Lower Bound Estimates of the Eigenvalue of The Smallest

Modulus Associated with a General Weighted Regular Sturm-Liouville Problem with Mixed Boundary Conditions

By

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Declaration

The work described in this Master of Science (MSc) dissertation was carried out under the supervision of Dr. Mervis Kikonko , Department of Mathematics, Statistics and Actuarial Science, University of Zambia, Lusaka.

The MSc dissertation represents original work by the author and has not otherwise been submitted in any form for any degree or diploma to any other University. Where use has been made of the work of others it is duly acknowledged in the text.

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Approval

This dissertation of **Boas Chisha** has been approved as fulfilling the requirements for the award of the Master of Science in Mathematics by the University of Zambia.

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Abstract

This study aims to obtain a lower bound on the eigenvalue of the smallest modulus in a general weighted regular Sturm-Liouville problem with mixed boundary conditions. The problem is expressed as

$$-u''(x) + q(x)u(x) = \lambda r(x)u(x)$$

over the interval $[a, b]$ with $u'(a) = u(b) = 0$. In 1988 Mingarelli [31] and in 2016 Kikonko & Mingarelli [23] focused on finding lower bounds under Dirichlet boundary conditions with assumptions that $q \in L^{\infty}(a, b)$ and $q \in L^{1}(a, b)$, respectively. However, to the best of our knowledge the case of mixed boundary conditions has not been covered.

In this study we obtain the lower bound on the eigenvalue of the smallest modulus in the case of mixed boundary conditions. We consider two different assumptions on q, and obtain a bound in each case. To obtain the bounds we use the Fredholm integral operators alongside solutions of related Cauchy problem to compute the bounds. We then give examples to verify the results and compare them to those obtained under Dirichlet boundary conditions to show insights into the eigenvalue behavior.

This study uses the "Root Finding Analytic" package in Maple software for eigenvalue calculations and contributes to the broader understanding of Sturm-Liouville problems with mixed boundary conditions, highlighting key differences from Dirichlet cases.

Finally, we raise some open questions for the future studies on general weighted regular Sturm-Liouville problems.

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Dedication

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Index of notation

Below is a list of symbols that will be frequently used and a brief indication of their meaning.

$[a, b]$	The set of real numbers x such that a$\leq x \leq b$
$C^n[a, b]$	Continuous function space on $[a, b]$
$L^p[a, b]$	Lebesgue space on $[a, b]$
$L^p_r[a, b]$	Weighted Lebesgue space on $[a, b]$
λ_n	Eigenvalue
$r(x)$	The weight function of the Sturm-Liouville equation
$u_n(x)$	The Eigenfunction of the Sturm-Liouville problem corresponding to λ_n
L	Linear differential operator
$\mathcal{L}$	Sturm-Liouville operator
T	A linear integral operator
$\sigma(\mathcal{L})$	The spectrum of the Sturm-Liouville operator
$\mathcal{H}$	Hilbert Space
$\mathbb{K}$	scalar field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$
$\mathbb{N}$	The set of natural numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{Q}$	The set of rational numbers
DE	Differential equation
$\Longleftrightarrow$	If and only if
SL	Sturm-Liouville
$SLPs$	Sturm-Liouville problems
BCs	Boundary conditions
IE	Integral equation
$\mathcal{L}D$	Left definite
$\mathcal{R}D$	Right definite

CHAPTER 1

Introduction

The Sturm-Liouville problem (SLP) is about finding all values of $\lambda \in \mathbb{C}$ and corresponding nonzero functions u which satisfy the differential equation

$$-\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + q(x)u = \lambda r(x)u \tag{1.0.0.1}$$

and the separated boundary conditions

$$\alpha_1 u(a) + \alpha_2 p(a)u'(a) = 0, \quad \beta_1 u(b) + \beta_2 p(b)u'(b) = 0 \tag{1.0.0.2}$$

defined on an interval (a, b), $-\infty \leq a \ < b \leq \infty$. Here, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ and are such that α_1 and α_2 do not vanish at the same time, similarly, for β_1 and β_2 (see ([37]) chapter 12). The parameter λ is called the eigenvalue and corresponding function u is called an eigenfunction of problem (1.0.0.1)-(1.0.0.2). The set comprising all eigenvalues of problem (1.0.0.1))-(1.0.0.2) is called the spectrum, [5], [20], [41]. Each eigenvalue may have atleast one eigenfunction. An eigenvalue with only one eigenfunction is called a simple eigenvalue, otherwise it is non-simple. The functions p',p, q, and r are continuous on (a, b) and p^{-1}, q, $r \in L_{loc}(a, b)$ see definition (2.1.2.2). The two functions $r(x)$ and $q(x)$ play an important role in the nature of the spectrum as shall be explained in chapter (2). By a solution, we mean a generally complex-valued function $u(x)$ of the real variable x such that u and pu' are absolutely contnuous on (a, b) that satisfies equation (1.0.0.1) and the boundary conditions (1.0.0.2). If the endpoints a and b are finite and $p(x) > 0$, the problem is called regular, otherwise it is singular if anyone of the two regularity condition(or both) are not satisfied, [19], [41].

Equation (1.0.0.1) can also be written in the form

$$\mathcal{L}u = \lambda r(x)u \tag{1.0.0.3}$$

where $\mathcal{L} = -\frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x)$ is called the Sturm-Liouville (SL) operator[41]. We prove in chapter (2) that any second order linear differential equation can be put into the form of the SL operator. This is one of the reasons why the study of SLPs is of great importance.

The SL theory is important in applied mathematics, where SLPs occur very commonly. The SLPs have been extensively used for solving a lot of problems including problems in physics and engineering. More often SLPs arise as a result of using the method of separation of variables to solve the classical partial differential equations such as Laplace's equation, the heat equation, and the wave equation [19]. According to Hinton and Schaefer [3], their use in problems of vibrations, heat treansfer, quantum mechanics and a host of other areas have proven successful for many years.

In the boundary conditions (1.0.0.2), when $\alpha_2 = 0 = \beta_2$, we get Dirichlet BCs; When $\alpha_1 = 0 = \beta_1$ leads to Neumann BCs, and a combination of the two BCs yields mixed BCs. In other words for mixed boundary conditions the matrix

$$\begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}$$

is nonsingular (See remark 9.2.9 in [1] page 193). In terms of the heat equation example, mixed boundary conditions may represent heat loss through convection to surrounding air or radiation from a hot surface, Dirichlet conditions correspond to maintaining a fixed temperature at the ends of the rod. The Neumann boundary conditions would correspond to no heat flow across the ends, or insulating conditions, as there would be no temperature gradient at those points. The more general boundary conditions allow for partially insulated boundaries [43].

Studies of SLPs were started by Charles François Sturm (1803-1855) and Joseph Liouville(1809-1882) in the 1830's and have thus been extensively studied. Studies on general weighted SLPs can be attributed to Haupt [16] and Richardson [36]. Other studies were carried out by Atkinson and Jabon [5] as well as Mingarelli, who has done quite a lot of studies in the general case of the SLPs [29] and Kikonko [22]. In 1988 Mingarelli [31] and in 2016 Kikonko and Mingarelli [23] focused on finding lower bound on the eigenvalue of the smallest modulus of equation (1.0.0.1) using Dirichlet boundary conditions with the assumption that $q \in L^{\infty}[a, b]$ and $q \in L^1[a, b]$ respectively. In this dissertation, we extend the study to obtaining the bounds in the case of mixed BCs.

In chapter (2), we give preliminary results and definitions that will help in understanding the main results. Also, classification of general weighted regular SLPs is discussed in this chapter, including the behavior of the spectrum, thereby giving a deeper understanding of the eigenvalue of the smallest modulus. Chapter (3) covers the work done by Mingarelli [31] and that by Kikonko and Mingarelli [23], in the case of Dirichlet BCs. The work is covered in detail for ease of reference. The extension of the two studies in chapter (3) to mixed BCs is covered in Chapter (4) which is the main study of this dissertation. In chapter (5) we discuss and conclude the study as well as give some open questions for future studies.

CHAPTER 2

Definitions and Preliminary Results

In this chapter, we recall some basic concepts and definitions that are used in this dissertation.

2.1. Function Spaces

2.1.1 The C^n Spaces

The notation

$$C^n[a, b]$$

represents spaces of functions with continuous derivatives up to order n. They are defined as follows:

(i) **Continuous Functions:**

$$C[a, b] = \{f : [a, b] \to \mathbb{R} \mid f \text{ is continuous on } [a, b]\}.$$

(ii) **Continuously Differentiable Functions:**

$$C^1[a, b] = \{f : [a, b] \to \mathbb{R} \mid f \text{ and } f' \text{ are continuous on } [a, b]\}.$$

(iii) **Twice Continuously Differentiable Functions:**

$$C^2[a,] = \{f : [a, b] \to \mathbb{R} \mid f, f', f'' \text{ are continuous on } [a, b]\}.$$

(iv) **Higher-Order Continuously Differentiable Functions:**

$$C^n[a, b] = \{f : (a, b) \to \mathbb{R} \mid f, f', f'', \dots, f^{(n)} \text{ are continuous on } [a, b]\}.$$

We adapted the above definitions from [40].

2.1.2 L^p Space

Normed function spaces, particularly L^p spaces, play an important role in mathematical analysis, functional analysis, and differential equations. Among these, the L^1 and L^∞ norms are widely used to study integrability and boundedness properties of functions. These norms are important in real and functional analysis ([11, 27, 39]).

Definition 2.1.2.1. [39] Let $[a, b]$ be an interval in $\mathbb{R}$, the **Lebesgue space** $L^p[a, b]$ (or simply L^p) is defined as:

$$L^p[a, b] = \{f : [a, b] \to \mathbb{C} \text{ measurable} \mid \|f\|_{L^p} < \infty\},$$

Definition 2.1.2.2. [13] Let $(a, b) \subset \mathbb{R}$ be an open interval. A function f is said to belong to the local L^p-space, denoted by $L^p_{\text{loc}}(a, b)$, if for every compact subset $K \subset (a, b)$,

$$\int_K |f(x)|^p \, dx < \infty.$$

Definition 2.1.2.3 (Weighted L^p Space)**.** [13] Let $[a, b]$ be an interval in $\mathbb{R}$, and let $r : [a, b] \to (0, \infty)$ be a measurable function, called the *weight function.* The **weighted Lebesgue space** $L^p_r[a, b]$ (or simply L^p_r) is defined as:

$$L^p_r[a, b] = \{f : [a, b] \to \mathbb{C} \text{ measurable} \mid \|f\|_{L^p_r} < \infty\},$$

where the **weighted norm** is given by:

$$\|f\|_{L^p_r} = \left(\int_a^b |f(x)|^p |r(x)| dx\right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

For $p = 1$, this simplifies to the weighted L^1-norm:

$$\|f\|_{L^1_r} = \int_a^b |f(x)||r(x)| dx.$$

For $p = \infty$, we define:

$$\|f\|_{L^\infty_r} = \operatorname{ess} \sup_{x \in (a,b)} |f(x)||r(x)|.$$

These spaces generalize the standard Lebesgue spaces by incorporating the weight function $r(x)$, which can emphasize or de-emphasize certain regions of the domain.

Definition 2.1.2.4 (Inner Product)**.** [13] Let $L^2_r[a, b]$ be the weighted Hilbert space $\mathcal{H}$ defined as:

$$\mathcal{H} = L^2_r[a, b] = \left\{ f : [a, b] \to \mathbb{C} \mid \int_a^b |f(x)|^2 |r(x)| \, dx < \infty \right\}.$$

The inner product on $\mathcal{H}$ is given by:

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}|r(x)| \, dx, \quad f, g \in L^2_r[a, b],$$

and the associated norm $\| \cdot \|$ is defined as:

$$\|f\| = \left(\int_a^b |f(x)|^2 \, |r(x)| \, dx\right)^{1/2}.$$

2.2. The Wronskian

Definition 2.2.0.5. [2] For any two functions $y_1, y_2 \in C^1$, the determinant

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_2(x)y_1'(x)$$

is called the *Wronskian* of y_1 and y_2. The symbol $W(y_1, y_2)(x)$ is sometimes abbreviated to $W(x)$.

The Wronskian derives its significance in the study of differential equations from the following lemmas.

Lemma 2.2.0.6. *[2] If y_1 and y_2 are solutions of the homogeneous equation*

$$y'' + h(x)y' + q(x)y = 0, \quad x \in [a, b], \tag{2.2.0.1}$$

where $q \in C[a, b]$, then either $W(y_1, y_2)(x) = 0$ for all $x \in [a, b]$, or $W(y_1, y_2)(x) \neq 0$ for all $x \in [a, b]$.

We adapt the proof from [2]

Proof. Let y_1 and y_2 be two solutions of the differential equation (2.2.0.1). We define the Wronskian of y_1 and y_2 as

$$W(y_1, y_2)(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x).$$

The derivative of the Wronskian is given by

$$W'(y_1, y_2)(x) = y_1'y_2' + y_1y_2'' - (y_1''y_2 + y_1'y_2') = y_1(x)y_2''(x) - y_1''(x)y_2(x).$$

Substituting the differential equation (2.2.0.1) for y_1 and y_2, we get:

$$y_1'' + h(x)y_1' + q(x)y_1 = 0, \quad y_2'' + h(x)y_2' + q(x)y_2 = 0.$$

This gives us

$$W'(y_1, y_2)(x) = y_1(x)\left(-h(x)y_2'(x) - q(x)y_2(x)\right) - \left(-h(x)y_1'(x) - q(x)y_1(x)\right)y_2(x).$$

Simplifying, we obtain:

$$W'(y_1, y_2)(x) = -h(x)\left(y_1(x)y_2'(x) - y_1'(x)y_2(x)\right) = -h(x)W(y_1, y_2)(x).$$

Thus, the derivative of the Wronskian satisfies the equation

$$W'(y_1, y_2)(x) = -h(x)W(y_1, y_2)(x). \tag{2.2.0.2}$$

This is a first-order linear differential equation for $W(y_1, y_2)(x)$.

Dividing both sides of (2.2.0.2) by $W(y_1, y_2)(x)$ and integrating we get

$$W(y_1, y_2)(x) = Ce^{-\int h(x)dx},$$

where C is a constant.

Now, we consider two cases:

Case 1: If $C = 0$, then $W(y_1, y_2)(x) = 0$ for all $x \in [a, b]$.

Case 2: If $C \neq 0$, then $W(y_1, y_2)(x) \neq 0$ for all $x \in [a, b]$, since the exponential term $e^{-\int h(x)dx}$ is never zero.

Therefore, we conclude that either $W(y_1, y_2)(x) = 0$ for all $x \in [a, b]$, or $W(y_1, y_2)(x) \neq 0$ for all $x \in [a, b]$. □

From the lemma above we see that the wronskian of two independent solutions of a second order differential equation does not vanish in general.

Lemma 2.2.0.7. *[2] Any two solutions y_1 and y_2 of $y'' + h(x)y' + q(x)y = 0, \quad x \in [a, b]$, are linearly independent if, and only if, $W(y_1, y_2)(x) \neq 0$ on $I = [a, b]$.*
We adapt the proof from [2]

Proof. Suppose y_1 and y_2 are linearly dependent. Then they exist constants c_1 and c_2, not both zero, such that:

$$c_1y_1 + c_2y_2 = 0.$$

which can be written as

$$y_1 = -\frac{c_2}{c_1}y_2, \quad c_1 \neq 0$$

let $c = -\frac{c_2}{c_1}$,
then we have

$$y_1 = cy_2 \tag{2.2.0.3}$$

differentiating (2.2.0.3) we get

$$y_1' = cy_2' \tag{2.2.0.4}$$

thus

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} cy_2(x) & y_2(x) \\ cy_2'(x) & y_2'(x) \end{vmatrix} = cy_2(x)y_2'(x) - cy_2(x)y_2'(x) = 0 \tag{2.2.0.5}$$

and therefore $W(y_1, y_2)(x) = 0$ for all $x \in [a, b]$ but this contradicts the assumption that that $W(y_1, y_2)(x) \neq 0$. Hence y_1 and y_2 must be linearly independent. □

The Wronskian is a very important tool in finding the Greens function, to be presented in the next section.

2.3. The Green's Function

Let L be a linear differential operator acting on a function $u(x)$ over a domain $[a, b]$, with specified boundary conditions. The **Green's function** $G(x, \xi)$ associated with the operator L is a function that satisfies:

$$L[G(x, \xi)] = \delta(x - \xi),$$

subject to the same boundary conditions imposed on $u(x)$.

Here:

(i) x is the variable of the solution domain, and ξ is the source point.

(ii) $\delta(x - \xi)$ is the **Dirac delta function**, a generalized function (or distribution) with the property that

$$\int_a^b \delta(x - \xi)\phi(x)\, dx = \phi(\xi)$$

for any test function $\phi(x)$. It represents an "infinitely sharp spike" at $x = \xi$ and zero elsewhere, effectively isolating the effect of a point source at ξ.

Once the Green's function $G(x, \xi)$ is determined, the solution $u(x)$ to the inhomogeneous differential equation

$$L[u(x)] = f(x)$$

can be written as:

$$u(x) = \int_a^b G(x, \xi) f(\xi)\, d\xi.$$

This integral expresses the solution as a superposition of the responses to point sources located at ξ, weighted by the values of the function $f(\xi)$. We can find more information on Greens function in [42].

2.4. Linear Differential Operators

Definition 2.4.0.8. [45] A **linear differential operator** is an operator that acts on a function u by taking its derivatives and combining them linearly with coefficient functions. Formally, an nth-order linear differential operator L is defined as:

$$L = p_n(x)\frac{d^n}{dx^n} + p_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + p_1(x)\frac{d}{dx} + p_0(x),$$

and when it acts on the function u, we have

$$Lu = p_n(x)\frac{d^n u}{dx^n} + p_{n-1}(x)\frac{d^{n-1}u}{dx^{n-1}} + \cdots + p_1(x)\frac{du}{dx} + p_0(x)u,$$

where:

(i) $u = u(x)$ is the function being acted upon,

(ii) $p_0(x), a_1(x), \ldots, p_n(x)$ are given coefficient functions that may depend on x,

(iii) $\frac{d^n u}{dx^n}$ denotes the nth derivative of u.

A common example is the **second-order linear differential operator**, L defined by

$$L = \frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x).$$

Theorem 2.4.0.9. [2] Let $L : L^2[a,b] \cap C^2[a,b] \to L^2[a,b]$ represent a second-order linear differential operator

$$L = p_2(x)\frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + p_0(x), \quad x \in [a,b],$$

where $p_2 \in C^2[a,b]$, $p_1 \in C^1[a,b]$, and $p_0 \in C[a,b]$. Then:

(i) The operator L is formally self-adjoint ($Ł^* = L$) if p_2, p_1, p_0 are real-valued functions and $p_2' = p_1$, where L^* is the formal adjoint of L.

(ii) L is self-adjoint ($L^\dagger = L$) provided that it is formally self-adjoint and satisfies Equation $v(f'g - fg')\Big|_a^b = 0$ for all $f, g \in C^2(I) \cap L^2(I)$, where $L^\dagger$ is the adjoint operator of L.

(iii) If L is self-adjoint, the eigenvalues of the equation $Lu + \lambda r u = 0$ are real. Moreover, any two eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to the weight function $r(x)$ in $L^2[a,b]$.

We adopt the proof from [2]

Proof. (i) By definition, the adjoint L^* satisfies the relation:

$$\langle Lf, g\rangle = \langle f, L^* g\rangle, \quad \forall f, g \in C^2(I) \cap L^2(I). \tag{2.4.0.1}$$

Using integration by parts,

$$\langle Lf, g\rangle = \int_a^b (pf'' + qf' + rf)\bar{g}dx \tag{2.4.0.2}$$

$$= [pf'\bar{g} - f(p\bar{g})']_a^b + \int_a^b f(p\bar{g})''dx + \int_a^b qf\bar{g}dx - \int_a^b f(q\bar{g})'dx + \int_a^b fr\bar{g}dx. \tag{2.4.0.3}$$

Thus, we obtain

$$\langle Lf, g\rangle = \langle f, (\bar{p}g)'' - (\bar{q}g)' + \bar{r}g\rangle + [p(f'\bar{g} - f\bar{g}') + (q - p')f\bar{g}]_a^b. \tag{2.4.0.4}$$

From this, the formal adjoint of L, L^*, is defined by

$$L^* g = \bar{p}g'' + (2\bar{p}' - \bar{q})g' + (\bar{p}'' - \bar{q}' + \bar{r})g. \tag{2.4.0.5}$$

L is formally self-adjoint if $L^* = L$, i.e.,

$$\begin{aligned}\bar{p} &= p,\\ 2\bar{p}' - \bar{q} &= q,\\ \bar{p}'' - \bar{q}' + \bar{r} &= r.\end{aligned}$$

These conditions are satisfied if and only if p, q, r are real and $q = p'$ and so,

$$Lf = (pf')' + rf. \tag{2.4.0.6}$$

(ii) Comparing Equations (2.4.0.1) and (2.4.0.4), we find that a formally self-adjoint operator L is self-adjoint if

$$p(f'\bar{g} - f\bar{g}')\Big|_a^b = 0, \quad \forall f, g \in C^2(I) \cap L^2(I), \tag{2.4.0.7}$$

and this condition must hold at the boundary.

(iii) Suppose $\lambda \in \mathbb{C}$ is an eigenvalue of $-L$, so that there exists a function $f \in L^2(a,b) \cap C^2(a,b)$, $f \neq 0$, satisfying

$$Lf + \lambda f = 0. \tag{2.4.0.8}$$

Taking the inner product with f, we obtain

$$\lambda \|f\|^2 = \langle \lambda f, f\rangle = -\langle Lf, f\rangle. \tag{2.4.0.9}$$

Since L is self-adjoint,

$$-\langle Lf, f\rangle = -\langle f, Lf\rangle = \langle f, \lambda f\rangle = \bar{\lambda}\|f\|^2. \tag{2.4.0.10}$$

Thus, $\lambda\|f\|^2 = \bar{\lambda}\|f\|^2$, implying $\lambda = \bar{\lambda}$ since $\|f\|^2 \neq 0$. Hence, all eigenvalues are real. If λ and μ are distinct eigenvalues of $-L$ with eigenfunctions f and g, then

$$\lambda\langle f, g\rangle = -\langle Lf, g\rangle = -\langle f, Lg\rangle = \mu\langle f, g\rangle. \tag{2.4.0.11}$$

Thus,

$$(\lambda - \mu)\langle f, g\rangle = 0. \tag{2.4.0.12}$$

Since $\lambda \neq \mu$, it follows that $\langle f, g\rangle = 0$, proving that eigenfunctions corresponding to distinct eigenvalues are orthogonal. Thus, under this assumption, L is self-adjoint.

□

In the next subsection we introduce a self-adjoint linear differential operator, the SL operator, $\mathcal{L}$.

2.4.1 Sturm-Liouville Operator $\mathcal{L}$

Definition 2.4.1.10. [47] A **Sturm-Liouville operator** $\mathcal{L}$, is a second-order linear differential operator of the form:

$$\mathcal{L} = -\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x) \tag{2.4.1.1}$$

where $p(x)$ and $q(x)$ are given functions.

Theorem 2.4.1.11. [[41] page 187] Any second order linear differential equation

$$p_2(x)u'(x) + p_1(x)u'(x) + p_0(x)u(x) = f(x) \tag{2.4.1.2}$$

can be put into the Sturm-Liouville form

$$\mathcal{L}u = F(x)$$

which can be written as

$$-\frac{d}{dx}\left(P(x)\frac{du}{dx}\right) + Q(x)u = F(x) \quad \text{or} \quad (P(x)u')' + Q(x)u = F(x) \tag{2.4.1.3}$$

We adapt the proof from [41].

Proof. Consider the equation

$$p_2(x)u'' + p_1(x)u' + p_0(x)u = f(x) \tag{2.4.1.4}$$

If $p_1(x) = p_2'(x)$, then we can write the equation in the form

$$p_2(x)u'' + p_1(x)u' + p_0(x)u = (P(x)u')' + Q(x)u = F(x) \tag{2.4.1.5}$$

and this is the correct form. However, in a case where $p_2'(x) \neq p_1(x)$, we divide through equation (2.4.1.2) by $p_2(x)$, to get

$$u'' + \frac{p_1(x)}{p_2(x)}u' + \frac{p_0(x)}{p_2(x)}u = \frac{f(x)}{p_2(x)}$$

Now, we multiply the differential equation by μ, the integrating, such that:

$$\mu(x)u'' + \mu(x)\frac{p_1(x)}{p_2(x)}u' + \mu(x)\frac{p_0(x)}{p_2(x)}u = \mu(x)\frac{f(x)}{p_2(x)}$$

. The first two terms can now be combined into an exact derivative $(\mu u')'$ if μ satisfies

$$\frac{d\mu}{dx} = \mu(x)\frac{p_1(x)}{p_2(x)}$$

This is solved to give

$$\mu(x) = e^{\int \frac{p_1(x)}{p_2(x)}dx}$$

. To turn it into a Sturm-Liouville form. Therefore equation (2.4.1.2) can be put in the form,

$$\frac{d}{dx}\left(P(x)\frac{du}{dx}\right) + Q(x)u = F(x)$$

where

$$P(x) = e^{\int \frac{p_1(x)}{p_2(x)} dx}$$
$$Q(x) = \frac{p_0(x)}{p_2(x)} e^{\int \frac{p_1(x)}{p_2(x)} dx}$$
$$F(x) = \frac{f(x)}{p_2(x)} e^{\int \frac{p_1(x)}{p_2(x)} dx}$$

□

Example 2.4.1.12. [41] Consider the equation,

$$x^2 y'' + xy' + 2y = 0.$$

We see that

$$p_2(x) = x^2,\ p_2'(x) = 2x \neq p_1(x) = x$$

Thus, the above equation can be multiplied by the factor

$$\mu(x) = \frac{1}{p_2(x)} e^{\int \frac{p_1(x)}{p_2(x)} dx} = \frac{1}{x^2} e^{\int \frac{dx}{x}} = \frac{1}{x}$$

to put the equation in Sturm-Liouville form:

$$0 = xy'' + y' + \frac{2}{x} y = (xy')' + \frac{2}{x} y.$$

2.5. Integral Equations

An **integral equation** is an equation in which an unknown function appears under an integral sign. Integral equations have various applications in mathematical physics and engineering. In certain integral equations, the parameter λ can represent an eigenvalue of the integral operator. In this section we define two types of linear integral equations.

Definition 2.5.0.13. [24] A **Fredholm integral equation** is an integral equation in which the limits of integration are fixed constants. It is generally expressed as:

$$f(x) = g(x) + \lambda \int_a^b K(x,t) f(t)\, dt, \tag{2.5.0.1}$$

where:

(i) $f(x)$ is the unknown function,

(ii) $g(x)$ is a given function,

(iii) λ is a parameter,

(iv) $K(x,t)$ is the **kernel**, which defines the interaction between x and t,

(v) The limits of integration, a and b, are fixed.

Definition 2.5.0.14. [24] A **Volterra integral equation** differs from a Fredholm integral equation in that the upper limit of integration is variable rather than fixed. It has the general form:

$$f(x) = g(x) + \lambda \int_a^x K(x,t) f(t)\, dt. \tag{2.5.0.2}$$

Here, the upper limit of integration is x, meaning that the integral depends on the current value of x.

More information on integral equations is found in [34] and [17].
In this study we will use the integral operator $(Tf)(x) = \int_a^b G(x,t)f(t)r(t)dt$ to obtain our main results, where $G(x,t)$ is the Green's function associated with the SLP.

2.5.1 Linear Integral Operator T

Definition 2.5.1.15. [38] A **linear integral operator** is an operator that transforms a function into another function through integration while preserving linearity. It is defined by an integral expression of the form:

$$(Tf)(x) = \int_a^b K(x,t) f(t)\, dt,$$

where:

(i) T is the integral operator,

(ii) $f(t)$ is the function being acted upon,

(iii) $K(x,t)$ is a given function called the **kernel** of the operator,

(iv) a and b are the limits of integration.

2.5.2 Properties of Linear Integral Operators

1. **Linearity**: If T is an integral operator and f_1, f_2 are functions with constants c_1, c_2, then:
$$T(c_1 f_1 + c_2 f_2) = c_1 T(f_1) + c_2 T(f_2).$$

2. **Kernel Function**: The nature of $K(x,t)$ determines the properties of T, such as whether it represents convolution or other transformations.

3. **Fredholm vs. Volterra Operators**:

   (i) If the limits of integration are fixed (i.e., a, b are constants), the operator is called a **Fredholm integral operator**.

(ii) If the upper limit of integration is variable (e.g., x), the operator is called a **Volterra integral operator**:

$$(Tf)(x) = \int_a^x K(x,t)f(t)\,dt.$$

Lemma 2.5.2.16. *([31], [23]) The linear operator T defined by $(Tf)(x) = \int_a^b G(x,t)f(t)r(t)dt$ maps $\mathcal{H}$ into $\mathcal{H}$ and is a bounded (compact but not necessarily self-adjoint) operator on $\mathcal{H}$ whose operator norm, $\|T\|$, is given by;*

$$\|T\| = \left(\int \int_I |G(x,t)|^2|r(x)||r(t)|dxdt\right)^{\frac{1}{2}}. \tag{2.5.2.1}$$

Proof. We want to show that T maps $\mathcal{H}$ into $\mathcal{H}$ and is a bounded operator with operator norm given by

$$\|T\| = \left(\int_I \int_I |G(x,t)|^2|r(x)||r(t)|dxdt\right)^{\frac{1}{2}}.$$

Let $f \in \mathcal{H}$. We need to show that $Tf \in \mathcal{H}$, i.e., Tf is square-integrable see definition 2.1.2.4 .

We have that:

$$\begin{aligned}\|Tf\|^2 &= \langle Tf, Tf\rangle \\ &= \int_a^b Tf(x)\overline{Tf(x)}r(x)dx \\ &= \int_a^b |(Tf)(x)|^2 r(x)dx.\end{aligned}$$

Using the definition of the operator T, we write:

$$|Tf(x)|^2 = \left|\int_a^b G(x,t)f(t)r(t)dt\right|^2.$$

Applying the Cauchy-Schwarz inequality for integrals, which states that:

$$\left|\int_a^b u(t)v(t)dt\right|^2 \le \left(\int_a^b |u(t)|^2dt\right)\left(\int_a^b |v(t)|^2dt\right),$$

where we take:

$$u(t) = G(x,t)\sqrt{|r(t)|}, \quad v(t) = f(t)\sqrt{|r(t)|},$$

we obtain:

$$|Tf(x)|^2 \le \left(\int_a^b |G(x,t)|^2|r(t)|dt\right)\left(\int_a^b |f(t)|^2|r(t)|dt\right).$$

Thus,

$$|Tf(x)|^2 \leq \left(\int_a^b |G(x,t)|^2|r(t)|dt\right)\|f\|^2. \tag{2.5.2.2}$$

Multiplying both sides of 2.5.2.2 by $r(x)$ we get

$$|Tf(x)|^2 r(x) \leq \left(\int_a^b |G(x,t)|^2|r(t)|dt\right) r(x)\|f\|^2. \tag{2.5.2.3}$$

Integrating 2.5.2.3 over [a,b] gives us:

$$\|Tf\|^2 \leq \int_a^b \left(\int_a^b |G(x,t)|^2|r(t)|dt\right) r(x)dx \cdot \|f\|^2.$$

Using Fubini's theorem to interchange the order of integration and simplify, we get:

$$\|Tf\|^2 \leq \left(\int_a^b \int_a^b |G(x,t)|^2|r(t)|r(x)dxdt\right)\|f\|^2. \tag{2.5.2.4}$$

Define:

$$C = \left(\int_a^b \int_a^b |G(x,t)|^2|r(t)|r(x)dxdt\right)^{1/2}.$$

Taking the square root on both sides of 2.5.2.4 gives:

$$\|Tf\| \leq C\|f\|.$$

Thus, T is a bounded operator with norm $\|T\| \leq C$. we see that $C < \infty$, since $r(x) \in L^\infty(a,b)$ and G is continuous on $[a,b] \times [a,b]$. Thus T is bounded as an operator on $\mathcal{H}$ and $Tf \in \mathcal{H}$ with

$$\|T\| \leq \left(\int_a^b \int_a^b |G(x,t)|^2|r(t)||r(x)|dxdt\right)^{\frac{1}{2}}.$$

T is compact because the operator defined by an integral kernel $G(x,t)$ with square-integrable kernel is known to be compact on L^2 spaces. The proof of compactness generally relies on approximating $G(x,t)$ by finite-rank operators, which are compact, and using the fact that limits of compact operators are also compact.

Next, we show the equality for the operator norm $\|T\|$

Assume that f is such that $\|f\| = 1$. Then the operator norm $\|T\|$ is given by:

$$\|T\| = \sup_{\|f\|=1} \|Tf\| = \sup_{\|f\|=1} \left(\int_a^b |(Tf)(x)|^2 r(x)dx\right)^{\frac{1}{2}}.$$

From the earlier inequality, we know:

$$\|Tf\| \leq \left(\int_a^b \int_a^b |G(x,t)|^2|r(t)||r(x)|dxdt\right)^{\frac{1}{2}}.$$

Equality is attained when $f(t)$ aligns with the eigenfunction corresponding to the largest eigenvalue of T^*T (this is a typical argument for operators defined by integral kernels).

Thus, the operator norm $\|T\|$ is:

$$\|T\| = \left(\int_a^b \int_a^b |G(x,t)|^2 |r(t)||r(x)| dx dt \right)^{\frac{1}{2}}.$$

□

Theorem 2.5.2.17. [7] Suppose that the boundary value problem for $\mathcal{L}u = f(x)$ has a unique solution for each $f \in L_r^2[a,b]$, and that the associated Green's function $G(x,t)$ exists. Define the integral operator

$$(Tu)(x) = \int_a^b G(x,t)r(t)u(t)\,dt.$$

Then λ is an eigenvalue of the Sturm–Liouville problem $\mathcal{L}u = \lambda r(x)u$ if and only if $\frac{1}{\lambda}$ is an eigenvalue of the compact integral operator T, i.e.,

$$u = \lambda T u.$$

We adapt the proof from [7].

Proof. If $\mathcal{L}u = \lambda r(x)u$, then using the Green's function:

$$u(x) = \int_a^b G(x,t)\lambda r(t)u(t)\,dt = \lambda(Tu)(x),$$

so $u = \lambda T u$, and hence $\frac{1}{\lambda}$ is an eigenvalue of T. Conversely, suppose $u = \lambda T u$, then

$$u(x) = \lambda \int_a^b G(x,t)r(t)u(t)\,dt,$$

which implies $\mathcal{L}u = \lambda r(x)u$ by the definition of Green's function. □

Theorem 2.5.2.18. [7] Let T be the compact integral operator defined by $(Tu)(x) = \int_a^b G(x,t)r(t)u(t)\,dt$ on the Hilbert space $L_r^2[a,b]$, with operator norm $\|T\|$. Let λ be a nonzero eigenvalue of the Sturm–Liouville problem

$$-u''(x) + q(x)u(x) = \lambda r(x)u(x), \quad a \le x \le b$$

$$u'(a) = u(b) = 0.$$

Then

$$|\lambda| \ge \frac{1}{\|T\|}.$$

In particular, if λ_0 is an eigenvalue of smallest modulus, then

$$|\lambda_0| \ge \frac{1}{\|T\|}.$$

We adapt the proof from [7].

Proof. From Theorem 2.5.2.17, we have that $\frac{1}{\lambda}$ is an eigenvalue of the compact operator T. From operator theory, the spectral radius of a compact operator is bounded above by its norm, i.e., $\left|\frac{1}{\lambda}\right| \leq \|T\|$. Taking reciprocals gives the desired result:

$$|\lambda| \geq \frac{1}{\|T\|}.$$

□

2.6. Sturm-Liouville Problems

2.6.1 Regular Sturm-Liouville Problems

Definition 2.6.1.19. [44] A **second-order regular Sturm-Liouville problem** is a problem composed of the differential equation:

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = \lambda r(x)u, \tag{2.6.1.1}$$

and BCs,

$$\alpha_1 u(a) + \alpha_2 u'(a) = 0, \quad \beta_1 u(b) + \beta_2 u'(b) = 0, \tag{2.6.1.2}$$

where:

(i) $p(x), q(x), r(x)$ are given real-valued functions,

(ii) $p(x) > 0$ and $r(x) > 0$ and the interval $[a, b]$ is finite.

When $r(x) > 0$, all the eigenvalues of the SLP are real because the SL operator $\mathcal{L}$ is self-adjoint. This is the classical case, which is widely covered in undergraduate courses. The general case in which $r(x)$ has no sign restrictions is covered in detail in section 2.8.

Corollary 2.6.1.20. *[2] If L is self-adjoint and $r > 0$ is a continuous function on $[a, b]$, the eigenvalues of $Lu + \lambda ru = 0$ are real, and eigenfunctions associated with distinct eigenvalues are orthogonal in $L^2_r(a, b)$.*

Example 2.6.1.21. [2] Consider the Sturm-Liouville operator

$$\mathcal{L} = -\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x),$$

on the interval $[0, \pi]$ with weight function $r(x) \equiv 1$, $p(x) \equiv 1$ and boundary conditions $u(0) = u(\pi) = 0$. The eigenvalue problem

$$\mathcal{L}u + \lambda ru = 0$$

reduces to

$$-u'' + q(x)u + \lambda u = 0.$$

For $q(x) = 0$, the eigenfunctions are $u_n(x) = \sin(nx)$ with eigenvalues $\lambda_n = n^2$. These eigenfunctions satisfy

$$\langle u_m, u_n \rangle = \int_0^\pi \sin(mx)\sin(nx)dx = 0 \quad \text{for } m \neq n.$$

Thus, the eigenfunctions are orthogonal in $L^2(0, \pi)$, illustrating the corollary.

2.6.2 Properties of Eigenvalues and Eigenfunctions in the classical case

Remark 2.6.2.22. [1] If $\phi(t)$ is an eigenfunction corresponding to an eigenvalue λ, so is $c\phi(t)$ for all $c \neq 0$.

We adapt the proof from [1]

Proof. Consider the Sturm-Liouville problem:

$$-\frac{d}{dt}\left(p(t)\frac{d\phi}{dt}\right) + q(t)\phi = \lambda r(t)\phi,$$

$$\alpha_1\phi(a) + \alpha_2\phi'(a) = 0, \quad \beta_1\phi(b) + \beta_2\phi'(b) = 0.$$

Now, define $\psi(t) = c\phi(t)$ for some constant $c \neq 0$. Then:

$$\frac{d\psi}{dt} = c\frac{d\phi}{dt}, \quad \frac{d}{dt}\left(p(t)\frac{d\psi}{dt}\right) = c\frac{d}{dt}\left(p(t)\frac{d\phi}{dt}\right). \tag{2.6.2.1}$$

From (2.6.2.1) we have:

$$-\frac{d}{dt}\left(p(t)\frac{d\psi}{dt}\right) + q(t)\psi = c\left[-\frac{d}{dt}\left(p(t)\frac{d\phi}{dt}\right) + q(t)\phi\right] = c(\lambda r(t)\phi) = \lambda r(t)(c\phi) = \lambda r(t)\psi.$$

Thus, $\psi(t) = c\phi(t)$ satisfies the same differential equation with eigenvalue λ.

Next, we verify the boundary conditions:

$$\alpha_1\psi(a) + \alpha_2\psi'(a) = \alpha_1(c\phi(a)) + \alpha_2(c\phi'(a)) = c(\alpha_1\phi(a) + \alpha_2\phi'(a)) = 0.$$

Similarly,

$$\beta_1\psi(b) + \beta_2\psi'(b) = c(\alpha_1\phi(b) + \alpha_2\phi'(b)) = 0.$$

Thus, $\psi(t) = c\phi(t)$ satisfies the boundary conditions.

Since $c\phi(t)$ satisfies both the Sturm-Liouville equation and the boundary conditions, it is also an eigenfunction corresponding to the same eigenvalue λ. □

Theorem 2.6.2.23. [1] If $r(t) > 0$, then the eigenvalues of $(pu')' + \lambda ru = 0$, $u'(a) = u(b) = 0$, are all positive.

We adapt the proof from [1].

Proof. Let λ be an eigenvalue of $(pu')'+\lambda ru=0$, $u'(a)=u(b)=0$. Multiplying the equation by $u(t)$ and integrating over $[a,b]$, we find

$$\int_a^b (p(t)u'(t))'u(t)\,dt+\lambda\int_a^b r(t)u^2(t)\,dt=0. \tag{2.6.2.2}$$

Integrating by parts, the first integral becomes

$$\int_a^b (p(t)u'(t))'u(t)\,dt=p(b)u'(b)u(b)-p(a)u'(a)u(a)-\int_a^b p(t)(u'(t))^2\,dt.$$

Since $u'(a)=u(b)=0$, we get

$$\int_a^b (p(t)u'(t))'u(t)\,dt=-\int_a^b p(t)(u'(t))^2\,dt.$$

Since $p(t)>0$ and $u(t)\neq 0$, this integral is strictly negative. From (2.6.2.2), it follows that

$$-\int_a^b p(t)(u'(t))^2\,dt+\lambda\int_a^b r(t)u^2(t)\,dt=0.$$

$$\implies \lambda\int_a^b r(t)u^2(t)\,dt=\int_a^b p(t)(u'(t))^2\,dt \tag{2.6.2.3}$$

Since the right hand side of (2.6.2.3) is strictly positive, then

$$\lambda\int_a^b r(t)u^2(t)\,dt>0.$$

Taking into account that $r(t)>0$ and $u(t)\neq 0$, it follows that $\lambda>0$. $\square$

Theorem 2.6.2.24. (see [1] page 188). Let $\lambda_1\neq\lambda_2$ be two distinct eigenvalues of $(pu')'+\lambda ru=0$, $u'(a)=u(b)=0$ and denote by $\varphi_1(t)$ and $\varphi_2(t)$ their corresponding eigenfunctions. Then

$$\int_a^b r(t)\varphi_1(t)\varphi_2(t)\,dt=0.$$

We adapt the proof from [1].

Proof. Multiplying $(p\varphi_1')'+\lambda_1 r\varphi_1=0$ by φ_2 and integrating by parts over $[a,b]$, we obtain

$$\int_a^b p\varphi_1'\varphi_2'\,dt=\int_a^b \lambda_1\varphi_1(t)\varphi_2(t)r(t)\,dt.$$

Similarly, multiplying $(p\varphi_2')'+\lambda_2 r\varphi_2=0$ by φ_1 and integrating by parts, we obtain

$$\int_a^b p\varphi_1'\varphi_2'\,dt=\int_a^b \lambda_2\varphi_1(t)\varphi_2(t)r(t)\,dt.$$

Therefore,

$$\int_a^b \lambda_1\varphi_1(t)\varphi_2(t)r(t)\,dt=\int_a^b \lambda_2\varphi_1(t)\varphi_2(t)r(t)\,dt,$$

$$(\lambda_1 - \lambda_2) \int_a^b \varphi_1(t)\varphi_2(t)r(t)\,dt = 0$$

which implies

$$\int_a^b \varphi_1(t)\varphi_2(t)r(t)\,dt = 0,$$

since $\lambda_1 \neq \lambda_2$. $\square$

The theorem above demonstrates that eigenvalues associated with distinct eigenfunctions are orthogonal with respect to the weight function $r(t)$.

2.6.3 Sturm Oscillation Theory

In Sturm-Liouville theory the interest is in answering the following questions, among others[19]; for the equation:

$$(p(x)u'(x))' + q(x)u(x) = 0 \tag{2.6.3.1}$$

1. How often does a solution oscillate in an interval (a,b) i.e, how many zeros does it have in (a,b)?
2. How many maxima and minima does it have between a pair of consecutive zero?
3. What happens to these zeros when one changes $p(x)$ or $q(x)$?

Theorem 2.6.3.25 (Sturm Oscillation Theorem)**.** [[19]] Theorem 2.2.9]: The boundary value problem

$$\begin{aligned} -(p(x)u'(x))' + q(x)u(x) &= \lambda r(x)u(x),\ [a,b] \\ \alpha_1 u(a) + \beta_1 u'(a) &= 0, \\ \alpha_2 u(b) + \beta_2 u'(b) &= 0, \end{aligned}$$

with $r(x) > 0$ has an infinite sequence of eigenvalues $\{\lambda_n\}_0^\infty$ with $\lambda_n < \lambda_{n+1}$ and $\lambda_n \to \infty$ as $n \to \infty$. The eigenvalues are labeled in such a way that an eigenfunction u_n corresponding to λ_n has precisely n zeros in the open interval (a,b).

Definition 2.6.3.26. [19] The number of zeros that an eigenfunction has within the interval (a,b) is called the oscillation number of the corresponding eigenvalue, i.e. If $u_n(x) = u(x, \lambda_n)$ then the oscillation number for λ_n is n. We call λ_n, $u_n(x, \lambda_n)$ an eigen pair.

Theorem 2.6.3.25 can be verified by using the example below:

Example 2.6.3.27. (see [19] page 17) Consider the problem below. We need to find the eigenvalues and corresponding eigenfunctions and plot their graphs to view how many interior zeros each has on the interval $[1, 4]$.

$$x^2u'' + xu' + \lambda u = 0 \tag{2.6.3.2}$$

$$u'(1) = 0 = u'(4). \tag{2.6.3.3}$$

The equation above can be written in Sturm-Liouville form:
Using theorem 2.4.1.11 we see that $p_2(x) = x^2$ and $p_2(x) = 2x \neq p_1(x) = x$, hence we use

$$\mu = \frac{1}{p_2(x)} e^{\int \frac{p_1(x)}{p_2(x)}} = \frac{1}{x^2} e^{\int \frac{x}{x^2} dx} = \frac{1}{x}$$

multiplying this to 2.6.3.2 gives us

$$xu'' + u' + \frac{\lambda u}{x} = 0$$

which the can be written as

$$(xu')' + \frac{\lambda}{x} u = 0. \tag{2.6.3.4}$$

Here $p_2(x) = x > 0$, $r(x) = \frac{1}{x} > 0$, on $[1, 4]$ and $q(x) = 0$.
Solving 2.6.3.2 by substituting $u(x) = x^r$, $u'(x) = rx^{r-1}$ and $u''(x) = r(r-1)x^{r-2}$ we obtain the characteristic equation:

$$r^2 + \lambda = 0 \Rightarrow r = \pm i\sqrt{\lambda}.$$

Thus the general solution is:

$$u(x) = A\cos(\sqrt{\lambda}\ln x) + B\sin(\sqrt{\lambda}\ln x).$$

From $u'(1) = 0$, we get $B = 0$, so:

$$u(x) = A\cos(\sqrt{\lambda}\ln x).$$

From $u'(4) = 0$, we get:

$$\sin(\sqrt{\lambda}\ln 4) = 0 \Rightarrow \sqrt{\lambda}\ln 4 = n\pi, \quad n \in \mathbb{Z}.$$

Thus, the eigenvalues are:

$$\lambda_n = \left(\frac{n\pi}{\ln 4}\right)^2, \quad n \geq 0$$

The corresponding eigenfunctions are:

$$u_n(x) = A_n \cos\left(\frac{n\pi}{\ln 4}\ln x\right). \tag{2.6.3.5}$$

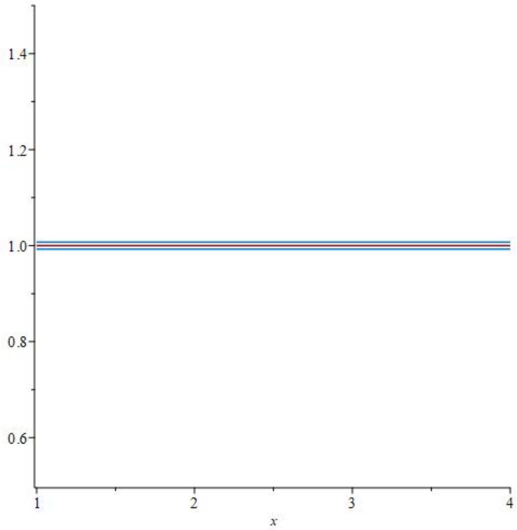
where the constants A_n are arbitrary and nonzero, so without loss of generality, we assume $A_n = 1$, so $u_n(x) = \cos\left(\frac{n\pi}{\ln(4)}\ln(x)\right), \quad n > 0$. We find the zeroes of 2.6.3.5 that is

$$\begin{aligned}
&u_n(x) = 0 \\
\implies &\cos\left(\frac{n\pi}{ln4}\ln x\right) = 0 \\
&\frac{n\pi}{ln4}\ln x = \cos^{-1}(0) \\
&\frac{n\pi}{ln4}\ln x = \frac{\pi}{2} + m\pi, \quad \text{were}\, m \,\text{is any integer} \\
&lnx = \frac{ln4}{n\pi}\left(\frac{\pi}{2} + n\pi\right) \\
&x = e^{\frac{ln4}{n\pi}\left(\frac{\pi}{2}+n\pi\right)}
\end{aligned}$$

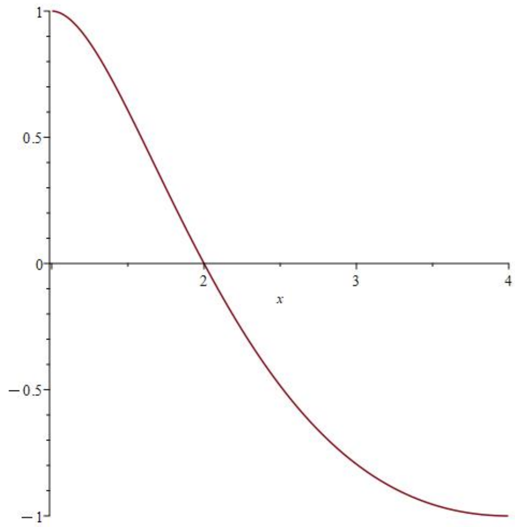
Then:

1. For n=0 we have $\lambda_0 = 0$, $u_0(x) = 1$, and it has no zeros on $(1, 4)$.
2. $\lambda_1 = \frac{\pi^2}{\ln(4)^2}$, $u_1(x) = \cos\left(\frac{\pi}{\ln(4)} \ln(x)\right)$, and it has one interior zero on $[1, 4]$.
3. $\lambda_2 = \frac{4\pi^2}{\ln(4)^2}$, $u_2(x) = \cos\left(\frac{2\pi}{\ln(4)} \ln(x)\right)$, and it has two interior zeros on $[1, 4]$.
4. $\lambda_3 = \frac{9\pi^2}{\ln(4)^2}$, $u_3(x) = \cos\left(\frac{3\pi}{\ln(4)} \ln(x)\right)$, and it has three interior zeros on $[1, 4]$.
5. $\lambda_4 = \frac{16\pi^2}{\ln(4)^2}$, $u_4(x) = \cos\left(\frac{4\pi}{\ln(4)} \ln(x)\right)$, and it has four interior zeros on $[1, 4]$.
6. $\lambda_5 = \frac{25\pi^2}{\ln(4)^2}$, $u_5(x) = \cos\left(\frac{5\pi}{\ln(4)} \ln(x)\right)$, and it has five interior zeros on $[1, 4]$.

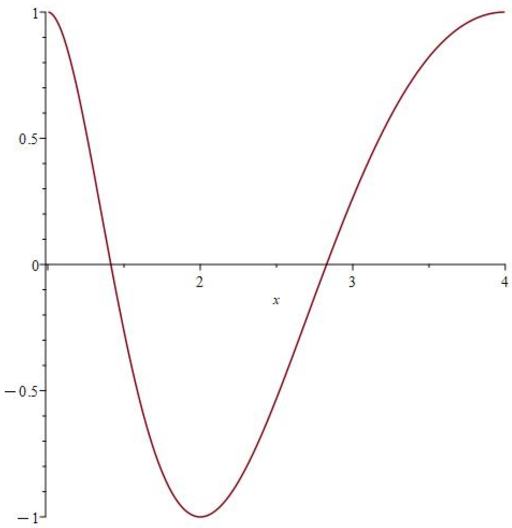
See the graphs below to verify the above example
(The graphs were plotted by Maple software 2-D plot).



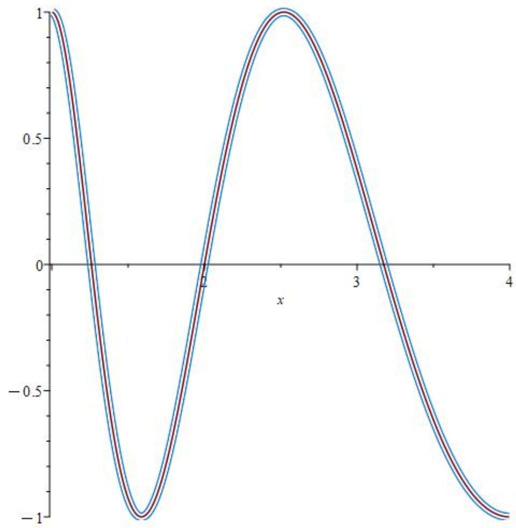
(a) Graph of $u_0(x)$



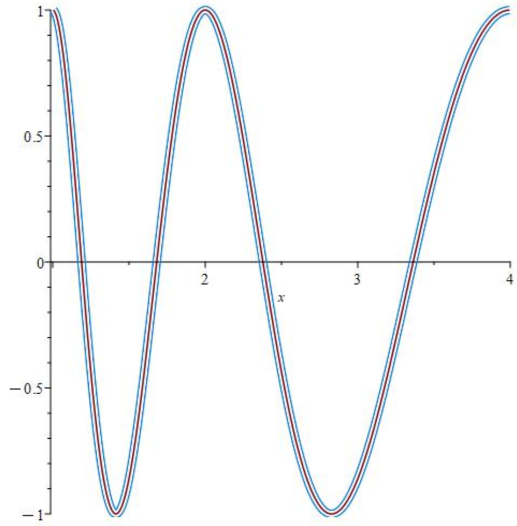
(b) Graph of $u_1(x)$



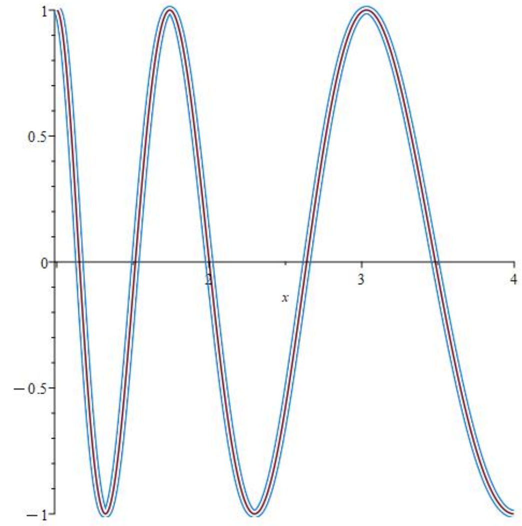
(c) Graph of $u_2(x)$



(d) Graph of $u_3(x)$



(e) Graph of $u_4(x)$



(f) Graph of $u_5(x)$

Figure 2.1: Graphs of the first 6 Eigenfunctions

Theorem 2.6.3.28 (Sturm Separation Theorem)**.** ([19] Theorem 2.2.6 or see [2]): If u_1 and u_2 are linearly independent solutions of the problem,

$$-(p(x)u'(x))' + q(x)u(x) = \lambda r(x)u(x),\ [a.b],$$
$$\alpha_1 u(a) + \beta_1 u'(a) = 0,$$
$$\alpha_2 u(b) + \beta_2 u'(b) = 0,$$

then the zeros of u_1 are distinct from those of u_2, and the two sequences of zeros alternate; that is, u_1 has exactly one zero between any two successive zeros of u_2, and vice versa in other words, their zeros separate each other on the interval (a, b).

Example 2.6.3.29. [19] Using the problem considered in example (2.6.3.27), we demonstrate separation theorem using eigenfunctions $u_4(x)$ and $u_5(x)$ as shown in figure 2.2 below

(The graphs were plotted by Maple software 2-D plot)

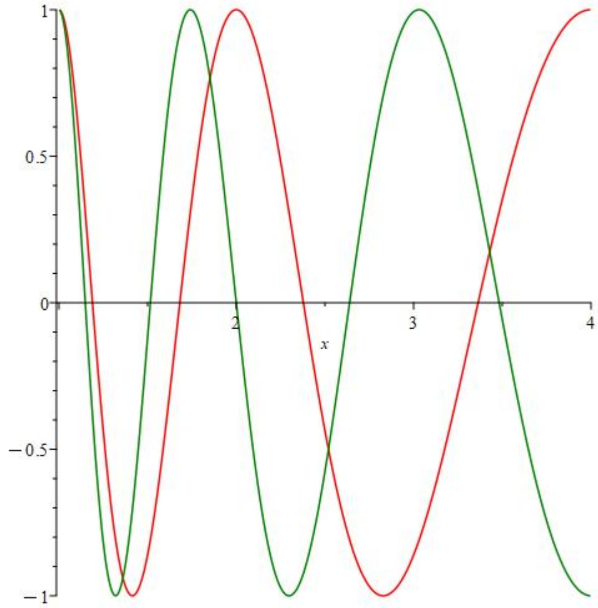


Figure 2.2: graphs of $u_4(x)$ and $u_5(x)$

Lemma 2.6.3.30. *[19] In the equation $(P(x)u'(x))' + Q(x)u(x) = 0$, if $Q(x) > 0$, then $u(x)$ has exactly one maximum (or minimum) between two successive zeros of a given solution. Thus, an infinite sequence of maxima and minima above (or below) the x-axis is impossible if $Q(x) > 0$ in $[a, b]$.*

Proof. [7] The differential equation $(P(x)u'(x))' + Q(x)u(x) = 0$ is of the Sturm-Liouville form, which governs the oscillatory properties of its solutions. A key result from Sturm's theory states that between two successive zeros of any nontrivial solution, there is exactly one extremum. An extremum of $u(x)$ occurs at points x_c where the first derivative vanishes (at the turning point), that is

$$u'(x_c) = 0.$$

Differentiating the term under the derivative by applying the product rule we get:

$$P(x)u''(x) + P'(x)u'(x) + Q(x)u(x) = 0.$$

Rearranging for $u''(x)$:

$$u''(x) = -\frac{P'(x)}{P(x)}u'(x) - \frac{Q(x)}{P(x)}u(x).$$

Since at an extremum x_c, we have $u'(x_c) = 0$, this simplifies to:

$$u''(x_c) = -\frac{Q(x_c)}{P(x_c)}u(x_c).$$

Since $Q(x) > 0$ and $P(x) > 0$, the sign of $u''(x)$ at x_c depends on the sign of $u(x_c)$: If $u(x_c) > 0$, then $u''(x_c) < 0$, implying x_c is a local maximum. If $u(x_c) < 0$, then $u''(x_c) > 0$, implying x_c is a local minimum. Thus, there is exactly **one** extremum between any two successive zeros of $u(x)$.

If an infinite sequence of maxima and minima were possible while staying above (or below) the x-axis, then there would exist multiple extrema between two successive zeros. However, this contradicts the previous conclusion that exactly one extremum exists in such an interval. Therefore, an infinite sequence of oscillations is impossible when $Q(x) > 0$.

By using Sturm's separation theorem and analyzing the second derivative, we have established that between any two successive zeros of $u(x)$, there is exactly one extremum. Consequently, an infinite sequence of maxima and minima above (or below) the x-axis is impossible if $Q(x) > 0$ in $[a, b]$. □

2.7. Greens Function for a Sturm-Liouville Problem

Suppose we want to find the solution of the differential equation

$$\mathcal{L}[u] = f, \;\; a \leq x \leq b$$

.

where $\mathcal{L}$ is a Sturm-Liouville (SL) differential operator and $u(x)$ satisfies boundary conditions at $x = a$ and $x = b$. The solution is formally given by

$$u = \mathcal{L}^{-1}[f].$$

The inverse of the differential operator is an integral operator, which we seek to write in the form:

$$u(x) = \int_a^b G(x,t) f(t) dt.$$

The function $G(x,t)$ is referred to as the kernel of the integral operator and called the Green's function. The Green's function, also known as Green's operator or influence function is a mathematical tool used in the solution of linear differential equations with specific boundary conditions or initial conditions. It is named after the British mathematician George Green, who introduced this in the 1830s. The specific form of $G(x,t)$ depends on the particular operator and the boundary conditions associated with the problem. The Green's function $G(x,t)$ for a Sturm-Liouville problem is a piecewise defined function:

$$G(x,t) = \begin{cases} \dfrac{y(x)z(t)}{p(t)W(y,z)}, & \text{if } a \leq x < t, \\ \dfrac{y(t)z(x)}{p(t)W(y,z)}, & \text{if } t < x \leq b, \end{cases} \tag{2.7.0.1}$$

where $y(x)$ and $z(x)$ are two linearly independent solutions to the SLP involving $(-p(x)u'(x))' + q(x)u(x) = 0$ and Cauchy conditions, chosen in such a way that the Green's function $G(x,t)$ satisfies the BCs of the SLP, $W(y,z)$ is the Wronskian of y and z, and $p(x)$ is the coefficient of $y'(x)$ in the SL equation. We adapt the construction of the Greens function (2.7.0.1) from

[42],

we consider the Sturm-Liouville problem:

$$\frac{\partial}{\partial x}(-p(x)G_x) + q(x)G = \delta(x-t), \quad a \le x \le b, \tag{2.7.0.2}$$

subject to the mixed boundary conditions:

$$G_x(a,t) = 0, \quad G(b,t) = 0. \tag{2.7.0.3}$$

To construct the Green's function, we first solve the associated homogeneous equation:

$$(-p(x)u'(x))' + q(x)u(x) = 0. \tag{2.7.0.4}$$

We select two linearly independent solutions, $y(x)$ and $z(x)$, such that:

(i) $y(x)$ satisfies the Neumann condition at $x = a$, i.e., $y'(a) = 0$.

(ii) $z(x)$ satisfies the Dirichlet condition at $x = b$, i.e., $z(b) = 0$.

The Wronskian of these solutions is given by:

$$W(y,z)(x) = y(x)z'(x) - z(x)y'(x). \tag{2.7.0.5}$$

We construct the Green's function piecewise as:

$$G(x,t) = \begin{cases} c_1 y(x), & a \le x < t, \\ c_2 z(x), & t < x \le b. \end{cases} \tag{2.7.0.6}$$

Continuity at $x = t$
Since $G(x,t)$ must be continuous at $x = t$, we require:

$$c_1 y(t) = c_2 z(t). \tag{2.7.0.7}$$

Jump Condition at $x = t$
Integrating the differential equation across $x = t$, we obtain:

$$-p(t)\left[G_x(t^+) - G_x(t^-)\right] = 1. \tag{2.7.0.8}$$

Substituting $G(x,t)$,

$$-p(t)\left[c_2 z'(t) - c_1 y'(t)\right] = 1. \tag{2.7.0.9}$$

Solving for c_1 and c_2, we obtain:

$$c_1 = \frac{z(t)}{p(t)W(y,z)}, \quad c_2 = \frac{y(t)}{p(t)W(y,z)}. \tag{2.7.0.10}$$

Thus, the Green's function is given by:

$$G(x,t) = \begin{cases} \frac{y(x)z(t)}{p(t)W(y,z)}, & a \le x < t, \\ \frac{y(t)z(x)}{p(t)W(y,z)}, & t < x \le b. \end{cases} \tag{2.7.0.11}$$

The constructed Green's function satisfies the mixed boundary conditions and the required jump condition. □

More information on the greens function is found in [2], [11], [14] and [42].

2.7.1 The Inverse ($\mathcal{L}^{-1}$) of the Sturm-Liouville Operator

Consider the Sturm-Liouville problem of the form [9]:

$$\mathcal{L}[u] = -\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = \lambda r(x)u, \quad x \in [a,b], \tag{2.7.1.1}$$

$$u'(a) = u(b) = 0. \tag{2.7.1.2}$$

Suppose the operator $\mathcal{L}$ is **invertible**, meaning that for any function $f(x)$, there exists a unique solution $u(x)$ to the inhomogeneous problem:

$$\mathcal{L}[u] = f(x).$$

The inverse operator $\mathcal{J} = \mathcal{L}^{-1}$ is then defined as:

$$\mathcal{J}f = \mathcal{L}^{-1}[f(x)]. \tag{2.7.1.3}$$

This means that if $u(x)$ satisfies $\mathcal{L}[u] = f(x)$, then we can rewrite the problem in terms of an integral operator representation using **Green's function**, as follows:

$$\mathcal{L}^{-1}[f(x)] = u(x) = \int_a^b G(x,s)r(s)f(s)\,ds.$$

2.8. Classification of General Weighted Regular Sturm-Liouville Problems

The results that have been presented in the prequel are all based on the classical SL theory, in which the weight function $r(x) > 0$ for all $x \in [a,b]$, and $q(x)$ takes on any sign. In this section we introduce the general weighted regular SLP, which is the case in which $r(x)$ can take both positive and negative signs on the interval [a,b]. Therefore, three cases arise.

We give a brief outline of the three different cases that arise as one varies the signs of the weight function $r(x)$ and the coefficient function $q(x)$.

When $r(x) > 0$ a.e, and $q(x)$ takes on both positive and negative values on the interval [a,b], problem (1.0.0.1)-(1.0.0.2) is right definite (which is the classical case already covered). When $q(x) \geq 0$ a.e. and $r(x)$ takes on both signs then problem (1.0.0.1)-(1.0.0.2) is left definite. When $q(x) < 0$ a.e and $r(x)$ changes sign, problem (1.0.0.1)-(1.0.0.2) is non-definite (or indefinite).

2.8.1 Right Definite Case

This is the classical case discussed in section (2.6). We list the properties without proof for most of them, as some are proved in section (2.6). More details can be found in [2], [6], [10], [12], [15], [28], [33], [41] and the reference listed there.

1. The eigenvalues are real, countable, ordered, and there is a smallest eigenvalue, and so we can write them as $\lambda_0 < \lambda_1 < \lambda_2 < \dots$. However, there is no largest eigenvalue and so $\lambda_n \to \infty$ as $n \to \infty$. In this case the eigenvalue of the smallest modulus is λ_0, which is the smallest eigenvalue if all eigenvalues are positive. If it happens that $\lambda_0 < 0$, then the eigenvalue of the smallest modulus could be negative or positive. In the right definite case λ_0 is the smallest eigenvalue with the corresponding eigenfunction having no zero in the interval (a,b).

2. For each eigenvalue λ_n, there exists an eigenfunction u_n with n zeros in $[a, b]$.

3. The set of eigenfunctions is complete; i.e., any piecewise smooth function can be represented by a generalized Fourier series expansion of the eigenfunctions,

$$f(x) \sim \sum_{n=1}^{\infty} c_n u_n(x),$$

where

$$c_n = \frac{\langle f, u_n \rangle}{\langle u_n, u_n \rangle}.$$

Actually, one needs $f(x) \in L_r^2[a, b]$, the set of square integrable functions over $[a, b]$ with weight function $r(x)$. By square integrable, we mean that $\langle f, f \rangle < \infty$.

4. The Rayleigh Quotient [43],

$$\lambda_n = -\frac{\int_a^b pu_n \frac{du_n}{dx}\Big|_a^b - \int_a^b \left(p\frac{du_n}{dx}\right)^2 - qu_n^2 \, dx}{\langle u_n, u_n \rangle}.$$

The quotient is derived below:
Multiplying both sides of the equation

$$\mathcal{L}u_n = -\lambda_n r(x) u_n$$

by $u_n(x)$:

$$u_n(x)\mathcal{L}u_n(x) = -\lambda_n r(x) u_n(x)^2.$$

Next, we integrate both sides over the interval $[a, b]$:

$$\int_a^b u_n(x)\mathcal{L}u_n(x)\,dx = -\lambda_n \int_a^b r(x)u_n(x)^2\,dx.$$

Now, assuming the operator $\mathcal{L}$ has the form:

$$\mathcal{L}u_n(x) = -\frac{d}{dx}\left(p(x)\frac{du_n(x)}{dx}\right) + q(x)u_n(x),$$

we substitute this into the left-hand side:

$$\int_a^b u_n(x)\left(-\frac{d}{dx}\left(p(x)\frac{du_n(x)}{dx}\right) + q(x)u_n(x)\right)\,dx.$$

To handle the first term, we apply integration by parts:

$$\int_a^b u_n(x)\left(-\frac{d}{dx}\left(p(x)\frac{du_n(x)}{dx}\right)\right)\,dx = -\left[p(x)u_n(x)\frac{du_n}{dx}\right]_a^b + \int_a^b p(x)\left(\frac{du_n(x)}{dx}\right)^2\,dx.$$

For the second term, we have:

$$\int_a^b u_n(x)q(x)u_n(x)\,dx = \int_a^b q(x)u_n(x)^2\,dx.$$

Combining both terms, we get:

$$-\left[p(x)u_n(x)\frac{du_n}{dx}\right]_a^b + \int_a^b p(x)\left(\frac{du_n}{dx}\right)^2\,dx + \int_a^b q(x)u_n(x)^2\,dx = -\lambda_n\int_a^b r(x)u_n(x)^2\,dx.$$

Finally, solving for λ_n, we divide both sides by $\langle u_n, u_n\rangle = \int_a^b r(x)u_n(x)^2\,dx$, giving the Rayleigh Quotient:

$$\lambda_n = -\frac{\left[p(x)u_n(x)\frac{du_n}{dx}\right]_a^b - \int_a^b p(x)\left(\frac{du_n}{dx}\right)^2\,dx - \int_a^b q(x)u_n(x)^2\,dx}{\int_a^b r(x)u_n(x)^2\,dx}.$$

$$\lambda_n = -\frac{\int_a^b pu_n\frac{du_n}{dx}\Big|_a^b - \int_a^b \left(p\frac{du_n}{dx}\right)^2 - qu_n^2\,dx}{\langle u_n, u_n\rangle}.$$

The Rayleigh quotient is useful for getting estimates of eigenvalues and proving some properties.

2.8.2 The Left Definite Case

In this case, two sequences of real eigenvalues $\lambda_n^{\pm}$ exist where $\lambda_n^{\pm} \to \pm\infty$ as $n \to \infty$ one positive and the other negative, and the eigenvalue of the smallest modulus, λ_0 is then the first positive (or negative) eigenvalue (whose corresponding eigenfunction has no zero in (a,b). In the left definite case the eigenvalue of the smallest modulus may not be unique, because of symmetry. Infact, one would expect atleast two eigenvalues to have the same oscillation number, unlike in the RD case where each eigenvalue has a unique oscillation number. The rest of the properties in (2.8.1), hold in the LD case, too. [25]

2.8.3 The Non-Definite Case

In the non-definite case, the spectrum consists of a discrete, doubly infinite sequence of real eigenvalues, with at most a finite and even number of non-real eigenvalues (occurring in complex conjugate pairs). The real eigenvalues do not behave like in the RD and LD cases. The smallest real eigenvalue can have oscillation number which is atleast one, unlike in the RD and LD cases in which the smallest eigenvalue has an oscillation number of zero. In the non-definite case, non-real eigenvalues may exist and so the eigenvalue of the smallest modulus, λ_0, may be either real or non-real. In this non-definite case, if λ_0 is real, then the corresponding eigenfunction can have any number of zeros on the interval (a,b) in contrast with the other two cases [4], [20], [21], [26], [29], [30], [46].

Remark 2.8.3.31. If the problem

$$-(p(x)u'(x))' + q(x)u(x) = \lambda u(x), \lambda \neq 0$$

$$\alpha_1 u(a) + \alpha_2 p(a)u'(a) = 0, \;\; \beta_1 u(b) + \beta_2 p(b)u'(b) = 0$$

has N distinct negative eigenvalues, then the number of distinct pairs of non-real eigenvalues of the non-definite problem

$$-(p(x)u'(x))' + q(x)u(x) = \lambda r(x)u(x), \lambda \neq 0$$

and the separated boundary conditions

$$\alpha_1 u(a) + \alpha_2 p(a)u'(a) = 0, \;\; \beta_1 u(b) + \beta_2 p(b)u'(b) = 0$$

cannot exceed N.

This means that, when non-real eigenvalues exist, their number is finite, unlike the real eigenvalues [Theorem 4.2.1 in [32], Corollary 1.7 in [8] .

Definition 2.8.3.32. An eigenfunction φ (real or not) for which

$$\langle r\varphi, \varphi\rangle = \int_a^b r|\varphi|^2 \, dx = 0$$

is called a ghost state (see [29].)

Proposition 2.8.3.33. *Any eigenfunction corresponding to a non-real eigenvalue of a regular Sturm-Liouville problem (2.8.3.31)-(2.8.3.31) is a ghost state [19].*

Proof. First, we observe that non-real eigenvalues and eigenfunctions occur in complex-conjugate pairs. That is, if λ is a non-real eigenvalue, then $\overline{\lambda}$ is also an eigenvalue. Let $\overline{u}$ be an eigenfunction corresponding to the non-real eigenvalue $\overline{\lambda}$, then we have

$$-(p(x)\overline{u}')' + q(x)\overline{u} = \overline{\lambda} r(x)\overline{u}. \tag{2.8.3.1}$$

Multiplying the given equation for u,

$$-(p(x)u')' + q(x)u = \lambda r(x)u, \tag{2.8.3.2}$$

by $\overline{u}$, and (2.8.3.1) by u, we obtain:

$$-(p(x)u')'\overline{u} + q(x)|u|^2 = \lambda r(x)|u|^2, \tag{2.8.3.3}$$

and

$$-(p(x)\overline{u}')'u + q(x)|u|^2 = \overline{\lambda} r(x)|u|^2, \tag{2.8.3.4}$$

respectively. Subtracting (2.8.3.4) from (2.8.3.3) yields:

$$-(p(x)u')'\overline{u} + (p(x)\overline{u}')'u = (\lambda - \overline{\lambda})r(x)|u|^2. \tag{2.8.3.5}$$

Next, we integrate both sides of (2.8.3.5) over $[a, b]$. Using integration by parts on the left-hand side, we focus on the terms:

$$-\int_a^b (p(x)u')'\overline{u}\,dx \quad \text{and} \quad \int_a^b (p(x)\overline{u}')'u\,dx.$$

Applying the integration by parts formula,

$$-\int_a^b (p(x)u')'\overline{u}\,dx = [-p(x)u'\overline{u}]_a^b + \int_a^b p(x)u'\overline{u}'\,dx,$$

and similarly for the second term,

$$\int_a^b (p(x)\overline{u}')'u\,dx = [p(x)\overline{u}'u]_a^b - \int_a^b p(x)\overline{u}'u'\,dx.$$

Substituting these into (2.8.3.5), the boundary terms cancel due to the boundary conditions:

$$\alpha_1 u(a) + \alpha_2 p(a)u'(a) = 0 \quad \text{and} \quad \beta_1 u(b) + \beta_2 p(b)u'(b) = 0,$$

and this reduces the integral to:

$$\int_a^b (p(x)u'\overline{u}' - p(x)\overline{u}'u')\,dx = (\lambda - \overline{\lambda})\int_a^b r(x)|u|^2 dx.$$

The left-hand side simplifies to zero because

$$p(x)u'\overline{u}' - p(x)\overline{u}'u' = 0$$

since,

$$\int_a^b 0\,dx = 0.$$

Thus, we have:

$$0 = (\lambda - \overline{\lambda})\int_a^b r(x)|u|^2 dx.$$

Since λ is not real, we have $\lambda - \overline{\lambda} \neq 0$, implying that

$$\int_a^b r(x)|u|^2 dx = \langle ru, u\rangle = 0 \,\text{and}\, u \in L^2[a, b].$$

□

Theorem 2.8.3.34 (Richardson's Oscillation Theorem [35])**.** Let r be continuous and not vanish identically in any right neighborhood of $x = a$. If $r(x)$ changes its sign precisely once in (a, b), then the roots of the real and imaginary parts ϕ and φ of any non-real eigenfuction $u = \phi + i\varphi$, corresponding to a non-real eigenvalue, separate one another (or interlace).

CHAPTER 3

Lower Bound Estimates of the Eigenvalue of the Smallest Modulus Associated with General Weighted Regular Sturm-Liouville Problem in the Case of Dirichlet Boundary Conditions

3.1. Introduction

In this chapter, we review the works done by Mingarelli [31], and Kikonko and Mingarelli [23]. The existence of the eigenvalue of the smallest modulus, which we shall label λ_0, is guaranteed by the discreteness of the spectrum and the assumption that $\lambda = 0$ is not an eigenvalue of the problem (3.2.0.1)-(3.2.0.2). Moreover, the eigenvalue λ_0 is not unique, since there exist problems where the real spectrum is symmetric about zero in the left definite case. In fact even when nonreal eigenvalues exist, they appear in conjugate pairs; hence the nonreal spectrum is symmetric about the x-axis.

3.2. Lower Bound Estimates for the Case $q \in L^{\infty}[a, b]$

In the paper[31], Mingarelli obtained some lower bounds on the eigenvalues of the smallest modulus associated with the Sturm-Liouville problem:

$$-u^{''}(x) + q(x)u(x) = \lambda r(x)u(x), \quad a \leq x \leq b \tag{3.2.0.1}$$

$$u(a) = u(b) = 0, \tag{3.2.0.2}$$

in the case that $q, r : [a, b] \to \mathbb{R}$, $q \in L^{\infty}(a, b)$, $\int_a^b |r(x)| > 0$ a.e on [a,b].
To obtain the bounds we use associated Green's function and the integral operator $(Tu)(x) = \int_a^b G(x, t)u(t)r(t)\, dt$ discussed in chapter (2, lemma 2.5.2.16).
The assumption is that $\lambda = 0$ is not an eigenvalue of (3.2.0.1)-(3.2.0.2), which is equivalent to the fact that the boundary value problem,

$$-u^{''}(x) + q(x)u(x) = 0 \tag{3.2.0.3}$$

$$u(a) = u(b) = 0, \tag{3.2.0.4}$$

admits a unique Green's function, $G(x, \tau)$, defined and continuous over $[a, b] \times [a, b] \equiv D$.

Lemma 3.2.0.35. *Let y and z be linearly independent solutions of the problems*

$$\begin{aligned} -y'' + q(x)y &= 0 \\ y(a) = 0,\ y'(a) &= 1, \end{aligned} \tag{3.2.0.5}$$

$$\begin{aligned} -z'' + q(x)z &= 0 \\ z(b) = 0, z'(b) &= 1, \end{aligned} \tag{3.2.0.6}$$

respectively.

Then, for the problem (3.2.0.3)- (3.2.0.4), the Green's function is given by:

$$G(x, t) = \begin{cases} \dfrac{y(x)z(t)}{y(b)}, & \text{if } a \le x < t, \\ \dfrac{y(t)z(x)}{y(b)}, & \text{if } t < x \le b. \end{cases} \tag{3.2.0.7}$$

Proof. The Green's function of a SLP is given by:

$$G(x, t) = \begin{cases} \frac{y(x)z(t)}{p(t)W(y,z)}, & a \le x < t, \\ \frac{y(t)z(x)}{p(t)W(y,z)}, & t < x \le b, \end{cases} \tag{3.2.0.8}$$

from which we obtain $G(x, t)$ for the problem (3.2.0.3)- (3.2.0.4), which reduces to

$$G(x, t) = \begin{cases} \frac{y(x)z(t)}{W(y,z)}, & a \le x < t, \\ \frac{y(t)z(x)}{W(y,z)}, & t < x \le b, \end{cases} \tag{3.2.0.9}$$

where

$$W(y, z)(x) = p(x)[y(x)z'(x) - y'(x)z(x)].$$

We evaluate $W(y, z)$ at $x = a$ or $x = b$.

At $x = b$,

$$\begin{aligned} W(y, z)(b) &= [y(b)z'(b) - y'(b)z(b)], \\ &= [y(b) \cdot 1 - y'(b) \cdot 0,] \\ &= y(b) \end{aligned}$$

Thus, we have

$$W(y, z) = y(b). \tag{3.2.0.10}$$

as our Wronskian.

Substituting (3.2.0.10) in (3.2.0.11), we obtain the Greens function:

$$G(x, t) = \begin{cases} \dfrac{y(x)z(t)}{y(b)}, & \text{if } a \le x < t, \\ \dfrac{y(t)z(x)}{y(b)}, & \text{if } t < x \le b. \end{cases} \tag{3.2.0.11}$$

□

We note that the Greens function (3.2.0.11) satisfies the conditions (3.2.0.4) as $G(a,t) = y(a)z(t) = 0$ and $G(b,t) = y(t)z(b) = 0$, by (3.2.0.5) and (3.2.0.6), respectively.

Lemma 3.2.0.36. *[31] Let $y(x,q)$ denote the solution of the Cauchy problem:*

$$\begin{aligned}
&-y''(x) + q(x)y(x) = 0, \quad [a,b]\\
&y(a) = 0, \quad y'(a) = 1.\\
&B(c) = \{q : [a,b] \to \mathbb{R}, q \in L^\infty[a,b], \|q\|_\infty = c\}.
\end{aligned}$$

Then for fixed $x \in [a,b]$,

$$\sup_{q \in B(c)} |y(x,q)| \leq \frac{\sinh(\sqrt{c}(x-a))}{\sqrt{c}}. \tag{3.2.0.12}$$

Similarly, if $z(x,q)$ denotes the solution of

$$-z''(x) + q(x)z(x) = 0, \quad a \leq x \leq b,$$

satisfying:

$$z(b) = 1, \quad z'(b) = 0,$$

then for fixed $x \in [a,b]$,

$$\sup_{q \in B(c)} |z(x,q)| \leq \frac{\sinh(\sqrt{c}(b-x))}{\sqrt{c}}. \tag{3.2.0.13}$$

To prove lemma (3.2.0.36), we use the integral representation of the problems (3.2.0.5) and (3.2.0.6), respectively, as given in [31].
The integral representation of (3.2.0.5) is given by:

$$y(x) = x - a + \int_a^x (x-s)q(s)y(s)ds. \tag{3.2.0.14}$$

In this section we are using the integral equations as given in [31] and [23], without deriving them. However, for the main results of this dissertation, covered in chapter (4) the integral equations are derived.

One method for solving an integral equation is the use of Neumann series, so, we obtain the terms $A_n(x)'s$ of the Neumann series expansion of $y(x)$, the solution to the problem (3.2.0.5) and find the upper bounds on their modulus.
Thus

$$y(x) = \sum_{n=0}^{\infty} A_n(x),$$

where

$$A_0(x) = x - a$$

and for $n \geq 1$ we have:

$$A_n(x) = \int_a^x (x-s)q(s)A_{n-1}(s)ds,$$

$$
\begin{aligned}
A_1(x) &= \int_a^x (x-s)q(s)A_0(s)\,ds, \\
|A_1(x)| &= \left|\int_a^x (x-s)q(s)A_0(s)\,ds\right|, \\
&\leq \int_a^x |x-s||q(s)||A_0|ds, \\
&\leq \max_{a\leq s\leq x}|q(s)|\int_a^x |(x-s)||(s-a)|\,ds, \\
&= \|q\|_\infty \int_a^x |(x-s)(s-a)|\;ds.
\end{aligned}
$$

Using integration by parts we let,

$$
u = x-s \implies du = -ds \quad \text{and} \quad dv = (s-a)\,ds \implies v = \frac{(s-a)^2}{2}
$$

$$
\begin{aligned}
\implies \int_a^x udv = uv|_a^x - \int_a^x v\,du &= 0 + \int_a^x \frac{(s-a)^2}{2}ds, \\
&= \frac{(x-a)^3}{6}
\end{aligned}
$$

$$
\begin{aligned}
\implies |A_1| &\leq \|q\|_\infty \frac{(x-a)^3}{6}, \\
A_2(x) &= \int_a^x (x-s)q(s)A_1(s)\,ds, \\
|A_2(x)| &= \left|\int_a^x (x-s)q(s)A_1(s)\,,ds\right| \\
&\leq \int_a^x |x-s||q(s)||A_1(s)|ds, \\
&\leq \max_{a\leq s\leq x}|q(s)|\int_a^x |(x-s)|\|q\|_\infty \frac{(s-a)^3}{6}\;ds.
\end{aligned}
$$

Again, using integration by parts, we let

$$
u = x-s \implies \quad du = -ds \quad \text{and} \quad dv = \frac{(s-a)^3}{6}ds \implies v = \frac{(s-a)^4}{24}
$$

$$
\begin{aligned}
\implies \int_a^x udv = uv|_a^x - \int_a^x v\,du &= 0 + \int_a^x \frac{(s-a)^4}{24}ds, \\
&= \frac{(x-a)^5}{120}
\end{aligned}
$$

$$
\therefore |A_2| \leq \|q\|_\infty^2 \frac{(x-a)^5}{120}.
$$

Proceeding in a similar manner we get the upper bound of A_n given by:

$$
|A_n| \leq \|q\|_\infty^n \frac{(x-a)^{2n+1}}{(2n+1)!}, \quad n \geq 1,
$$

$$
\implies y(x) = x - a + \sum_{n=1}^{\infty} A_n,
$$

and

$$|y(x)| = \left| x - a + \sum_{n=1}^{\infty} A_n \right| \leq |x-a| + \sum_{n=1}^{\infty} |A_n|.$$

Let $c = \|q\|_\infty$, then

$$|y(x)| \leq x - a + \sum_{n=1}^{\infty} c^n \frac{(x-a)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} c^n \frac{(x-a)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(\sqrt{c})^{2n+1}}{\sqrt{c}} \frac{(x-a)^{2n+1}}{(2n+1)!}$$

$$= \frac{1}{\sqrt{c}} \sinh(\sqrt{c}(x-a))$$

$$\therefore \sup_{q \in B(c)} |y(x,q)| \leq \frac{\sinh(\sqrt{c}(x-a))}{\sqrt{c}}, \tag{3.2.0.15}$$

establishing the bound (3.2.0.12) in lemma (3.2.0.36), [31]. Similarly, the integral equation of

$$-z''(x) + q(x)z(x) = 0$$

$$z'(a) = 0, \ z(a) = 1$$

is given by;

$$z(x) = b - x + \int_x^b (s-x)q(s)z(s)\, ds, \ [31].$$

Using the terms $B_n(x)'s$, we have:

$$z(x) = \sum_{n=0}^{\infty} B_n(x),$$

where

$$B_0 = b - x,$$

$$B_n(x) = \int_x^b (s-x)q(s)B_{n-1}(s)ds.$$

Similar procedure as in the case of finding the bounds for $y(x)$, we get,

$$|B_n(x)| \leq \sum_{n=0}^{\infty} c^n \frac{(b-x)^{2n+1}}{(2n+1)!}.$$

Hence we have,

$$\sup_{q \in B(c)} |z(x,q)| \leq \frac{\sinh(\sqrt{c}(b-x))}{\sqrt{c}}, \tag{3.2.0.16}$$

thus establishing the bound (3.2.0.13) in lemma (3.2.0.36), [31].

Lemma 3.2.0.37. *[31] Let y, z denote the two linearly independent solutions of*

$$-y'' + q(x)y = 0$$

satisfying

$$y(a) = 0\ , y'(a) = 1, \quad \textit{and} \quad z(b) = 0\ , z'(b) = 1,$$

respectively. Then

1.
$$\sup_{q\in B(c)} \|y(x,q)\| \leq \left(c^{-1}\int_a^b \sinh^2(\sqrt{c}(x-a))|r(x)|dx\right)^{\frac{1}{2}} = \left\|\frac{\sinh(\sqrt{c}(x-a))}{\sqrt{c}}\right\|, \tag{3.2.0.17}$$

2.
$$\sup_{q\in B(c)} \|z(x,q)\| \leq \left(c^{-1}\int_a^b \sinh^2(\sqrt{c}(b-x))|r(x)|dx\right)^{\frac{1}{2}} = \left\|\frac{\sinh(\sqrt{c}(b-x))}{\sqrt{c}}\right\|, \tag{3.2.0.18}$$

We adapt the proof from [31].

Proof. We prove the result by calculating the $\mathcal{H}$ norm of y ans z, and use the bounds in lemma (3.2.0.36) as shown below:

$$\begin{aligned}\|y(x,q)\| = \langle y, y\rangle &= \left(\int_a^b y\overline{y}|r(x)|dx\right)^{\frac{1}{2}}\\ &= \left(\int_a^b |y(x,q)|^2|r(x)|dx\right)^{\frac{1}{2}}.\end{aligned}$$

Taking the supremum on $\|y(x,q)\|$ we have

$$\begin{aligned}\sup_{q\in B(c)} \|y(x,q)\| &\leq \left(\int_a^b \left|\frac{\sinh(\sqrt{c}(x-a))}{\sqrt{c}}\right|^2 |r(x)|dx\right)^{\frac{1}{2}}\\ &= \left\|\frac{\sinh(\sqrt{c}(x-a))}{\sqrt{c}}\right\|,\end{aligned}$$

thereby establishing (3.2.0.17).
Similarly,

$$\begin{aligned}\|z(x,q)\| = \langle z, z\rangle &= \left(\int_a^b z\overline{z}|r(x)|dx\right)^{\frac{1}{2}}\\ &= \left(\int_a^b |z(x,q)|^2|r(x)|dx\right)^{\frac{1}{2}}.\end{aligned}$$

Taking the supremum on $\|z(x,q)\|$ we have

$$\begin{aligned}\sup_{q\in B(c)} \|z(x,q)\| &\leq \left(\int_a^b \left|\frac{\sinh(\sqrt{c}(b-x))}{\sqrt{c}}\right|^2 |r(x)|dx\right)^{\frac{1}{2}}\\ &= \left\|\frac{\sinh(\sqrt{c}(x-a))}{\sqrt{c}}\right\|,\end{aligned}$$

thereby establishing (3.2.0.18). $\square$

Lemma 3.2.0.38. *An eigenvalue of the smallest modulus λ_0 of (3.2.0.1) satisfies*

$$|\lambda_0 \geq |y(b)|\frac{|y(b)|}{\sqrt{2}\|y\|\|z\|}.$$

Proof. We prove the lemma using the definition of the norm of the operator $(Tf)(x) = \int_a^b G(x,t)f(t)r(t)\,dt$ given by

$$\|T\| = \left(\int_a^b \int_a^b |G(x,t)|^2|r(x)||r(t)|\,dx\,dt\right)^{\frac{1}{2}},$$

where

$$G(x,t) = \begin{cases} \dfrac{y(x)z(t)}{y(b)}, & \text{if } a \leq x < t, \\ \dfrac{y(t)z(x)}{y(b)}, & \text{if } t < x \leq b, \end{cases}$$

and $r(t)$ is the weight function.

Thus,

$$\begin{aligned}
\|T\| &= \left(\int_a^b \left[\int_a^t \left|\frac{y(x)z(t)}{y(b)}\right|^2 |r(x)||r(t)|\,dx + \int_t^b \left|\frac{y(t)z(x)}{y(b)}\right|^2 |r(x)||r(t)|\,dx\right] dt\right)^{\frac{1}{2}} \\
&= \left(\int_a^b \left[\frac{|z(t)|^2|r(t)|}{|y(b)|^2}\int_a^t |y(x)|^2|r(x)|\,dx \right.\right. \\
&\qquad \left.\left. + \frac{|y(t)|^2|r(t)|}{|y(b)|^2}\int_t^b |z(x)|^2|r(x)|\,dx\right] dt\right)^{\frac{1}{2}} \\
&\leq \left(\int_a^b \left[\frac{|z(t)|^2|r(t)|}{|y(b)|^2}\|y(x)\|^2 + \frac{|y(t)|^2|r(t)|}{|y(b)|^2}\|z(x)\|^2\right] dt\right)^{\frac{1}{2}} \\
&= \left(\frac{\|y(x)\|^2}{|y(b)|^2}\int_a^b |z(t)|^2|r(t)|\,dt + \frac{\|z(x)\|^2}{|y(b)|^2}\int_a^b |y(t)|^2|r(t)|\,dt\right)^{\frac{1}{2}} \\
&= \left(\frac{\|y\|^2\|z\|^2}{|y(b)|^2} + \frac{\|z\|^2\|y\|^2}{|y(b)|^2}\right)^{\frac{1}{2}} \\
&= \left(2\frac{\|y\|^2\|z\|^2}{|y(b)|^2}\right)^{\frac{1}{2}} \\
&= \sqrt{2}\frac{\|y\|\|z\|}{|y(b)|}
\end{aligned}$$

$$\therefore \|T\| \leq \frac{\sqrt{2}\|y\|\|z\|}{|y(b)|} \implies \frac{1}{\|T\|} \geq \frac{|y(b)}{\sqrt{2}\|y\|\|z\|}.$$

By theorem (2.5.2.17), we get:

$$|\lambda_0| \geq \frac{1}{\|T\|} \geq \frac{|y(b)|}{\sqrt{2}\|y\|\|z\|}$$

$$\therefore |\lambda_0| \geq \frac{|y(b)|}{\sqrt{2}\|y\|\|z\|}.$$

□

Applying the results of lemma (3.2.0.37) to the lower bounds above (lemma (3.2.0.38)) Mingarelli established the main result as stated in the theorem below:

Theorem 3.2.0.39. [31] Let $\|q\|_\infty = c$, then an eigenvalue λ_0, of smallest modulus may be estimated by;

$$|\lambda_0| \geq |y(b)|c\left(\sqrt{2}\|\sinh(\sqrt{c}(x-a))\|\|\sinh(\sqrt{c}(b-x))\|\right)^{-1}. \tag{3.2.0.19}$$

Proof. Since

$$|\lambda_0| \geq \frac{|y(b)|}{\sqrt{2}\|y\|\|z\|}, \tag{3.2.0.20}$$

by the above lemma, and

$$\sup_{q\in B(c)} \|y(x,q)\| \leq \left\|\frac{\sinh(\sqrt{c}(x-a))}{\sqrt{c}}\right\|, \tag{3.2.0.21}$$

$$\sup_{q\in B(c)} \|z(x,q)\| \leq \left\|\frac{\sinh(\sqrt{c}(b-x))}{\sqrt{c}}\right\|. \tag{3.2.0.22}$$

By lemma (3.2.0.37), substituting (3.2.0.21) and (3.2.0.22) yields

$$|\lambda_0| \geq |y(b)|c\left(\sqrt{2}\|\sinh(\sqrt{c}(x-a))\|\|\sinh(\sqrt{c}(b-x))\|\right)^{-1}$$

□

3.3. Lower Bound Estimates for the Case $q \in L^1[a, b]$

In this section, we discuss the works done by Kikonko and Mingarelli [23] which is an extension of the results obtained in [31], (in other words this is an extension of the results obtained in the previous section (3.2) to the case where the assumption on the coefficient function $q(x)$ is replaced by the more general assumption that $q(x) \in L^1[a, b]$.

We present the work as presented in [23] for ease of reference. Like in section (3.2), the Greens function of

$$\begin{aligned} -u''(x) + q(x)u(x) &= 0 \\ u(a) = u(b) &= 0, \end{aligned}$$

is given by,

$$G(x,t) = \begin{cases} \dfrac{y(x)z(t)}{y(b)}, & \text{if } a \leq x < t, \\ \dfrac{y(t)z(x)}{y(b)}, & \text{if } t < x \leq b, \end{cases} \tag{3.3.0.1}$$

where y, z are the solutions of the Cauchy problems:

$$\begin{aligned} -y''(x) + q(x)y(x) = 0, &\quad a \leq x \leq b, \\ y(a) = 0; \quad y'(a) = 1, & \end{aligned} \tag{3.3.0.2}$$

$$-z''(x) + q(x)z(x) = 0, \quad a \leq x \leq b,$$

$$z(b) = 0; \quad z'(b) = 1, \tag{3.3.0.3}$$

respectively.

Lemma 3.3.0.40. *[23] Let $y(x,q)$ denote the solution of the Cauchy problem*

$$-y''(x) + q(x)y(x) = 0, \quad a \leq x \leq b, \tag{3.3.0.4}$$

$$y(a) = 0; \quad y'(a) = 1, \tag{3.3.0.5}$$

$$B(c) = \{q : [a,b] \to \mathbb{R}, q \in L^1[a,b], \|q\|_1 = c\}.$$

Then for fixed $x \in [a,b]$,

$$\sup_{q \in B(c)} |y(x,q)| \leq (x-a)e^{c(x-a)}. \tag{3.3.0.6}$$

Similarly, if $z(x,q)$ denotes the solution of (3.3.0.4) *satisfying*

$$z(b) = 0; \quad z'(b) = 1, \tag{3.3.0.7}$$

then for fixed $x \in [a,b]$,

$$\sup_{q \in B(c)} |z(x,q)| \leq (b-x)e^{c(b-x)}. \tag{3.3.0.8}$$

Like in the previous section (3.2), to prove the above lemma, we obtain the terms of the Neumann series expansion of $y(x)$ and $z(x)$ and find the upper bounds on their modulus. We adapt the proof from [23].

The integral representation of the associated Cauchy problem,

$$-y^{''}(x) + q(x)y(x) = 0$$

$$y(a) = 0, \; y'(a) = 1$$

is

$$y(x) = x - a + \int_a^x (x-s)q(s)y(s)ds. \tag{3.3.0.9}$$

Using the Neumann series expansion, we have

$$y(x) = \sum_{n=0}^{\infty} A_n(x),$$

where

$$A_0(x) = x - a,$$

and for $n \geq 1$ we have:

$$A_n(x) = \int_a^x (x-s)q(s)A_{n-1}(s)ds.$$

$$A_1(x) = \int_a^x (x-s)q(s)A_0(s)\,ds.$$

$$|A_1(x)| = \left| \int_x^a (x-s)q(s)A_0(s) \right|$$

$$\leq \int_a^x |(x-s)||q(s)||A_0|ds$$

$$\leq \max_{a\leq s\leq x}|(x-s)(s-a)|\int_a^x |q(s)|\,ds \leq \max_{a\leq s\leq x}|(x-s)(s-a)|\|q\|_1.$$

Using calculus, we obtain maximum value of $|(x-s)(s-a)|$. Clearly the maximum is obtained in the interior of $[a,x]$, since if $f(s)=(x-s)(s-a)$, then $f(a)=f(x)=0$ anf f is continuous, meaning that we apply Rolle's theorem. Hence, we use the second derivative test as follows. Let $s=t$ then $f_1(t)=(x-t)(t-a)$, so that

$$f_1'(t) = -(t-a)+(x-t) = 0 \quad \Rightarrow t = \frac{x+a}{2}.$$

We see that $f_1''(t) = -2 < 0$, so $f_1(t)$ attains its maximum value at $t=\frac{x+a}{2}$.

Therefore, the maximum value of $f_1(t)$ is:

$$f_1\left(\frac{x+a}{2}\right) = \left(\left(x-\frac{x+a}{2}\right)\left(\frac{x+a}{2}-a\right)\right) = \frac{(x-a)^2}{4},$$

and so

$$|A_1| \leq \frac{(x-a)^2}{2^2}\|q\|_1.$$

$$A_2 = \int_a^x (x-s)q(s)A_1(s)\,ds$$

$$|A_2| = \left|\int_a^x (x-s)q(s)A_1(s)\,ds\right|$$

$$\leq \int_a^x |(x-s)||q(s)||A_1|\,ds$$

$$\leq \max_{a\leq s\leq x}\left\{|(x-s)|\left|\frac{(s-a)^2}{2^2}\right\}\right|\|q\|_1\int_a^x |q(s)|\,ds$$

$$\leq \max_{a\leq s\leq x}\left|(x-s)\frac{(s-a)^2}{2^2}\right|\|q\|_1^2.$$

Proceeding like before, we let $f_2(s)=(x-s)\frac{(s-a)^2}{2^2}$ and $s=t$, then

$$f_2(t) = (x-t)\frac{(t-a)^2}{2^2},$$

and $f_2(x)=f_2(a)=0$, and so the maximum value of $f_2(t)$ is attained in (a,x).

$$f_2'(t) = -\frac{(t-a)^2}{4}+2\frac{(t-a)}{2^2}(x-t) = \frac{t-a}{4}\,(2x-3t+a)=0$$

$$\Rightarrow t=a \quad \text{or} \quad t=\frac{2x+a}{3},$$

but $t=a$ is a boundary point, so we check what happens at $t=\frac{2x+a}{3}$,

$$f_2''(t) = \frac{-2(t-a)}{4}+\frac{2(x-t)}{4}-\frac{2(t-a)}{4} = \frac{1}{2}(x-t)-(t-a)$$

and

$$f_2''\left(\frac{2x+a}{3}\right) = \frac{1}{2}(-x+a) < 0, \text{ in } (a,x).$$

Hence $f_2(t)$ attains it's maximum when $t = \frac{2x+a}{3}$, which is

$$f_2\left(\frac{2x+a}{3}\right) = \left[\left(x - \frac{2x+a}{3}\right)\frac{\left(\frac{2x+a}{3} - a\right)^2}{4}\right] = \frac{(x-a)^3}{27}.$$

Thus,

$$|A_2| \leq \frac{(x-a)^3}{3^3}\|q\|_1^2$$

$$A_3 = \int_a^x (x-s)q(s)A_2(s)\,ds$$

$$\begin{aligned}
|A_3| &= \left|\int_a^x (x-s)q(s)A_2(s)\,ds\right| \\
&\leq \int_a^x |(x-s)||q(s)||A_2(s)|\,ds \\
&\leq \int_a^x |(x-s)||q(s)|\left|\frac{(s-a)^3}{27}\right| \|q\|_1^2\,ds \\
&\leq \max_{a\leq s\leq x}|(x-s)|\left|\frac{(s-a)^3}{3^3}\right| \|q\|_1^2 \int_a^x |q(s)|\,ds \\
&= \max_{a\leq s\leq x}\left|(x-s)\frac{(s-a)^3}{3^3}\right| \|q\|_1^2 \int_a^x |q(s)|\,ds \\
&= \max_{a\leq s\leq x}\left|(x-s)\frac{(s-a)^3}{3^3}\right| \|q\|_1^3.
\end{aligned}$$

Proceeding like before, we let $f_3(s) = (x-s)\frac{(s-a)^3}{3^3}$ and $t = s$, then

$$f_3(t) = (x-t)\frac{(t-a)^3}{3^3},$$

and

$$f_3'(t) = -\frac{(t-a)^3}{27} + \frac{3}{27}(t-a)^2(x-t) = 0.$$

$$\implies t = a \quad \text{or} \quad t = \frac{3x+a}{4}.$$

Using the second derivative test we find that: $f_3(t)$ attains it's maximum at $t = \frac{3x+a}{4}$, which is

$$f_3\left(\frac{3x+a}{4}\right) = \left[\left(x - \frac{3x+a}{4}\right)\frac{\left(\frac{3x+a}{4} - a\right)^3}{27}\right] = \frac{(x-a)^4}{256}.$$

Thus,

$$|B_3| \leq \frac{(x-a)^4}{4^4}\|q\|_1^3.$$

$$\vdots$$

$$\therefore |A_n| \leq \frac{(x-a)^{n+1}}{n^{n+1}} \|q\|_1^n, \quad n = 1, 2, 3, \ldots$$

Let $c = \|q\|_1$,

$$\Rightarrow |A_n| \leq c^n \frac{(x-a)^{n+1}}{(n+1)^{n+1}}, \quad n = 1, 2, 3, \ldots.$$

Now, $y(x) = x - a + \sum_{n=1}^{\infty} A_n(x)$, and so,

$$\begin{aligned}
|y(x)| = |x - a + \sum_{n=1}^{\infty} A_n| &\leq |x-a| + \sum_{n=1}^{\infty} |A_n| \\
&\leq x - a + \sum_{n=1}^{\infty} c^n \frac{(x-a)^{n+1}}{(n+1)^{n+1}} \\
&\leq x - a + \sum_{n=1}^{\infty} c^n \frac{(x-a)^{n+1}}{n!} \\
&= x - a + (x-a) \sum_{n=1}^{\infty} c^n \frac{(x-a)^n}{n!} \\
&= (x-a) \left[1 + \sum_{n=1}^{\infty} c^n \frac{(x-a)^n}{n!} \right] \\
&= (x-a) \sum_{n=0}^{\infty} \frac{(c(x-a))^n}{n!},
\end{aligned}$$

and the series

$$\sum_{n=0}^{\infty} \frac{(c(x-a))^n}{(n)!} = e^{c(x-a)}$$

converges, so the Neumann series converges too. This yields

$$|y(x)| \leq (x-a) e^{c(x-a)}.$$

Thus, for fixed $x \in [a, b]$ we have

$$\sup_{q \in B(c)} |y(x, q)| \leq (x-a) e^{c(x-a)}, \tag{3.3.0.10}$$

and the bound in (3.3.0.6) is established.

Similarly, the integral equation of $-z''(x) + q(x)z(x) = 0$, $z'(b) = 1$, $y(b) = 0$ is given by;

$$z(x) = b - x + \int_x^b (s-x) q(s) z(s)\, ds,$$

so that

$$z(x) = \sum_{n=0}^{\infty} B_n(x),$$

where

$$B_0(x) = b - x,$$

$$B_n(x) = \int_x^b (s-x)q(s)B_{n-1}(s)ds.$$

Proceeding like before, we get:

$$|B_n(x)| \leq \sum_{n=1}^{\infty} \frac{c^n(b-x)^{n+1}}{(n+1)^{n+1}}, n \geq 1.$$

Hence,

$$|z(x)| = \left| b - x + \sum_{n=1}^{\infty} B_n \right|$$

$$\leq |b-x| + \sum_{n=1}^{\infty} |B_n|$$

$$\leq b - x + \sum_{n=1}^{\infty} \frac{c^n(b-x)^{n+1}}{(n+1)^{n+1}} \leq b - x + \sum_{n=1}^{\infty} c^n \frac{(b-x)^{n+1}}{n!}$$

$$= b - x + (b-x)\sum_{n=1}^{\infty} c^n \frac{(b-x)^n}{n!} = (b-x)\left[1 + \sum_{n=1}^{\infty} c^n \frac{(b-x)^n}{n!}\right]$$

$$= (b-x)\sum_{n=0}^{\infty} c^n \frac{(b-x)^n}{n!}.$$

Hence for fixed $x \in [a,b]$ we have

$$|z(x,q)| \leq (b-x)e^{c(b-x)}, \tag{3.3.0.11}$$

thereby establishing the bound in (3.3.0.8).

The results of lemma (3.3.0.40) lead to the estimation of the $\mathcal{H}$ norm of y ans z, which lead to the following lemma.

Lemma 3.3.0.41. *[23] Let y and z denote the two linearly independent solutions of* (3.3.0.2) *and* (3.3.0.3)*, respectively, then:*

1.

$$\sup_{q \in B(c)} \|y(x,q)\| \leq \left(\int_a^b ((x-a)e^{c(x-a)})^2 |r(x)| dx\right)^{\frac{1}{2}} = \|(x-a)e^{c(x-a)}\|, \tag{3.3.0.12}$$

2.

$$\sup_{q \in B(c)} \|z(x,q)\| \leq \left(\int_a^b ((b-x)e^{c(b-x)})^2 |r(x)| dx\right)^{\frac{1}{2}} = \|(b-x)e^{c(b-x)}\|. \tag{3.3.0.13}$$

We adapt the proof from [23]

Proof. We have

$$\|y(x,q)\| = \langle y, y \rangle = \int_a^b y\overline{y}|r(x)|$$

$$= \left(\int_a^b |y(x,q)|^2 |r(x)|\, dx \right)^{\frac{1}{2}}.$$

Taking the supremum on $\|y(x,q)\|$ yields

$$\sup_{q \in B(c)} \|y(x,q)\| \leq \left(\int_a^b \left((x-a)e^{c(x-a)} \right)^2 |r(x)|\, dx \right)^{\frac{1}{2}} = \|(x-a)e^{c(x-a)}\|.$$

Similarly

$$\|(x,q)\| = \langle z, z \rangle = \int_a^b z\bar{z}|r(x)|$$

$$= \left(\int_a^b |z(x,q)|^2 |r(x)|\, dx \right)^{\frac{1}{2}}.$$

Taking the supremum on $\|z(x,q)\|$ yields

$$\sup_{q \in B(c)} \|z(x,q)\| \leq \left(\int_a^b \left((b-x)e^{c(b-x)} \right)^2 |r(x)|\, dx \right)^{\frac{1}{2}} = \|(b-x)e^{c(b-x)}\|.$$

□

Applying the above lemma (3.3.0.41) to the lower bound in (lemma 3.2.0.38) Kikonko and Mingarelli obtain the main result, theorem (3.3.0.42) which is a variant of theorem, (3.2.0.39):

Theorem 3.3.0.42. [23] Let $\|q\|_1 = c$. Then for problem (3.2.0.1)-(3.2.0.2), an eigenvalue λ_0 of the smallest modulus may be estimated by

$$|\lambda_0| \geq |y(b)| \left(\sqrt{2} \|(x-a)e^{c(x-a)}\| \|(b-x)e^{c(b-x)}\| \right)^{-1}, x \in (a,b), \tag{3.3.0.14}$$

where $y(b)$ is the solution of the Cauchy problem (3.3.0.4)-(3.3.0.5) evaluated at $x = b$.

We adapt the the proof from [23].

Proof. Substituting the results of lemma (3.3.0.41) into

$$|\lambda_0| \geq \frac{|y(b)|}{\sqrt{2}\|y\|\|z\|},$$

yields

$$|\lambda_0| \geq |y(b)| \left(\sqrt{2} \|(x-a)e^{c(x-a)}\| \|(b-x)e^{c(b-x)}\| \right)^{-1}, x \in (a,b).$$

□

3.3.1 Examples

Here we give the examples as given in [23] to illustrate that the inequality (3.3.0.14) really holds. We consider the case where $q(x) = q_0 \in \mathbb{R}$ on the interval $[-1,1]$. In this case, equation (3.3.0.14) becomes

$$|\lambda_0| \geq |y(1)| \left(\sqrt{2} \|(x+1)e^{c(x+1)}\| \|(1-x)e^{c(1-x)}\| \right)^{-1}, x \in (a,b), \tag{3.3.1.1}$$

where $c = \left(\int_{-1}^{1} |q_0|\, dx\right)^{1/2} = \left(x|q_0|\Big|_{-1}^{1}\right)^{\frac{1}{2}} = \sqrt{2|q_0|}$. The eigenvalues are found using the Maple© package `RootFinding[Analytic]`.

Example 3.3.1.43. (the case $q_0 = -6\pi^2$ and $r(x)$ changes sign). This is an example of a non-definite SLP.

We consider the problems

$$\begin{cases} y'' + 6\pi^2 y = 0, \\ y(-1) = 0, \\ y'(-1) = 1, \end{cases} \tag{3.3.1.2}$$

and

$$\begin{cases} y'' + 6\pi^2 y = -\lambda r(x) y, \\ y(-1) = 0 = y(1), \end{cases} \tag{3.3.1.3}$$

where

$$r(x) = \begin{cases} -1, & \text{if } x \in (-1, 0), \\ 1, & \text{if } x \in (0, 1). \end{cases} \tag{3.3.1.4}$$

The solution to the problem (3.3.1.2) is

$$y(x) = \frac{\sin\left(\sqrt{6}\pi(x+1)\right)}{\sqrt{6}\pi}, \tag{3.3.1.5}$$

and thus

$$y(1) = \frac{\sin(2\sqrt{6}\pi)}{\sqrt{6}\pi}. \tag{3.3.1.6}$$

Substituting into inequality (3.3.1.1) yields that $\lambda_0 \geq 3.048 \times 10^{-11}$, and solving problem (3.3.1.3) gives the spectrum to be

$$\ldots, -157, -67.2, -52.2, -14.3 \pm 11.8i, 14.3 \pm 11.8i, 52.2, 67.2, 157.8, \ldots \tag{3.3.1.7}$$

From the spectrum, we see that λ_0 is not unique in this particular case since $|-14.3 - 11.8i|$ and $|14.3 + 11.8i|$ all give the smallest modulus, which is 18.5. Therefore, $|\lambda_0| = 18.5$ and equation (3.3.1.1) is satisfied.

Example 3.3.1.44. (the case $q_0 = 6\pi^2$ and $r(x)$ changes sign). This is an example of a left definite case.

We consider the problems

$$\begin{cases} -y'' + 6\pi^2 y = 0, \\ y(-1) = 0, \\ y'(-1) = 1, \end{cases} \tag{3.3.1.8}$$

and

$$\begin{cases} -y'' + 6\pi^2 y = \lambda r(x) y, \\ y(-1) = 0 = y(1), \end{cases} \tag{3.3.1.9}$$

where $r(x)$ is as given in Example 1.

The solution to the problem in equation (3.3.1.8) is

$$y(x) = \frac{\sinh\left(\sqrt{6}\pi(x+1)\right)}{\sqrt{6}\pi}, \tag{3.3.1.10}$$

and so

$$y(1) = \frac{\sinh(2\sqrt{6}\pi)}{\sqrt{6}\pi}. \tag{3.3.1.11}$$

Substituting into equation (3.3.1.1) yields that $\lambda_0 \geq 0.000236$, and solving problem (3.3.1.9) gives the spectrum to be

$$\ldots, -137.7, -93.3, -67.6, -59.2, 67.6, 93.3, 137.7, \ldots \tag{3.3.1.12}$$

From the spectrum, we see that $|\lambda_0| = 59.2$ and equation (3.3.1.1) is satisfied.

Example 3.3.1.45. (the case $q_0 = 6\pi^2$ and $r(x) \equiv 1$). This is an example of the right definite case.

Here we consider problem (3.3.1.8) and the problem

$$\begin{cases} y'' + (\lambda - 6\pi^2) y = 0, \\ y(-1) = 0 = y(1). \end{cases} \tag{3.3.1.13}$$

Solving problem (3.3.1.13) gives the spectrum to be $61.7, 81.4, 120.9, 180.1, \ldots,$

and clearly, equation (3.3.1.1) is satisfied since $\lambda_0 = 61.7$.

CHAPTER 4

Lower Bound Estimates of the Eigenvalue of the Smallest Modulus Associated with General Weighted Regular Sturm-Liouville Problems with Mixed Boundary Conditions

4.1. Introduction

We now extend the results obtained in [23] or (section (3.3)) and [31] (or section (3.2)) to mixed boundary conditions. We wish to obtain the lower bound on the eigenvalue of the smallest modulus (labelled λ_0) of the problem,

$$-u''(x) + q(x)u(x) = \lambda r(x)u(x), \quad a \le x \le b \tag{4.1.0.1}$$

$$u'(a) = u(b) = 0, \tag{4.1.0.2}$$

by considering two cases: namely, case 1: $q \in L^1[a,b]$ and case 2: $q \in L^\infty[a,b]$.

Like in chapter 3 we begin by writing down the Green's function of the problem

$$-u''(x) + q(x)u(x) = 0 \tag{4.1.0.3}$$

$$u'(a) = u(b) = 0, \tag{4.1.0.4}$$

which is given by

$$G(x,t) = \begin{cases} \dfrac{y(x)z(t)}{y(b)}, & \text{if } a \le x < t, \\ \dfrac{y(t)z(x)}{y(b)}, & \text{if } t < x \le b, \end{cases} \tag{4.1.0.5}$$

where y, z are linearly independent solutions of equation (4.1.0.3) satisfying the initial conditions

$$y'(a) = 0,\, y(a) \quad = 1. \tag{4.1.0.6}$$

$$z'(b) = 1, z(b) \quad = 0. \tag{4.1.0.7}$$

It should be noted here that the Green's function, $G(x,t)$ for a SLP is derived in chapter (3), and that $G(x,t)$ satisfies the BCs (4.1.0.4) as follows:

$\frac{\partial G}{\partial x}|_{x=a} = y'(a)z(t) = 0$ by (4.1.0.6), and $G(b,t) = y(t)z(b) = 0$, by (4.1.0.7).

We present and prove lemma (4.1.0.46) using mixed boundary conditions (this is a variant of lemma from [23]

Lemma 4.1.0.46. *[23], [31] An eigenvalue, λ_0, of the problem*

$$-u''(x) + q(x)u(x) = \lambda r(x)u(x), \quad a \le x \le b \tag{4.1.0.8}$$

$$u'(a) = u(b) = 0, \tag{4.1.0.9}$$

of the smallest modulus satisfies

$$|\lambda_0| \ge \frac{|y(b)|}{\sqrt{2}\|y\|\|z\|}, \tag{4.1.0.10}$$

where y and z are solutions of (4.1.0.8)-(4.1.0.9), satisfying the conditions $y'(a) = 0$, $y(a) = 1$, and $z'(b) = 1$, $z(b) = 0$, respectively and $\|y\|$, $\|z\|$ are their respective $\mathcal{H}$-norms.

Proof. We use the Greens function (4.1.0.5) and the operator $(Tf)(x) = \int_a^b G(x,t)f(t)r(t)\,dt$ to prove the lemma.

$$\|T\| = \left(\int_a^b \int_a^b |G(x,t)|^2 |r(x)||r(t)|\,dx\,dt\right)^{\frac{1}{2}}$$

We replace $G(x,t)$ with (4.1.0.5), to have:

$$\begin{aligned}
\|T\| &= \left(\int_a^b \left[\int_a^t \left|\frac{y(x)z(t)}{y(b)}\right|^2 |r(x)||r(t)|\,dx + \int_t^b \left|\frac{y(t)z(x)}{y(b)}\right|^2 |r(x)||r(t)|\,dx\right] dt\right)^{\frac{1}{2}} \\
&= \left(\int_a^b \left[\frac{|z(t)|^2|r(t)|}{|y(b)|^2} \int_a^t |y(x)|^2|r(x)|\,dx + \frac{|y(t)|^2|r(t)|}{|y(b)|^2} \int_t^b |z(x)|^2|r(x)|\,dx\right] dt\right)^{\frac{1}{2}} \\
&\le \left(\int_a^b \left[\frac{|z(t)|^2|r(t)|}{|y(b)|^2}\|y(x)\|^2 + \frac{|y(t)|^2|r(t)|}{|y(b)|^2}\|z(x)\|^2\right] dt\right)^{\frac{1}{2}} \\
&= \left(\frac{\|y(x)\|^2}{|y(b)|^2} \int_a^b |z(t)|^2|r(t)|\,dt + \frac{\|z(x)\|^2}{|y(b)|^2} \int_a^b |y(t)|^2|r(t)|\,dt\right)^{\frac{1}{2}} \\
&= \left(\frac{\|y\|^2\|z\|^2}{|y(b)|^2} + \frac{\|z\|^2\|y\|^2}{|y(b)|^2}\right)^{\frac{1}{2}} = \left(2\frac{\|y\|^2\|z\|^2}{|y(b)|^2}\right)^{\frac{1}{2}} = \sqrt{2}\frac{\|y\|\|z\|}{|y(b)|}
\end{aligned}$$

$\therefore \|T\| \le \dfrac{\sqrt{2}\|y\|\|z\|}{|y(b)|}$ taking the reciprocal we have

$\dfrac{1}{\|T\|} \ge \dfrac{|y(b)|}{\sqrt{2}\|y\|\|z\|}$. By theorem 2.5.2.17,which states that $|\lambda_0| \ge \dfrac{1}{\|T\|}$

we get $|\lambda_0| \ge \dfrac{|y(b)|}{\sqrt{2}\|y\|\|z\|}$

(4.1.0.11)

□

4.2. Integral Equations

In this section, we find the integral representations of the two Cauchy problems:

$$-y''(x) + q(x)y(x) = 0 \quad y'(a) = 0, \quad y(a) = 1, \tag{4.2.0.1}$$

$$-z''(x) + q(x)z(x) = 0 \quad z'(b) = 1\,, \quad z(b) = 0. \tag{4.2.0.2}$$

We convert problem (4.2.0.1) to an integral equation:

Integrating we get:

$$\begin{aligned} \int_a^x y''(s)\,ds &= \int_a^x q(s)y(s)\,ds \\ y'(x) - y'(a) &= \int_a^x q(s)y(s)\,ds \\ y'(x) &= \int_a^x q(s)y(s)\,ds. \end{aligned}$$

Let $x = t$, integrating again:

$$\int_a^x y'(t)\,dt = \int_a^x \int_a^t q(s)y(s)\,ds\,dt,$$

which yields:

$$\begin{aligned} y(x) - y(a) &= \int_a^x \int_a^t q(s)y(s)\,ds\,dt \\ y(x) = y(a) &+ \int_a^x \int_a^t q(s)y(s)\,ds\,dt \\ y(x) = 1 &+ \int_a^x \int_a^t q(s)y(s)\,ds\,dt. \end{aligned}$$

Changing the order of integration yields:

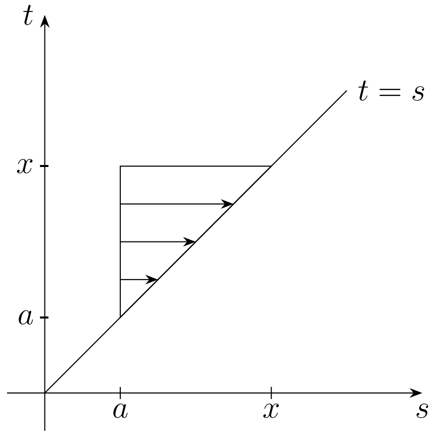


Figure 4.1

$$\Rightarrow y(x) = 1 + \int_a^x \int_s^x q(s)y(s)\,dt\,ds,$$

which simplifies to:

$$y(x) = 1 + \int_a^x (x-s)q(s)y(s)ds. \tag{4.2.0.3}$$

Next, we obtain the integral representation of (4.2.0.2):

This implies that

$$\begin{aligned}
\int_x^b z''(s)\,ds &= \int_x^b q(s)z(s)\,ds \\
z'(b) - z'(x) &= \int_x^b q(s)z(s)\,ds \\
1 - z'(x) &= \int_x^b q(s)z(s)\,ds \\
z'(x) &= 1 - \int_x^b q(s)z(s)\,ds
\end{aligned}$$

Integrating again yields

$$\begin{aligned}
z(b) - z(x) &= b - x - \int_x^b \int_t^b q(s)z(s)\,ds\,dt \\
z(x) &= x - b + \int_x^b \int_t^b q(s)z(s)\,ds\,dt
\end{aligned}$$

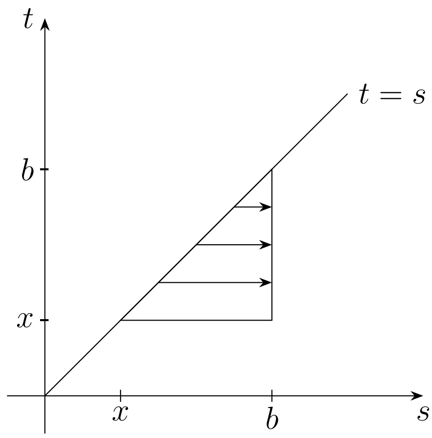


Figure 4.2

$$z(x) = x - b + \int_x^b \int_x^s q(s)z(s)\,dt\,ds$$

$$\implies z(x) = x - b + \int_x^b (s-x)q(s)z(s)\,ds. \tag{4.2.0.4}$$

Next, we look at the two spaces in which $q(x)$ is being considered, and obtain the bounds on $y(x)$ and $z(x)$.

4.3. Obtaining the Bounds when $q \in L^1[a, b]$

We now give a variant of Lemma 4 in [31]:

Lemma 4.3.0.47. *Let $y(x, q)$ denote the solution of the Cauchy problem:*

$$\begin{aligned}&-y''(x) + q(x)y(x) = 0, \quad a \le x \le b,\\&y'(a) = 0, \quad y(a) = 1.\\&B(c) = \{q : [a, b] \to \mathbb{R}, q \in L^1[a, b], \|q\|_1 = c, \}\end{aligned}$$

then for fixed $x \in [a, b]$,

$$\sup_{q \in B(c)} |y(x, q)| \le e^{c(x-a)}. \tag{4.3.0.1}$$

Similarly, if $z(x, q)$ denotes the solution of

$$-z''(x) + q(x)z(x) = 0, \quad a \le x \le b,$$

satisfying:

$$z(b) = 0, \quad z'(b) = 1,$$

then for fixed $x \in [a, b]$,

$$\sup_{q \in B(c)} |z(x, q)| \le (b - x)e^{c(b-x)}. \tag{4.3.0.2}$$

Like in chapter (3) to prove lemma (4.3.0.47), we find the terms, $A_n(x)$'s of the Neumann series expansion of the integral equation in (4.2.0.3). These $A_n(x)$ are the successive approximations to the solution of the problem (4.2.0.1) that are derived iteratively using the integral equation (4.2.0.3),

$$y(x) = 1 + \int_a^x (x - s)q(s)y(s)ds.$$

The solution $y(x)$ can be written as an infinite sum of the Neumann terms $A_n(x)$'s ([18], [24]), that is,

$$y(x) = \sum_{n=0}^{\infty} A_n(x),$$

where each term $A_n(x)$ is obtained iteratively as follows: $A_0(x) = 1$.

$$A_n(x) = \int_a^x (x - s)q(s)A_{n-1}(s)\,ds,\ n \ge 1.$$

Thus, we can write as:

$$y(x) = 1 + \sum_{n=1}^{\infty} A_n(x).$$

$$A_1(x) = \int_a^x (x - s)q(s)A_0(s)\,ds$$

$$
\begin{aligned}
\Rightarrow |A_1| &= \left| \int_a^x (x-s)q(s)A_0(s)\,ds \right| \\
&\leq \int_a^x |(x-s)||q(s)||A_0(s)|\,ds \\
&\leq \max_{a\leq s\leq x} |(x-s)| \int_a^x |q(s)|\,ds
\end{aligned}
$$

$$
\therefore |A_1| \leq (x-a)\|q\|_1\,, \text{since } q \in L^1(a,b).
$$

$$
\begin{aligned}
A_2(x) &= \int_a^x (x-s)q(s)A_1(s)\,ds \\
|A_2(x)| &= \left| \int_a^x (x-s)q(s)A_1(s)\,ds \right| \\
&\leq \int_a^x |(x-s)||q(s)||A_1(s)|\,ds \\
&\leq \int_a^x |(x-s)||q(s)||(s-a)|\|q\|_1\,ds \\
&\leq \max_{a\leq s\leq x} |(x-s)(s-a)|\|q\|_1 \int_a^x |q(s)|\,ds \\
&= \max_{a\leq s\leq x} |(x-s)(s-a)|\,\|q\|_1^2.
\end{aligned}
$$

Let $s = t$ and $f_1(t) = (x-t)(t-a)$, then we obtain the maximum value of $f_1(t)$ using calculus, since $f_1(a) = 0 = f_1(x)$. This means that the maximum is attained in the interior of $[a,x]$, by Rolle's theorem,

$$
f_1'(t) = -(t-a) + (x-t) = 0 \quad \Rightarrow t = \frac{x+a}{2}
$$

We see that $f_1''(t) = -2 < 0$, so $f_1(t)$ attains its maximum value at $t = \frac{x+a}{2}$.

Therefore, the maximum value of $f_1(t)$ is:

$$
f_1\left(\frac{x+a}{2}\right) = \left(\left(x - \frac{x+a}{2}\right)\left(\frac{x+a}{2} - a\right)\right) = \frac{(x-a)^2}{4}
$$

and so

$$
|A_2| \leq \frac{(x-a)^2}{2^2}\|q\|_1^2
$$

$$
\begin{aligned}
A_3 &= \int_a^x (x-s)q(s)A_2(s)\,ds \\
|A_3| &= \left| \int_a^x (x-s)q(s)A_2(s)\,ds \right| \\
&\leq \int_a^x |(x-s)||q(s)||A_2|\,ds \\
&\leq \int_a^x |(x-s)||q(s)|\left|\frac{(s-a)^2}{2^2}\|q\|_1^2\right|\,ds
\end{aligned}
$$

$$\leq \max_{a\leq s\leq x}\left|(x-s)\frac{(s-a)^2}{2^2}\|q\|_1^2\right|\int_a^b |q(s)|\,ds$$

$$= \max_{a\leq s\leq x}\left|(x-s)\frac{(s-a)^2}{2^2}\right|\|q\|_1^3,$$

let $f_2(s) = \frac{(x-s)(s-a)^2}{4}$, $f_2(a) = 0 = f_2(x)$, hence maximum value is attained in (a,x). Let $s = t$, then

$$f_2(t) = (x-t)\frac{(t-a)^2}{2^2}$$

and

$$f_2'(t) = -\frac{(t-a)^2}{4} + 2\frac{(t-a)}{2^2}(x-t) = \frac{t-a}{4}(2x-3t+a) = 0$$

$$\Rightarrow t = a \quad \text{or} \quad t = \frac{2x+a}{3},$$

but a is at the boundary, which is already ruled off. So we consider $t = \frac{2x+a}{3}$.

$$f_2''(t) = \frac{x-3t+2a}{2},$$

and

$$f_2''\left(\frac{2x+a}{3}\right) = -\frac{1}{2}(x-a) < 0.$$

Hence the maximum value is attained at $t = \frac{2x+a}{3}$ which is

$$f_2\left(\frac{2x+a}{3}\right) = \left[\left(x - \frac{2x+a}{3}\right)\frac{\left(\frac{2x+a}{3}-a\right)^2}{4}\right] = \frac{(x-a)^3}{27}$$

Thus,

$$|A_3| \leq \frac{(x-a)^3}{3^3}\|q\|_1^3.$$

$$A_4 = \int_a^x (x-s)q(s)A_3(s)\,ds$$

$$|A_4| = \left|\int_a^x (x-s)q(s)A_3(s)\,ds\right| \leq \int_a^x |(x-s)q(s)||A_3(s)|\,ds$$

$$\leq \int_a^x |(x-s)||q(s)|\left|\frac{(s-a)^3}{27}\right|\|q\|_1^3\,ds$$

$$\leq \max_{a\leq s\leq x}|(x-s)|\left|\frac{(s-a)^3}{3^3}\right|\|q\|_1^3\int_a^x |q(s)|\,ds$$

$$= \max_{a\leq s\leq x}\left|(x-s)\frac{(s-a)^3}{3^3}\right|\|q\|_1^3\int_a^b |q(s)|\,ds$$

$$= \max_{a\leq s\leq x}\left|(x-s)\frac{(s-a)^3}{3^3}\right|\|q\|_1^4.$$

Let $t = s$, then

$$f_3(t) = (x-t)\frac{(t-a)^3}{3^3}$$

and

$$f_3'(t) = -\frac{(t-a)^3}{27} + \frac{3}{27}(t-a)^2(x-t) = 0.$$

$$\Rightarrow t = a \quad \text{or} \quad t = \frac{3x+a}{4}.$$

Proceeding in a similar manner like before we find that $f_3(t)$ attains it's maximum at $t = \frac{3x+a}{4}$, which is:

$$f_3\left(\frac{3x+a}{4}\right) = \left[\left(x - \frac{3x+a}{4}\right)\frac{\left(\frac{3x+a}{4} - a\right)^3}{27}\right] = \frac{(x-a)^4}{256}.$$

Thus,

$$|A_4| \leq \frac{(x-a)^4}{4^4}\|q\|_1^4,$$

and

$$\vdots$$

$$|A_n| \leq \frac{(x-a)^n}{n^n}\|q\|_1^n, \quad n = 1, 2, 3, \dots .$$

Let $c = \|q\|_1$,

$$\Rightarrow |A_n| \leq c^n \frac{(x-a)^n}{n^n}, \quad n = 1, 2, 3, \dots .$$

Using

$$y(x) = 1 + \sum_{n=1}^{\infty} A_n(x),$$

we have

$$|y| = \left|1 + \sum_{n=1}^{\infty} A_n\right| \leq 1 + \sum_{n=1}^{\infty} |A_n|$$

$$\leq 1 + \sum_{n=1}^{\infty} c^n \frac{(x-a)^n}{n^n} \leq 1 + \sum_{n=1}^{\infty} c^n \frac{(x-a)^n}{n!}$$

$$= \sum_{n=0}^{\infty} c^n \frac{(x-a)^n}{n!},$$

which is Taylor series of

$$e^{c(x-a)}.$$

Therefore,

$$|y(x,q)| \leq e^{c(x-a)}, \tag{4.3.0.3}$$

and the bound in (4.3.0.1) is established.

By using a similar approach as above and using the integral equation 4.2.0.4 given by

$$z(x) = x - b + \int_x^b (s-x)q(s)z(s)\,ds$$

we have:

$$z(x) = \sum_{n=0}^{\infty} B_n(x),$$

where

$$B_0(x) = x - b,$$

and

$$B_n(x) = \int_x^b (x-s)q(s)B_{n-1}(s)\,ds,\ n \geq 1.$$

Thus we have:

$$|B_0(x)| \leq \max_{x \leq s \leq b} |s-b| = |x-b| = b-x$$

$$B_1(x) = \int_x^b (s-x)q(s)B_0(s)\,ds$$

$$|B_1(x)| = \left| \int_x^b (s-x)q(s)B_0(s)\,ds \right|$$

$$\leq \int_x^b |(s-x)||q(s)||B_0(s)|\,ds \leq \int_x^b |(s-x)||q(s)|(b-s)\,ds$$

$$\implies |B_1(x)| \leq \max_{x \leq s \leq b} |(s-x)(b-s)| \int_x^b |q(s)|\,ds$$

$$\leq \max_{x \leq s \leq b} |(s-x)(b-s)| \|q\|_1.$$

Let $g_1(t) = (t-x)(b-t)$, $x \leq t \leq b$, $g_1(x) = 0 = g_1(b) \implies$ maximum is attained in the interior of $[x, b]$.

$$g_1'(t) = -(t-x) + (b-t) = 0 \Rightarrow t = \frac{x+b}{2}$$

$$g_1''(t) = -2 < 0 \Rightarrow t = \frac{x+b}{2} \quad \text{gives maximum value.}$$

Hence,

$$\max_{x \leq s \leq b} |(s-x)(b-s)| = g_1\left(\frac{x+b}{2}\right) = \left(\frac{x+b}{2} - x\right)\left(b - \frac{x+b}{2}\right) = \frac{(b-x)^2}{4}$$

$$\implies |B_1| \leq \frac{(b-x)^2}{4} \|q\|_1.$$

$$B_2(x) = \int_x^b (s-x)q(s)B_1(s)\,ds$$

$$|B_2(x)| = \left| \int_x^b (s-x)q(s)B_1(s)\,ds \right| \leq \int_x^b |(s-x)||q(s)||B_1(s)|\,ds$$

$$\leq \int_x^b |(s-x)| \left| \frac{(b-s)^2}{4} \|q\|_1 \right| |q(s)|\,ds$$

$$\leq \max_{x \leq s \leq b} \left| (s-x)\frac{(b-s)^2}{4} \right| \|q\|_1 \int_x^b |q(s)|\,ds$$

$$\leq \max_{x \leq s \leq b} \left| (s-x)\frac{(b-s)^2}{4} \right| \|q\|_1^2.$$

Let $g_2(t) = (t-x)\dfrac{(b-t)^2}{4}$, $x \le t \le b$, then

$$g_2'(t) = \frac{(b-t)^2}{4} - \frac{2}{4}(b-t)(t-x) = 0 \Rightarrow t = \frac{2x+b}{3}.$$

We see that $g_2''(t) < 0$, so g_2 attains its maximum value at $t = \dfrac{2x+b}{3}$, which is

$$g_2\left(\frac{2x+b}{3}\right) = \left(\frac{2x+b}{3} - x\right)\left(\frac{b - \left(\frac{2x+b}{3}\right)}{4}\right) = \frac{(b-x)^3}{27}.$$

Thus,

$$|B_2(x)| \le \frac{(b-x)^3}{27}\|q\|_1^2.$$

$$B_3(x) = \int_x^b (s-x)q(s)B_2(s)\,ds$$

$$\begin{aligned}
|B_3(x)| = \left|\int_x^b (s-x)q(s)B_2(s)\,ds\right| &\le \int_x^b |(s-x)||q(s)||B_2(s)|\,ds \\
&\le \int_x^b \left|(s-x)\frac{(b-s)^3}{27}\right| \|q\|_1^2 |q(s)|\,ds \\
&\le \max_{x\le s\le b}\left|(s-x)\frac{(b-s)^3}{27}\right| \|q\|_1^2 \int_x^b |q(s)|\,ds \\
&= \max_{x\le s\le b}\left|(s-x)\frac{(b-s)^3}{27}\right| \|q\|_1^3.
\end{aligned}$$

Let $g_3(t) = (t-x)\dfrac{(b-t)^3}{27}$, $x \le t \le b$, then

$$\begin{aligned}
g_3'(t) &= \frac{(b-t)^3}{27} - \frac{3}{27}(b-t)^2(t-x) = 0 \\
&\Rightarrow \frac{(b-t)^2}{27}\left[(b-t) - 3(t-x)\right] = 0,
\end{aligned}$$

giving $t = b$ or $t = \dfrac{3x+b}{4}$, and $g_3''\left(\dfrac{3x+b}{4}\right) < 0$.

Thus, the maximum value of $g_3(x)$ is

$$g_3\left(\frac{3x+b}{4}\right) = \left(\frac{3x+b}{4} - x\right)\left(\frac{b - \left(\frac{3x+b}{4}\right)}{27}\right) = \frac{(b-x)^4}{4^4}.$$

Therefore,

$$|B_3| \le \frac{(b-x)^4}{4^4}\|q\|_1^3.$$

Proceeding in a similar manner, we find:

$$|B_4| \leq \frac{(b-x)^5}{5^5} \|q\|_1^4$$

and in general:

$$|B_n| \leq \frac{(b-x)^{n+1}}{(n+1)^{n+1}} \|q\|_1^n \, , n = 1, 2, 3, \cdots .$$

Let $c = \|q\|_1$, we have:

$$|B_n| \leq c^n \frac{(b-x)^{n+1}}{(n+1)^{n+1}}, \; n = 1, 2, 3, \ldots . \tag{4.3.0.4}$$

The series solution of z is given by

$$z(x) = x - b + \sum_{n=1}^{\infty} B_n(x),$$

and so,

$$\begin{aligned}
|z(x)| = \left| x - b + \sum_{n=1}^{\infty} B_n(x) \right| &\leq b - x + \sum_{n=1}^{\infty} |B_n(x)| \\
\leq b - x + \sum_{n=1}^{\infty} c^n \left(\frac{b-x}{n+1} \right)^{n+1} &\leq b - x + \sum_{n=1}^{\infty} c^n \frac{(b-x)^{n+1}}{n!} \\
&= b - x + (b-x) \sum_{n=1}^{\infty} c^n \frac{(b-x)^n}{n!} \\
&= (b-x) \left[1 + \sum_{n=1}^{\infty} c^n \frac{(b-x)^n}{n!} \right]
\end{aligned}$$

which yields

$$|z(x,q)| \leq (b-x) e^{c(b-x)}. \tag{4.3.0.5}$$

Thus establishing the bound (4.3.0.2).

Lemma 4.3.0.48. *Let y and z denote the two linearly independent solutions of*

$$\begin{aligned}
-y''(x) + q(x)y(x) = 0 \\
y'(a) = 0 \quad y(a) = 1
\end{aligned}$$

and

$$\begin{aligned}
-z''(x) + q(x)z(x) = 0 \\
z(b) = 0 \quad z'(b) = 1,
\end{aligned}$$

respectively.

Then,

1.

$$\sup_{q \in B(c)} \|y(x,q)\| \leq \left(\int_a^b \left(e^{c(x-a)} \right)^2 |r(x)| \, dx \right)^{\frac{1}{2}}, \tag{4.3.0.6}$$

2.
$$\sup_{q\in B(c)} \|z(x,q)\| \leq \left(\int_a^b \left((b-x)e^{c(b-x)}\right)^2 |r(x)|\, dx\right)^{\frac{1}{2}}. \tag{4.3.0.7}$$

Proof. We have

$$\|y(x,q)\| = \langle y(x,q), y(x,q)\rangle = \int_a^b y(x,q)\overline{y(x,q)}|r(x)|dx = \left(\int_a^b |y(x,q)|^2|r(x)|\, dx\right)^{\frac{1}{2}}$$
$$\leq \left(\int_a^b \left(e^{c(x-a)}\right)^2 |r(x)|\, dx\right)^{\frac{1}{2}}.$$

Taking the supremum of $\|y(x,q)\|$ yields the bound in (4.3.0.6).
Similarly,

$$\|z(x,q)\| = \langle z(x,q), z(x,q)\rangle = \int_a^b z(x,q)\overline{z(x,q)}|r(x)|dx = \left(\int_a^b |z(x,q)|^2|r(x)|\, dx\right)^{\frac{1}{2}}$$
$$\leq \left(\int_a^b \left((b-x)e^{c(b-x)}\right)^2 |r(x)|\, dx\right)^{\frac{1}{2}},$$

and taking the supremum of $\|z(x,q)\|$ gives the bound (4.3.0.7). □

4.4. Obtaining the Bounds when $q \in L^\infty[a,b]$

Lemma 4.4.0.49. *Let $y(x,q)$ denote the solution of the Cauchy problem:*

$$\begin{aligned} &-y''(x) + q(x)y(x) = 0, \quad a \leq x \leq b,\\ &y'(a) = 0, \quad y(a) = 1.\\ &B(c) = \{q : [a,b] \to \mathbb{R},\ q \in L^\infty[a,b],\ \|q\|_\infty = c\}. \end{aligned}$$

Then for fixed x in [a,b]

$$\sup_{q\in B(c)} |y(x,q)| \leq \cosh(\sqrt{c}(x-a)). \tag{4.4.0.1}$$

Similarly if $z(x,q)$ denotes the solution of the Cauchy problem,

$$\begin{aligned} &-z''(x) + q(x)z(x) = 0, \quad a \leq x \leq b,\\ &z(b) = 0,\ z'(b) = 1, \end{aligned}$$

then for fixed x in [a,b],

$$\sup_{q\in B(c)} |z(x,q)| \leq \frac{\sinh(\sqrt{c}(b-x))}{\sqrt{c}}. \tag{4.4.0.2}$$

Using a similar approach in the previous section and using the integral equation in (4.2.0.2), i.e

$$y(x) = 1 + \int_a^x (x-s)q(s)y(s)\, ds,$$

we have:

$$y(x) = \sum_{n=0}^{\infty} A_n(x),$$

where

$$A_0(x) = 1,$$

and

$$A_n(x) = 1 + \int_a^x (x-s)q(s)A_{n-1}(s)\,ds, \quad n \geq 1$$

$$\Rightarrow A_1(x) = \int_a^x (x-s)q(s)A_0(s)\,ds$$

$$|A_1(x)| = \left|\int_a^x (x-s)q(s)\,ds\right| \leq \int_a^x |(x-s)||q(s)|\,ds$$

$$\leq \max_{a \leq s \leq x} |q(s)| \int_a^x |x-s|\,ds$$

$$|A_1(x)| \leq \|q\|_\infty \left(sx - \frac{s^2}{2}\right)\Bigg|_a^x = \|q\|_\infty \left(\frac{x^2}{2} - ax + \frac{a^2}{2}\right) = \|q\|_\infty \frac{(x-a)^2}{2}.$$

$$\Rightarrow |A_1(x)| \leq \|q\|_\infty \frac{(x-a)^2}{(2 \times 1)!}.$$

$$A_2(x) = \int_a^x (x-s)q(s)A_1(s)ds$$

$$|A_2(x)| = \left|\int_a^x (x-s)q(s)A_1(s)\right|$$

$$|A_2(x)| \leq \int_a^x |(x-s)||q(s)||A_2(s)|ds$$

$$|A_2(x)| \leq \int_a^x |(x-s)||q(s)|\|q\|_\infty \left|\frac{(x-a)^2}{2}\right| ds$$

$$\leq \max_{a \leq s \leq x} |q(s)| \|q\|_\infty \int_a^x \left|(x-s)\frac{(x-a)^2}{2}\right| ds.$$

Using integration by parts, let

$$u = x - s \quad \Rightarrow \quad du = -1\,ds,$$

$$dv = \frac{(s-a)^2}{2}\,ds \quad \Rightarrow \quad v = \frac{(s-a)^3}{6}.$$

Then

$$\int_a^x u\,dv = uv\Big|_a^x - \int_a^x v\,du = (x-s)\frac{(s-a)^3}{6}\Bigg|_a^x - \int_a^x \frac{(s-a)^3}{6}(-1)ds$$

$$= (x-s)\frac{(s-a)^3}{6}\Bigg|_a^x + \int_a^x \frac{(s-a)^3}{6}\,ds = \frac{(x-a)^4}{24}$$

Thus:

$$|A_2| \leq \frac{(x-a)^4}{(2 \times 2)!}\|q\|_\infty^2.$$

$$A_3(x) = \int_a^x (x-s)q(s)A_2(s)\,ds$$

$$|A_3(x)| = \left|\int_a^x (x-s)q(s)A_2(s)ds\right| \le \int_a^x |(x-s)||q(s)||A_2(s)|ds$$

$$\le \int_a^x |(x-s)||q(s)|\|q\|_\infty^2 \frac{(x-a)^4}{24}\,ds$$

$$\le \|q\|_\infty^3 \int_a^x |x-s|\left|\frac{(s-a)^4}{24}\right|ds.$$

Thus, using integration by parts again we obtain

$$|A_3(x)| \le \frac{(x-a)^6}{(2\times 3)!}\|q\|_\infty^3.$$

Proceeding similarly we get:

$$|A_n(x)| \le \|q\|_\infty^n \frac{(x-a)^{2n}}{(2n)!},$$

and

$$y(x) = 1 + \sum_{n=1}^{\infty} A_n(x).$$

$$|y(x)| = \left|1 + \sum_{n=1}^{\infty} A_n(x)\right| \le 1 + \sum_{n=1}^{\infty} |A_n(x)|$$

Let $c = \|q\|_\infty$, then

$$|y(x)| \le 1 + \sum_{n=1}^{\infty} c^n \frac{(x-a)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} c^n \frac{(x-a)^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(\sqrt{c}(x-a))^{2n}}{(2n)!} = \cosh(\sqrt{c}(x-a)).$$

$$\implies |y(x)| \le \cosh(\sqrt{c}(x-a)). \tag{4.4.0.3}$$

This establishes the bound in (4.4.0.1).
Similarly, using the integral equation

$$z(x) = x - b + \int_x^b (s-x)q(s)z(s)\,ds$$

we have

$$z(x) = \sum_{n=0}^{\infty} B_n(x)$$

where

$$B_0 = x - b,$$

and

$$B_n(x) = \int_x^b (s-x)q(s)B_{n-1}(s)ds,\ n \ge 1.$$

$$|B_0(x)| = |x-b| = b-x.$$

$$B_1(x) = \int_x^b (s-x)q(s)B_0(s)\,ds$$

$$|B_1(x)| = \left|\int_x^b (s-x)(s-b)q(s)\,ds\right| \le \int_x^b |(s-x)||(s-b)||q(s)|\,ds$$

$$\le \max_{x\le s\le b}|q(s)| \int_x^b |(s-x)||(s-b)|\,ds.$$

Using integration by parts

$$u = s-x \quad \Rightarrow \quad du = ds,$$

$$dv = (s-b)\,ds \quad \Rightarrow \quad v = \frac{(s-b)^2}{2}.$$

Then

$$\int_x^b u\,dv = uv\Big|_x^b - \int_x^b v\,du = (s-x)\frac{(s-b)^2}{2}\Big|_x^b - \int_x^b \frac{(s-b)^2}{2}\,ds$$

$$= \frac{(x-b)^3}{6}.$$

Thus

$$|B_1(x)| \le \frac{(x-b)^3}{6}\|q\|_\infty.$$

Proceeding similarly, we get

$$|B_n(x)| \le \frac{(x-b)^{2n+1}}{(2n+1)!}\|q\|_\infty^n.$$

$$|B_n| \le \frac{(x-b)^{2n+1}}{(2n+1)!}\|q\|_\infty^n, \quad n \ge 1.$$

$$z(x) = x-b+\sum_{n=1}^{\infty} B_n$$

$$|z(x)| = \left|x-b+\sum_{n=1}^{\infty} B_n\right| \le b-x+\sum_{n=1}^{\infty}|B_n(x)| \le b-x+\sum_{n=1}^{\infty} c^n\frac{(b-x)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} c^n\frac{(b-x)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(\sqrt{c}(b-x))^{2n+1}}{\sqrt{c}(2n+1)!}$$

$$= \frac{\sinh(\sqrt{c}(b-x))}{\sqrt{c}}, \tag{4.4.0.4}$$

and the bound in (4.4.0.2) is established.

Lemma 4.4.0.50. *Let $y(x,q)$ and $z(x,q)$ denote the two linearly independent solutions of the problems*

$$-y''(x)+q(x)y(x)=0, \quad a\le x\le b,$$
$$y'(a)=0, \quad y(a)=1$$

and

$$-z''(x)+q(x)z(x)=0, \quad a\le x\le b$$
$$z(b)=0, \;\; z'(b)=1,$$

respectively, then for a fixed $x\in[a,b]$ we have

1.
$$\sup_{q\in B(c)}\|y(x,q)\|\le\left(\int_a^b\cosh^2(\sqrt{c}(x-a))|r(x)|dx\right)^{\frac{1}{2}}=\|\cosh(\sqrt{c}(x-a))\|, \quad (4.4.0.5)$$

2.
$$\sup_{q\in B(c)}\|z(x,q)\|\le\left(c^{-1}\int_a^b\sinh^2(\sqrt{c}(b-x))|r(x)|dx\right)^{\frac{1}{2}}=\left\|\frac{\sinh(\sqrt{c}(b-x))}{\sqrt{c}}\right\|. \quad (4.4.0.6)$$

Proof.

$$\|y(x,q)\|=\langle y(x,q),y(x,q)\rangle=\left(\int_x^b y(x,q)\overline{y(x,q)}|r(x)|dx\right)^{\frac{1}{2}}$$
$$=\left(\int_x^b\cosh^2(\sqrt{c}(x-a))|r(x)|dx\right)^{\frac{1}{2}}=\|\cosh(\sqrt{c}(x-a))\|,$$

taking the supremum of $\|y(x,q)\|$ over all $x\in[a,b]$ yields (4.4.0.5).
Similarly,

$$\|z(x,q)\|=\langle z(x,q),z(x,q)\rangle=\left(\int_x^b z(x,q)\overline{z(x,q)}|r(x)|dx\right)^{\frac{1}{2}}$$
$$=\left(c^{-1}\int_x^b\sinh^2(\sqrt{c}(b-x))|r(x)|dx\right)^{\frac{1}{2}}=\left\|\frac{\sinh(\sqrt{c}(b-x))}{\sqrt{c}}\right\|,$$

taking the supremum of $\|z(x,q)\|$ over all $x\in[a,b]$ yields (4.4.0.6). □

We now present the main results of the dissertation when $q\in L^1$ and $q\in L^\infty$, respectively.

Theorem 4.4.0.51. Let $\|q\|_1=c$. Then for the problem

$$-u''(x)+q(x)u(x)=\lambda r(x)u(x), \quad a\le x\le b,$$
$$u'(a)=u(b)=0,$$

an eigenvalue λ_0 of the smallest modulus satisfies:

$$|\lambda_0|\ge|y(b)|\left(\sqrt{2}\|e^{c(x-a)}\|\|(b-x)e^{c(b-x)}\|\right)^{-1}, \quad (4.4.0.7)$$

where $x\in[a,b]$.

This is a variant of Theorem (3.3.0.42).

Proof. By lemma (4.1.0.46), an eigenvalue of the smallest modulus satisfies

$$|\lambda_0| \geq \frac{|y(b)|}{\sqrt{2}\|y\|\|z\|} = |y(b)| \left(\sqrt{2}\|y\|\|z\|\right)^{-1}. \tag{4.4.0.8}$$

Substituting the results of lemma (4.3.0.47) given by

1.
$$\sup_{q\in B(c)} \|y(x,q)\| \leq \left(\int_a^b \left(e^{c(x-a)}\right)^2 |r(x)|\, dx\right)^{\frac{1}{2}} = \|e^{c(x-a)}\|,$$

2.
$$\sup_{q\in B(c)} \|z(x,q)\| \leq \left(\int_a^b \left((b-x)e^{c(x-a)}\right)^2 |r(x)|\, dx\right)^{\frac{1}{2}} = \|(b-x)e^{c(b-x)}\|,$$

into (4.4.0.8) yields

$$|\lambda_0| \geq |y(b)| \left(\sqrt{2}\|e^{c(x-a)}\|\|(b-x)e^{c(b-x)}\|\right)^{-1}. \tag{4.4.0.9}$$

□

Similarly, We give a variant of Theorem (3.2.0.39).

Theorem 4.4.0.52. Let $\|q\|_\infty = c$. Then for the problem

$$-u''(x) + q(x)u(x) = \lambda r(x)u(x), \quad a \leq x \leq b,$$

$$u'(a) = u(b) = 0,$$

an eigenvalue λ_0 of the smallest modulus satisfies

$$|\lambda_0| \geq |y(b)| \left(\sqrt{2}\|\cosh(\sqrt{c}(x-a))\| \left\|\frac{\sinh(\sqrt{c}(b-x))}{\sqrt{c}}\right\|\right)^{-1}, \tag{4.4.0.10}$$

for $x \in (a,b)$.

Proof. Substituting the results of Lemma (4.4.0.50), given by

1.
$$\sup_{q\in B(c)} \|y(x,q)\| \leq \left(\int_a^b \cosh^2(\sqrt{c}(x-a))|r(x)|dx\right)^{\frac{1}{2}} = \|\cosh(\sqrt{c}(x-a))\|,$$

2.
$$\sup_{q\in B(c)} \|z(x,q)\| \leq \left(c^{-1}\int_a^b \sinh^2(\sqrt{c}(b-x))|r(x)|dx\right)^{\frac{1}{2}} = \left\|\frac{\sinh(\sqrt{c}(b-x))}{\sqrt{c}}\right\|,$$

into (4.4.0.8) gives us

$$|\lambda_0| \geq |y(b)| \left(\sqrt{2}\|\cosh(\sqrt{c}(x-a))\| \left\|\frac{\sinh(\sqrt{c}(b-x))}{\sqrt{c}}\right\|\right)^{-1}.$$

□

Here we give examples to illustrate theorems in (4.4.0.51) and (4.4.0.52). We consider the case where $q(x) = q_0 \in \mathbb{R}$ on the interval [-1,1]. In this case, (4.4.0.7) becomes,

$$|\lambda_0| \geq |y(1)| \left(\sqrt{2}\|e^{c(x+1)}\|\|(1-x)e^{c(1-x)}\|\right)^{-1}, \tag{4.5.0.1}$$

where $c = (\int_{-1}^{1} |q_0|dx)^{\frac{1}{2}} = \left(x|q_0|\Big|_{-1}^{1}\right)^{\frac{1}{2}} = \sqrt{2|q_0|}$.

The eigenvalues are found using the Maple© package `RootFinding[Analytic]` software (See Appendix 1 for the details).

Example 4.5.0.53. (the case $q_0 = -4\pi^2$ and $r(x)$ changes sign). This is an example of the non-definite SLP.

$$y'' + 4\pi^2 y = 0, \quad y(-1) = 1, \quad y'(-1) = 0, \tag{4.5.0.2}$$

$$y'' + 4\pi^2 y = -\lambda r(x)y, \quad y'(-1) = 0, \quad y(1) = 0. \tag{4.5.0.3}$$

We take

$$r(x) = \begin{cases} -1, & \text{if } x \in [-1, 0), \\ 1, & \text{if } x \in (0, 1]. \end{cases}$$

We find the solution of (4.5.0.2)

$$y(x) = c_1 \cos 2\pi x + c_2 \sin 2\pi x.$$

Applying BCs yields

$$c_1 = 1 \text{ and } c_2 = 0,$$

$$\therefore y = \cos 2\pi x.$$

(a) By theorem (4.4.0.51) the L^1 bound becomes,

$$|\lambda_0| \geq |y(1)| \left(\sqrt{2}\|e^{c(x-1)}\|\|(b-x)e^{c(1-x)}\|\right)^{-1}, \tag{4.5.0.4}$$

$$\text{where} \quad y(1) = \cos 2\pi(1) = 1, \tag{4.5.0.5}$$

and

$$c = \sqrt{2|q_0|} = \sqrt{2|-4\pi^2|} = 2\pi\sqrt{2},$$

$$\|e^{c(x-a)}\| = \|e^{2\pi\sqrt{2}(x+1)}\| = \langle e^{2\pi\sqrt{2}(x+1)}, e^{2\pi\sqrt{2}(x+1)}\rangle = \left(\int_{-1}^{1} |e^{2\pi\sqrt{2}(x+1)}|^2| \pm 1|dx\right)^{\frac{1}{2}}$$

$$= \left(\int_{-1}^{1} e^{4\pi\sqrt{2}(x+1)}dx\right)^{\frac{1}{2}} = \left(e^{4\pi\sqrt{2}} \int_{-1}^{1} e^{4\pi\sqrt{2}x}dx\right)^{\frac{1}{2}} = \left(e^{4\pi\sqrt{2}}\left[\frac{e^{4\pi\sqrt{2}} - e^{-4\pi\sqrt{2}}}{4\pi\sqrt{2}}\right]\right)^{\frac{1}{2}}$$

$$= \sqrt{\frac{e^{8\pi\sqrt{2}} - 1}{4\pi\sqrt{2}}}$$

$$\approx 12394121.35650267, \tag{4.5.0.6}$$

$$\|(1-x)e^{c(1-x)}\| = \|(1-x)e^{2\pi\sqrt{2}(1-x)}\| = \left(\int_{-1}^{1}(1-x)^2 e^{4\pi\sqrt{2}(1-x)}dx\right)^{\frac{1}{2}}.$$

Let

$$u = (1-x)^2 \implies du = -2(1-x),$$

and

$$dv = e^{4\pi\sqrt{2}(1-x)}dx \implies v = -\frac{e^{4\pi\sqrt{2}(1-x)}}{4\pi\sqrt{2}}.$$

Thus

$$\int_{-1}^{1} udv = uv|_{-1}^{1} - \int_{-1}^{1} vdu,$$

$$= -(1-x)^2\frac{e^{4\pi\sqrt{2}(1-x)}}{4\pi\sqrt{2}}\bigg|_{-1}^{1} - 2\int_{-1}^{1}\frac{e^{4\pi\sqrt{2}(1-x)}}{4\pi\sqrt{2}}(1-x)dx, \tag{4.5.0.7}$$

let $u = 1-x$, $du = -dx$, $\quad dv = e^{4\pi\sqrt{2}(1-x)}dx$, $v = -\frac{e^{4\pi\sqrt{2}(1-x)}}{4\pi\sqrt{2}}$, then

$$\int_{-1}^{1} udv = uv|_{-1}^{1} - \int_{-1}^{1} vdu = -(1-x)\frac{e^{4\pi\sqrt{2}(1-x)}}{4\pi\sqrt{2}}\bigg|_{-1}^{1} - \int_{-1}^{1}\frac{e^{4\pi\sqrt{2}(1-x)}}{4\pi\sqrt{2}}dx$$

$$= -(1-x)^2\frac{e^{4\pi\sqrt{2}(1-x)}}{4\pi\sqrt{2}}\bigg|_{-1}^{1} - \frac{2}{4\pi\sqrt{2}}\left[-(1-x)\frac{e^{4\pi\sqrt{2}(1-x)}}{4\pi\sqrt{2}}\bigg|_{-1}^{1} - \int_{-1}^{1}\frac{e^{4\pi\sqrt{2}(1-x)}}{4\pi\sqrt{2}}dx\right]$$

$$= -(1-x)^2\frac{e^{4\pi\sqrt{2}(1-x)}}{4\pi\sqrt{2}}\bigg|_{-1}^{1} - \frac{2}{4\pi\sqrt{2}}\left[-(1-x)\frac{e^{4\pi\sqrt{2}(1-x)}}{4\pi\sqrt{2}}\bigg|_{-1}^{1} + \frac{e^{4\pi\sqrt{2}(x-1)}}{32\pi^2}\bigg|_{-1}^{1}\right]$$

$$= \frac{e^{8\pi\sqrt{2}}}{\pi\sqrt{2}} - \frac{e^{8\pi\sqrt{2}}}{8\pi^2} + \frac{e^{8\pi\sqrt{2}}}{64\pi^3\sqrt{2}} - \frac{1}{64\pi^3\sqrt{2}} \approx 580854397752491.1$$

$$\implies \sqrt{580854397752491.1} \approx 24100921.09759482. \tag{4.5.0.8}$$

Substituting (4.5.0.5), (4.5.0.6) and (4.5.0.8) into (4.5.0.4) yields,

$$|\lambda_0| \geq \left(\sqrt{2}(12394121.35650267)(24100921.09759482)\right)^{-1}$$

$$= \left(422439366775469.75\right)^{-1} = 2.367 \times 10^{-15}.$$

Thus,

$$|\lambda_0| \geq 2.367 \times 10^{-15}. \tag{4.5.0.9}$$

(b) Similarly by theorem (4.4.0.52) given by,

$$|\lambda_0| \geq |y(b)|\left(\sqrt{2}\|\cosh(\sqrt{c}(x-a))\|\left\|\frac{\sinh(\sqrt{c}(b-x))}{\sqrt{c}}\right\|\right)^{-1}, \tag{4.5.0.10}$$

where

$$c = \|4\pi^2\|_\infty = 4\pi^2 \quad \text{and} \quad y(1) = \cos 2\pi(1) = 1,$$

$$\|\cosh(\sqrt{c}(x-a))\|_\infty = \|\cosh 2\pi(x+1)\|_\infty = \sup_{x\in[-1,1]} \cosh\left(2\pi(x+1)\right)$$

$$= \cosh 4\pi \approx 143375.65657. \tag{4.5.0.11}$$

$$\left\|\frac{\sinh(\sqrt{c}(b-x))}{\sqrt{c}}\right\|_\infty = \frac{\sinh 4\pi}{2\pi} = 22818.94. \tag{4.5.0.12}$$

Substituting (4.5.0.11) and (4.5.0.12)) in (4.5.0.10) yields,

$$|\lambda_0| \geq 2.1612 \times 10^{-10}. \tag{4.5.0.13}$$

We solve problem (4.5.0.3) as follows:

$$\begin{cases} y_1''(x) + (4\pi^2 - \lambda)y_1(x) = 0, & \text{if } x \in [-1,0], \\ y_2''(x) + (4\pi^2 + \lambda)y_2(x) = 0, & \text{if } x \in [0,1], \end{cases} \tag{4.5.0.14}$$

$$y_1(x) = c_1 \cos\left(\sqrt{4\pi^2-\lambda}x\right) + c_2 \sin\left(\sqrt{4\pi^2-\lambda}x\right),$$

$$y_1'(x) = -c_1\left(\sqrt{4\pi^2-\lambda}\right)\sin\left(\sqrt{4\pi^2-\lambda}x\right) + c_2\left(\sqrt{4\pi^2-\lambda}\right)\cos\left(\sqrt{4\pi^2-\lambda}x\right),$$

$$y_1'(-1) = c_1\left(\sqrt{4\pi^2-\lambda}\right)\sin\left(\sqrt{4\pi^2-\lambda}\right) + c_2\left(\sqrt{4\pi^2-\lambda}\right)\cos\left(\sqrt{4\pi^2-\lambda}\right) = 0,$$

$$\implies c_2 = \frac{-c_1 \sin\left(\sqrt{4\pi^2-\lambda}\right)}{\cos\left(\sqrt{4\pi^2-\lambda}\right)},$$

$$y_1(x) = c_1\cos\left(\sqrt{4\pi^2-\lambda}x\right) + \frac{-c_1\sin\left(\sqrt{4\pi^2-\lambda}\right)}{\cos\left(\sqrt{4\pi^2-\lambda}\right)}\sin\left(\sqrt{4\pi^2-\lambda}x\right)$$

$$\text{which reduces to } y_1(x) = \frac{c_1}{\cos\left(\sqrt{4\pi^2-\lambda}\right)}\cos\left(\sqrt{4\pi^2-\lambda}(x+1)\right),$$

$$\text{Similarly,}\quad y_2(x) = d_1\cos\left(\sqrt{4\pi^2+\lambda}x\right) + d_2\sin\left(\sqrt{4\pi^2+\lambda}x\right),$$

$$y_2(0) = d_1, \quad y_2'(0) = d_2\sqrt{4\pi^2+\lambda}$$

$$y_1(0) = \frac{c_1\cos\left(\sqrt{4\pi^2-\lambda}\right)}{\cos\left(\sqrt{4\pi^2-\lambda}\right)} = c_1$$

$$y_1'(0) = \frac{-c_1\sqrt{4\pi^2-\lambda}\sin\left(\sqrt{4\pi^2-\lambda}\right)}{\cos\left(\sqrt{4\pi^2-\lambda}\right)}.$$

For continuity we require $y_1'(0) = y_2'(0)$ and $y_1(0) = y_2(0)$.

$$y_1(0) = y_2(0) \implies d_1 = c_1,\ y_1'(0) = y_2'(0) \implies d_2 = \frac{-c_1\sqrt{4\pi^2-\lambda}\sin\left(\sqrt{4\pi^2-\lambda}\right)}{\sqrt{(4\pi^2+\lambda)}\cos\left(\sqrt{4\pi^2-\lambda}\right)}.$$

$$y_2(x) = c_1 cos\sqrt{\lambda+4\pi^2}x - \frac{c_1\sqrt{4\pi^2-\lambda}\sin\left(\sqrt{4\pi^2-\lambda}\right)}{\sqrt{(4\pi^2+\lambda)}\cos\left(\sqrt{4\pi^2-\lambda}\right)}\sin\left(\sqrt{4\pi^2+\lambda}x\right).$$

$$y_2(x) = \frac{c_1}{\sqrt{\lambda+4\pi^2}\cos\sqrt{4\pi^2-\lambda}}\Bigg(\sqrt{\lambda+4\pi^2}\cos\sqrt{4\pi^2-\lambda}\cos\sqrt{\lambda+4\pi^2}x$$

$$-\sqrt{4\pi^2-\lambda}\sin\sqrt{4\pi^2-\lambda}\sin\sqrt{\lambda+4\pi^2}x\Bigg)$$

Applying the BC at $x = 1$ we have,

$$y_2(1) = \frac{c_1}{\sqrt{\lambda+4\pi^2}\cos\sqrt{4\pi^2-\lambda}}\Bigg(\sqrt{\lambda+4\pi^2}\cos\sqrt{4\pi^2-\lambda}\cos\sqrt{\lambda+4\pi^2}$$

$$-\sqrt{4\pi^2-\lambda}\sin\sqrt{4\pi^2-\lambda}\sin\sqrt{\lambda+4\pi^2}\Bigg) = 0.$$

This leads to the dispersion relation

$$\sqrt{\lambda+4\pi^2}\cos\sqrt{4\pi^2-\lambda}\cos\sqrt{\lambda+4\pi^2} - \sqrt{4\pi^2-\lambda}\sin\sqrt{4\pi^2-\lambda}\sin\sqrt{\lambda+4\pi^2} = 0,$$

which is solved using Root-finding analytic package in Maple to give the spectrum

$$\cdots -477.4, -343.8, -229.7, -134.8, -57.1, -39.5, -30.8, -7.315\pm 14.59i, 24.18\pm 9.767i, 36.45, 94.27,$$

$$179.9, 284.3, 408.2, \cdots$$

and so $|-7.315 \pm 14.59i| = 16.3$ gives the smallest modulus and (4.5.0.9) and (4.5.0.13) are satisfied.

Example 4.5.0.54. (the case $q_0 = 4\pi^2$ and $r(x)$ changes sign). This is an example of the left definite case.

We consider the problem

$$-y'' + 4\pi^2 y = 0, \quad y(-1) = 1, \quad y'(-1) = 0, \tag{4.5.0.15}$$

$$-y'' + 4\pi^2 y = \lambda r(x) y, \quad y'(-1) = 0, \quad y(1) = 0, \tag{4.5.0.16}$$

where

$$r(x) = \begin{cases} -1, & \text{if } x \in [-1, 0), \\ 1, & \text{if } x \in (0, 1]. \end{cases}$$

We find the solution of (4.5.0.15), giving the general solution as

$$y(x) = c_1 \cosh x + c_2 \sinh x.$$

Applying BCs gives

$$y(x) = \cosh(2\pi(1+x))$$

$$\implies y(1) = \cosh(4\pi) \approx 143375.65657 \tag{4.5.0.17}$$

(a) For $x \in [-1, 1]$, then from theorem (4.4.0.7), the $L^1[-1, 1]$ bound becomes,

$$|\lambda_0| \geq |y(1)| \left(\sqrt{2}\|e^{c(x+1)}\|\|(1-x)e^{c(1-x)}\|\right)^{-1}, \tag{4.5.0.18}$$

$$c = \sqrt{2|q_0|} = \sqrt{2|4\pi^2|} = 2\pi\sqrt{2}. \tag{4.5.0.19}$$

From (4.5.0.6) we have,

$$\|e^{c(x-a)}\|_1 = \|e^{2\pi\sqrt{2}(x+1)}\|_1 \approx 12394121.35650267, \tag{4.5.0.20}$$

similarly, from (4.5.0.8), we get

$$\|(1-x)e^{c(1-x)}\|_1 = \|(1-x)e^{2\pi\sqrt{2}(1-x)}\|_1 \approx 24100921.09759482. \tag{4.5.0.21}$$

Substituting (4.5.0.17), (4.5.0.20) (4.5.0.21) into (4.5.0.18) gives us

$$|\lambda_0| \geq 3.394 \times 10^{-10}. \tag{4.5.0.22}$$

(b) Taking a similar approach, for $x \in [-1,1]$, the $L^\infty[-1,1]$ bound becomes,

$$|\lambda_0| \geq |y(1)| \left(\sqrt{2} \| \cosh(\sqrt{c}(x+1)) \| \left\| \frac{\sinh(\sqrt{c}(1-x))}{\sqrt{c}} \right\| \right)^{-1}, \tag{4.5.0.23}$$

where

$$c = \|4\pi^2\|_\infty = 4\pi^2,$$

$$y(1) = \cosh(4\pi) = 143375.65657. \tag{4.5.0.24}$$

$$\| \cosh(\sqrt{c}(x+1))\|_\infty = \| \cosh(2\pi(x+1))\|_\infty = \cosh 4\pi \approx 143375.65657, \tag{4.5.0.25}$$

$$\left\| \frac{\sinh(\sqrt{c}(1-x))}{\sqrt{c}} \right\|_\infty = \frac{\sinh 4\pi}{2\pi} \approx 22818.94. \tag{4.5.0.26}$$

Substituting (4.5.0.24), (4.5.0.26) into (4.5.0.23), yields

$$|\lambda_0| \geq 3.0987 \times 10^{-5}. \tag{4.5.0.27}$$

Next, we solve (4.5.0.16)

$$\begin{cases} y_1''(x) + (-4\pi^2 - \lambda) y_1(x) = 0, & \text{if } x \in [-1, 0), \\ y_2''(x) + (\lambda - 4\pi^2) y_2(x) = 0, & \text{if } x \in (0, 1]. \end{cases} \tag{4.5.0.28}$$

Thus we have

$$\begin{aligned}
y_1(x) &= c_1 \cos\left(\sqrt{-4\pi^2 - \lambda} x\right) + c_2 \sin\left(\sqrt{-4\pi^2 - \lambda} x\right). \\
y_1'(x) &= -c_1 \left(\sqrt{-4\pi^2 - \lambda}\right) \sin\left(\sqrt{-4\pi^2 - \lambda} x\right) + c_2 \left(\sqrt{-4\pi^2 - \lambda}\right) \cos\left(\sqrt{-4\pi^2 - \lambda} x\right), \\
y_1'(-1) &= c_1 \left(\sqrt{-4\pi^2 - \lambda}\right) \sin\left(\sqrt{-4\pi^2 - \lambda}\right) + c_2 \left(\sqrt{-4\pi^2 - \lambda}\right) \cos\left(\sqrt{-4\pi^2 - \lambda}\right) = 0, \\
\implies c_2 &= \frac{-c_1 \sin\left(\sqrt{-4\pi^2 - \lambda}\right)}{\cos\left(\sqrt{-4\pi^2 - \lambda}\right)}.
\end{aligned}$$

$$y_1(x) = c_1 \cos\left(\sqrt{-4\pi^2 - \lambda} x\right) - \frac{c_1 \sin\left(\sqrt{-4\pi^2 - \lambda}\right)}{\cos\left(\sqrt{-4\pi^2 - \lambda}\right)} \sin\left(\sqrt{-4\pi^2 - \lambda} x\right),$$

which reduces to $y_1(x) = \dfrac{c_1}{\cos\left(\sqrt{-4\pi^2 - \lambda}\right)} \cos\left(\sqrt{-4\pi^2 - \lambda}(x+1)\right).$

$$
\begin{aligned}
\text{Similarly,}\quad & y_2(x) = d_1 \cos\left(\sqrt{\lambda - 4\pi^2}x\right) + d_2 \sin\left(\sqrt{\lambda - 4\pi^2}x\right),\\
& y_2(0) = d_1, \quad y_2'(0) = d_2\sqrt{\lambda - 4\pi^2},\\
& y_1(0) = \frac{c_1 \cos\left(\sqrt{-4\pi^2-\lambda}\right)}{\cos\left(\sqrt{-4\pi^2-\lambda}\right)} = c_1,\\
& y_1'(0) = \frac{-c_1\sqrt{-4\pi^2-\lambda}\sin\left(\sqrt{-4\pi^2-\lambda}\right)}{\cos\left(\sqrt{-4\pi^2-\lambda}\right)}.\\
& y_1(0) = y_2(0) \implies d_1 = c_1,\ y_1'(0) = y_2'(0)
\end{aligned}
$$

$$
\implies \quad d_2 = \frac{-c_1\sqrt{-4\pi^2-\lambda}\sin\left(\sqrt{-4\pi^2-\lambda}\right)}{\sqrt{(\lambda-4\pi^2)}\cos\left(\sqrt{-4\pi^2-\lambda}\right)}.
$$

$$
y_2(x) = c_1\cos\sqrt{\lambda-4\pi^2}x - \frac{c_1\sqrt{-4\pi^2-\lambda}\sin\left(\sqrt{-4\pi^2-\lambda}\right)}{\sqrt{(\lambda-4\pi^2)}\cos\left(\sqrt{-4\pi^2-\lambda}\right)}\sin\left(\sqrt{\lambda-4\pi^2}x\right).
$$

$$
\begin{aligned}
y_2(x) = \frac{c_1}{\sqrt{\lambda-4\pi^2}\cos\sqrt{-4\pi^2-\lambda}}\Bigg(&\sqrt{\lambda-4\pi^2}\cos\sqrt{-4\pi^2-\lambda}\cos\sqrt{\lambda-4\pi^2}x\\
&-\sqrt{-4\pi^2-\lambda}\sin\sqrt{-4\pi^2-\lambda}\sin\sqrt{\lambda-4\pi^2}x\Bigg).
\end{aligned}
$$

Applying the BC at $x = 1$ we have

$$
\begin{aligned}
y_2(1) = \frac{c_1}{\sqrt{\lambda-4\pi^2}\cos\sqrt{-4\pi^2-\lambda}}\Bigg(&\sqrt{\lambda-4\pi^2}\cos\sqrt{-4\pi^2-\lambda}\cos\sqrt{\lambda-4\pi^2}\\
&-\sqrt{-4\pi^2-\lambda}\sin\sqrt{-4\pi^2-\lambda}\sin\sqrt{\lambda-4\pi^2}\Bigg) = 0.
\end{aligned}
$$

This leads to the dispersion relation,

$$
\sqrt{\lambda-4\pi^2}\cos\sqrt{-4\pi^2-\lambda}\cos\sqrt{\lambda-4\pi^2} - \sqrt{-4\pi^2-\lambda} - \sin\sqrt{-4\pi^2-\lambda}\sin\sqrt{\lambda-4\pi^2} = 0,
$$

which is solved to find the spectrum of the problem.

Using the maple package rootfinding analytic gives the spectrum $-426.8, -313.6, -220.2, -146.5, -92.60$ $-57.98, -41.48, 39.48, 47.58, 72.93, 117.1, 180.89, 264.45, 367.7, 490.9, \cdots$. Here $\lambda_0 = 39.48$ and satisfies (4.5.0.22) and (4.5.0.27).

Example 4.5.0.55. (the case $q_0 = 4\pi^2$ and $r(x) \equiv 1$). This is an example of the right definite case.
We consider the problems

$$
-y'' + 4\pi^2 y = 0, \quad y(-1) = 1, \quad y'(-1) = 0, \tag{4.5.0.29}
$$

$$
-y'' + 4\pi^2 y = \lambda r(x) y, \quad y'(-1) = 0, \quad y(1) = 0. \tag{4.5.0.30}
$$

We find the solution of (4.5.0.29) to be

$$
\begin{aligned}
& y(x) = \cosh(2\pi(x+1)),\\
\implies & y(1) = \cosh 4\pi = 143375.65657,
\end{aligned} \tag{4.5.0.31}
$$

(a) For $x \in [-1,1]$, from theorem (4.4.0.7), the $L^1[-1,1]$ bound becomes,

$$|\lambda_0| \geq |y(1)| \left(\sqrt{2}\|e^{c(x+1)}\|\|(1-x)e^{c(1-x)}\|\right)^{-1}, \tag{4.5.0.32}$$

$$c = \sqrt{2|q_0|} = \sqrt{2|4\pi^2|} = 2\pi\sqrt{2}, \tag{4.5.0.33}$$

$$\|e^{c(x-a)}\|_1 = \|e^{2\pi\sqrt{2}(x+1)}\|_1 \approx 12394121.35650267, \tag{4.5.0.34}$$

$$\|(1-x)e^{c(1-x)}\|_1 = \|(1-x)e^{2\pi\sqrt{2}(1-x)}\|_1 \approx 24100921.09759482. \tag{4.5.0.35}$$

Substituting (4.5.0.31), (4.5.0.34), (4.5.0.35) into (4.5.0.32) gives us

$$|\lambda_0| \geq 3.394 \times 10^{-10}. \tag{4.5.0.36}$$

(b) Taking a similar approach, for $x \in [-1,1]$, the $L^\infty[-1,1]$ bound becomes,

$$|\lambda_0| \geq |y(1)| \left(\sqrt{2}\|\cosh(\sqrt{c}(x+1))\| \left\|\frac{\sinh(\sqrt{c}(1-x))}{\sqrt{c}}\right\|\right)^{-1}, \tag{4.5.0.37}$$

Where,

$$c = \|4\pi^2\|_\infty = 4\pi^2,$$

$$y(1) = \cosh(4\pi) = 143375.65657, \tag{4.5.0.38}$$

$$\|\cosh(\sqrt{c}(x+1))\|_\infty = \|\cosh(2\pi(x+1))\|_\infty = \cosh 4\pi \approx 143375.65657, \tag{4.5.0.39}$$

$$\left\|\frac{\sinh(\sqrt{c}(1-x))}{\sqrt{c}}\right\|_\infty = \frac{\sinh 4\pi}{2\pi} \approx 22818.94, \tag{4.5.0.40}$$

substituting (4.5.0.38), (4.5.0.39) and (4.5.0.40) into (4.5.0.37) yields,

$$|\lambda_0| \geq 3.0988 \times 10^{-5}. \tag{4.5.0.41}$$

Next, we solving problem (4.5.0.30)

,

$$-y'' + 4\pi^2 y = \lambda y, \quad y'(-1) = 0, \quad y(1) = 0.$$

We are given that $r(x) = 1$, so the equation simplifies to:

$$\begin{aligned}
& y'' - (4\pi^2 - \lambda)y = 0. \\
& y(x) = c_1 \cos\left(\sqrt{\lambda - 4\pi^2}x\right) + c_2 \sin\left(\sqrt{\lambda - 4\pi^2}x\right), \\
& y(1) = c_1 \cos\left(\sqrt{\lambda - 4\pi^2}\right) + c_2 \sin\left(\sqrt{\lambda - 4\pi^2}\right) = 0, \\
\implies & c_2 = \frac{-c_1 \cos\left(\sqrt{\lambda - 4\pi^2}\right)}{\sin\left(\sqrt{\lambda - 4\pi^2}\right)},
\end{aligned}$$

$$\begin{aligned}
y(x) &= c_1 \cos\left(\sqrt{\lambda - 4\pi^2}x\right) - \frac{c_1 \cos\left(\sqrt{\lambda - 4\pi^2}\right)}{\sin\left(\sqrt{\lambda - 4\pi^2}\right)} \sin\left(\sqrt{\lambda - 4\pi^2}x\right), \\
&= \frac{-c_1}{\sin\sqrt{\lambda - 4\pi^2}}\left(\sin(\sqrt{\lambda - 4\pi^2}(1-x))\right)
\end{aligned}$$

$$
\begin{aligned}
y'(x) &= \frac{c_1\sqrt{\lambda - 4\pi^2}}{\sin\left(\sqrt{4\pi^2 - \lambda}\right)} \cos\left(\sqrt{4\pi^2 - \lambda}(1-x)\right), \\
y'(-1) &= \frac{c_1\sqrt{\lambda - 4\pi^2}}{\sin\left(\sqrt{\lambda - 4\pi^2}\right)} \cos\left(2\sqrt{\lambda - 4\pi^2}\right) = 0 \\
&\implies \sqrt{\lambda - 4\pi^2}\cos(2\sqrt{\lambda - 4\pi^2}) = 0,
\end{aligned}
$$

is our dispersion relation. Solving this yields the spectrum below

$$39.48, 40.09, 45.03, 54.90, 69.70, 89.44, 114.1, \cdots.$$

Here, $\lambda_0 = 39.48$ which satisfies (4.5.0.36) and (4.5.0.41).

CHAPTER 5

Discussion and Conclusion

5.1. Discussion

5.1.1 Comparison of Eigenvalue Estimates in L^1 and L^∞

We begin by discussing our main results:

1. For $q \in L^1$ the eigenvalue of the smallest modulus can be estimated by :

$$|\lambda_0| \geq |y(b)| \left(\sqrt{2}\|e^{c(x-a)}\|\|(b-x)e^{c(b-x)}\|\right)^{-1}. \tag{5.1.1.1}$$

   where $\|q\|_1 = c$.

2. For $q \in L^\infty$:

$$|\lambda_0| \geq |y(b)| \left(\sqrt{2}\|\cosh(\sqrt{c}(x-a))\| \left\|\frac{\sinh(\sqrt{c}(b-x))}{\sqrt{c}}\right\|\right)^{-1}, \tag{5.1.1.2}$$

   where $\|q\|_\infty = c$.

We see that the two estimates, (5.1.1.1) and (5.1.1.2) differ in their mathematical functions and the norm of q. (5.1.1.1) has exponential functions, which can grow exponentially. In contrast, (5.1.1.2) has hyperbolic functions, which are symmetric.

Below are some of the differences in the behavior of the functions:

(i) The exponential functions in (5.1.1.1) can grow rapidly or decay rapidly, making them sensitive to the boundary conditions and the sum effect of q in L^1.

(ii) The hyperbolic functions in (5.1.1.2) offer symmetry and bounded growth properties, making them well-suited for cases where q is controlled by its essential supremum in L^∞.

We further see that the choice of norm for q significantly impacts the eigenvalue estimates:

(a) When $q \in L^1$, the norm $\|q\|_1 = \int |q(x)|\,dx$ measures the cumulative magnitude of q. This makes (5.1.1.1) sensitive to the total distribution of q, allowing us to cover bigger regions as long as the total integral remains finite.

(b) When $q \in L^\infty$, the norm $\|q\|_\infty = \operatorname{ess\,sup} |q(x)|$ measures the maximum value of q. This makes (5.1.1.2) focus on the worst-case behavior of q, providing a stricter and more uniform bound.

While results for $q \in L^\infty$ may provide sufficient information for uniformly bounded functions, they do not necessarily cover cases where q is only in L^1. The cumulative effects of q in L^1 may show us smaller details not captured by the L^∞ norm. [47] and [11]

5.1.2 Comparison of Results

We first compare results obtained in this dissertation using mixed boundary conditions with those obtained by Kikonko and Mingarelli using Dirichlet boundary conditions with the assumption that $q \in L^1$.
Kikonko and Mingarelli [23] used Dirichlet boundary conditions and considered $q \in L^1$ and estimated the lower bound estimate of the eigenvalue of the smallest modulus $|\lambda_0|$ to be

$$|\lambda_0| \geq |y(b)| \left(\sqrt{2}\|(x-a)e^{c(x-a)}\|\|(b-x)e^{c(b-x)}\|\right)^{-1}, \quad x \in (a,b), \tag{5.1.2.1}$$

while in this dissertation we used mixed boundary conditions and considered q to be in both L^1 and L^∞. When $q \in L^1$ we established the results to be:

$$|\lambda_0| \geq |y(b)| \left(\sqrt{2}\|e^{c(x-a)}\|\|(b-x)e^{c(b-x)}\|\right)^{-1}. \tag{5.1.2.2}$$

We see that the supremum of the $\|y\|$ in (5.1.2.1) has a factor $(x-a)$ when compared to the supremum of $\|y\|$ in (5.1.2.2), which reflects the symmetry of Dirichlet boundary conditions and makes the bound sensitive to the position of x within the interval (a, b). This factor affects the growth of the exponential term $e^{c(x-a)}$, potentially weakening the bound for small intervals where $(x-a)$ is small. (5.1.2.2) relies on the norm of the exponential term $e^{c(x-a)}$, which can provide a tighter bound in cases where the dependence on $(x-a)$ has no significance.

Overall, (5.1.2.1) is well-suited for problems with Dirichlet boundary conditions due to its symmetry, while (5.1.2.2) aligns better with the asymmetry of mixed boundary conditions. The comparison between the eigenvalue bounds obtained for Dirichlet and mixed boundary conditions highlights key differences.

Next, we compare Mingarelli's result, obtained for Dirichlet boundary conditions when he considered $q \in L^\infty$ and the result for this dissertation when we considered $q \in L^\infty$ and used mixed boundary conditions. Mingarelli's result is given by:

$$|\lambda_0| \geq |y(b)|c \left(\sqrt{2}\|\sinh(\sqrt{c}(x-a))\|\|\sinh(\sqrt{c}(b-x))\|\right)^{-1}. \tag{5.1.2.3}$$

And the result, in this dissertation is given by:

$$|\lambda_0| \geq |y(b)| \left(\sqrt{2}\|\cosh(\sqrt{c}(x-a))\| \left\|\frac{\sinh(\sqrt{c}(b-x))}{\sqrt{c}}\right\|\right)^{-1}. \tag{5.1.2.4}$$

We see that Mingarelli's result (5.1.2.3) benefits from its symmetry, which is well-suited to Dirichlet boundary conditions. But it completely depends on $\sinh x$, which grows slower than $\cosh x$. The result in (5.1.2.4) has both $\sinh x$ and $\cosh x$, making it more appropriate for the asymmetry of mixed boundary conditions. Additionally, the presence of $\frac{sinh(\sqrt{c}(b-x))}{\sqrt{c}}$ in (5.1.2.4) moderates the sensitivity to small values of c, but it may weaken the bound in such cases.

For small values of c, (5.1.2.3) is sharper due to the explicit scaling by c. Conversely, for larger c, (5.1.2.4) provides a tighter bound because of the faster growth of $\cosh x$. This shows that each bound is most effective within its respective context: (5.1.2.3) for Dirichlet boundary conditions and (5.1.2.4) for mixed boundary conditions. In general, understanding both cases is useful for detailed insights into the behavior of eigenvalues under different assumptions about the function q.

5.2. Conclusion

In this dissertation, we extended the results obtained by Kikonko and Mingarelli [23] and Mingarelli[31], which established lower bounds for an eigenvalue of the smallest modulus corresponding to the problem (3.2.0.1)-(3.2.0.2) in the case of Dirichlet BCs.

Our main work was to extend these results to mixed boundary conditions, while maintaining the assumptions on q in both $L^\infty(a,b)$ and $L^1(a,b)$. The lower bounds for the eigenvalue of the smallest modulus were obtained in both cases, thus consolidating the results previously obtained under Dirichlet boundary conditions. This work provides a significant step towards a more comprehensive understanding of eigenvalue problems with mixed boundary conditions. Future work may involve further investigation into higher-order differential equations.

The eigenvalue estimates in (5.1.1.1) and (5.1.1.2) are both useful in their respective contexts. If the problem involves exponential growth or decay or relies on the integral properties of q, the L^1-based result in (5.1.1.1) may be more informative. However, if the problem requires uniform bounds or focuses on the maximum effect of q, the L^∞-based result in (5.1.1.2) is likely more appropriate.

5.3. Open Questions

Below are some of the many open questions that Mingarelli in [29] raises on non-definite or indefinite Sturm-Liouville problems, which we raise here too.

1. What are the sufficient conditions for the existence of at least one non-real eigenvalue of a SLP?
2. Is Richardson's oscillation theorem (2.8.3.34) for non-real eigenfunctions true in general?
3. To what extent is Richardson's theorem for non-real eigenfunctions true?

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Appendix

> $restart$: *the case of general parameters* $\lambda = \lambda$:

> $Digits := 10$: *accuracy* :

> $\lambda = \lambda$:

> $f := \lambda \to \sqrt{\lambda - 4\cdot\pi^2}\cdot\cos\left(2\cdot\sqrt{\lambda - 4\cdot\pi^2}\right);$

$$f := \lambda \mapsto \sqrt{\lambda - 4\cdot\pi^2}\cdot\cos\left(2\cdot\sqrt{\lambda - 4\cdot\pi^2}\right)$$

> $f(\lambda);$

$$\sqrt{-4\,\pi^2 + \lambda}\,\cos\left(2\sqrt{-4\,\pi^2 + \lambda}\right)$$

> $with(RootFinding);$

[*Analytic, AnalyticZerosFound, BivariatePolynomial, EnclosingBox, EvaluateAtRoot, HasRealRoots, Homotopy, Isolate, NextZero, Parametric, RefineRoot, WitnessPoints*]

> $Analytic(f(\lambda), re = -1000\,..1000, im = -1000\,..1000);$

69.70408110, 40.09526786, 89.44328990, 114.1173009, 795.1200045, 883.9464440, 558.2494990, 54.89967450, 143.7261142, 178.2697294, 262.1613669, 45.03007010, 977.7076860, 217.7481472, 365.7922132, 311.5093895, 425.0098397, 489.1622690, 632.2715320, 711.2283670, 39.47841284

> $h := \lambda \to \sqrt{\lambda - 4\cdot\pi^2}\cdot\cos\left(\sqrt{-\lambda - 4\cdot\pi^2}\right)\cdot\cos\left(\sqrt{\lambda - 4\cdot\pi^2}\right) - \sqrt{-\lambda - 4\cdot\pi^2}\cdot\sin\left(\sqrt{-\lambda - 4\cdot\pi^2}\right)\cdot\sin\left(\sqrt{\lambda - 4\cdot\pi^2}\right);$

$$h := \lambda \mapsto \sqrt{\lambda - 4\cdot\pi^2}\cdot\cos\left(\sqrt{-\lambda - 4\cdot\pi^2}\right)\cdot\cos\left(\sqrt{\lambda - 4\cdot\pi^2}\right) - \sqrt{-\lambda - 4\cdot\pi^2}\cdot\sin\left(\sqrt{-\lambda - 4\cdot\pi^2}\right)\cdot\sin\left(\sqrt{\lambda - 4\cdot\pi^2}\right)$$

> $h(\lambda);$

$$\sqrt{-4\,\pi^2 + \lambda}\,\cos\left(\sqrt{-4\,\pi^2 - \lambda}\right)\cos\left(\sqrt{-4\,\pi^2 + \lambda}\right) - \sqrt{-4\,\pi^2 - \lambda}\,\sin\left(\sqrt{-4\,\pi^2 - \lambda}\right)\sin\left(\sqrt{-4\,\pi^2 + \lambda}\right)$$

> $with(RootFinding);$

[*Analytic, AnalyticZerosFound, BivariatePolynomial, EnclosingBox, EvaluateAtRoot, HasRealRoots, Homotopy, Isolate, NextZero, Parametric, RefineRoot, WitnessPoints*]

> $Analytic(h(\lambda), re = -1000\,..1000, im = -1000\,..1000);$

978.9436825, −885.2433230, 796.4836910, −712.6656170, 633.7900680, −559.8581680, 490.8712096, −426.8306598, −92.60485555, 72.93061140, −313.5952404, 47.57569262, −41.48016123, −57.98110825, 117.1172332, −146.5299814, 180.8740704, 367.7381224, 264.4034524, −220.1634534, 39.47841284

> $g := \lambda \to \sqrt{\lambda + 4\cdot\pi^2}\cdot\cos\left(\sqrt{-\lambda + 4\cdot\pi^2}\right)\cdot\cos\left(\sqrt{\lambda + 4\cdot\pi^2}\right) - \sqrt{-\lambda + 4\cdot\pi^2}\cdot\sin\left(\sqrt{-\lambda + 4\cdot\pi^2}\right)\cdot\sin\left(\sqrt{\lambda + 4\cdot\pi^2}\right);$

$$g := \lambda \mapsto \sqrt{\lambda + 4\cdot\pi^2}\cdot\cos\left(\sqrt{-\lambda + 4\cdot\pi^2}\right)\cdot\cos\left(\sqrt{\lambda + 4\cdot\pi^2}\right) - \sqrt{-\lambda + 4\cdot\pi^2}\cdot\sin\left(\sqrt{-\lambda + 4\cdot\pi^2}\right)\cdot\sin\left(\sqrt{\lambda + 4\cdot\pi^2}\right)$$

> $g(\lambda);$

$$\sqrt{4\pi^2 + \lambda}\,\cos\left(\sqrt{4\pi^2 - \lambda}\right)\cos\left(\sqrt{4\pi^2 + \lambda}\right) - \sqrt{4\pi^2 - \lambda}\,\sin\left(\sqrt{4\pi^2 - \lambda}\right)\sin\left(\sqrt{4\pi^2 + \lambda}\right)$$

> *with*(*RootFinding*);

[*Analytic*, *AnalyticZerosFound*, *BivariatePolynomial*, *EnclosingBox*, *EvaluateAtRoot*, *HasRealRoots*, *Homotopy*, *Isolate*, *NextZero*, *Parametric*, *RefineRoot*, *WitnessPoints*]

> *Analytic*($g(\lambda)$, *re* = −1000 ..1000, *im* = −1000 ..1000);

−57.12824005, 179.9154822, −996.1708095, 897.4034070, −803.5619545, 551.5718420, −630.6489845, 36.44611042, −477.4087603, −343.7966350, 94.26892500, −30.84516860, −229.7113032, 714.6446150, 24.18345792 − 9.766676525 I, 24.18345787 + 9.766676520 I, −134.8463270, −39.47840328, −7.315173110 − 14.58594486 I, 284.3238875, 408.1534529, −7.315173110 + 14.58594487 I

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