 have non-negative eigenvalues.

Infact by theorem 3 of chapter 1, the hopscotch process (2.25), 2.26) is stable if the matrix H is non-singular, has a full set of eigenvectors and satisfies the properties (1.40) in section (1.2) of chapter 1.
But H is a non-singular Stieltjes matrix. Therefore it satisfies all the conditions in theorem 3. Hence the proof.

Example 2.1

A thin rectangular steel plate is of dimensions 10cm x 20cm. If on of the 10cm edges is held at 0°c, what are the steady - state temperatures at interior points?

This problem is stated mathematically in the following way:

Find $U(x,y)$ such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (2.40)$$

with boundary conditions

$$u(x,0) = 0$$
$$u(x,10) = 0$$
$$u(0,y) = 0$$
$$u(20,y)=100$$

This is the so called Dirichlet boundary value problem (i.e a problem with known boundary values).

Solution

Equation (2.40) gives us the computational algorithms

$$U^{m+1}_{i,j} = U^{m}_{i,j} + \frac{w}{4}\{U^{m}_{i+1,j} + U^{m}_{i-1,j} + U^{m}_{i,j-1} + U^{m}_{i,j+1} - 4U^{m}_{i,j}\} \quad -(2.41)$$

$$U^{m+1}_{i,j} = U^{m}_{i,j} + \frac{w}{4-4w}\{U^{m+1}_{i+1,j} + U^{m+1}_{i-1,j} + U^{m+1}_{i,j-1} + U^{m+1}_{i,j+1} - 4U^{m+1}_{i,j}\} \quad -(2.42)$$

and $$U^{m+2}_{i,j} = 2U^{m+1}_{i,j} - U^{m}_{i,j} \quad (2.43)$$

In the first iteration use equation (2.41) when $(m+i+j)$ is odd and use equation (2.42) when $(m+i+j)$ is even.

For the second iteration apply equation (2.43) for points with $(m+i+j+1)$ odd, otherwise use equation (2.42).

Letting $h = \Delta x = \Delta y = 2.5$, the solution is sought at 21 interior mesh points as in fig. 2.1

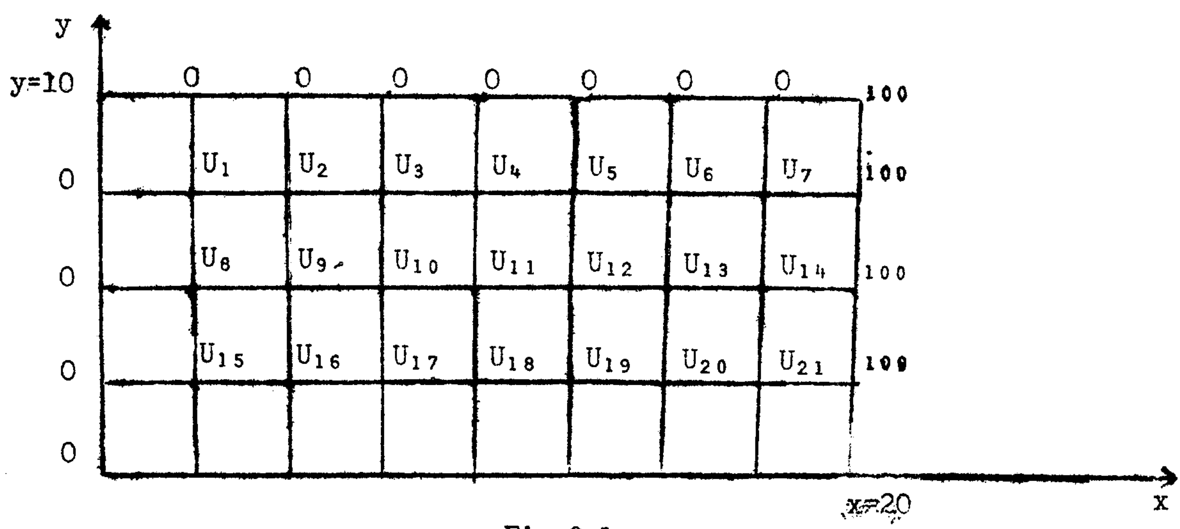

Fig 2.1

The problem arising at this stage is that of estimating reasonably good starting values for the 21 mesh points. (Collatz (1966) for instance, give a very good discussion on this subject).

To do this we first start with a course mesh of $h = 5$cm. This gives us only three unknown values of $U_{i,j}$ shown in fig.2.2, which satisfy the general equation

$$-4U_{i,j}+U_{i+1,j}+U_{i-1,j}+U_{i,j-1}+U_{i,j+1} = 0 \qquad (2.44)$$

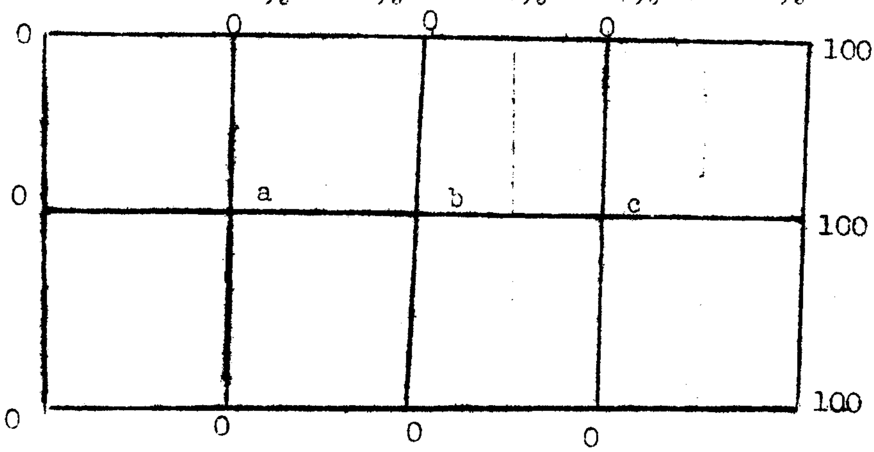

Fig. 2.2

Application of equation (2.44) to the three mesh points a,b,c in fig. 2 yields three equations

$$\begin{aligned} -4a + b &= 0 \\ -4b + c + a &= 0 \\ -4c + b + 100 &= 0 \end{aligned} \qquad (2.45)$$

Solving the set of equations (2.45) simultaneously gives $a = 1.7857$, $b = 7.1429$ and $c = 26.7857$. Next we subdivide the rectangular region into mesh points of size $h = {}^{10}/_{3}$ cm. This gives us 10 unknown values of $U_{i,j}$ (fig. 2.3), which also satisfy the general equation (2.44)

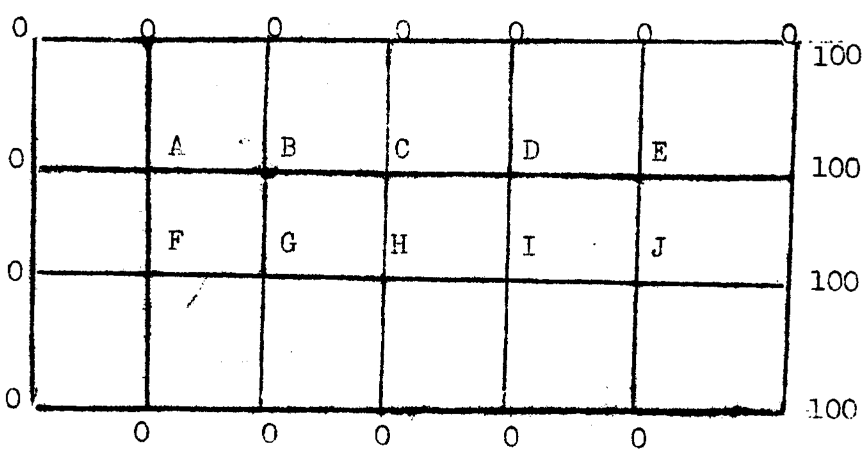

Fig. 2.3

Application of equation (2.44) to the 10 mesh points gives rise to 10 simultaneous equations in A,B,C, D,E,F,G,H,I and J unknowns. However, on account of symmetry, only the five equations in A,B,C,D and E unknowns may be solved. These are

$$
\begin{aligned}
-4A + A + B &= 0 \\
-4B + B + A + C &= 0 \\
-4C + C + B + D &= 0 \\
-4D + D + C + E &= 0 \\
-4E + E + D + 100 &= 0
\end{aligned}
\qquad (2.46)
$$

Solving the set of simultaneous equations (2.46) yields the results

$$
\begin{aligned}
A = F &= 0.6945 \\
B = G &= 2.0834 \\
C = H &= 5.5556 \\
D = I &= 14.5833 \\
E = J &= 38.1944
\end{aligned}
$$

With the above results at hand, we can now estimate the other remaining starting values in figure 2.1 by interpolation. The reader is refered to Curtis (1978) for a thorough discussion on the subject of interpolation. Application of the Newton-Gregory forward and backward interpolation polynomials to the mesh points in fig. 2.1 and using the values obtained in fig. 2.3, we obtain the starting values along the first and third mesh rows

$$
\begin{aligned}
U_1 = U_{15} &= 0.6728 \qquad & U_4 = U_{18} &= 5.5546 \\
U_2 = U_{16} &= 1.1285 \qquad & U_5 = U_{19} &= 11.8967 \\
U_3 = U_{17} &= 2.4307 \qquad & U_7 = U_{21} &= 50.0651
\end{aligned}
$$

For the values along the middle mesh row in fig. 2.1, U_9, U_{11} and U_{13} are known, as these are respectively the same points as a, b, and c in fig. 2.2. The remaining points, namely U_8, U_{10}, U_{12} and U_{14} can be obtained by application of equation (2.44). This gives the results

$U_8 = 0.7828$ $\quad U_{12} = 14.4305$

$U_9 = 1.7857$ $\quad U_{13} = 26.7857$

$U_{10} = 3.2739$ $\quad U_{14} = 50.7936$

$U_{11} = 7.1429$

With the full set of starting values thus obtained, calculations using equations (2.41), (2.42) and (2.43) gave the following results:

$U_1 = 0.3530$	$U_8 = 0.4989$	$U_{15} = 0.3530$
$U_2 = 0.9132$	$U_9 = 1.2894$	$U_{16} = 0.9132$
$U_3 = 2.0103$	$U_{10} = 2.8324$	$U_{17} = 2.0103$
$U_4 = 4.2957$	$U_{11} = 6.0194$	$U_{18} = 4.2957$
$U_5 = 9.1532$	$U_{12} = 12.6536$	$U_{19} = 9.1532$
$U_6 = 19.6632$	$U_{13} = 26.2894$	$U_{20} = 19.6632$
$U_7 = 43.2101$	$U_{14} = 53.1774$	$U_{21} = 43.2101$

Table 2.1 shows the analytical and hopscotch results at the points U_9 U_{11} and U_{13}.

Table 2.

	Analytical	Hopscotch	Absolute Percent Error
U_9	1.0943	1.2894	8.5
U_{11}	5.4885	6.0194	9.7
U_{13}	26.0944	26.2894	0.4

Comparison of these results shows that the error is reasonably small at Points near the source of high temperature, for example at U_{13}, but becomes quite unreasonably large as one moves further away from the source, for example at points U_9 and U_{11}. This phenomena is rather difficult to explain. Whether the large percentage error is caused by the crudeness of the starting values or is inherent in the method itself remai matters of conjecture.

However, the question which remains to be answered is whether the results are acceptable or not. The only way one could make a judgement was to compare the hopscotch results with those obtained by applying the already established numerical methods. For example the same problem under discussion was solved using the elimination and Liebmann methods discussed by Curtis (1978). Table 2.2 shows some of the results.

Table 2.2

	Hopscotch	Elimination	Liebmann
U_9	1.2894	1.2894	1.291
U_{11}	6.0194	6.0193	6.021
U_{13}	26.2894	26.2893	26.290

Comparison of the results in Table 2.2 indicates a strong agreement between the hopscotch method on one hand and the elimination and Liebmann methods on the other. It follows therefore that if the elimination and Liebmann methods are acceptable schemes for approximating the analytical solutions of elliptic partial differential equations, then the hopscotch method should even be more acceptable. This is because of the advantages the latter method has over the other two.

For instance in the elimination method the storage capacity in the computer increases and becomes unbearable as the mesh width is reduced inorder to improve the accuracy of the solution. The problem of storage is fully discussed by Curtis (1978). The hopscotch method takes care of this problem by overwriting the preceeding values. On the other hand, the drawback in the Liebmann method is that as the mesh width is decreased, convergence becomes very slow. The hopscotch method again takes care of this drawback. Infact, in the example above, 21 iterations were required for the solution to converge in the case of the Liebmann method whereas solution converged after only 13 iterations in the case of the hopscotch method.

It is evident from chapters 1 and 2 that the hopscotch method yields better results when it is used to solve parabolic equations than elliptic equations. The author would therefore, recommend the hopscotch method without any reservations when the problem to be solved is parabolic.

CHAPTER 3

Nonlinear Hyperbolic Equations

The universal applicability nature of the hopscotch algorithm has been mentioned in the general introduction of this work. However, the emphasis so far has been placed on parabolic and elliptic equations of the linear type. To break this trend, we shall now examine hyperbolic equations of the nonlinear type in this chapter. Application of the method to linear hyperbolic equations follows precisely those already considered in the previous cases.

Work has been done over the past few years and several finite difference schemes have been devised for the solution of nonlinear hyperbolic systems of the form $\frac{\partial u}{\partial t} - \frac{\partial f}{\partial x} = 0$, where u is a vector function of the unknowns $(u_1, u_2, u_3, ----, u_N)$ and f is a nonlinear function of the components of u. To a larger extent such methods have been tested exhaustively on smooth solutions although increasing attention is being paid to the schemes" capabilities of representing schock-like phenomena that arise in the solutions of nonlinear hyperbolic systems.

The explicit methods that have been considered fall into two distinct categories, namely, first order methods with added viscosity terms, and sec order methods which have no additional viscosity terms but are of higher order accuracy than the former methods.

Of the two types of methods, Gourlay and Morris(1972) contend, partly through Van Lear's analysis and partly out of their own analysis, that first order methods with an optimal pseudoviscosity term seem to do better on problems with discontinuities than do the higher order accurate methods. In the same paper by Gourlay and Morris reference is further made to existing works which consider schemes of third order accuracy. The method we shall consider here falls in the first category.

3.1 DEVELOPMENT OF THE ALGORITHM

Consider the first order system of conservation laws

$$\frac{\partial u}{\partial t} - \frac{\partial f}{\partial x} = 0 , \qquad (3.1)$$

satisfying the initial and boundary conditions

$$u(x,0) = g(x) \qquad \text{for } a\leq x\leq b$$

$$u(0,t) = h(t), \qquad t\geq 0$$

where f is a vector function of the components of u and u is an unknown vector function of x and t.

For a full discussion on conservation laws the reader is refered to, for instance, Mitchell (1969). We note that $\frac{\partial f}{\partial x}$ can be written in non-conservation form as $\frac{\partial f}{\partial x} = A\frac{\partial u}{\partial x}$ so that equation (3.1) becomes

$$\frac{\partial u}{\partial t} - A\frac{\partial u}{\partial x} = 0, \qquad (3.2)$$

where A is the Jacobian matrix of the components of f with respect to the components of u.

Let us illustrate the foregoing with a practical example.

Example 3.1

Conservation of mass, energy and momentum per unit volume gives rise to equation (3.1).

i.e $$\frac{\partial u}{\partial t} - \frac{\partial f}{\partial x} = 0$$

with

$$u = \begin{pmatrix} \rho \\ e \\ m \end{pmatrix}, \quad f(u) = \begin{pmatrix} -m \\ \dfrac{-em}{\rho} + \tfrac{1}{2}(\gamma-1)\dfrac{m^3}{\rho^2} \\ -(\gamma-1)e + \tfrac{1}{2}(\gamma-3)\dfrac{m^2}{\rho} \end{pmatrix}$$

where

ρ is the mass per unit volume

$m = \rho v$ is the momentum per unit volume

v is the velocity in the x-direction

γ is the ratio of specific heats

$e = \frac{1}{2}\rho v^2 + \dfrac{p}{(\gamma-1)}$ is the energy per unit volume

p is the pressure.

The nonconservation form (3.2) will therefore be given by

$$\frac{\partial u}{\partial t} - A(u)\frac{\partial u}{\partial x} = 0,$$

where $A(u) = \dfrac{\partial(f_1, f_2, f_3)}{\partial(\rho, e, m)}$,

where $f_1 = -m$, $f_2 = \dfrac{-\gamma em}{\rho} + \frac{1}{2}(\gamma-1)\dfrac{m^3}{\rho^2}$

and $f_3 = -(\gamma-1)e + \frac{1}{2}(\gamma-3)\dfrac{m^2}{\rho}$.

In this example it is easily seen that

$$A = \begin{pmatrix} 0 & 0 & -1 \\ -\gamma EM - (\gamma-1)M^3 & -\gamma M & -\gamma E + {}^3/_2(\gamma-1)M^2 \\ -\frac{1}{2}(\gamma-3)M^2 & -(\gamma-1) & (\gamma-3)M \end{pmatrix}$$

where $M = {}^m/_\rho$ and $E = {}^e/_\rho$.

Equation (3.2) is called hyperbolic if the eigenvalues of A are real and A has n-linearly independent eigenvectors.

Another good discussion regarding difference schemes for solving equation (3.2) for various forms of the matrix A is given by Mc Guire and Morris (1973).

For the present we assume A is a constant matrix with real eigenvalues and n-linearly independent eigenvectors.

Let $\Delta x = h$, $\Delta t = k$ and $U_i^m = U(ih, mk)$ be the grid parameters with the grid ratio $r = k/h$ held constant.

Define the finite difference operators Hx, $\delta^2 x$, and μ_x by

$$H_x U_i^m = U_{i+1}^m - U_{i-1}^m \tag{3.3}$$

$$\delta^2 x\ U_i^m = U_{i+1}^m - 2U_i^m + U_{i-1}^m \tag{3.4}$$

$$\mu_x U_i^m = \tfrac{1}{2}(U_{i+1}^m + U_{i-1}^m) \text{ - the average difference operator} \tag{3.5}$$

With these notations the classic first order finite difference scheme due to Lax (see Ames (1977)) is given by

$$U_i^{m+1} = U_i^m - \frac{r}{2} H_x f_i^m + \alpha\delta^2 x\ U_i^m \tag{3.6}$$

where α is a constant coefficient of the pseudoviscous term $\delta^2{}_x U_i^m$ which is chosen to obtain best shock resolutions, and $\alpha \geq 0$. Introducing the odd-even function θ defined

as
$$\theta_i^m = \begin{cases} 1 & (m+i) \quad \text{odd} \\ 0 & (m+i) \quad \text{even} \end{cases} \tag{3.7}$$

the resulting hopscotch scheme is given by

$$U_i^{m+1} + \theta_i^{m+1}(\frac{r}{2} H_x f_i^{m+1} - \alpha\delta^2{}_x U_i^{m+1})$$

$$= U_i^m - \theta_i^m(\frac{r}{2} H_x f_i^m - \alpha\delta^2{}_x U_i^m) \cdot \tag{3.8}$$

When $(m+i)$ is odd, equation (3.8) reduces to equation (3.6), and when it is even, we have

$$U_i^{m+1} + \frac{r}{2} H_x f_i^{m+1} - \alpha\delta^2 x U_i^{m+1} = U_i^m$$

or
$$U_i^{m+1} = U_i^m - \frac{r}{2} H_x f_i^{m+1} + \alpha\delta^2{}_x U_i^{m+1} \tag{3.9}$$

Equation (3.9) appears to be an implicit scheme although this is in actual fact not so because all the required values on the right-hand side have already been calculated in equation (3.6). Therefore, as in the previous cases, the scheme (3.8) is explicit

Incrementing m by 1 in equation (3.8) we have the equation

$$U_i^{m+2} + \theta_i^{m+2}(\frac{r}{2} H_x f_i^{m+2} - \alpha\delta^2{}_x U_i^{m+2})$$

$$= U_i^{m+1} - \theta_i^{m+1}(\frac{r}{2} H_x f_i^{m+1} - \alpha\delta^2{}_x U_i^{m+1}) \tag{3.10}$$

Writing the right-hand side of (3.10) in the form

$$2U_i^{m+1} - \{U_i^{m+1} + \theta_i^{m+1}(\frac{r}{2} H_x f_i^{m+1} - \alpha\delta^2{}_x U_i^{m+1})\}$$

and using equation (3.8), equation (3.10) can be written as

$$U_i^{m+2} + \theta_i^{m+2}(\frac{r}{2} H_x f_i^{m+2} - \alpha\delta^2{}_x U_i^{m+2}$$

$$= 2U_i^{m+1} - \{U_i^m - \theta_i^m (\frac{r}{2} H_x f_i^m - \alpha\delta^2{}_x U_i^m)\} \tag{3.11}$$

When $(m+i)$ is odd, equation (3.6) simplifies to

$$U_i^{m+2} = 2U_i^{m+1} - U_i^m \ . \tag{3.12}$$

Equations (3.8) and (3.12) give us the complete version of the hopscotch algorithm similar to that described in the parabolic and elliptic cases. Note that the relation (3.12) does not apply at all grid points $(m+i)$ with i fixed. This property eliminates the possible manifestation of the inherent linear instability of the two term recursion. The computational details remain similar to those outlined in the previous chapters. We shall now turn to the analysis of the stability criterion.

3.2 STABILITY

The analysis for the stability of the difference schemes (3.8) and (3.11) could be carried out in the same spirit as that of McGuire and Morris (1975). However, because of the complexity of the procedure, we propose to follow that suggested by Gourlay and Morris (1972). According to Gourlay and Morris the stability of the scheme (3.8) and (3.11) may be studied away from boundaries and from the initial line by using the equivalence of the process to a three-level scheme due to Du Fort and Frankel (see Gourlay (1971)) on two interlacing grids. Thus (3.8) and (3.11) are equivalent to the three-level scheme

$$(1+2\alpha)U_i^{m+1} = (1-2\alpha)U_i^{m-1} - rH_x f_i^m + 4\alpha\mu_x U_i^m \ . \tag{3.13}$$

The stability of (3.13) is analysed by using the standard Von Neumann approach to its linearization. Linearizing equation (3.13), we obtain a matrix equation:

$$I(1+2\alpha)U_i^{m+1} = I(1-2\alpha)U_i^{m-1} - rA(U_{i+1}^m - U_{i-1}^m) + I2\alpha(U_{i+1}^m + U_{i-1}^m). \quad (3.14)$$

Now for the system (3.14) to be stable, it is required that all the eigenvalues of the amplification matrix (see Mitchell (1969)) of (3.14) be less than unity in modulus. According to McGuire and Morris (1975), the eigenvalues ρ of the amplification matrix are given by replacing the matrix A in equation (3.14) by one of its eigenvalues, say λ, U_i^{m+1} by ρ^2, U_i^{m-1} by 1, $U_{i+1}^m - U_{i-1}^m$ by $2i\rho \sin\theta$ and $U_{i+1}^m + U_{i-1}^m$ by $2\rho\cos\theta$. This then gives

$$(1+2\alpha)\rho^2 = (1-2\alpha) - 2ir\lambda\rho \sin\theta + 4\alpha\rho\cos\theta$$

or $$(1+2\alpha)\rho^2 + (2ir\lambda \sin\theta - 4\alpha\cos\theta)\rho - (1 - 2r) = 0 \quad (3.15)$$

where θ is the variable in the Fourier space corresponding to ih. (A more elaborate way of obtaining the polynomial (3.15) is the method by Von Neumann in Mitchell (1969). The condition for stability thus amounts to requiring that the roots ρ of the quadratic equation (3.15) lie inside or on the unit circle.

We shall apply the following theorem due to Miller and given by McGuire and Morris (1975).

Theorem 3.1

Let f be a polynomial of degree n, and f' its derivativ Also let

$$f_1(\xi) = \{f^*(0)f(\xi) - f(0)f^*(\xi)\}/\xi$$

be the reduced polynomial where

$$f^*(\xi) \equiv \bar{f}\left(\frac{1}{\xi}\right).$$

Then f is a Von Neumann polynomial (all its roots lie on, or inside the unit circle) iff either

(i) $|f^*(0)| \geq |f(0)|$, and f_1 is a Von Neumann polynomial, or

(ii) $f_1 \equiv 0$, and f' is a Von Neumann polynomial.

We omit the proof and simply apply the theorem. Now when

$$f(\rho) = (1+2\alpha)\rho^2 + (2ir\lambda\sin\theta - 4\alpha\cos\theta)\rho + (2\alpha - 1) \tag{3.16}$$

then

$$f(\frac{1}{\rho}) = (2\alpha - 1)\rho^2 + (-4\alpha\cos\theta + 2ir\lambda\sin\theta) + (1+2\alpha) \tag{3.17}$$

and

$$f^*(\rho) = \bar{f}(\frac{1}{\rho}) = (2\alpha-1)\rho^2 + (-4\alpha\cos\theta - 2ir\lambda\sin\theta)\rho + (1+2\alpha). \tag{3.18}$$

Differentiating the polynomial (3.16), we obtain

$$f'(\rho) = 2(1+2\alpha)\rho + (-4\alpha\cos\theta + 2ir\lambda\sin\theta). \tag{3.19}$$

From the polynomials (3.16) and (3.18) we have respectively

$$f(0) = (2\alpha-1) \tag{3.20}$$

and

$$f^*(0) = (1+2\alpha) \tag{3.21}$$

The function $f_1(\rho)$ is given by

$$f_1(\rho) = \{f^*(0)f(\rho) - f(0)f^*(\rho)\}/\rho$$

$$= 8\alpha\rho + 8ir\alpha\sin\theta - 8\alpha\sin\theta \tag{3.22}$$

Applying condition (i) of Theorem 3.1, it is clearly seen from (3.20) and (3.21) that

$$|f^*(0)| \geq |f(0)|, \quad \text{for all } \alpha \geq 0$$

and also from (3.22) that $f_1(\rho)$ is a Von Neumann polynomial, for

$r|\lambda| \leq 1$. This clearly follows from the definition of the Von Neumann polynomial.

Since $f_1(\rho) = 8r\rho + 8ir\alpha\sin\theta - 8\alpha\cos\theta$

then $f_1(\rho) = 0$ implies that

$$8\alpha\rho + 8ir\alpha\sin\theta - 8\alpha\cos\theta = 0,$$

from which, upon solving for ρ, we obtain

$$\rho = \cos\theta - ir\lambda\sin\theta. \tag{3.23}$$

It is evident from (3.23) that ρ will lie on, or inside the unit circle if and only if $r|\lambda| \leq 1$, for all $\alpha \geq 0$.

Therefore the Courant-Friedrichs-Lewy condition for stability of explicit difference replacements of hyperbolic equations is satisfied by the hopscotch scheme (3.8) and (3.11).

3.3 C O N V E R G E N C E

We shall not go in detail of the analysis of convergence here but merely make remarks based on existing results to that effect.

The first observation we make is that the scheme (3.13) is an explicit three-level scheme and as such according to McGuire and Morri (1975), it is subject to the Courant-Friedrichs-Lewy condition for convergence, namely, $r|\lambda| \leq 1$.

Secondly we quote, without proof, Lax's equivalence theorem given by Mitchell (1969).

Theorem 3.2

> Given a properly posed linear initial value problem and a finite difference approximation to its solution that satisf the consistency condition, stability is the necessary and sufficient condition for convergence.

Now using the first observation and Theorem 3.2 plus the fact that scheme (3.13) is consistent (see Mitchell (1969) and that the linearized system of the partial differenctial equation (3.1) is well posed (see McGuire and Morris (1973), we conclude that scheme (3. and hence the hopscotch scheme (3.8) and (3.11) is convergent provide $r|\lambda| \leq 1$ for all $\alpha \geq 0$.

Example 3.2

Solve the nonlinear hyperbolic partial differential equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\tfrac{1}{2}u^2\right) = 0$$

satisfying the initial and boundary conditions

$$u(x,0) = x \quad 0 \leq x \leq 1$$

$$u(0,t) = 0, \quad t \geq 0.$$

Compare the result with the analytical solution

$$u(x,t) = {}^{x}/1+t$$

at time $t = 0.1$ seconds.

Solution

With $\Delta x = h = 0.1$ and $r = 0.5$,

$$\Delta t = k = (0.1)(0.5)$$

$$= 0.05$$

we have 11 mesh points altogether in the interval $0 \leq x \leq 1$, including those on the Boundaries.

At mesh points with $(m+i)$ odd we use

$$U_i^{m+1} = U_i^m - {}^{r}/2\ H_x f_i^m + \alpha\delta^2{}_x U_i^m$$

Thus along the time level $t = 0.05$ seconds, we use the above formula to obtain $U_{2,1}$, $U_{4,1}$, $U_{6,1}$, $U_{8,1}$ and $U_{10,1}$

From $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(\frac{1}{2}u^2) = 0$, we have $f = \frac{1}{2}u^2$.

With the choice of $\alpha = \frac{1}{2}$

$$U_{2,1} = U_{2,0} - \tfrac{1}{4}\{f_{3,0} - f_{1,0}\} + \tfrac{1}{8}\{U_{3,0} - 2U_{2,0} + U_{1,0}\};$$

From the mesh values along the time step $t = 0$

$$U_{1,0} = 0.1, \quad U_{2,0} = 0.2, \quad U_{3,0} = 0.3.$$

Thus $f_{1,0} = \frac{1}{2}\{(U_{1,0})^2\} = \frac{1}{2}(0.1)^2 = 0.005$

$$f_{3,0} = \tfrac{1}{2}\{(U_{3,0})^2\} = \tfrac{1}{2}(0.3)^2 = 0.045$$

Therefore

$$U_{2,1} = 0.2 - \tfrac{1}{4}\{0.045 - 0.005\} + \tfrac{1}{8}\{0.3 - 2(0.2) + 0.1\}$$

$$U_{2,1} = 0.1900$$

In the same way we obtain $U_{4,1}$, $U_{6,1}$, $U_{8,1}$ and $U_{10,1}$.

Perhaps it should be noted here that the values of the mesh points along the boundary $x = 1$ require the use of a fictitious point outside the region. To avoid this we define the operations H_x and $\delta^2{}_x$ in the same way but about the point U^m_{i-1}. So that we have

$$H_x f^m_{i-1} = f^m_i - f^m_{i-2}$$

and $$\delta^2{}_x\, U^m_{i-1} = U^m_i - 2U^m_{i-1} + U^m_{i-2}.$$

This redefinition will only apply to the points on the boundary $x = 1$.

At mesh points with $(i+m)$ an even integer, we use

$$U_i^{m+1} = U_i^m - {}^r/_2\, H_x f_i^{m+1} + \alpha\delta^2{}_x\, U_i^{m+1} .$$

Thus along the same time-level $t = 0.05$, we obtain the values of $U_{1,1}$, $U_{3,1}$, $U_{5,1}$, $U_{7,1}$ and $U_{9,1}$ with the help of the above equation.

$$\text{e.g.} \quad U_{1,1} = \tfrac{1}{2} U_{1,0} - \frac{1}{8}\{f_{2,1} - f_{0,1}\} + \tfrac{1}{4}\{U_{2,1} + U_{1,0}\}$$

$$= \tfrac{1}{2}(0.1) - \frac{1}{8}(0.0180) + \tfrac{1}{4}(0.19)$$

$$U_{1,1} = 0.0953$$

The rest of the points are obtained in a similar manner.

Having calculated values of all the mesh points on the first time step, increment m by 1, and use the equation

$$U_i^{m+2} = 2U_i^{m+1} - U_i^m$$

for all those mesh points with $(m+i+1)$ odd. Use equation

$$U_i^{m+2} = U_i^{m+1} - {}^r/_2\, H_x f_i^{m+2} + \alpha\delta^2{}_x U_i^{m+2}$$

otherwise

The results are given in table 3.1 along with the analytical solution at time $t = 0.1$.

Table 3.1

	Time = 0.1		
	Analytical	Numerical	Absolute Percent Error
0.1	0.0909	0.0906	0.33
0.2	0.1818	0.1814	0.22
0.3	0.2727	0.2714	0.48
0.4	0.3636	0.3624	0.33
0.5	0.4545	0.4506	0.86
0.6	0.5454	0.5333	0.39
0.7	0.6363	0.6814*	7.1

Comparison of the results in Table 3.1 shows that the hopscotch algorithm gives good results at internal mesh points. The problem arises when it comes to calculating the values of the points along the boundary at $x = 1$, where the finite difference scheme gives rise to fictitious points. The poor results which are obtained at these points may in the long run affect the entire solution as we advance forward in time. Perhaps a better method other than the one employed in the example is required to handle the points on the boundary if better results are to be realised. Therefore, if such methods exist but the author has not been able to come across any one of them, it is suggested they be employed. Otherwise the problem remains an open question for investigation.

CONCLUSION

The hopscotch algorithm has been tested on a number of examples and the results have been compared with analytical solutions. However, it would appear as though the work was strictly devoted to testing the accuracy of the method at the expense of its efficiency. It has been assumed here that the reader is already farmiliar with implicit schemes so that appriciaton of the advantages underlying the hopscotch method as against the former would be realised immediately one is introduced to the latter. Vast literature exists on implicit schemes and the interested reader is only refered to books by Mitchell (1969), Ames (1969), Smith (1975) and Curtis (1978) for details on these methods which require more programming effort.

In chapter 3 similarity was drawn between the hopscotch alogorithm and the Du Fort-Frankel method, both of which are explicit. The question which may be asked then is that, if the hopscotch process is the same as the Du Fort-Frankel method, why bother about the former? There are obviously a number of advantages in using the hopscotch method. The great advantage of the hopscotch method is its complete generality with respect to partial differential equations. It is readily programmed without a great deal of effort and is self-starting

The main disadvantage in the Du Fort-Frankel method is that it is a three-level algorithm and this requires an additional level of starting data. The reader is refered to Gourlay (1971) for a full discussion on the comparison of the two methods.

In conclusion therefore one can say the hopscotch method is a gene purpose algorithm. Its computationally explicit nature makes it easie to programme. Finally, because it over writes the previous values, i cuts drastically on the storage requirements in the computer.

APPENDIX

In this appendix an attempt is made to try and summarise some of the important aspects of this work which, due to time constrients, it was found unnecessary to discuss them in detail.

4.1 n - space dimensions

In the development of the hopscotch algorithm in the last three chapters we have tended to restrict ourselves to either one or two space dimensional problems in one or two variables. This was intended to make the work simple. The algorithm is infact a general purpose one and its application can be extended to multidimensional problems.

4.1.1 Linear parabolic equations

Consider the linear equation

$$\frac{\partial u}{\partial t} = L_h + g(x_i, t) \qquad (4.1)$$

where L is a second order linear elliptic differential operator in the space variables x_i, $i = 1, 2, ---, n$. The solution is required in $R \times [0, T]$ where R is an n-space dimensional region with boundary ∂R. Appropriate initial and boundary data are given on $t = 0$ and $\partial R \times [0, T]$ respectively. A square grid of size h is placed on R giving

$$R_h = \{ih;\ ih \in R,\ i = (i_1, i_2, ----- i_n),\ \text{an integer vector}\}.$$

A time step of size k is taken with $t_m = mk$. Then with U_i^m an approximation to $u(ih, mk)$, the values U_i^m are required at the points (ih, mk). With the odd-even function θ_i^m defined as

$$\theta_i^m = \begin{cases} 1 & \text{if} \quad m + \sum_{j=1}^{n} i_j \quad \text{odd} \\ 0 & \text{if} \quad m + \sum_{j}^{n} i_j \quad \text{even} \end{cases}$$

the hopscotch algorithm for equation (4.1) becomes

$$U_i^{m+1} - k\theta_i^{m+1} L_h U_i^{m+1} = U_i^m + k\theta_i^m L_h U_i^m + k\theta_i^{m+1} g_i^{m+1} + k\theta_i^m g_i^m \quad (4$$

where L_h is an E-operator. In n-space dimensions an E operator is defined as a (2n+1) point replacement for the second-order elliptic differential operator L. In one space dimension for example, the E-operator L_h is the normal three point defference replacement of L. In two space dimensions L_h is five point replacement of L, and so on, leading to the general formula (2n + 1) in n-space dimensions. Notice that the explicit nature of the hopscotch algorithm is not lost in the general n-space dimensional case so long as L_h is an E-operator

4.1.2 Linear elliptic equations

We recall from chapter 2 that the hopscotch algorithm for the two space dimensional elliptic equation is given by

$$U_{i,j}^{m+1} - \theta_{i,j}^{m+1} \frac{w}{4} L_h U_{i,j}^{m+1} = U_{i,j}^m + \theta_{i,j}^m \frac{w}{4} L_h U_i^m + \theta_{i,j}^{m+1} g_{i,j}^{m+1} + \theta_{i,j}^m g_{i,j}^m$$

—(4.3)

Infact Gourlay and McGuire [1971] suggest a slight modification on the factor $\frac{w}{4}$ on the left-hand side for faster convergence.
Replacing $\frac{w}{4}$ by $\frac{w - 4w}{4}$ on the left-hand side of (4.3) we obtain

$$U_{i,j}^{m+1} - \theta_{i,j}^{m+1} \frac{w}{4 - 4w} L_h U_{i,j}^{m+1} = U_{i,j}^m + \theta_{i,j}^m \frac{w}{4} L_h U_{i,j}^m + \theta_{i,j}^{m+1} g_{i,j}^{m+1} + \theta_{i,j}^m g_{i,j}^m$$

—(4.4)

An extension of equation (4.4) to n-space dimensional problems simply involves the generalization of the space subscripts.

Denoting the space subscripts by $i = 1, 2, ----, n$, equation (4.4) becomes

$$U_i^{m+1} - \theta_i^{m+1} \frac{w}{4 - 4w} L_h U_i^{m+1} = U_i^m + \theta_i^m \frac{w}{4} L_h U_i^m + \theta_i^{m+1} f_i^{m+1} + \theta_i^m g_i^m \qquad (4.5)$$

which defines the hopscotch scheme for the n-space dimensional elliptic equation

$$Lu = g(x_i, t)$$

where L is a second order elliptic partial differential operator in the s variables x_i, $i = 1, 2, ---n$.

4.1.3 First order nonlinear hyperbolic equations.

The hopscotch algorithm can be extended to first order nonlinear hyperbolic equations in n-space dimensions.

First consider the algorithm in one - space dimension given by

$$U_i^{m+1} + \theta_i^{m+1}\left(\frac{r}{2} H_x f_i^{m+1} - \alpha\delta^2_x U_i^{m+1}\right) = U_i^m - \theta_i^m\left(\frac{r}{2} H_x f_i^m - \alpha\delta_x^2 U_i^m\right) \qquad (4.6)$$

This equation can be extended to a two space dimensional case in the form

$$U_{i,j}^{m+1} + \theta_{i,j}^{m+1}\left\{\frac{r}{2}(H_x + H_y) f_{i,j}^{m+1} - \alpha(\delta_x^2 + \delta_y^2) U_{i,j}^{m+1}\right\}$$

$$= U_{i,j}^m - \theta_{i,j}^m \left\{\frac{r}{2}(H_x + H_y) f_{i,j}^m - \alpha(\delta_x^2 + \delta_y^2) U_{i,j}^m\right\} \qquad (4.7)$$

where $\Delta x = \Delta y$, r is the mesh ratio and α is a constant.

Thus generalising equation (4.7) to n-space dimensions we get

$$U_i^{m+1} + \theta_i^{m+1}\left\{\frac{r}{2}\left(\sum_{i=1}^n Hx_i\right) f_i^{m+1} - \alpha\left(\sum_{i=1}^n \delta_{xi}^2\right) U_i^{m+1}\right\}$$

$$= U_i^m - \theta_i^m\left\{\frac{r}{2}\left(\sum_{i=1}^n Hx_i\right) f_i^{m+1} - \alpha\left(\sum_{i=1}^n \delta^2 x_i\right) U_i^m\right\} \qquad (4.8)$$

where $i = 1, 2, \dots, n$, $\Delta x_1 = \Delta x_2 = \dots = \Delta x_n$, r is the mesh ratio and α is a constant.

4.2 Nonlinear parabolic and elliptic equations

Suggestions have been put forward by Gourlay (1970) as to how nonlinear parabolic and elliptic equations could be tackled. Although there are no numerical examples to back up these suggestions, there is every hope that the suggested approach can produce the desired results. Suffice here to further suggest that more effort be put in this area in future work of this nature.

Gourlay (1970) considers the hopscotch process applied to a non-linear E-operator L_h in two definite cases:

(i) mild non-linearity when

$$L(u) = \nabla^2 u + g(x,y,u) \tag{4.9}$$

(ii) strong non-linearity when

$$L(u) = \nabla[a(u)\,\nabla u] + g(x,y,u). \tag{4.10}$$

It is important to note that of the steps of the computational algorithm in the linear case, only step (3) is affected by non-linearity. In the case of a mildly nonlinear operator such as equation (4.9), the step requires the point-wise solution of an equation of the type

$$\alpha u + g(x,y,u) = \beta$$

where α, β are known. Application of the Newton iteration would provide rapid convergence in this case. As starting value one could take the average of the nearest neighbours to the point u, as these are already known. Occasionally such iteration may not be required, as for example the common problem

$$\nabla^2 u = u^2$$

when tackled by hopscotch gives rise to a pointwise quadratic equation in u . We notice here that although the implicitness is not elliminated, it will only occur at roughly half the points in the grid since step (3) is only used when $(m + 1+i + j)$ is odd.

Apart from the above process, an alternative method which is also applicable to the strongly non-linear problems such as equation (4.10) has been suggested. This involves the use of the technique outlined above but without iteration. That is to say, instead of solving, for instance

$$1/_{h^2}\,(\delta_x^2 + \delta_y^2)u_{i,j} = (u_{i,j})^2$$

solve

$$1/_{h^2}(\delta_x^2 + \delta_y^2)u_{i,j} = \{\tfrac{1}{4}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1})\}^2$$

which happens to be explicit.

Notice that both these techniques enable one to keep storage to a minimum.

4.3 Second order linear hyperbolic equations.

It has been remarked in chapter 3 that application of the hopscotch algorithm to linear hyperbolic systems easily carries over from that of linear parabolic and elliptic cases. True as this may be, care must be taken however, especially in the handling of the extra term which arises in the second order linear hyperbolic equations of the form

$$\frac{\partial^2 u}{\partial t^2} = Lu + g$$

where L is a second order linear elliptic partial differential operator a is a function of the space and time variables. An idea as to how such terms can be handled will be found in a book by Curtis (1978).

R E F E R E N C E S

Ames, F. W. — Numerical Methods for Partial Differential Equations, Academic Press, London 1977

Collatz, L. — Numerical Treatment of Differential Equations, Springer, New York, 1966.

Collatz, L. — Functional analysis and Numerical Mathematics, Springer, New York 1966.

Curtis, F.G. — Applied Numerical Analysis, Addison-Wesley Publishing Company, 1978

Forsythe, G.E. and Wasow R. W. — Finite Difference Methods for Partial Differentia Equations, Wiley, New York, 1960.

Fox,,L. — Numerical Solution of Ordinary and Partial Differential Equations, Addison-Wesley, Reading, Mass., 1962.

Gourlay, A.R. — Journal of the Institute of Mathematics and its Applications, Volume 6, p. 375, (1970)

Gourlay, A.R. — Proceedings of the Royal Society of London, A. 323, P. 219, (1971).

Gourlay, A. R. and McGuire, R.G. — Journal of the Institute of Mathematics and its Application, Volume 7, p. 216, (1971).

Gourlay, A.R. and Morris, LI.J. — IBM Journal of Research and Development, Volume 16, p. 349, (1972).

Gourlay, A.R. and Morris, Ll. J — Mathematics of Computation, Volume 22, p. 549, (1968).

Gourlay, A.R. and Morris, Ll.J. — Mathematics of Computation, Volume 22, p.28, (1968) McGuire, R.G. and Morris, Ll.J. — Journal of Computational Physics, Volume 11, p. 531, (1973).

McGuire, R.G. and Morris, Ll. J — Mathematics of Computation, Volume 29, p. 407, (1975).

Mitchell, R. A. — Computational Methods in Partial Differential Equations, John Wiley and Sons, 1969.

Smith, D.G. — Numerical Solution for Partial Differential Equations, Oxford University Press, 1975.

Varga, R.S. — Matrix Iterature Analysis, Prentice Hall, 1965.