ISBELL CONVEXITY IN FUZZY QUASI-METRIC SPACES

By

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DECLARATION

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ABSTRACT

The concept of hyperconvexity in metric spaces was introduced by Aronszajn and Panichpakdi in 1956. This concept was then generalised to the framework of quasi-metric spaces by John Isbell in 1964, which he called Isbell convexity. In 2019, Yiǵit and Efe generalised this concept of hyperconvexity to the framework of fuzzy metric spaces and they called this new concept fuzzy hyperconvexity. In this MSc thesis, we introduce the concept of Isbell convexity in fuzzy quasi-metric spaces, which we call fuzzy Isbell convexity. This idea extends Isbell convexity in quasi-metric spaces to fuzzy quasi-metric spaces. We prove that a fuzzy quasi-metric space is fuzzy Isbell convex if and only if it is fuzzy metrically convex and has a mixed binary intersection property. Furthermore, we present the concept of a compatible quasi-metric, which generalises the concept of the compatible metric introduced by Radu, to the assymetric setting. We then use this new concept to generalise some fixed point theorems in quasi-metric spaces to the framework of fuzzy quasi-metric spaces. Finally, we introduce a t-nonexpansive map and show that the fixed point set of a t-nonexpansive map in an F-bounded fuzzy Isbell convex space is fuzzy Isbell convex.

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DEDICATION

I dedicate this work to Jehovah God for allowing me such a rare opportunity to study at the University of Zambia, I also dedicate this work to my father Benson Chama Malama, my mother Maureen Malama, my siblings; Chongo Malama, Kabole Malama, Chibale Malama, Muteke Malama, Grace Mumbi Malama, my fiancee Felistus Kunda, my friend Keith Muzingwani and my cousin Ben Kabwe for their support.

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INDEX OF NOTATION

Below is a list of symbols that will be frequently used and a brief indication of their meaning.

(X, d)	A metric space
$(X, M, *)$	A fuzzy metric space or a fuzzy quasi-metric space
$B_d(x, r)$	Open ball of radius r centred at x
$C_d(x, r)$	Closed ball of radius r centred at x
$B_M(x, r, t)$	Open ball of radius r centred at x in a fuzzy metric space or fuzzy quasi-metric space
$C_M(x, r, t)$	Closed ball of radius r centred at x in a fuzzy metric space or fuzzy quasi-metric space
$\mathrm{Fix}(T)$	Fixed point set of T
$\mathbb{N}$	The set of natural numbers
$\mathbb{R}$	The set of real numbers
$\max(\vee)$	Maximum
$\min(\wedge)$	Minimum
$\mathscr{A}(X)$	The collection of admissible subsets of a metric space
$\mathscr{E}(X)$	The collection of externally hyperconvex subsets of a metric space
$\mathcal{H}(X)$	The collection of hyperconvex subsets of a metric space

CHAPTER 1

INTRODUCTION

1.1. Background

The theory of fuzzy metric spaces was introduced by Kramosil and Michalek in [22]. According to Kramosil and Michalek [22], a fuzzy metric on a nonempty set X is a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ such that for all $x, y, z \in X$ and $t, s \geq 0$, the axioms; $M(x, y, 0) = 0$, $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$, $M(x, y, t) = M(y, x, t)$, $M(x, y, t) * M(y, z, s) \leq M(x, z, t+s)$ and $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous are satisfied. The 3-tuple $(X, M, *)$ is called a fuzzy metric space. This concept of fuzzy metric spaces is related to the class of probabilistic metric spaces (or generalised Menger spaces). In [13], George and Veeramani studied a stronger form of fuzziness. According to George and Veeramani [13], a fuzzy metric on a nonempty set X is a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times (0, \infty)$ such that for all $x, y, z \in X$ and $t, s > 0$, the axioms; $M(x, y, t) > 0$, $M(x, y, t) = 1$ if and only if $x = y$, $M(x, y, t) = M(y, x, t)$, $M(x, y, t) * M(y, z, s) \leq M(x, z, t+s)$ and $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous are satisfied. The 3-tuple $(X, M, *)$ is called a fuzzy metric space.
Recently, Gregori and Romaguera [36] introduced two definitions of fuzzy quasi-metric spaces that generalise the corresponding notions of fuzzy metric spaces by Kramosil and Michalek and by George and Veeraamani to the asymmetric setting and several properties were obtained. According to Gregori and Romaguera [36], a fuzzy quasi-metric on a nonempty set X that generalise the notion of a fuzzy metric by Kramosil and Michalek to the asymmetric setting is a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ such that for all $x, y, z \in X$ and $t, s \geq 0$, the axioms; $M(x, y, 0) = 0$, $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$, $M(x, y, t) * M(y, z, s) \leq M(x, z, t+s)$ and $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous are satisfied. The 3-tuple $(X, M, *)$ is called a fuzzy quasi-metric space. Similarly, a fuzzy quasi-metric on a nonempty set X that generalise the notion of a fuzzy metric by George and Veeramani to the asymmetric setting is the pair $(M, *)$ such that $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times (0, \infty)$ such that for all $x, y, z \in X$ and $t, s > 0$, the axioms; $M(x, y, t) > 0$, $M(x, y, t) = M(y, x, t) = 1$ if and only if $x = y$, $M(x, x, t) = 1$, $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ and $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous are satisfied. The 3-tuple $(X, M, *)$ is called a fuzzy quasi-metric space.
In this dissertation, we study the concept of hyperconvexity in fuzzy quasi-metric spaces, which we call fuzzy Isbell convexity. The notion of a hyperconvex metric space is due to Aronszajn and Panichpakdi and was introduced in 1956. According to Aronszajn and Panichpakdi [2], a metric space (X, d) is called hyperconvex if for any indexed collection of

closed balls $\{C_d(x_i, r_i)\}_{i\in I}$ of X which satisfy $d(x_i, x_j) \leq r_i + r_j$ where $i, j \in I$, it is necessarily the case that $\bigcap_{i\in I} C_d(x_i, r_i) \neq \emptyset$. Since then, a number of results on hyperconvexity in metric spaces have been obtained. Authors such as Khamsi, Kirk, Espinola,Yañez, Sine and Soardi have studied fixed point theory in hyperconvex metric spaces, (see [12], [19], [34], [35]). For instance, in 1979, Sine [34] and Soardi [35] proved, independently, that the fixed point property for non-expansive mappings holds in bounded hyperconvex spaces.

This concept was generalised to the framework of quasi-metric spaces by Isbell in 1964 (see [15]). A quasi-metric on a nonempty set X is a function $q : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$ the axioms $q(x, y) \geq 0$, $q(x, x) = 0$, $q(x, y) = 0 = q(y, x)$ implies that $x = y$, and $q(x, y) \leq q(x, z) + q(z, y)$ are satisfied. The pair (X, q) is called a quasi-metric space (see [17]). If further, the symmetry property is satisfied, that is, $q(x, y) = q(y, x)$ for all $x, y \in X$, then q is called a metric and (X, q) is called a metric space. According to Isbell [15], a quasi-metric space (X, q) is said to be Isbell convex if for any collection of indexed points $\{x_i\}_{i\in I}$ in X and collections of non-negative real numbers $\{r_i\}_{i\in I}$ and $\{s_i\}_{i\in I}$ such that $q(x_i, x_j) \leq r_i + s_j$ for any $i, j \in I$, $\bigcap_{i\in I} (C_q(x_i, r_i) \cap C_{q^{-1}}(x_i, s_i)) \neq \emptyset$. And after this generalisation, authors such as Elisabeth Kemajou, Hans-Peter A. Künzi, Olivier Olela Otafudu and Hope Sabao have studied the concept of hyperconvexity in quasi-metric spaces which is called Isbell convexity (see [17], [25], [28]). From these studies, hyperconvexity has played an important role in the study of fixed points and best appoximations in quasi-metric spaces. For instance, in 2012, Künzi and Otafudu [25] proved that if (X, q) is a bounded Isbell convex quasi-metric space and if $T : (X, q) \to (X, q)$ is a nonexpansive map, then the fixed point set $\mathrm{Fix}(T)$ of T in (X, q) is nonempty and Isbell convex.

Recently, Yiǵit and Efe [38] studied the concept of hyperconvexity in fuzzy metric spaces which they called fuzzy hyperconvexity. According to Yiǵit and Efe [38] a fuzzy metric space $(X, M, *)$ is said to be fuzzy hyperconvex if for any collection $\{C_M(x_i, r_i, t_i)\}_{i\in I}$ of closed balls in X such that $M(x_i, x_j, t_i+t_j) \geq (1-r_i)*(1-r_j)$ for all $i, j \in I$, $\bigcap_{i\in I} C_M(x_i, r_i, t_i) \neq \emptyset$. Yiǵit and Efe [38] proved that a fuzzy metric space $(X, M, *)$ is said to be fuzzy hyperconvex if and only if it has the ball intersection property and is fuzzy metrically convex.

In this MSc thesis, we present the concept of Isbell convexity in fuzzy quasi-metric spaces. We then introduce the concept of a compatible quasi-metric and use this concept to deduce some fixed point theorems in fuzzy Isbell convex fuzzy quasi-metric spaces.

1.2. Outline of the MSc Thesis

Chapter 1. This chapter presents a background on the concept of hyperconvexity in metric spaces, Isbell convexity in quasi-metric spaces and fuzzy hyperconvexity in fuzzy metric spaces as investigated by different scholars and the outline of this MSc dissertation.

Chapter 2. In this chapter, we recall some important definitions and results in hyperconvex metric spaces. In the first section, we present a summary of metric spaces and uniform spaces. In the second section of this chapter, we present a summary of hyperconvexity in metric spaces, in the third section, we present a summary of subsets of hyperconvex spaces and in the fourth section, we present a summary of some fixed point theorems in hyperconvex metric spaces.

Chapter 3. In this chapter, we recall the concept of Isbell convexity in quasi-metric spaces which will be generalised to fuzzy quasi-metric spaces later in chapter 5. In the first section, we present quasi-metric spaces and quasi-uniform spaces. In the second section of this chapter, we present the concept of Isbell convexity in quasi-metric spaces and in the third section, we present a summary of q-admissible subsets.

Chapter 4. In this chapter, we recall the concept of fuzzy hyperconvex fuzzy metric spaces. In the first section, we present triangular norms. In the second section of this chapter, we present fuzzy metric spaces. In the third section, we present a summary of a compatible metric and in the fourth section, we present fuzzy hyperconvexity in fuzzy quasi-metric spaces.

Chapter 5. In this chapter, we present fuzzy Isbell convex fuzzy quasi-metric spaces. In the first section, we present fuzzy quasi-metric spaces. In the second section of this chapter, we present the concept of fuzzy Isbell convexity in fuzzy quasi-metric spaces and in the third section, we present the concept of fuzzy admissible subsets.

Chapter 6. In this chapter, we present some fixed point theorems in fuzzy Isbell convex fuzzy quasi-metric spaces, we first introduce the concept of a compatible quasi-metric and then use this quasi-metric to deduce some fixed point theorems in fuzzy quasi-metric spaces.

Chapter 7. In this chapter, we present a discussion of the results.

Chapter 8. In this chapter, we give a conclusion of the MSc thesis.

CHAPTER 2

HYPERCONVEX METRIC SPACES

In this chapter, we recall the concept of hyperconvexity in metric spaces. This chapter plays an important role as a background to our study. We begin by giving a summary of metric spaces and uniform spaces, thereafter, we present a summary of hyperconvex metric spaces, subsets of hyperconvex spaces and some fixed point theorems in hyperconvex spaces. For more details, see [5], [12], [18], [19], [23] and [28].

2.1. Metric spaces

In this section, we recall the definition of a metric space, a pseudometric space and give some of their examples.

Definition 2.1.1 ([28])**.** *Let X be a nonempty set and $d : X \times X \longrightarrow [0, \infty)$ be a function mapping from $X \times X$ into the set of nonnegative real numbers. Then d is called a pseudometric on X if and only if the following axioms are satisfied;*

(i) $d(x, x) = 0$ *for all* $x \in X$

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

The pair (X, d) is called a pseudometric space. Furthermore, if for any $x, y \in X$ we have that

$$d(x, y) = 0 \quad \textit{implies that} \quad x = y,$$

then d is a metric on X and the pair (X, d) is called a metric space.

This means that every metric space is a pseudometric space, but the converse need not be true.

Example 2.1.1 ([23])**.** The space l^p is the space of sequences $x = (x_1, x_2, ...)$ of numbers (complex or real) such that $\sum_{j=1}^{\infty} |x_j|^p < \infty$, where $p \geq 1$ is a fixed real number. The metric between two such sequences $x = (x_1, x_2, ...)$ and $y = (y_1, y_2, ...)$ is defined by

$$d(x, y) = \left(\sum_{j=1}^{\infty} |x_j - y_j|^p \right)^{\frac{1}{p}}.$$

Thus the pair (l^p, d) is a metric space.

Example 2.1.2 ([23])**.** Let X be a nonempty set of all real-valued functions $x, y, \cdots$ which are functions of an independent real variable t and are well defined and continuous on a given closed interval $F = [a, b]$. Then

$$d(x, y) = \max_{t \in F} |x(t) - y(t)|$$

is a metric and the pair (X, d) is a metric space.

Example 2.1.3. ([37]) Let $X = \mathbb{R}^2$ and consider the function $d : X \times X \to [0, \infty)$ given by

$$d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1|.$$

Then $d(x, x) = |x_1 - x_1| = 0$, $d(x, y) = |x_1 - y_1| = |y_1 - x_1| = d(y, x)$ and the triangle inequality follows from the triangle inequality on $\mathbb{R}$, so (X, d) satisfies the defining conditions of a pseudometric space. Note, however, that this is not an example of a metric space, since we can have two distinct points that are distance 0 from each other, for example

$$d((2, 3), (2, 5)) = |2 - 2| = 0.$$

Before defining an open set and a closed set, we first give the following definition of an open ball and a closed ball.

Definition 2.1.2. *([37]) Let (X, d) be a metric space and x be a point of X. For $\epsilon > 0$, we define an open ball of radius ϵ centered at x to be the set*

$$B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}.$$

Similarly, we define a closed ball of radius ϵ centered at x to be the set

$$C_d(x, \epsilon) = \{y \in X : d(x, y) \leq \epsilon\}.$$

We now define an open set and a closed set using an open ball.

Definition 2.1.3. *([37]) A subset E of a metric space (X, d) is open if for each $x \in E$ there exist an open ball $B_d(x, \epsilon)$ about x contained in E. A set is closed if and only if it is the complement of an open set.*

Definition 2.1.4. *Let (X, d) be a metric space and let M be a subset of X. A point $x \in M$ is said to be a limit point of M if every open ball containing x contains atleast an element of M distinct from x.*

Definition 2.1.5. *([18]]) A sequence (x_n) in a metric space (X, d) is said to converge to a point $x \in X$, if for every $\epsilon > 0$ there exists an integer $N \in \mathbb{N}$ such that $d(x_n, x) \leq \epsilon$ whenever $n \geq N$.*

In this case we say that x is the limit point of the sequence (x_n) in X and we write $\lim\limits_{n\to\infty} x_n = x$.

Definition 2.1.6. *([37]) A sequence (x_n) in a metric space (X, d) is Cauchy if and only if for each $\epsilon > 0$ there exists a positive integer N such that $d(x_n, x_m) < \epsilon$ whenever $m, n \geq N$.*

Definition 2.1.7. *([37]) A metric space (X, d) is complete if and only if every Cauchy sequence in X converges to a point in X.*

Definition 2.1.8. *([37]) Let (X, d) be a metric space. The collection of open sets in X form a topology on X called the metric topology $\tau(d)$.*

We now recall the definition of a uniform space. For more details see [5] and [37].

Definition 2.1.9. *([37]) Let X be a nonempty set, then the diagonal of $X \times X$, denoted by $\triangle$, is the set defined by*

$$\triangle = \{(x, x) : x \in X\}.$$

Definition 2.1.10. *([37]) If $U \subset X \times X$ and $V \subset X \times X$, then*

$$U \circ V = \{(x, y) : \text{for some } z, \quad (x, z) \in V \quad \text{and} \quad (z, y) \in U\}$$

and

$$V^{-1} = \{(y, x) : (x, y) \in V\}.$$

Definition 2.1.11. *([37]) A filter $\mathcal{F}$ or $\{F\}$ on a set X is a nonempty collection of nonempty subsets of X satisfying the following properties:*

(i) If $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$.

(ii) If $E \in \mathcal{F}$ and $E \subset F$, then $F \in \mathcal{F}$.

Definition 2.1.12. *([5]) A diagonal uniformity or a uniform structure on a set X is a filter $\mathcal{F}$ on $X \times X$ consisting of subsets of $X \times X$ called entourages or surroundings, satisfying the following properties:*

(i) If $F \in \mathcal{F}$, then $\triangle \subset \mathcal{F}$.

(ii) If $F \in \mathcal{F}$, then there exists an entourage $E \in \mathcal{F}$ such that $E^{-1} \subset F$.

(iii) If $F \in \mathcal{F}$, then there exists an entourage $E \in \mathcal{F}$ such that $E \circ E \subset F$.

A set X together with a uniform structure $\mathcal{F}$ is called a uniform space and is denoted by $(X, \mathcal{F})$.

Definition 2.1.13. *([37]) A subcollection $\mathcal{B}$ is called a base for the uniformity $\mathcal{F}$ if and only if $\mathcal{B} \subset \mathcal{F}$ and each $F \in \mathcal{F}$ contains some $B \in \mathcal{B}$.*

Remark 2.1.1. ([37]) Any metric d on a set X generates a metric uniformity $\mathcal{F}_d$ on X, uniformities that can be generated from metrics are called metrizable uniformities.

Example 2.1.4. ([37]) The usual uniformity on $\mathbb{R}$ is the uniformity, having for a base the collection of sets $D_\epsilon, \epsilon > 0$, where

$$D_\epsilon = \{(x, y) : |x - y| < \epsilon\}.$$

Definition 2.1.14. *([5]) Let X be a nonempty set, then for any $x \in X$ and for any subset F in the filter $\mathcal{F}$ of X, we define*

$$F[x] = \{y \in X : (x, y) \in F\}.$$

Definition 2.1.15. *([5]) Let $(X, \mathcal{F})$ be a uniform space. The topology τ defined by the uniformity $\mathcal{F}$ which is also called the uniform topology is the collection of all subsets A of X*

such that for each $x \in A$ there exists an entourage $F \in \mathcal{F}$ with

$$F[x] = \{y \in X : (x, y) \in F\} \subset A.$$

The notion of a Cauchy sequence extends to uniform spaces, and a uniformly continuous map sends Cauchy sequences to Cauchy sequences.

Definition 2.1.16. *([5]) A filter $\{\mathcal{F}_a\}$ in the uniform space $(X, \mathcal{F})$ is said to converge to a point $x_0 \in X$ if for each $F \in \mathcal{F}$, there exists $\mathcal{F}_a$ such that $(x_a, x_0) \in F$ for all $x_a \in \mathcal{F}_a$.*

Definition 2.1.17. *([5]) A filter $\{\mathcal{F}_a\}$ in the uniform space $(X, \mathcal{F})$ is called a Cauchy filter if for each entourage $F \in \mathcal{F}$, there exist a $\mathcal{F}_a$ such that $(x_a, y_a) \in F$ for all $(x_a, y_a) \in \mathcal{F}_a \times \mathcal{F}_a$.*

Definition 2.1.18. *([5]) A uniform space $(X, \mathcal{F})$ is said to be complete if and only if every Cauchy filter in the space converges to a point in the space.*

2.2. Hyperconvex metric spaces

The notion of hyperconvex metric spaces was introduced by Aronszajn and Panichpakdi [2] in 1956. Since then, a number of results on hyperconvexity in metric spaces have been obtained. Authors such as Khamsi, Kirk, Espinola,Yañez, Sine and Soardi have studied fixed point theory in hyperconvex metric spaces, (see [12], [19], [34] and [35]). For instance, in 1979, Sine [34] and Soardi [35] proved, independently, that the fixed point property for non-expansive mappings holds in bounded hyperconvex spaces.

In this section, we recall the definition of a hyperconvex metric space, its examples and give some properties. We begin by recalling the definition of a metrically convex metric space.

Definition 2.2.1. *([28]) A metric space (X, d) is said to be metrically convex if for any points $x, y \in X$ and any positive real numbers r_1 and r_2 such that $d(x, y) \leq r_1 + r_2$, there exists $z \in X$ such that $d(x, z) \leq r_1$ and $d(z, y) \leq r_2$ or equivalently $z \in C_d(x, r_1) \cap C_d(y, r_2)$.*

We now recall the definition of a hyperconvex metric space.

Definition 2.2.2. *([27]) A metric space (X, d) is called hyperconvex if for any indexed collection of closed balls $\{C_d(x_i, r_i)\}_{i \in I}$ of X which satisfy $d(x_i, x_j) \leq r_i + r_j$ where $i, j \in I$, it is necessarily the case that*

$$\bigcap_{i \in I} C_d(x_i, r_i) \neq \emptyset.$$

Definition 2.2.3. *([28]) Let (X, d) be a metric space. A collection of closed balls $\{C_d(x_i, r_i)\}_{i \in I}$, where each two intersect, is said to have a binary ball intersection property if for all $i \in I$,*

$$\bigcap_{i \in I} C_d(x_i, r_i) \neq \emptyset.$$

Definition 2.2.4. *([17]) A metric space (X, d) is said to be hypercomplete if every collection of closed balls, $\{C_d(x_i, r_i)\}_{i \in I}$, where $r_i \in [0, \infty)$ and $x_i \in X$ whenever $i \in I$, having a binary* *intersection property satisfies*

$$\bigcap_{i\in I} C_q(x_i, r_i) \neq \emptyset.$$

The following proposition shows that a metrically convex space with the binary intersection property is hyperconvex and that the converse is also true.

Proposition 2.2.5. *([18]) A metric space (X, d) is hyperconvex if and only if it is metrically convex and has the binary ball intersection property.*

Proof. Let (X, d) be a metrically convex metric space with the binary ball intersection property, then for any two closed balls $C_d(x_i, r_i)$ and $C_d(x_j, r_j)$ in X, $C_d(x_i, r_i) \cap C_d(x_j, r_j) \neq \emptyset$. By the metric convexity of (X, d), there exists $z \in C_d(x_i, r_i) \cap C_d(x_j, r_j)$ such that

$$d(x_i, x_j) \leq d(x_i, z) + d(z, x_j).$$

Now $C_d(x_i, r_i) = \{z \in X : d(x_i, z) \leq r_i\}$ and $C_d(x_j, r_j) = \{z \in X : d(x_j, z) \leq r_j\}$.
Thus,

$$d(x_i, x_j) \leq d(x_i, z) + d(z, x_j) \leq r_i + r_j.$$

And since (X, d) has the binary ball intersection property, we have that $\bigcap_{i\in I} C_d(x_i, r_i) \neq \emptyset$. Hence (X, d) is a hyperconvex metric space. Conversely suppose that (X, d) is hyperconvex then for any collection of closed balls $\{C_d(x_i, r_i)\}_{i\in I}$ of X and for any collection of positive reals $\{r_i\}_{i\in I}$ such that $d(x_i, x_j) \leq r_i + r_j$, it is necessary that $\bigcap_{i\in I} C_d(x_i, r_i) \neq \emptyset$, thus for $i, j \in I$ we have that $C_d(x_i, r_i) \cap C_d(x_j, r_j) \neq \emptyset$ and this implies that (X, d) is metrically convex. Also if $C_d(x_i, r_i) \cap C_d(x_j, r_j) \neq \emptyset$ for any two closed balls from a collection $\{C_d(x_i, r_i)\}_{i\in I}$ of closed balls, then by the hyperconvexity of (X, d), we have that $\bigcap_{i\in I} C_d(x_i, r_i) \neq \emptyset$. Thus (X, d) has the binary ball intersection property. □

We now prove that every hyperconvex metric space (X, d) is complete by showing that every Cauchy sequence in (X, d) converges in (X, d).

Proposition 2.2.6. *([12]) Every hyperconvex metric space (X, d) is complete.*

Proof. Let $\{x_i\}$ be a Cauchy sequence in X and let $\{C_d(x_i, r_i)\}_{i\in\mathbb{N}}$ be a collection of closed balls where $r_i = \sup_{j\geq i} d(x_i, x_j)$. By Proposition 2.2.5, X has the binary ball intersection property, thus $\bigcap_{i\in\mathbb{N}} C_d(x_i, r_i) \neq \emptyset$ and since $\lim_{i\to\infty} r_i = 0$, $\bigcap_{i=1}^{\infty} C_d(x_i, r_i)$ will contain one point $z \in X$ which is the limit of the Cauchy sequence (x_i). □

Proposition 2.2.7. *([18]) The set $\mathbb{R}$ of real numbers has the binary intersection property. That is for a collection $\{I_\alpha\}_{\alpha\in A}$ of bounded closed intervals of $\mathbb{R}$ each two of which intersect, we have that*

$$\bigcap_{\alpha\in A} I_\alpha \neq \emptyset.$$

Proof. Suppose that $\bigcap_{\alpha \in A} I_\alpha = \emptyset$. Then by compactness there exist

$$\{I_1, I_2, ..., I_{N+1}\} \subseteq \{I_\alpha\}$$

such that

$$I = \bigcap_{i=1}^{N} I_i \neq \emptyset \quad \text{while} \quad I_{N+1} = \bigcap_{i=1}^{N+1} I_i = \emptyset.$$

Thus, I and I_{N+1} are disjoint closed intervals in $\mathbb{R}$. Select any point x that lies strictly between them. (This is possible because the complement of $I \cup I_{N+1}$ is an open set.) Then by the fact that any two members of the original collection intersect,

$$x \in I_i \cap I_{N+1}, \quad i = 1, 2, ..., N.$$

This implies that

$$x \in \bigcap_{i=1}^{N+1} I_i,$$

which contradicts the assumption that $I_{N+1} = \emptyset$. Hence $\mathbb{R}$ has the binary intersection property. □

Example 2.2.1. ([28]) The set $\mathbb{R}$ of real numbers equipped with the usual metric $d(x, y) = |x - y|$ is hyperconvex.

Proof. By Proposition 2.2.7 and Definition 2.2.2, $(\mathbb{R}, d)$ is hyperconvex. □

We now recall the following properties of the space l_∞, whose elements consist of all bounded sequences $(x_1, x_2, ...)$ of real numbers, with distance $d_\infty(x, y)$ between two such sequences $x = (x_1, x_2, ...)$ and $y = (y_1, y_2, ...)$ taken as

$$d_\infty(x, y) = \sup_{i \in \mathbb{N}} |x_i - y_i|.$$

Theorem 2.2.8. *([18]) Let $\{C_{d_\infty}(x_i, r_i)\}_{i \in I}$ be a collection of closed balls in l_∞ each two of which intesect. Then*

$$\bigcap_{i \in I} C_{d_\infty}(x_i, r_i) \neq \emptyset.$$

Proof. Let

$$L_n = \{x = (x_1, x_2, ...) \in l_\infty : x_i = 0 \quad \text{if} \quad i \neq n\}.$$

We show that L_n is an isometric copy of $\mathbb{R}$. Indeed $L_1 = (x_1, 0, 0, 0, ...)$,
$L_2 = (0, x_2, 0, 0, 0, ...)$, $L_3 = (0, 0, x_3, 0, 0...)$, ... We define a map $f : L_n \to \mathbb{R}$ by

$$f\,(x = (x_1, x_2, x_3, ...)) = x_i \quad \text{for} \quad x_i \neq 0, \quad \text{where} \quad x \in L_n.$$

Then for $x = (0, 0, ...x_i, 0, 0, ...) \in L_n$ and $y = (0, 0, ...y_i, 0, 0, ...) \in L_n$, we have that

$$d_\infty(f(x), f(y)) = \sup_{i \in \mathbb{N}} |f(x_i) - f(y_i)| = |x_i - y_i| = d(x_i, y_i).$$

Thus, f is an isometry and so L_n is an isometric copy of $\mathbb{R}$. From Example 2.2.1, $\mathbb{R}$ is hyperconvex and since L_n is an isometric copy of $\mathbb{R}$, we have that for the collection $\{C_{d_\infty}(x_i, r_i) \cap L_n\}$ of closed intervals of L_n each two of which intersect,

$$\bigcap_{i \in I} (L_n \cap C_{d_\infty}(x_i, r_i)) \neq \emptyset, \quad \text{for each} \quad n \in \mathbb{N}$$

Select

$$x_n \in \bigcap_{i \in I} (L_n \cap C_{d_\infty}(x_i, r_i)), \quad n = 1, 2, 3, ...$$

Then if $x = (x_1, x_2, x_3, ...)$ it follows that

$$x \in \bigcap_{i \in I} C_{d_\infty}(x_i, r_i).$$

□

Proposition 2.2.9. *[18] Suppose $C_{d_\infty}(x_i, r_i)$ and $C_{d_\infty}(x_j, r_j)$ are two closed balls in l_∞. Then*

$$C_{d_\infty}(x_i, r_i) \cap C_{d_\infty}(x_j, r_j) \neq \emptyset$$

if and only if

$$d_\infty(x_i, x_j) \leq r_i + r_j.$$

***Proof*.** Let $z \in C_{d_\infty}(x_i, r_i) \cap C_{d_\infty}(x_j, r_j)$, then $d_\infty(x_i, x_j) \leq d_\infty(x_i, z) + d_\infty(z, x_j) \leq r_i + r_j$. Conversely, suppose $d_\infty(x_i, x_j) \leq r_i + r_j$ and if $x_i = (x_{i_1}, x_{i_2}, ...)$ and $x_j = (x_{j_1}, x_{j_2}, ...)$, then

$$|x_{i_k} - x_{j_k}| \leq r_i + r_j, \quad \text{for each} \quad k = 1, 2, ...$$

Let $z_k = \frac{r_j x_{i_k} + r_i x_{j_k}}{r_i + r_j}$. Then for each $k = 1, 2, ...$

$$|x_{i_k} - z_k| = \left| x_{i_k} - \frac{r_j x_{i_k} + r_i x_{j_k}}{r_i + r_j} \right| = \left| \frac{(x_{i_k} - x_{j_k}) r_i}{r_i + r_j} \right| \leq \frac{(r_i + r_j) r_i}{r_i + r_j} \leq r_i.$$

Similarly, $|z_k - x_{j_k}| \leq r_j$, which implies that

$$d_\infty(x_i, z) \leq r_i \quad \text{and} \quad d_\infty(x_j, z) \leq r_j, \quad z = (z_1, z_2, ...).$$

Therefore, $z \in C_{d_\infty}(x_i, r_i) \cap C_{d_\infty}(x_j, r_j)$. □

2.3. Subsets of hyperconvex Spaces

In this section we recall the definition of admissible subsets, externally hyperconvex subsets and proximinal subsets of metric spaces. We begin the section by considering admissible subsets.

Definition 2.3.1. *([28]) Let (X, d) be a metric space. The cover of a subset A of X denoted cov(A), is the set*

$$cov(A) = \bigcap \{C_d(x, r) : \textit{each} \quad C_d(x, r) \quad \textit{is a closed ball and} \quad A \subseteq C_d(x, r)\}.$$

Definition 2.3.2. *([12]) Let (X, d) be a metric space. We denote the collection of all subsets of X which are intersections of closed balls by $\mathcal{A}(X)$, that is*

$$\mathcal{A}(X) = \{A \subset X : A = cov(A)\}.$$

The elements of $\mathcal{A}(X)$ are called admissible subsets of X.

Next we recall the definition of an externally hyperconvex subset of a metric space.

Definition 2.3.3. *([19]) A subset E of a metric space (X, d) is said to be externally hyperconvex (relative to X) if given any collection $\{x_i\}_{i\in I}$ of points in X and any collection $\{r_i,\}_{i\in I}$ of positive real numbers satisfying*

$$d(x_i, x_j) \leq r_i + r_j \quad \textit{and} \quad dist(x_i, E) \leq r_i,$$

where $dist(x, E) = \inf\{d(x, y) : y \in E\}$, it follows that

$$\bigcap_{i\in I} C_d(x_i, r_i) \cap E \neq \emptyset.$$

The class of all the externally hyperconvex subsets of X will be denoted as $\mathcal{E}(X)$.

Before comparing the classes of subsets of metric spaces we first recall the concept of proximality.

Definition 2.3.4. *([28]) A subset E of a metric space (X, d) is said to be proximinal (with respect to X) if*

$$E \cap C_d(x, dist(x, E)) \neq \emptyset \quad \textit{for each} \quad x \in X.$$

Lemma 2.3.5. *([12]) Any admissible or externally hyperconvex subset E of a hyperconvex metric space (X, d) is proximinal.*

Proof. Let E be an admissible subset of X and set $E = \bigcap_{i\in I} C_d(x_i, r_i)$. Then for any $x \in X$ and for any $\epsilon > 0$, there exists $a_\epsilon \in E$ such that $d(x, a_\epsilon) \leq \text{dist}(x, E) + \epsilon$. Since X is hyperconvex, we have that $\bigcap_{i\in I} C_d(x_i, r_i) \cap C_d(x, \text{dist}(x, E) + \epsilon) \neq \emptyset$. Thus,

$$E \cap C_d(x, \text{dist}(x, E)) = \bigcap_{i\in I} C_d(x_i, r_i) \bigcap \left(\bigcap_{\epsilon>0} C_d(x, \text{dist}(x, E) + \epsilon) \right) \neq \emptyset,$$

where $C_d(x, \text{dist}(x, E)) = \bigcap_{\epsilon>0} C_d(x, \text{dist}(x, E) + \epsilon)$.

This shows that E is proximinal. The proof follows in a similar way, if E is taken to be an externally hyperconvex subset of X. □

We now recall the result that gives the relationship among admissible subsets, externally hyperconvex subsets and hyperconvex subsets of a metric space. In the following, we deonote the collection of hyperconvex subsets by $\mathcal{H}(X)$ and give an outline of the proof.

Theorem 2.3.6. *([28]) Let (X,d) be a hyperconvex metric space, then*

$$\mathcal{A}(X) \subseteq \mathcal{E}(X) \subseteq \mathcal{H}(X)$$

Proof. We first prove that $\mathcal{A}(X) \subseteq \mathcal{E}(X)$. Let E be an admissible subset of X, let $\{x_i\}_{i\in I}$ be a collection of points in X and let $\{r_i\}_{i\in I}$ be a collection of positive real numbers satisfying

$$d(x_i, x_j) \leq r_i + r_j \quad \text{and} \quad \operatorname{dist}(x_i, E) \leq r_i$$

for any $i, j \in I$. By lemma 2.3.5, E is proximinal. Thus for any $i \in I$, there exists a point $a_i \in E$ such that $d(x_i, a_i) = \operatorname{dist}(x_i, E)$, which gives $E \cap C_d(x_i, r_i) \neq \emptyset$. Now since X is hyperconvex, the conditions on both collections imply that $\bigcap\limits_{i\in I} C_d(x_i, r_i) \neq \emptyset$. And since E is admissible and $E \cap C_d(x_i, r_i) \neq \emptyset$, it follows that

$$E \cap \left(\bigcap_{i\in I} C_d(x_i, r_i) \right) \neq \emptyset,$$

which proves the first inclusion.
The second inclusion $\mathcal{E}(X) \subseteq \mathcal{H}(X)$ follows directly from the definition of an externally hyperconvex subset of (X, d). □

Next, we recall the following intersection property of hyperconvex metric spaces. In general, the intersection of two hyperconvex subsets of a given hyperconvex metric space need not be hyperconvex. Consider the following example in $\mathbb{R}^2_\infty$.
Example 2.3.1. ([28]) Let H_1 denote the line segment joining $(-1,0)$ to $(1,0)$ and let H_2 be the union of the line segment joining $(-1,0)$ to $(0,1)$ and the line segment joining $(0,1)$ to $(1,0)$. Then H_1 and H_2 are both hyperconvex subsets of $\mathbb{R}^2_\infty$, but $H_1 \cap H_2 = \{(-1,0),(1,0)\}$. Since $H_1 \cap H_2$ consists of only two distinct points, it follows that $H_1 \cap H_2$ is not a hyperconvex subset of $\mathbb{R}^2_\infty$.

We also recall the following result that shows that the intersection of nonempty hyperconvex subsets of a descending sequence of a bounded hyperconvex space is nonempty and hyperconvex.
Theorem 2.3.7. *([28]) Let (X,d) be a bounded hyperconvex metric space and let $\{H_\beta\}_{\beta\in\Gamma}$ be a descending sequence of nonempty hyperconvex subsets of (X,d). Then $H = \bigcap\limits_{\beta=1}^{\infty} H_\beta$ is nonempty and hyperconvex.*

Proof. Consider the collection

$$\mathcal{F} = \left\{ A = \prod_{\beta\in\Gamma} A_\beta : A_\beta \in \mathcal{A}(H_\beta) \quad \text{and} \quad \{A_\beta\} \quad \text{is decreasing and nonempty} \right\}.$$

Since (X,d) is bounded, then each H_β is bounded and since $\prod\limits_{\beta\in\Gamma} H_\beta \in \mathcal{F}$, we have that $\mathcal{F} \neq \emptyset$. Now since H_β is hyperconvex, $\mathcal{A}(H_\beta)$ is compact for every $\beta \in \Gamma$. Therefore, $\mathcal{F}$ satisfies the assumptions of Zorn's lemma when ordered by set inclusion. Hence for every $D \in \mathcal{F}$ there exists a minimal element $A \in \mathcal{F}$ such that $A \subset D$. We claim that if $A = \prod_{\beta\in\Gamma} A_\beta$ is minimal then there exists $\beta_0 \in \Gamma$ such that $\operatorname{diam}(A_\beta) = 0$ for every $\beta \geq \beta_0$. Indeed, let $\beta \in \Gamma$ be fixed. For every $D \subset X$, set

$$\operatorname{cov}_\beta(D) = \bigcap_{x\in H_\beta} C_d(x, r_x(D)).$$

Consider $A' = \prod_{\alpha\in\Gamma} A'_\alpha$ where

$$\begin{cases} A'_\alpha = \operatorname{cov}_\beta(A_\beta) \cap A_\alpha & \text{if} \quad \alpha \leq \beta \\ A'_\alpha = A_\alpha & \text{if} \quad \alpha \geq \beta. \end{cases}$$

The collection $\{A'_\alpha\}_{\alpha\geq\beta}$ is decreasing since $A \in \mathcal{F}$. Let $\alpha \leq \gamma \leq \beta$. Then $A'_\gamma \subset A'_\alpha$ since $A_\gamma \subset A_\alpha$ and $A_\beta = \operatorname{cov}_\beta(A_\beta) \cap A_\beta$. Hence the collection $\{A'_\alpha\}$ is decreasing. On the other hand if $\alpha \leq \beta$, $\operatorname{cov}(A_\beta) \cap A_\alpha \in \mathcal{A}(H_\alpha)$ since $H_\beta \subset H_\alpha$. Hence $A'_\alpha \in \mathcal{A}(H_\alpha)$. Therefore, we have that $A' \in \mathcal{F}$. Since A is minimal, $A = A'$ which implies that $A_\alpha = \operatorname{cov}_\beta(A_\beta) \cap A_\alpha$, for every $\alpha \leq \beta$. Let $x \in H_\beta$ and $\alpha \leq \beta$. Since $A_\beta \subset A_\alpha$, then $r_x(A_\beta) \leq r_x(A_\alpha)$. Because $\operatorname{cov}_\beta(A_\beta) = \bigcap_{x\in H_\beta} C_d(x, r_x(A_\beta))$, then we have $\operatorname{cov}_\beta(A_\beta) \subset C_d(x, r_x(A_\beta))$ which implies $r_x(\operatorname{cov}_\beta(A_\beta)) \leq r_x(A_\beta)$. Additionally $A_\alpha \subset \operatorname{cov}_\beta(A_\beta)$ so $r_x(A_\beta) \leq r_x(A_\alpha) \leq r_x(\operatorname{cov}_\beta(A_\beta)) \leq r_x(A_\beta)$. Therefore, we have that $r_x(A_\alpha) = r_x(A_\beta)$ for every $x \in H_\beta$. Using the definition of r, we get that $r(A_\alpha) \leq r(A_\beta)$. Let $a \in A_\alpha$ and set $s = r_a(A_\alpha)$. Then $a \in \operatorname{cov}_\beta(A_\beta)$ since $A_\alpha \subset \operatorname{cov}_\beta(A_\beta)$. Hence

$$a \in \bigcap_{x\in A_\beta} C_d(x, s) \cap \operatorname{cov}_\beta(A_\beta).$$

So, from the hyperconvexity of H_β,

$$S_\beta = H_\beta \cap \left(\bigcap_{x\in A_\beta} C_d(x, s) \right) \cap \operatorname{cov}_\beta(A_\beta) \neq \emptyset.$$

Let $z \in S_\beta$, then $z \in \bigcap_{x\in A_\beta} C_d(x, s)$ and, since $A_\beta = H_\beta \cap \operatorname{cov}_\beta(A_\beta)$, it follows that $r_z(A_\beta) \leq s$, which implies that $r(A_\beta) \leq s = r_a(A_\alpha)$ for every $a \in A_\alpha$. Hence $r(A_\beta) \leq r(A_\alpha)$. Therefore we have that $r(A_\beta) = r(A_\alpha)$, for every $\alpha, \beta \in \Gamma$. Assume that $\operatorname{diam}(A_\beta) > 0$ for every $\beta \in \Gamma$. Set $A'' = C(A_\beta)$ for every$\beta \in \Gamma$. The collection $\{A''_\beta\}$ is decreasing. Indeed, let $\alpha \leq \beta$ and $x \in A''_\beta$. Then we have $r_x(A_\beta) = r(A_\beta)$. Since we proved that $r_z(A_\beta) = r_z(A_\alpha)$ for every $z \in H_\beta$, then $r_x(A_\alpha) = r_x(A_\beta) = r(A_\beta) = r(A_\alpha)$, which implies that $x \in A''_\alpha$. Therefore, we have that $A'' = \prod_{\beta\in\Gamma} A''_\beta \in \mathcal{F}$. Since $A'' \subset A$ and A is minimal, we get $A = A''$. Therefore, we have that $C(A_\beta) = A_\beta$ for every $\beta \in \Gamma$. This contradicts the fact that H_β is hyperconvex for every $\beta \in \Gamma$. Hence there exists $\beta_0 \in \Gamma$ such that $\operatorname{diam}(A_\beta) = 0$, for every $\beta \geq \beta_0$. Thus, the proof of our claim is complete since we have $A_\beta = \{a\}$ for every $\beta \geq \beta_0$ which implies that $a \in \bigcap_{\beta\in\Gamma} H_\beta$ which implies that $\bigcap_{\beta\in\Gamma} H_\beta \neq \emptyset$. We now show that $S = \bigcap_{\beta\in\Gamma} H_\beta$ is hyperconvex. Let $\{C_d(x_i, r_i)\}_{i\in I}$ be a collection of closed balls centered in S such that

$$\bigcap_{i\in I} C_d(x_i, r_i) \neq \emptyset.$$

Set

$$D_\beta = \bigcap_{i\in I} C_d(x_i, r_i) \cap H_\beta$$

for $\beta \in \Gamma$. Since H_β is hyperconvex and the collection $C_d(x_i, r_i)$ is centered in H_β, D_β is not empty and $D_\beta \in \mathcal{A}(H_\beta)$. Therefore, D_β is hyperconvex and the above proof shows that $\bigcap_{\beta\in\Gamma} D_\beta \neq \emptyset$ and this completes the proof. □

2.4. Some fixed point theorems in hyperconvex metric spaces

In this section, we recall some fixed point theorems in hyperconvex metric spaces and the approximation of fixed point theorems of a nonexpansive self-map in metric spaces. We begin this section with the following definition.

Definition 2.4.1. *([12]) Let (X_1, d_1) and (X_2, d_2) be metric spaces. A map $T : X_1 \to X_2$ is said to be Lipschitzian if there exists a constant $k \geq 0$ such that $d_2(T(x), T(y)) \leq kd_1(x, y)$ for any $x, y \in X$. If $k = 1$, the map is called nonexpansive (and contraction if $k < 1$).*

Definition 2.4.2. *([12]) Let X and Y be metric spaces and let A be a subset of X, then the metric space Y is said to be injective if for any given non-expansive mapping $T : A \to Y$, there exists a non-expansive extension $\tilde{T} : X \to Y$.*

We now recall the definition of a fixed point set.

Definition 2.4.3. *([27]) Let (X, d) be a metric space. If $T : X \to X$ is a map, then $x \in X$ is a fixed point of T if $T(x) = x$. We denote the fixed point set of T by Fix(T), where Fix(T) $= \{x \in X : T(x) = x\}$.*

Next we recall that fixed point set of a bounded hyperconvex space is hyperconvex. First we recall the following Proposition.

Proposition 2.4.4. *([18]) Suppose (X, d) is a hyperconvex metric space. Then each set $D \in \mathcal{A}(X)$ is itself hyperconvex*

Proof. Since D is admissible, we can write $D = \bigcap_{i\in I} C_d(x_i, r_i)$, where $x_i \in X$. Now let $\{C_d(x_\alpha, r_\alpha)\}_{\alpha\in A}$ be a collection of closed balls centered at points $x_\alpha \in D$ for which $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$, where $\alpha, \beta \in A$. By the hyperconvexity of X,

$$\bigcap_{i\in A} C_d(x_\alpha, r_\alpha) \neq \emptyset.$$

Now consider the collection of closed balls $\{C_d(x_\alpha, r_\alpha :) : \alpha \in A\} \bigcup \{C_d(x_i, r_i) : i \in I\}$. Then for $\alpha \in A$ and $i, j \in I$,

$$d(x_\alpha, x_i) \leq r_i \leq r_\alpha + r_i$$

and

$$d(x_i, x_j) \leq d(x_i, x_\alpha) + d(x_\alpha, x_j) \leq r_i + r_j,$$

so it again follows from the hyperconvexity of X that

$$\left(\bigcap_{\alpha \in A} C_d(x_\alpha, r_\alpha)\right) \bigcap \left(\bigcap_{i \in I} C_d(x_i, r_i)\right) = \left(\bigcap_{\alpha \in A} C_d(x_\alpha, r_\alpha)\right) \bigcap D \neq \emptyset$$

This proves that D is hyperconvex. □

Theorem 2.4.5. *([12]) Let (X, d) be a bounded hyperconvex metric space. Then any nonexpansive map $T : (X, d) \to (X, d)$ has a fixed point. Moreover, the fixed point set of T, $Fix(T)$, is hyperconvex.*

Proof. Since X is a bounded hyperconvex metric space, let $\mathcal{A}(X)$ be a collection of admissible subsets of X. Also set

$$\mathcal{F} = \{A \in A(X); \quad \text{with} \quad A \neq \emptyset \quad \text{and} \quad T(A) \subset A\}$$

then $X \in \mathcal{F}$. Since (X, d) is a bounded hyperconvex space, we have that the intersection of any collection of nonempty elements of $\mathcal{A}(X)$ is not empty and belongs to $\mathcal{A}(X)$. So by Zorn's lemma $\mathcal{F}$ has a minimal element A_0. By the nonexpansiveness of T, we have that $T(A_0) \subset A_0$, which implies that $\text{cov}(T(A_0)) \subset A_0$, which also implies that

$$T(\text{cov}(T(A_0))) \subset T(A_0) \subset \text{cov}(T(A_0)).$$

Thus $\text{cov}(T(A_0) \in \mathcal{F}$ and since $\text{cov}(T(A_0)) \subset A_0$, it must be that $\text{cov}(T(A_0)) = A_0$ by the minimality of A_0.

Now suppose $\text{diam}A_0 = r > 0$ and consider the Chebyshev center

$$C_{A_0}(A_0) = \bigcap \left\{C_d\left(x_0, \frac{r}{2}\right) : x_0 \in A_0\right\} \cap A_0.$$

Let $x \in A_0$ and $y \in C_{A_0}(A_0)$ then $d(T(x), T(y)) \leq d(x, y) \leq \frac{r}{2}$. Thus, $T(A_0) \subset C_d(T(y), \frac{r}{2})$ which also implies that $A_0 = \text{cov}(T(A_0)) \subset C_d(T(y), \frac{r}{2})$. Therefore $T(y) \in C_{A_0}(A_0)$ that is $T : C_{A_0}(A_0) \to C_{A_0}(A_0)$ which implies that $C_{A_0}(A_0) \in \mathcal{F}$ by the definition of $\mathcal{F}$. Hence $C_{A_0}(A_0) \neq A_0$ since $\text{diam}C_{A_0}(A_0) \leq \frac{\text{diam}A_0}{2} = \frac{r}{2}$, which follows from the definition of $C_{A_0}(A_0)$. This contradicts the minimality of A_0, thus $r = 0$, which implies that $A_0 = \{z\}$ and $T(z) = z$. Hence z is a fixed point of T.

We now show that $\text{Fix}(T)$ is hyperconvex. Let $\{x_i\}_{i \in I}$ be a collection of points in $\text{Fix}(T)$ such that

$$d(x_i, x_j) \leq r_i + r_j$$

for any $i, j \in I$, for some positive numbers $\{r_i\}_{i\in I}$. Set $H_0 = \bigcap_{i\in I} C_d(x_i, r_i)$ then by the hyperconvexity of (X, d), $H_0 \neq \emptyset$ and since the centers $\{x_i\}_{i\in I}$ are in Fix(T) and T is nonexpansive, then we have that $T(H_0) \subset H_0$. Moreover H_0 is a bounded hyperconvex metric space, so the above proof implies that T has a fixed point in H_0, which implies

$$\mathrm{Fix}(T) \cap \left[\bigcap_{i\in I} C_d(x_i, r_i)\right] \neq \emptyset.$$

This completes the proof by Theorem 2.3.6. □

Next we recall that the intersection of the fixed point set of any commuting collection of nonexpansive maps with a common fixed point is hyperconvex. We first give the following Theorem

Theorem 2.4.6. *([18]) Suppose (X, d) is a metric space which has the property that the fixed point set of every nonexpansive mapping of X into X is a nonempty nonexpansive retract of X, and suppose T and G are commuting nonexpansive mappings of X into X. Then $Fix(T) \cap Fix(G) \neq \emptyset$.*

Proof. First observe that if $x \in Fix(T)$ then $T \circ G(x) = G \circ T(x) = G(x)$. This proves $G : Fix(T) \to Fix(T)$. By assumption there is a nonexpansive retraction R of X onto the nonempty set $Fix(T)$. Thus $G \circ R : X \to Fix(T)$ is nonexpansive and by assumption has a nonempty fixed point set. However, since R is a retraction onto $Fix(T)$, it must be the case that $Fix(G \circ R) = Fix(G) \cap Fix(T)$. This proves that $Fix(T) \cap Fix(G) \neq \emptyset$. □

Theorem 2.4.7. *([12]) Let X be a bounded hyperconvex metric space. Any commuting collection of nonexpansive maps $\{T_i\}_{i\in I}$, with $T_i : X \to X$, has a common fixed point. Moreover, the common fixed point set*

$$\bigcap_{i\in I} Fix(T_i)$$

is hyperconvex.

Proof. Theorem 2.4.5 implies that for every $i \in I$, the set Fix(T_i) of common fixed point set of the mappings $T_i, i \in I$ is nonempty and hyperconvex. Now the collection $\{\mathrm{Fix}(T_i)\}_{i\in I}$ satisfies the finite intersection property and so by Theorem 2.3.7, we deduce that

$$\bigcap_{i\in I} \mathrm{Fix}(T_i)$$

is nonempty and hyperconvex. □

We now recall the definition of the ϵ-parallel set of a subset of a metric space.

Definition 2.4.8. *([28]) Let (X, d) be a metric space. Given a subset A of X, we define for $\epsilon > 0$ the ϵ-parallel set of A as*

$$N_\epsilon(A) = \bigcup_{a\in A} C_d(a, \epsilon).$$

Lemma 2.4.9. *([18]) Let (X, d) be a hyperconvex metric space and let $A \in \mathcal{A}(X)$, say $A = \bigcap_{i \in I} C_d(x_i, r_i)$. Then for each $\epsilon > 0$,*

$$N_\epsilon(A) = \bigcap_{i \in I} C_d(x_i, r_i + \epsilon).$$

Proof. Suppose that $y \in N_\epsilon(A)$. Then $y \in \bigcup_{a \in A} C_d(a, \epsilon)$ which implies that $d(y, a) \leq \epsilon$ for some $a \in A$. But for each $i \in I$, we have that

$$\begin{aligned} d(y, x_i) &\leq d(a, x_i) + d(a, y) \\ &\leq r_i + \epsilon \end{aligned}$$

and so $y \in \bigcap_{i \in I} C_d(x_i, r_i + \epsilon)$ and this proves that $N_\epsilon(A) \subseteq \bigcap_{i \in I} C_d(x_i, r_i + \epsilon)$.
Conversely, suppose that $y \in \bigcap_{i \in I} C_d(x_i, r_i + \epsilon)$ and let $i \in I$. Then $d(y, x_i) \leq r_i + \epsilon, i \in I$. Since $A \neq \emptyset$, we must have that

$$d(x_i, x_j) \leq r_i + r_j, \quad i, j \in I.$$

Thus, by the hyperconvexity of (X, d) we have that

$$A \cap C_d(y, \epsilon) = \left(\bigcap_{i \in I} C_d(x_i, r_i) \right) \cap C_d(y, \epsilon) \neq \emptyset,$$

which implies that $y \in N_\epsilon(A)$ and so $\bigcap_{i \in I} C_d(x_i, r_i + \epsilon) \subseteq N_\epsilon(A)$.
Hence $N_\epsilon(A) = \bigcap_{i \in I} C_d(x_i, r_i + \epsilon)$. □

Next we recall the definition of a nonexpansive retract of a metric space (X, d).
Definition 2.4.10. *([18]) A subset A of a metric space (X, d) is said to be a nonexpansive retract (of X) if there exists a nonexpansive retraction from X onto A, that is, a nonexpansive mapping $R : X \to A$ such that for all $x \in A$, $R(x) = x$.*
Lemma 2.4.11. *([28]) Let (X, d) be a hyperconvex metric space and let $A \in \mathcal{A}(X)$. Then for each $\epsilon > 0$ there is a nonexpansive retraction R of $N_\epsilon(A)$ into A which is such that for each $x \in N_\epsilon(A)$, $d(x, R(x)) \leq \epsilon$.*

Proof. Since $A \in \mathcal{A}(X)$, let $A = \bigcap_{i \in I} C_d(x_i, r_i)$ then by Lemma 2.4.9 we know that $N_\epsilon(A) \in \mathcal{A}(X)$, thus $N_\epsilon(A)$ is itself hyperconvex. Consider the collection $\mathcal{F}$ of all pairs (D, R_D), where

$$A \subseteq D \subseteq N_\epsilon(A)$$

and R_D is a nonexpansive retraction of D onto A for which $d(x, R_D(x)) \leq \epsilon$ for each $x \in D$. Now $(A, I_d) \in \mathcal{F}$ so $\mathcal{F} \neq \emptyset$. If one orders $\mathcal{F}$ in the usual way ($(D, R_D) \preceq (H, R_H)$ if and only if $D \subseteq H$ and R_H is an extension of R_D) then each chain in $(\mathcal{F}, \preceq)$ is bounded above, so by Zorn's Lemma $\mathcal{F}$ has a maximal element which we again denote (D, R_D). We need to show that $D = N_\epsilon(A)$. Suppose there exists $x \in N_\epsilon(A)$ such that $x \notin D$, and consider the set

$$C = \left(\bigcap_{w \in D} C_d\left(R_D(w), d(w, x)\right) \right) \cap \left(\bigcap_{i \in I} C_d(x_i, r_i) \right) \cap C_d(x, \epsilon).$$

First we show that $C \neq \emptyset$, and in order to do this we need only to show that each two members of the collections of balls used in defining C intersect. (This is because X is hyperconvex and thus has the binary ball intersection property.)
First if $w_1, w_2 \in D$ then

$$d(R_D(w_1), R_D(w_2)) \leq d(w_1, w_2) \leq d(w_1, x) + d(w_2, x),$$

which proves that

$$C_d\left(R_D(w_1), d(w_1, x)\right) \cap C_d\left(R_D(w_2), d(w_2, x)\right) \neq \emptyset,$$

so each two members of the first collection intersect. Also, for each $w \in D, R_D(w) \in A = \bigcap_{i \in I} C_d(x_i, r_i)$ so each ball in the first collection intersects each ball in the second collection. Since

$$x \in N_\epsilon(A) = \bigcap_{i \in I} C_d(x_i, r_i + \epsilon)$$

we know that $C_d(x, \epsilon) \cap C_d(x_i, r_i) \neq \emptyset$ for each $i \in I$. Finally, if $w \in D$, then

$$d(R_D(w), x) \leq d(R_D(w), w) + d(w, x) \leq \epsilon + d(w, x)$$

and this proves that

$$C_d(R_D(w)) \cap C_d(x, \epsilon) \neq \emptyset.$$

We conclude therefore that $A \supseteq C \neq \emptyset$. Now let $u \in C$ and define $R' : D \cap \{x\} \to A$ by setting $R'(w) = R_D(w)$ if $w \in D$ and $R'(x) = u$. Then for $w \in D$,

$$d(R'(x), R'(w)) = d(u, R_D(w)) \leq d(u, w)$$

so R' is nonexpansive. Also $d(R'(x), x) = d(u, x) \leq \epsilon$. With this we conclude that the pair $(D \cup \{x\}, R')$ contradicts the maximality of (D, R_D) in $(\mathcal{F}, \preceq)$. Therefore, $D = N_\epsilon(A)$ and the proof is complete. $\square$

We end this chapter by recalling the concept of approximate fixed points.

Definition 2.4.12. *([28]) Let (X, d) be a metric space. A map $f : (X, d) \to (X, d)$ is said to have approximate fixed points if*

$$\inf_{x \in X} d(x, f(x)) = 0.$$

Definition 2.4.13. *([28]) Let $f : (X, d) \to (X, d)$ be a map, where (X, d) is a metric space. For any $\epsilon > 0$, we denote by $F_\epsilon(f)$ the set of ϵ-approximate fixed points of f, that is,*

$$F_\epsilon(f) = \{x \in X : d(x, f(x)) \leq \epsilon\}.$$

Theorem 2.4.14. *([18]) Let (X, d) be a hyperconvex metric space and let $f : (X, d) \to (X, d)$ be nonexpansive. Then for any $\epsilon > 0$, the set*

$$F_\epsilon(f) = \{x \in X : d(x, f(x)) \leq \epsilon\}$$

is hyperconvex.

***Proof*.** Since f is nonexpansive, for any $\epsilon > 0$ there exists a point $x \in X$ such that $d(x, f(x)) \leq \epsilon$ and since X is hyperconvex, we may suppose that $F_\epsilon(f) \neq \emptyset$. Let $x_i \in F_\epsilon(f)$ and $r_i \geq 0$ such that

$$d(x_i, x_j) \leq r_i + r_j.$$

To show that $F_\epsilon(f)$ is hyperconvex, we need to show that

$$\left(\bigcap_{i \in I} C_d(x_i, r_i) \right) \cap F_\epsilon(f) \neq \emptyset.$$

Let $J = \bigcap_{i \in I} C_d(x_i, r_i)$, by the hyperconvexity of X, if $x \in J$, then for each $i \in I$

$$d(x_i, f(x)) \leq d(f(x), f(x_i)) + d(f(x_i), x_i) \leq d(x, x_i) + \epsilon \leq r_i + \epsilon.$$

This shows that $f(x) \in N_\epsilon(J)$. Now by Lemma 2.4.11, there is a nonexpansive retraction R of $N_\epsilon(J)$ onto J for which $d(Rx, x) \leq \epsilon$ for each $x \in N_\epsilon(J)$. Also, since $R \circ f$ is a nonexpansive map of J into J, it must have a fixed point by Theorem 2.4.5. Suppose $(R \circ f)(x_0) = x_0$ for some $x_0 \in J$. Then

$$d(x_0, f(x_0)) = d((R \circ f)(x_0), f(x_0)) \leq \epsilon,$$

which implies that $x_0 \in J \cap F_\epsilon(f)$. And this implies that

$$\left(\bigcap_{i \in I} C_d(x_i, r_i) \right) \cap F_\epsilon(f) \neq \emptyset.$$

□

CHAPTER 3

ISBELL CONVEX QUASI-METRIC SPACES

In this chapter, we recall the concept of Isbell convexity in quasi-metric spaces. The concept of Isbell convexity is a generalisation of hyperconvexity to the framework of quasi-metric spaces. This concept was introduced by Isbell in [15]. We begin this chapter by presenting a summary of quasi-metric spaces and quasi-uniform spaces. Thereafter, we present a summary of Isbell convex quasi-metric spaces and q-admissible subsets. Finally, we present some fixed point point theorems in Isbell convex quasi-metric spaces. For more details see [8], [17], [18], [25], [26],[27] and [32].

3.1. Quasi-metric spaces

In this section, we recall the concept of a quasi-metric space. This concept was introduced by Wilson [40] in 1931. A quasi-metric is a function q on $X \times X$ satisfying all the axioms of a metric with an exception of symmetry. It is possible to have $q(y,x) \neq q(x,y)$ for some $x, y \in X$. This modification of axioms of a metric space drastically changes the whole theory, mainly what concerns completeness and total boundedness. For more details on quasi-metric spaces see [8] and [32].

Definition 3.1.1 ([8]). *Let X be a nonempty set and $q : X \times X \longrightarrow [0, \infty)$ be a function mapping from $X \times X$ into the set of nonnegative real numbers. Then, q is called a quasi-pseudometric on X if and only if for all $x, y, z \in X$ the following properties are satisfied;*

(i) $q(x,y) \geq 0, \quad \textit{and} \quad q(x,x) = 0.$

(ii) $q(x,y) \leq q(x,z) + q(z,y).$

And the pair (X,q) is called a quasi-pseudometric space. If further, for each $x, y \in X$,

$$q(x,y) = 0 = q(y,x) \quad \textit{implies that} \quad x = y,$$

then q is called a quasi-metric and the pair (X,q) is called a quasi-metric space.

Remark 3.1.1. ([8]) Let q be a quasi-pseudometric on X. The conjugate q^{-1} of q is a mapping $q^{-1} : X \times X \longrightarrow [0, \infty)$ defined by $q^{-1}(x,y) = q(y,x)$ for all $x, y \in X$ and q^{-1} is also a quasi-pseudometric on X. If $q^{-1} = q$, then q is a pseudometric. Furthermore, the mapping $q^s : X \times X \to [0, \infty)$ defined by

$$q^s(x,y) = \max\{q(x,y), q^{-1}(x,y)\}$$

is a pseudometric on X which is a metric if and only if q is a quasi-metric.

Remark 3.1.2. ([8]) The following properties hold in quasi-metric spaces. Let (X, q) be a quasi-metric space, then for all $x, y \in X$, we have that

(i) $q(x, y) \leq q^s(x, y)$,

(ii) $q^{-1}(x, y) \leq q^s(x, y)$.

Example 3.1.1. ([8]) Let $X = [0, \infty)$ be a set of non-negative real numbers and define $q(a, b) = \max\{a - b, 0\}$ for $a, b \in X$, then q is a quasi-metric and the quasi-metric conjugate of q is $q^{-1}(a, b) = q(b, a) = \max\{b - a, 0\}$. Furthermore, $q^s(a, b) = |a - b|$ is a metric on X.

Example 3.1.2. ([32]) Let $q : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be defined by

$$q(x, y) = \begin{cases} y - x, & \text{if} \quad y \geq x, \\ 1 & \text{if} \quad y < x. \end{cases}$$

Then $(\mathbb{R}, q)$ is a quasi-metric space.

We now recall the definitions of the forward closed ball, forward open ball, backward open ball and backward closed ball.

Definition 3.1.2 ([8]). *Let q be a quasi-metric, then*

(i) the balls $B_q(x, r)$ and $C_q(x, r)$ with respect to the quasi-metric q defined by

$$B_q(x, r) = \{y \in X : q(x, y) < r\}$$

and

$$C_q(x, r) = \{y \in X : q(x, y) \leq r\}$$

are called forward open ball and forward closed ball respectively.

(ii) the balls $B_{q^{-1}}(x, r)$ and $C_{q^{-1}}(x, r)$ with respect to the quasi-metric q^{-1} defined by

$$B_{q^{-1}}(x, r) = \{y \in X : q^{-1}(x, y) < r\}$$

and

$$C_{q^{-1}}(x, r) = \{y \in X : q^{-1}(x, y) \leq r\}$$

are called backward open ball and backward closed ball respectively.

Definition 3.1.3 ([8]). *Let q be a quasi-metric on X, a set V is called a q-neighbourhood of an arbitrary point $x \in X$ if and only if there exists a $\delta > 0$ such that $B_q(x, \delta) = \{y \in X : q(x, y) < \delta\} \subset V$.*

Definition 3.1.4 ([8]). *The topology $\tau(q)$ of a quasi-metric space (X, q) can be defined as a family or collection of q-neighbourhoods of an arbitrary point $x \in X$ and is denoted by $\mathcal{V}_q(x)$.*

Definition 3.1.5 ([8]). *Let (X, q) be a quasi-metric space. A subset M of X is said to be $\tau(q)$-open if and only if for every $x \in M$ there exists $r > 0$ such that $B_q(x, r) \subseteq M$. We shall say that M is a q-neighborhood of x or that the set M is q-open.*

Remark 3.1.3 ([8])**.** Notice that $B_q(x,r)$ is $\tau(q)$-open, but $C_q(x,r)$ is not $\tau(q)$-closed in general. However, $C_q(x,r)$ is $\tau(q^{-1})$-closed and the following inclusions hold.

(i) $B_s(x,r) \subset B_q(x,r)$

(ii) $B_s(x,r) \subset B_{q^{-1}}(x,r)$

Remark 3.1.4. ([8]) The topology generated by a quasi-metric q is called a forward topology $\tau(q)$, while the conjugate quasi-metric q^{-1} generates another topology $\tau(q^{-1})$ called the backward topology and the third one is the topology $\tau(q^s)$ generated by the metric q^s. As a space with two topologies τ_q and τ_q^{-1}, a quasi-metric space (X,q) can be viewed as a bitopological space .

The topology $\tau(q^s)$ is finer than the topologies $\tau(q)$ and $\tau(q^{-1})$ in the sense of Kelly [16].

Definition 3.1.6. *([8]) Let q be a quasi-metric, then a sequence (x_n) that converges to a point x with respect to the topology $\tau(q)$ is called a q-convergence sequence and is denoted by $x_n \xrightarrow{q} x$, and is characterized in the following way:*

$$x_n \xrightarrow{q} x \iff q(x,x_n) \longrightarrow 0.$$

Also a sequence (x_n) that converges to a point x with respect to the topology $\tau(q^{-1})$ is called a q^{-1}-convergence sequence and is denoted by $x_n \xrightarrow{q^{-1}} x$, and is characterized in the following way:

$$x_n \xrightarrow{q^{-1}} x \iff q^{-1}(x,x_n) \longrightarrow 0 \iff q(x_n,x) \longrightarrow 0.$$

Definition 3.1.7 ([8])**.** *Let (X,q) be a quasi-metric space, then a sequence (x_n) in (X,q) is called q^s- Cauchy if it is a Cauchy sequence in the metric space (X,q^s), that is for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that*

$$q^s(x_n,x_k) < \epsilon \quad or \quad q(x_n,x_k) < \epsilon \quad for\ all \quad n,k \geq N.$$

Definition 3.1.8. *([8]) Let (X,q) be a quasi-metric space. The sequence (x_n) in X is a Cauchy sequence if $\lim_{n,m\to\infty} q(x_n,x_m) = 0$. A quasi-metric space (X,q) is bicomplete if every Cauchy sequence (x_n) converges with respect to $\tau(q)$ and with respect to $\tau(q^{-1})$ to a point $x \in X$.*

Remark 3.1.5 ([8])**.** A quasi-metric space (X,q) is called bicomplete if the associated metric space (X,q^s) is complete.

Definition 3.1.9 ([8])**.** *A quasi-metric space (X,q) is bicomplete if and only if the metric space (X,q^s) is complete.*

We now recall the definition of a quasi-uniform space. For more details see [8], [26].

Definition 3.1.10 ([8])**.** *A quasi-uniformity on a set X is a filter $\mathcal{F}$ on $X \times X$ consisting of subsets of $X \times X$ called entourages or surroundings, satisfying the following properties:*

(i) If $F \in \mathcal{F}$, then $\triangle \subset F$.

(ii) If $F \in \mathcal{F}$, then there exists an entourage $E \in \mathcal{F}$ such that $E \circ E \subset F$.

The pair $(X, \mathcal{F})$ is called a quasi-uniform space.

Definition 3.1.11. *([26]) Let $\mathcal{F}$ be a quasi-uniformity on $X \times X$, then the set $\mathcal{F}^{-1}$ defined by*

$$\mathcal{F}^{-1} = \{(y, x) \in X \times X : (x, y) \in F\},$$

where $F \in \mathcal{F}$ is called the conjugate quasi-uniformity of $\mathcal{F}$.

Remark 3.1.6. ([26]) A quasi-uniformity $\mathcal{F}$ is called a uniformity if $\mathcal{F}^{-1} = \mathcal{F}$.

Definition 3.1.12. *([8]) The uniformity $\mathcal{F}^s$ defined by $\mathcal{F}^s = \mathcal{F} \vee \mathcal{F}^{-1}$ is the coarsest uniformity finer than both $\mathcal{F}$ and $\mathcal{F}^{-1}$.*

Definition 3.1.13. *([26]) Let $(X, \mathcal{F})$ be a quasi-uniform space, then every quasi-uniformity on X induces a topology $\tau(\mathcal{F})$ which is a collection of neighbourhood filters, $\mathcal{F}(x) = \{F[x] : F \in \mathcal{F}\}$ where $x \in X$ and $F[x] = \{y \in X : (x, y) \in F\}$.*

Remark 3.1.7. ([8]) If (X, q) is a quasi-metric space, then for every $\epsilon > 0$

$$V_\epsilon = \{(x, y) \in X \times X : q(x, y) < \epsilon\}$$

is a basis for a quasi-uniformity $\mathcal{F}_q$ on X. For every $\epsilon > 0$ the family

$$V_\epsilon^- = \{(x, y) \in X \times X : q(x, y) \leq \epsilon\}$$

generates the same quasi-uniformity.

Note that $V_\epsilon(x) = \{x \in X : q(x, y) < \epsilon\}$ and $V_\epsilon^-(x) = \{x \in X : q(x, y) \leq \epsilon\}$, which implies that $V_\epsilon(x) = B_q(x, \epsilon)$ and $V_\epsilon^-(x) = C_q(x, \epsilon)$. Thus, it follows that the topologies generated by the quasi-metric q and by the quasi-uniformity $\mathcal{F}_q$ agree, that is;

$$\tau_q = \tau(\mathcal{F}_q).$$

Definition 3.1.14. *([8]) Let $M(X)$ be a collection of quasi-metrics, then the family of sets V_ϵ with $q \in M(X)$ and $\epsilon > 0$, generates a quasi-uniformity on X, where for every $\epsilon > 0$ the set*

$$V_\epsilon = \{(x, y) \in X \times X : q(x, y) < \epsilon\}$$

is a base for this quasi-uniformity.

3.2. Isbell convexity in quasi-metric spaces

In this section we recall the concept of Isbell convexity which will be later generalized to fuzzy quasi-metric setting.

Definition 3.2.1. *([25]) Let (X, q) be a quasi-metric space, then a pair $(C_q(x, r); C_{q^{-1}}(x, s))$ with $x \in X$ and non-negative reals r, s is called a double ball.*

Definition 3.2.2. *([17]) A quasi-metric space (X, q) is said to be q-hyperconvex or Isbell convex if for any collection of indexed points $\{x_i\}_{i \in I}$ in X and collections of non-negative real numbers $\{r_i\}_{i \in I}$ and $\{s_i\}_{i \in I}$ such that $q(x_i, x_j) \leq r_i + s_j$ for any $i, j \in I$,*

$$\bigcap_{i \in I} (C_q(x_i, r_i) \cap C_{q^{-1}}(x_i, s_i)) \neq \emptyset.$$

Example 3.2.1. ([17]) Let the set $\mathbb{R}$ of real numbers be equipped with the quasi-metric $q(x, y) = x \dot{-} y = \max\{x - y, 0\}$ whenever $x, y \in \mathbb{R}$. Then $(\mathbb{R}, q)$ is Isbell-convex.

Proof. If $\mathbb{R}$ is Isbell-convex then by definition we have to show that

$$\bigcap_{i \in I} (C_q(x_i, r_i) \cap C_{q^{-1}}(x_i, s_i)) \neq \emptyset$$

for any collection of points $\{x_i\}_{i \in I}$ in $\mathbb{R}$ and collections $\{r_i\}_{i \in I}$ and $\{s_i\}_{i \in I}$ of non-negative real numbers.
Thus, proving that there exists a finite subset $J \subset I$ such that

$$\bigcap_{i \in J} (C_q(x_i, r_i) \cap C_{q^{-1}}(x_i, s_i)) = \emptyset,$$

implies then that $\mathbb{R}$ is not Isbell-convex, so we prove this by contradiction.

We note that for $(\mathbb{R}, q)$ with the usual quasi-metric q, we have that $C_q(x_i, r_i) = [x_i - r_i, \infty)$ and $C_{q^{-1}}(x_i, s_i) = (-\infty, x_i + s_i]$. Suppose that

$$\bigcap_{i \in J} (C_q(x_i, r_i) \cap C_{q^{-1}}(x_i, s_i)) = \bigcap_{i \in J} ([x_i - r_i, \infty) \cap (-\infty, x_i + s_i]) = \emptyset,$$

then we have that $\max\{x_i - r_i\}_{i \in J} > \min\{x_i + s_i\}_{i \in J}$. Thus, there exists $i_0, j_0 \in J$ such that $x_{i_0} - r_{i_0} > x_{j_0} + s_{j_0}$, implying that $C_q(x_{i_0}, r_{i_0}) \cap C_{q^{-1}}(x_{j_0}, s_{j_0}) = \emptyset$. Particularly $x_{i_0} > x_{j_0}$. Thus,

$$q(x_{i_0}, x_{j_0}) = x_{i_0} - x_{j_0} > r_{i_0} + s_{j_0},$$

which contradicts the assumption that

$$\bigcap_{i \in J} (C_q(x_i, r_i) \cap C_{q^{-1}}(x_i, s_i)) = \emptyset.$$

Hence, we conclude that $\bigcap_{i \in J} (C_q(x_i, r_i) \cap C_{q^{-1}}(x_i, s_i)) \neq \emptyset$ for any $J \subset I$.
Now since for any $i \in I$, $(C_q(x_i, r_i) \cap C_{q^{-1}}(x_i, s_i)$ is compact with respect to the topology $\tau(q^s)$ on $\mathbb{R}$, we conclude that

$$\bigcap_{i \in I} (C_q(x_i, r_i) \cap C_{q^{-1}}(x_i, s_i)) \neq \emptyset.$$

Hence $(\mathbb{R}, q)$ is Isbell-convex. □

Example 3.2.2. ([17]) Let $\mathbb{R}$ be equipped with its standard metric $q^s(x, y) = |x - y|$ whenever $x, y \in \mathbb{R}$. Then $(\mathbb{R}, q^s)$ is not Isbell-convex.

Proof. For any $i \in [0, 1]$ set $r_i = \frac{1}{4}$ and $s_i = \frac{3}{4}$. Then for any $i, j \in [0, 1]$ we have that $q^s(i, j) \leq 1 = r_i + s_j$. But

$$\bigcap_{i \in [0,1]} (C_{q^s}(i, r_i) \cap C_{q^s}(i, s_i)) \subseteq C_{q^s}(0, \tfrac{1}{4}) \cap C_{q^s}(1, \tfrac{1}{4}) = [\tfrac{-1}{4}, \tfrac{1}{4}] \cap [\tfrac{3}{4}, \tfrac{5}{4}] = \emptyset.$$

□

Example 3.2.3. ([17]) Consider the product of $(\mathbb{R}, q)$ and $(\mathbb{R}, q^{-1})$, that is, $\mathbb{R}^2$ equipped with the quasi-metric $D((x_1, y_1), (x_2, y_2)) = \max\{x_1 - x_2, y_1 - y_2, 0\}$ whenever $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. Then the diagonal $\{(x, x) : x \in \mathbb{R}\}$ in this product quasi-metric space is isometric to $(\mathbb{R}, q^s)$.

Definition 3.2.3. *([17]) A quasi-metric space (X, q) is said to be metrically convex if for any points $x, y \in X$ and any non-negative real numbers r_1 and r_2 such that $d(x, y) \leq r_1 + r_2$, there exists $z \in X$ such that $q(x, z) \leq r_1$ and $q(z, y) \leq r_2$.*

Example 3.2.4. ([17]) The Sorgenfrey quasi-metric q on $\mathbb{R}$ defined by

$$q(x,y) = \begin{cases} x - y, & \text{if} \quad x \geq y, \\ 1 & \text{otherwise.} \end{cases}$$

is not metrically convex since $q(\frac{1}{2}, 1) = 1 \leq \frac{1}{2} + \frac{1}{2}$ and there exists no $z \in \mathbb{R}$ such that $q(\frac{1}{2}, z) \leq \frac{1}{2}$ and $q(z, 1) \leq \frac{1}{2}$, since such a z would satisfy $z \leq \frac{1}{2}$ and $z \leq 1$.

Definition 3.2.4. *([25]) A quasi-metric space (X, q) is said to have the mixed binary intersection property if for all $i, j \in I$, $C_q(x_i, r_i) \cap C_{q^{-1}}(x_i, s_i) \neq \emptyset$, for any collection of double balls $\left[(C_q(x_i, r_i))_{i\in I}\,; (C_{q^{-1}}(x_i, s_i))_{i\in I}\right]$ with $r_i, s_i \in [0, \infty)$ and $x_i \in X$ whenever $i \in I$.*

Definition 3.2.5. *([17]) A quasi-metric space (X, q) is said to be q-hypercomplete or Isbell-complete if every collection $\left[(C_q(x_i, r_i))_{i\in I}\,; (C_{q^{-1}}(x_i, s_i))_{i\in I}\right]$ of double balls, where $r_i, s_i \in [0, \infty)$ and $x_i \in X$ whenever $i \in I$, having the mixed binary intersection property satisfies*

$$\bigcap_{i\in I} (C_q(x_i, r_i) \cap C_{q^{-1}}(x_i, s_i)) \neq \emptyset.$$

Proposition 3.2.6. *([17]) A quasi-metric space (X, q) is Isbell-convex if and only if it is metrically convex and Isbell complete.*

Proof. Suppose that (X, q) is Isbell-convex. And let $x_1, x_2 \in X, r_1, s_2 \in [0, \infty)$ such that $q(x_1, x_2) \leq r_1 + s_2$. Now since (X, q) is Isbell-convex, it follows that $C_q(x_1, r_1) \cap C_{q^{-1}}(x_2, s_2) \neq \emptyset$, which implies there exists $x \in X$ such that $x \in C_q(x_1, r_1) \cap C_{q^{-1}}(x_2, s_2)$. Hence (X, q) is metrically convex. Now let the collection $[C_q(x_i, r_i); C_{q^{-1}}(x_i, s_i)]_{i\in I}$ of double balls have the mixed binary intersection property. Then $d(x_i, x_j) \leq r_i + s_j$ whenever $i, j \in I$. And so there exists $x \in \bigcap_{i\in I} (C_q(x_i, r_i) \cap C_{q^{-1}}(x_i, s_i))$ by Isbell-convexity of (X, q). Thus, (X, q) is Isbell-complete.

Conversely assume (X, q) is both metrically convex and Isbell-complete. And let $\{x_i\}_{i\in I}$ be a collection of points in X and let $\{r_i\}_{i\in I}$ and $\{s_i\}_{i\in I}$ be collections of non-negative real numbers such that $q(x_i, x_j) \leq r_i + s_j$ whenever $i, j \in I$. Then $[C_q(x_i, r_i); C_{q^{-1}}(x_i, s_i)]_{i\in I}$ has the mixed binary intersection property. Thus there exists $x \in \bigcap_{i\in I} C_q(x_i, r_i), C_{q^{-1}}(x_i, s_i)$ by Isbell-completeness of X. Hence (X, q) is Isbell-convex. □

Definition 3.2.7. *([25]) A subset A of a quasi-metric space (X, q) is said to be bounded if and only if there exists $x \in X$ and $r, s \in [0, \infty)$ such that*

$$A \subseteq C_q(x, r) \cap C_{q^{-1}}(x, s).$$

Proposition 3.2.8. *([17])*

(i) Let (X,q) be an Isbell convex quasi-metric space, then (X,q^{-1}) is Isbell convex.

(ii) Let (X,q) be an Isbell convex quasi-metric space, then (X,q^{s}) is Hyperconvex.

(iii) Let (X,q) be an Isbell complete quasi-metric space, then (X,q^{s}) is Hypercomplete.

Proof. (i) The statement follows immediately from the definition of Isbell convexity.

(ii) Suppose that (X,q) is an Isbell convex quasi-metric space and let $\{x_i\}_{i\in I}$ be a collection of points in X and let $\{x_i\}_{i\in I}$ be a collection of non-negative real numbers. Now suppose that $q^s(x_i,x_j)\le r_i+r_j$ for any $i,j\in I$. Then by the Isbell convexity of (X,q) we have that

$$\emptyset\neq\bigcap_{i\in I}(C_q(x_i,r_i)\cap C_{q^{-1}}(x_i,r_i))=\bigcap_{i\in I}C_{q^s}(x_i,r_i).$$

Thus (X,q^s) is hyperconvex.

(iii) Suppose that (X,d) is an Isbell complete quasi-metric space. And let the collection of closed balls $[C_{q^s}(x_i,r_i)]_{i\in I}$, where $x_i\in X$ and $r_i\ge 0$ for any $i\in I$ have the binary intersection property. Then $(C_q(x_i,r_i),C_{q^{-1}}(x_i,r_i))_{i\in I}$ has the mixed binary intersection property. Consequently,

$$\emptyset\neq\bigcap_{i\in I}C_{q^s}(x_i,r_i).$$

Thus, (X,q^s) is hypercomplete.

□

Proposition 3.2.9. *([17]) Every Isbell-convex quasi-metric space (X,q) is bicomplete.*

Proof. Since (X,q) is Isbell convex, by Proposition 3.2.8 (X,q^s) is hyperconvex. Since hyperconvex metric spaces are complete by Proposition 2.2.6, we conclude that the quasi-metric space (X,q) is bicomplete. □

3.3. q-Admissible subsets

In this section we recall the concept of q-admissible subsets which is also studied later in fuzzy quasi-metric setting. We begin by recalling a bicover of a subset A of a quasi-metric space (X,q).

Definition 3.3.1. *Let (X,q) be a quasi-metric space and let A be a non-empty bounded subset of a quasi-metric space (X,q). Then we set*

$$cov(A)_q=\bigcap\{C_q(x,r): A\subseteq C_q(x,r), x\in X, r\ge 0\}$$
$$cov(A)_{q^{-1}}=\bigcap\{C_q(x,r): A\subseteq C_{q^{-1}}(x,s), x\in X, s\ge 0\}.$$

We define the bicover of A by

$$bicov(A) := cov(A)_q \cap bicov(A)_{q^{-1}}.$$

Definition 3.3.2. *([25]) Let (X, q) be a quasi-metric space and let A be a nonempty bounded subset in (X, q). Then the cover of A denoted by $cov(A)$ is defined as*

$$cov(A) = \bigcap\{C_{q^s}(x, r) : A \subseteq C_{q^s}(x, r), x \in X\}.$$

Thus, $A \subseteq \text{bicov}(A) \subseteq \text{cov}(A)$.

Definition 3.3.3. *([25]) A nonempty bounded subset A of a quasi-metric space (X, q) is called q-admissible if $A = bicov(A)$.*

The collection of all q-admissible subsets of a quasi-metric space (X, q) will be denoted by $\mathcal{A}_q(X)$.

Example 3.3.1. ([25]) Let $X = [0, 1] \times [\frac{1}{4}, \frac{3}{4}]$ be equipped with the quasi-metric defined by $D((\alpha, \beta), (\alpha', \beta')) = (\alpha \dot{-} \alpha') \vee (\beta \dot{-} \beta')$ whenever $(\alpha, \beta), (\alpha', \beta') \in X$.

Consider $A = \{(0, \frac{1}{2}), (1, \frac{1}{2})\} \subseteq X$. Then $\text{bicov}(A)$ is equal to the line segment from $x = (0, \frac{1}{2})$ to $y = (1, \frac{1}{2})$. This follows from the fact that for each $\epsilon \in [0, \frac{1}{4}]$, $y \in C_D(x, \epsilon) = [0, 1] \times [\frac{1}{2} - \epsilon, \frac{3}{4}]$ and $x \in C_{D^{-1}}(y, \epsilon) = [0, 1] \times [\frac{1}{4}, \frac{1}{2} + \epsilon]$ and that segment is a subset of any set of the form $C_D(a, r) \cap C_{D^{-1}}(b, s)$ for which $\{x, y\} \subseteq C_D(a, r) \cap C_{D^{-1}}(b, s)$. Indeed assume that z belongs to this segment. Then $D(z, y) = 0 = D(x, z)$ and therefore $z \in C_D(a, r) \cap C_{D^{-1}}(b, s)$ by the triangle inequality.

On the other hand $\text{cov}(A) = X$, since $\{x, y\} \subseteq C_{D^s}(z, \epsilon)$ with $z \in X$ implies that $\epsilon \leq \frac{1}{2}$. Indeed assume that $z = (a, b)$. Then $a \leq D^s((a, b), (0, \frac{1}{2})) \leq \epsilon$ and $1 - a \leq D^s((a, b), (\frac{1}{2}, 1)) \leq \epsilon$. Thus $\epsilon \geq \max\{a, 1 - a\} \geq \frac{1}{2}$. It follows that $X \subseteq C_{D^s}(z, \epsilon)$, because the interval $[\frac{1}{4}, \frac{3}{4}]$ has length $\frac{1}{2}$. Therefore, $\text{cov}(A) = X$.

Next, we show that every admissible subset of an Isbell convex quasi-metric space is Isbell convex and this result will be generalised to the fuzzy quasi-metric setting (see Proposition 5.3.4).

Proposition 3.3.4. *([25]) Let (X, q) be an Isbell convex quasi-metric space. Then $D \in \mathcal{A}_q(X)$ is Isbell convex.*

Proof. Since $D \in \mathcal{A}_q(X)$, then $D = \text{bicov}(D)$, which implies that there exists a collection $\{x_i\}_{i \in I}$ of points in X, collections $\{r_i\}_{i \in I}$ and $\{s_i\}_{i \in I}$ of non-negative real numbers such that

$$\bigcap_{i \in I} C_q(x_i, r_i) \cap C_{q^{-1}}(x_i, s_i) \neq \emptyset.$$

Now let $\{C_q(x_\alpha, r_\alpha); C_{q^{-1}}(x_\alpha, s_\alpha)\}_{\alpha \in \Gamma}$ be a collection of double balls, where $x_\alpha \in D$ and $r_\alpha, s_\alpha \in (0, \infty)$ whenever $\alpha \in \Gamma$ such that

$$q(x_\alpha, x_\beta) \leq r_\alpha + s_\beta$$

whenever $\alpha, \beta \in \Gamma$. Then by Isbell convexity of X, we have that

$$\bigcap_{\alpha \in \Gamma} (C_q(x_\alpha, r_\alpha) \cap C_{q^{-1}}(x_\alpha, s_\alpha)) \neq \emptyset.$$

Now consider the collection
$[\{C_q(x_\alpha, r_\alpha)\}_{\alpha\in\Gamma}, \{C_{q^{-1}}(x_\alpha, s_\alpha)\}_{\alpha\in\Gamma}, \{C_q(x_i, r_i)\}_{i\in I}, \{C_{q^{-1}}(x_i, s_i)\}_{i\in I}]$ of closed balls. We have for each $\alpha \in \Gamma$ and $i \in I$,

$$q(x_\alpha, x_i) \leq s_i \leq r_\alpha + s_i$$

and

$$q(x_i, x_\alpha) \leq r_i \leq r_i + s_\alpha$$

Furthermore, for all $i, j \in I$ and $\alpha \in \Gamma$, we have that

$$q(x_i, x_j) \leq q(x_i, x_\alpha) + q(x_\alpha, x_j) \leq r_i + s_j.$$

It follows from Isbell convexity of (X, q) that

$$\left(\bigcap_{\alpha\in\Gamma} C_q(x_\alpha, r_\alpha) \cap C_{q^{-1}}(x_\alpha, s_\alpha)\right) \cap \left(\bigcap_{i\in I} C_q(x_i, r_i) \cap C_{q^{-1}}(x, s_i)\right)$$
$$= \left(\bigcap_{\alpha\in\Gamma} C_q(x_\alpha, r_\alpha) \cap C_{q^{-1}}(x_\alpha, s_\alpha)\right) \cap D \neq \emptyset.$$

Thus, the q-admissible subspace D of (X, q) is Isbell convex. □

We now recall the following theorem which shows the conditions for which the fixed point set described below is Isbell convex.

Theorem 3.3.5. *([25]) Let (X, q) be a bounded Isbell convex quasi-metric space. If $T : (X, q) \to (X, q)$ is nonexpansive, then the fixed point set Fix(T) of T in (X, q) is nonempty and Isbell convex.*

Proof. First we show that Fix$(T) \neq \emptyset$. Now since $T : (X, q) \to (X, q)$, is a nonexpansive map, we have that $q(Tx, Ty) \leq q(x, y) \leq q^s(x, y)$ and $q^{-1}(Tx, Ty) \leq q^{-1}(x, y) \leq q^s(x, y)$ for all $x, y \in X$, which implies that $q^s(Tx, Ty) \leq q^s(x, y)$. Hence the map $T : (X, q^s) \to (X, q^s)$ is a nonexpansive map and since (X, q) is bounded, (X, q^s) is also bounded. Now by Proposition 3.2.8 (ii) (X, q^s) is metrically hyperconvex. Since (X, q^s) is a bounded hyperconvex space and $T : (X, q^s) \to (X, q^s)$ is a nonexpansive map, then by Theorem 2.4.5, we have that Fix$(T) \neq \emptyset$. Now we show that Fix(T) is Isbell convex. Let $\{C_q(x_i, r_i), C_{q^{-1}}(x_i, s_i)\}_{i\in I}$ be a collection of closed double balls, where $x_i \in$ Fix(T) such that $q(x_i, x_j) \leq r_i + s_j$ for $i, j \in I$ and also let

$$X_0 = \bigcap_{i\in I} (C_q(x_i, r_i) \cap C_{q^{-1}}(x_i, s_i)).$$

Now since X is Isbell convex, the set

$$X_0 = \bigcap_{i \in I} (C_q(x_i, r_i) \cap C_{q^{-1}}(x_i, s_i)) \neq \emptyset.$$

Let $x \in X_0$. Then $x \in C_q(x_i, r_i)$ and $x \in C_{q^{-1}}(x_i, s_i)$ for all $i \in I$. Thus, $q(x_i, x) \leq r_i$ and $q^{-1}(x_i, x) \leq s_i$. Now since $x_i \in \mathrm{Fix}(T)$, we have that

$$q(x_i, Tx) = q(Tx_i, Tx) \leq q(x_i, x) \leq r_i$$

and

$$q^{-1}(x_i, Tx) = q^{-1}(Tx_i, Tx) \leq q^{-1}(x_i, x) \leq s_i,$$

which implies that $Tx \in C_q(x_i, r_i)$ and $Tx \in C_{q^{-1}}(x_i, s_i)$ for every $i \in I$. Thus, $Tx \in X_0$ and therefore, $T(X_0) \subseteq X_0$.

Furthermore, X_0 is a bounded Isbell convex quasi-metric space by proposition 3.3.4.

Now in the first part of the proof we showed that $\mathrm{Fix}(T) \neq \emptyset$ and this implies T has a fixed point in X_0, which implies that

$$\mathrm{Fix}(T) \cap \left[\bigcap_{i \in I} (C_q(x_i, r_i) \cap C_{q^{-1}}(x_i, s_i)) \right] \neq \emptyset.$$

Hence, $\mathrm{Fix}(T)$ is Isbell convex. □

In the next theorem we recall that the intersection of a descending collection of Isbell convex spaces is as well Isbell convex.

Theorem 3.3.6. *([27]) Let (X, q) be a bounded quasi-metric space and let $\{H_i\}_{i \in I}$ be a descending collection of nonempty Isbell convex subsets of X, where we assume that I is totally ordered such that for all $i, j \in I$, $i \leq j$ hold if and only if $H_j \subseteq H_i$. Then $\bigcap_{i \in I} H_i$ is nonempty and Isbell convex.*

Proof. We first show that $\bigcap_{i \in I} H_i \neq \emptyset$. Now let $\{(H_i, q)\}_{i \in I}$ be a descending collection of nonempty bounded Isbell convex subsets of (X, q) such that $i \leq j \Leftrightarrow H_j \subseteq H_i$ whenever $i, j \in I$. By proposition 3.2.8 (H_i, q^s) is a bounded hyperconvex metric space, whenever $i, j \in I$.

Now by Theorem 2.3.7, we have that $H = \bigcap_{i \in I} H_i \neq \emptyset$. We now show that $H = \bigcap_{i \in I} H_i$ is Isbell convex.

Let $\{x_\alpha\}_{\alpha \in \Gamma}$ be a collection of points in $H = \bigcap_{i \in I} H_i$ and $\{r_\alpha\}_{\alpha \in \Gamma}$ be collections of non-negative real numbers such that $q(x_\alpha, x_\beta) \leq r_\alpha + s_\beta$ whenever $\alpha, \beta \in \Gamma$. Then since (H_i, q) is an Isbell convex space for each $i \in I$ and $x_\alpha \in H_i$ whenever $\alpha \in \Gamma$, we have that

$$\mathcal{D}_i = \bigcap_{\alpha \in \Gamma} (C_q(x_\alpha, r_\alpha) \cap C_{q^{-1}}(x_\alpha, s_\alpha)) \cap H_i \neq \emptyset.$$

Since $i \leq j \Leftrightarrow H_j \subseteq H_i$ whenever $i, j \in I$ we have that $\mathcal{D}_j \subseteq \mathcal{D}_i$ whenever $i \leq j$. $\{\mathcal{D}_i\}_{i\in I}$ is a decreasing collection of subsets of X. Now by the first part of this proof, we have that

$$\begin{aligned}\emptyset \neq \bigcap_{i\in I} \mathcal{D}_i &= \bigcap_{i\in I}\left[\bigcap_{\alpha\in\Gamma}(C_q(x_\alpha, r_\alpha) \cap C_{q^{-1}}(x_\alpha, s_\alpha)) \cap H_i\right] \\ &= \bigcap_{\alpha\in\Gamma}(C_q(x_\alpha, r_\alpha) \cap C_{q^{-1}}(x_\alpha, s_\alpha)) \cap \bigcap_{i\in I} H_i,\end{aligned}$$

since $\{\mathcal{D}_{i\in I}\}$ is decreasing. This proves that $H = \bigcap_{i\in I} H_i$ is Isbell convex. □

Definition 3.3.7. *([25]) Let (X, q) be a quasi-metric space and let a collection of nonexpansive maps $\{T_i\}_{i\in I}$, with $T_i : (X, q) \to (X, q)$ be given. Then $\{T_i\}_{i\in I}$ is said to be a commuting collection if*

$$T_i \circ T_j = T_j \circ T_i \quad \textit{whenever} \quad i, j \in I.$$

The following lemma will be used in what follows.

Lemma 3.3.8. *([25]) Let $\{H_\alpha\}_{\alpha\in S}$ be a collection of bounded Isbell convex subsets of a quasi-metric space (X, q) with the finite intersection property, that is, $\bigcap_{\alpha\in F} H_\alpha \neq \emptyset$ where $F \subseteq S$, then the intersection $\bigcap_{\alpha\in S} H_\alpha$ is nonempty and Isbell convex.*

Proof. Let $V = \{I \subseteq S : \text{for all J finite} \quad J \subseteq S, \quad \bigcap_{I\cup J} H_\alpha \quad \text{is nonempty and Isbell convex}\}$ Obviously $\emptyset \in V$ and V satisfies the hypothesis of Zorn's lemma by Theorem 3.3.6. Let I be maximal in V. Then $I \cup \{\alpha\} \in V$ whenever $\alpha \in S$. Thus, by the maximality of I, $\alpha \in I$ whenever $\alpha \in S$. □

The next result is a consequence of Theorem 3.3.5 and Theorem 3.3.6 and it is a consequence of Theorem 6.2 in [18].

Theorem 3.3.9. *Let (X, q) be a bounded Isbell convex quasi-metric space. Any commuting collection of nonexpansive maps $\{T_i\}_{i\in I}$, with $T_i : (X, q) \to (X, q)$, has a common fixed point. Moreover, the common fixed point set $\bigcap_{i\in I} Fix(T_i)$ is Isbell convex.*

Proof. By Theorem 3.3.5, $\text{Fix}(T_i)$ is Isbell convex whenever $i \in I$. Thus, by Lemma 3.3.8, it suffices to show that $\bigcap_{i\in I} \text{Fix}(T_i) \neq \emptyset$ and is Isbell convex for any finite subset J of I. Suppose $J = \{1, 2, ..., n\}$. Then the collection of fixed point sets $\{\text{Fix}(T_i)\}_{i\in J} = \{\text{Fix}(T_1), ..., \text{Fix}(T_n)\}$. Since T_1 and T_2 commute, it is immediate that $T_2 : \text{Fix}(T_1) \to \text{Fix}(T_1)$. Thus $\text{Fix}(T_1) \cap \text{Fix}(T_2) \neq \emptyset$. By induction one concludes that

$$\bigcap_{i\in J} \text{Fix}(T_i) \neq \emptyset$$

and is Isbell convex. □

Definition 3.3.10. *([25]) Let (X, q) be a quasi-metric space. For a quasi-metric subspace A of X, we define for $\epsilon_1, \epsilon_2 \in (0, \infty)$ the ϵ_1, ϵ_2-parallel set of A as*

$$N_{\epsilon_1,\epsilon_2}(A) = \bigcup_{a \in A} C_q(a, \epsilon_2) \cap C_{q^{-1}}(a, \epsilon_1).$$

(Note that for each $\epsilon > 0$ in particular $N_{\epsilon,\epsilon}(A) = \bigcup_{a \in A}(C_{q^s}(a, \epsilon))$.

We now give the following Lemma which will be generalised to fuzzy quasi-metric spaces in the next chapter (see Lemma 5.3.6.)

Lemma 3.3.11. *([27]) Let (X, q) be an Isbell convex quasi-metric space. And let A be a q-admissible subset of X, say $\emptyset \neq A = \bigcap_{i \in I}(C_q(x_i, r_i) \cap C_{q^{-1}}(x_i, s_i))$ with $x_i \in X$ and r_i, s_i nonnegative reals whenever $i \in I \neq \emptyset$. Then for each $\epsilon_1, \epsilon_2 \geq 0$,*

$$N_{\epsilon_1,\epsilon_2}(A) = \bigcap_{i \in I}(C_q(x_i, r_i + \epsilon_2) \cap C_{q^{-1}}(x_i, s_i + \epsilon_1)).$$

Proof. Suppose $y \in N_{\epsilon_1,\epsilon_2}(A) = \bigcup_{a \in A} C_q(a, \epsilon_2) \cap C_{q^{-1}}(a, \epsilon_1)$, where $\epsilon_1, \epsilon_2 \geq 0$, for some $a \in A$. Then $y \in C_q(a, \epsilon_2)$ and $y \in C_{q^{-1}}(a, \epsilon_1)$, which implies that

$$q(a, y) \leq \epsilon_2 \quad \text{and} \quad q^{-1}(a, y) = q(y, a) = \epsilon_1.$$

But for each $i \in I$,

$$\begin{aligned} q(x_i, y) &\leq q(x_i, a) + q(a, y) \\ &\leq r_i + \epsilon_2 \end{aligned}$$

and

$$\begin{aligned} q(y, x_i) &\leq q(y, a) + q(a, x_i) \\ &\leq \epsilon_1 + s_i. \end{aligned}$$

Thus, for each $i \in I$, we have that $y \in C_q(x_i, r_i + \epsilon_2)$ and $y \in C_{q^{-1}}(x_i, s_i + \epsilon_1)$, which implies that $y \in \bigcap_{i \in I} C_q(x_i, r_i + \epsilon_2) \cap C_{q^{-1}}(x_i, s_i + \epsilon_1)$.

Hence,

$$N_{\epsilon_1,\epsilon_2}(A) \subseteq \bigcap_{i \in I} C_q(x_i, r_i + \epsilon_2) \cap C_{q^{-1}}(x_i, s_i + \epsilon_1).$$

Conversely, suppose

$$y \in \bigcap_{i \in I} C_q(x_i, r_i + \epsilon_2) \cap C_{q^{-1}}(x_i, s_i + \epsilon_1).$$

Then, for each $i \in I$, we have that $y \in C_q(x_i, r_i + \epsilon_2)$ and $y \in C_{q^{-1}}(x_i, s_i + \epsilon_1)$ which implies that

$$q(x_i, y) \leq r_i + \epsilon_2 \quad \text{and} \quad q(y, x_i) \leq s_i + \epsilon_1.$$

Since A is nonempty and by the definition of A, we must have that for any $i, j \in I$,

$$q(x_i, x_j) \leq q(x_i, a) + q(a, x_j) \leq r_i + s_j.$$

So by Isbell convexity of X, we have that

$$\begin{aligned}\emptyset &\neq \left(\bigcap_{i\in I} C_q(x_i,r_i)\cap C_q(y,\epsilon_1)\right)\cap\left(C_{q^{-1}}(x_i,s_i)\cap C_{q^{-1}}(y,\epsilon_2)\right)\\ &= \left(\bigcap_{i\in I} C_q(x_i,r_i)\cap C_{q^{-1}}(x_i,s_i)\right)\cap\left(C_q(y,\epsilon_1)\cap C_{q^{-1}}(y,\epsilon_2)\right)\\ &= A\cap C_q(y,\epsilon_1)\cap C_{q^{-1}}(y,\epsilon_2).\end{aligned}$$

Therefore, $a\in A$ such that $q(y,a)\leq\epsilon_1$ and $q(a,y)\leq\epsilon_2$.
Hence $y\in N_{\epsilon_1,\epsilon_2}(A)$, which implies that

$$\bigcap_{i\in I} C_q(x_i,r_i+\epsilon_2)\cap C_{q^{-1}}(x_i,s_i+\epsilon_1)\subseteq N_{\epsilon_1,\epsilon_2}(A).$$

Hence the proof is complete. □

Definition 3.3.12. *[27] Let Y be a subset of a quasi-metric space (X,q). A map $f:X\to Y$ is said to be a nonexpansive retraction if:*

(i) For each $x\in Y$, $f(x)=x$ that is, f is the identity function on its image, and

(ii) For any $x,y\in X$, $q(f(x),f(y))\leq q(x,y)$, that is, f is nonexpansive.

Lemma 3.3.13. *([25]) Let A be a non empty q-admissible subset of an Isbell convex quasi-metric space (X,q). Then, for each $\epsilon_1,\epsilon_2\geq 0$ there exists a nonexpansive retraction R of $N_{\epsilon_1,\epsilon_2}(A)$ onto A which has the property that $q(x,R(x))\leq\epsilon_1$ and $q(R(x),x)\leq\epsilon_2$ for each $x\in N_{\epsilon_1,\epsilon_2}(A)$.*

Proof. Assume $\emptyset\neq A=\bigcap_{i\in I}(C_q(x_i,r_i)\cap C_{q^{-1}}(x_i,s_i))$ with $I\neq\emptyset$. By Lemma 3.3.11 $N_{\epsilon_1,\epsilon_2}(A)$ is q-admissible in (X,q) and so $N_{\epsilon_1,\epsilon_2}(A)$ is Isbell convex by Proposition 3.3.4. Consider the collection $\mathcal{F}=\{(D,R_D):A\subseteq D\subseteq N_{\epsilon_1,\epsilon_2}(A)$ and $R_D:D\to A$ is a nonexpansive retraction such that $q(x,R(x))\leq\epsilon_1$ and $q(R(x),x)\leq\epsilon_2$ for each $x\in D\}$.
Note that $(A,I_A)\in\mathcal{F}$, where I_A is the identity map on A. So $\mathcal{F}\neq\emptyset$. If one orders $\mathcal{F}$ in the usual way ($(D,R_D)\preceq(H,R_H)$ if and only if $D\subseteq H$ and R_H is an extension of R_D) then each chain in $(\mathcal{F},\preceq)$ is bounded above, so by Zorn's Lemma $\mathcal{F}$ has a maximal element which we again denote by $(D:R_D)$. We need to show that $D=N_{\epsilon_1,\epsilon_2}(A)$. Suppose there exists $x\in N_{\epsilon_1,\epsilon_2}(A)$ such that $x\in D$, and consider the set

$$\begin{aligned}C=&\left[\bigcap_{w\in D} C_q(R_D(w),q(w,x))\cap C_{q^{-1}}(R_D(w),q(x,w))\right]\\ &\cap\left[\bigcap_{i\in I}(C_q(x_i,r_i)\cap C_{q^{-1}}(x_i,s_i))\right]\cap\left[C_q(x,\epsilon_1)\cap C_{q^{-1}}(x,\epsilon_2)\right].\end{aligned}$$

First we show that $C\neq\emptyset$, and in order to do this we need only to show that C has the mixed binary intersection property.

If $w_1, w_2 \in D$ then

$$q(R_D(w_1), R_D(w_2)) \leq q(w_1, w_2) \leq q(w_1, x) + q(x, w_2).$$

This proves that $C_q(R_D(w_1), q(w_1, x)) \cap C_{q^{-1}}(R_D(w_2), q(x, w_2)) \neq \emptyset$ by metric convexity of (X, q), so C has the mixed binary intersection property for the first collection. Also for each $w \in D$, $R_D(w) \in A = \bigcap_{i \in I}(C_q(x_i, r_i) \cap C_{q^{-1}}(x_i, s_i))$. So the mixed binary intersection property is satisfied for the second collection.
Since

$$N_{\epsilon_1,\epsilon_2}(A) = \bigcap_{i \in I}(C_q(x_i, r_i + \epsilon_2) \cap C_{q^{-1}}(x_i, s_i + \epsilon_1))$$

we know that $(C_q(x_i, r_i + \epsilon_2) \cap C_{q^{-1}}(x_i, s_i + \epsilon_1)) \neq \emptyset$ for each $i \in I$. Finally, if $w \in D$, then

$$q(R_D(w), x) \leq q(R_D(w), w) + q(w, x) \leq \epsilon_2 + q(w, x)$$

and

$$q(x, R_D(w)) \leq q(x, w) + q(w, R_D(w)) \leq d(x, w) + \epsilon_1.$$

Thus by metric convexity of (X, q) we have that

$$C_q(R_D(w), q(w, x)) \cap C_{q^{-1}}(x, \epsilon_2) \neq \emptyset,$$

as well as

$$C_{q^{-1}}(R_D(w), d(x, w)) \cap C_q(x, \epsilon_1) \neq \emptyset.$$

$C_q(x, \epsilon_1)$ and $C_{q^{-1}}(x, \epsilon_2)$ intersect. We have shown that the collection

$$[C_q(R_D(w), q(w, x))_{w \in D}, (C_q(x_i, r_i))_{i \in I}, C_q(x, \epsilon_1);$$
$$C_{q^{-1}}(R_D(w), q(x, w))_{w \in D}, (C_{q^{-1}}(x_i, s_i))_{i \in I}, C_{q^{-1}}(x, \epsilon_2)]$$

of double balls has the mixed binary intersection property.
We conclude therefore that $\emptyset \neq C \subseteq A$. Now let $u \in C$ and define $R' : D \cup \{x\} \to A$ by setting $R'(w) = R_D(w)$ if $w \in D$ and $R'(x) = u$. Then for $w \in D$,

$$q(R'(x), R'(w)) = q(u, R(w)) \leq q(u, w)$$

and

$$q(R'(w), R'(x)) = q(R(w), u) \leq q(w, u).$$

So R' is nonexpansive. Also $q(R'(x), x) = q(u, x) \leq \epsilon_2$ and $q(x, R'(x)) = q(x, u) \leq \epsilon_1$. With this we conclude that the pair $(D \cup \{x\} R')$ contradicts the maximality of (D, R_D) in $(\mathcal{F}, \preceq)$. Therefore, $D = N_{\epsilon_1,\epsilon_2}(A)$ and the proof is complete. □

Definition 3.3.14. *([25]) Let (X, q) be a quasi-metric space. A map $T : (X, q) \to (X, q)$ is said to have approximate fixed points if*

$$\inf_{x \in X} q^s(x, T(x)) = 0.$$

Definition 3.3.15. *([25]) Let* (X, q) *be a quasi-metric space. For a map* $T : (X, q) \to (X, q)$ *and for any* $\epsilon_1, \epsilon_2 \geq 0$*, we use* $F_{\epsilon_1,\epsilon_2}(T)$ *to denote the set of* ϵ_1, ϵ_2*-fixed points of* T*, that is,*

$$F_{\epsilon_1,\epsilon_2}(T) = \{x \in X : q(x, T(x)) \leq \epsilon_2 \quad \text{and} \quad q(T(x), x) \leq \epsilon_1\}.$$

Theorem 3.3.16. *([27]) Suppose that* (X, q) *is an Isbell convex quasi-metric space and that the map* $T : (X, q) \to (X, q)$ *is non-expansive. Furthermore, suppose that for some* $\epsilon_1, \epsilon_2 \geq 0$*,* $F_{\epsilon_1,\epsilon_2}(T) \neq \emptyset$*. Then, the set* $F_{\epsilon_1,\epsilon_2}(T)$ *is Isbell convex.*

Proof. For each $i \in I$, let $x_i \in F_{\epsilon_1,\epsilon_2}(T)$, and let $r_i \geq 0$ and $s_i \geq 0$ be such that $q(x_i, x_j) \leq r_i + s_j$. We need to show that

$$\left[\bigcap_{i \in I}(C_q(x_i, r_i) \cap C_{q^{-1}}(x_i, s_i))\right] \cap F_{\epsilon_1,\epsilon_2}(T) \neq \emptyset.$$

Since (X, q) is Isbell convex, by Proposition 3.3.4, $\emptyset \neq J = (C_q(x_i, r_i) \cap C_{q^{-1}}(x_i, s_i))$ is Isbell convex. Furthermore, J is bounded in (X, q). Also, if $x \in J$, then for each $i \in I$,

$$\begin{aligned} q(x_i, Tx) &\leq q(x_i, Tx_i) + q(Tx_i, Tx) \\ &\leq \epsilon_2 + q(x_i, x) \\ &\leq \epsilon_2 + r_i \end{aligned}$$

and

$$\begin{aligned} q(Tx, x_i) &\leq q(Tx, Tx_i) + q(Tx_i, x_i) \\ &\leq q(x, x_i) + \epsilon_1 \\ &\leq \epsilon_1 + s_i. \end{aligned}$$

This implies that $Tx \in C_q(x_i, r_i + \epsilon_2)$ and $Tx \in C_q(x_i, s_i + \epsilon_1)$, which implies that $Tx \in N_{\epsilon_1,\epsilon_2}(J)$ by Lemma 3.3.11. Now, by Lemma 3.3.13 there is a nonexpansive retraction R of $N_{\epsilon_1,\epsilon_2}(J)$ onto J for which $q(R(x), x) \leq \epsilon_2$ and $q(x, R(x)) \leq \epsilon_1$ whenever $x \in N_{\epsilon_1,\epsilon_2}(J)$. Also since $R \circ T$ is a nonexpansive map of J into J, it must have a fixed point by Theorem 3.3.5. Suppose that $R \circ Tx_0 = x_0$ for some $x_0 \in J$. Then, $q(x_0, Tx_0) = q(R \circ Tx_0, Tx_0) \leq \epsilon_2$, $q(Tx_0, x_0) = q(Tx_0, R \circ Tx) \leq \epsilon_1$ Thus, the proof is complete, since $x_0 \in J \cap F_{\epsilon_1,\epsilon_2}(T)$.

□

CHAPTER 4

FUZZY HYPERCONVEXITY

In this chapter, we recall the concept of hyperconvexity in fuzzy metric spaces which was introduced by Yiǵit and Efe [38]. This type of convexity is called fuzzy hyperconvexity and it is a generalisation of hyperconvexity in metric spaces to fuzzy metric space. Therefore, it is natural that we first recall the concept of fuzzy metric spaces. The concept of triangular norms (also called t-norms) plays an important role in the definition of a fuzzy metric space. Hence we start with this concept.

4.1. Triangular norms

In this section, we recall the definition of a triangular norm and give some of their examples.

Definition 4.1.1 ([7])**.** *A triangular norm (shortly, t-norm) is a binary operation on the unit interval* $[0,1]$*, that is, a function* $*:[0,1]\times[0,1]\to[0,1]$ *such that, for all* $a,b,c\in[0,1]$ *the following four axioms are satisfied:*

(i) $a*b=b*a$ *(commutativity);*

(ii) $a*(b*c)=(a*b)*c$ *(associativity);*

(iii) $a*1=a$ *and* $a*0=0$ *(boundary conditions);*

(iv) $a*b\le a*c$ *whenever* $b\le c$ *(monotonicity).*

The commutativity of (i), the boundary condition (iii) and the monotonicity (iv) imply that for each t-norm $*$ *and* $x\in[0,1]$*, the following boundary conditions are also satisfied.*

$$x*1=1*x=x,\quad x*0=0*x=0.$$

Furthermore, a triangular norm $*$ *is said to be continuous if* $*$ *is continuous, that is for all* $y\in[0,1]$ *the function* $\cdot*y:[0,1]\to[0,1], x\to x*y$ *is continuous.*

Example 4.1.1 ([21])**.** The Lukasiewicz $*_L$ defined by:

$$a*_L b=\max\{a+b-1,0\},$$

is a triangular norm, where $a,b\in[0,1]$.

Example 4.1.2 ([21])**.** The minimum $*_M$ defined by:

$$a*_M b=\min\{a,b\}$$

is a continuous triangular norm, where $a,b\in[0,1]$.

Example 4.1.3 ([21])**.** The product $*_P$ defined by:

$$a *_P b = a \cdot b$$

is a continuous triangular norm, where $a, b \in [0, 1]$.

4.2. Fuzzy metric spaces

In this section, we recall the definition of a fuzzy set and give an example, then we recall the definition of a fuzzy metric space. For details see [13] and [7].

Definition 4.2.1 ([39])**.** *Let X be a collection of objects denoted generically by x, then a fuzzy set $\tilde{A}$ in X is a set of ordered pairs:*

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\},$$

where $\mu_{\tilde{A}}(x)$ is called the membership function or grade of membership of x in $\tilde{A}$ that maps X to the membership space $M = [0, 1]$.

However, if the membership space $M = \{0, 1\}$, then $\tilde{A}$ is non-fuzzy and $\mu_{\tilde{A}}(x)$ is the same as the characteristic function of a non-fuzzy set.

Example 4.2.1 ([39])**.** A realtor wants to classify the house he offers to his clients. One indicator of comfort of these houses is the number of bedrooms in it. Let $X = \{1, 2, 3, ..., 10\}$ be the set of available types of houses described by $x =$ number of bedrooms in a house. Then the set indicating comfortable type of house for a four-person family described as

$$\tilde{A} = \{(1, 0.2), (2, 0.5), (3, 0.8), (4, 1), (5, 0.7), (6, 0.3)\}$$

is a fuzzy set.

Example 4.2.2. ([7]) The set $\tilde{A}$ of real numbers considerably larger than 10:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in \mathbb{R}\},$$

where $\mu_{\tilde{A}}(x) = \begin{cases} 0, & \text{if} \quad x \leq 10, \\ \frac{1}{1+(x-10)^2}, & \text{if} \quad x > 10. \end{cases}$

is a fuzzy set.

Take for instance the subset of real numbers $\{9, 10, 11, 12, 17\}$, then its fuzzy set with respect to the membership function $\mu_{\tilde{A}}(x)$ will be $\{(9, 0), (10, 0), (11, 0.5), (12, 0.2), (17, 0.02)\}$.

Definition 4.2.2. *([11]) Let X be a non-empty set, an intuitionistic fuzzy set is defined by:*

$$A = \{(x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)) : x \in X\},$$

where $\mu_{\tilde{A}}(x)$ is the degree of membership or the membership function of x to $\tilde{A}$, $\nu_{\tilde{A}}(x)$ is the degree of non-membership or the non-membership function of x to $\tilde{A}$, where

$$\mu_{\tilde{A}} : X \to [0, 1] \quad \textit{and} \quad \nu_{\tilde{A}} : X \to [0, 1]$$

and satisfy the condition:

$$0 \leq \mu_{\tilde{A}}(x) + \nu_{\tilde{A}}(x) \leq 1, \quad \textit{for all} \quad x \in X.$$

Furthermore, $\phi_{\tilde{A}}(x) = 1-\nu_{\tilde{A}}(x)-\mu_{\tilde{A}(x)}$ is called the intuitionistic fuzzy set index or hesitation margin of x in $\tilde{A}$. $\phi_{\tilde{A}}(x)$ is the degree of indeterminacy of $x \in X$ to the intuitionistic fuzzy set $\tilde{A}$ and $\phi_{\tilde{A}}(x) \in [0,1]$, that is, $\phi_{\tilde{A}}(x) :\rightarrow [0,1]$ and $0 \leq \phi_{\tilde{A}}(x) \leq 1$ and it expresses the lack of knowledge of whether x belongs to the intuitionistic fuzzy set or not.

Definition 4.2.3. *([7]) Let X be a nonempty set, $*$ be a continuous t-norm, and let $M : X^2 \times [0,\infty) \rightarrow [0,1]$ be a fuzzy set. Then the pair $(M,*)$ is called a KM-fuzzy pseudometric on a set X if for all $x,y,z \in X$ and $t,s \in (0,\infty)$ the following axioms are satisfied:*

(i) $M(x,y,0) = 0$

(ii) $M(x,x,t) = 1$

(iii) $M(x,y,t) = M(y,x,t)$

(iv) $M(x,y,\cdot) : [0,\infty) \rightarrow [0,1]$ *is left continuous.*

(v) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$

The 3-tuple $(X,M,*)$ is called a KM-fuzzy pseudometric space.
If further for all $t > 0$,

$$M(x,y,t) = M(y,x,t) = 1 \Longleftrightarrow x = y,$$

then the pair $(M,*)$ is called a KM-fuzzy metric and the 3-tuple $(X,M,*)$ is called a KM-fuzzy metric space.

Definition 4.2.4. *([7]) Let X be a non-empty set, $*$ be a continuous t-norm and M be a fuzzy set on $X^2 \times (0,\infty)$. Then the pair $(M,*)$ is called a GV-fuzzy pseudometric on a set X if for all $x,y,z \in X$ and $t,s \in (0,\infty)$ the following axioms are satisfied:*

(i) $M(x,y,t) > 0$

(ii) $M(x,x,t) = 1$

(iii) $M(x,y,t) = M(y,x,t)$

(iv) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$

(v) $M(x,y,\cdot) : (0,\infty) \rightarrow [0,1]$ *is continuous.*

The 3-tuple $(X,M,*)$ is called a GV-fuzzy pseudometric space.
If further for all $t > 0$,

$$M(x,y,t) = M(y,x,t) = 1 \Longleftrightarrow x = y,$$

then the pair $(M,*)$ is called a GV-fuzzy metric and the 3-tuple $(X,M,*)$ is called a GV-fuzzy metric space.

Remark 4.2.1. ([13]) $M(x, y, t)$ can be thought of as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$, for $t > 0$ and $M(x, y, t) = 0$ with ∞.

Example 4.2.3. ([20]) Let $X = \mathbb{R}$. Define $a *_P b = a \cdot b$ and

$$M(x, y, t) = \left[e^{\frac{|x-y|}{t}}\right]^{-1}$$

for all $x, y \in X$ and $t \in (0, \infty)$. Then $(X, M, *_P)$ is a GV-fuzzy metric space.

Proof. To prove we show that the axioms of a fuzzy metric space are satisfied.

(i) $M(x, y, t) > 0$
We have that $M(x, y, t) = \left[e^{\frac{|x-y|}{t}}\right]^{-1} > 0$ since $e^x > 0$ for any $x \in \mathbb{R}$.

(ii) $M(x, x, t) = 1$
We have that $M(x, x, t) = \left[e^{\frac{|x-x|}{t}}\right]^{-1} = 1$.

(iii) $M(x, y, t) = M(y, x, t)$
We have that $M(x, y, t) = \left[e^{\frac{|x-y|}{t}}\right]^{-1} = \left[e^{\frac{|y-x|}{t}}\right]^{-1} = M(y, x, t)$.

(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$
We have that

$$M(x, y, t) *_P M(y, z, s) = \left[e^{\frac{|x-y|}{t}}\right]^{-1} \cdot \left[e^{\frac{|y-z|}{s}}\right]^{-1} = \left[e^{\frac{|x-y|}{t}+\frac{|y-z|}{s}}\right]^{-1}.$$

Now
$$\frac{|x-y|}{t} + \frac{|y-z|}{s} \geq \frac{|x-y|}{t+s} + \frac{|y-z|}{t+s} = \frac{|x-y| + |y-z|}{t+s} \geq \frac{|x-z|}{t+s},$$
where the last inequality follows by the triangle inequality.
Thus,

$$\frac{|x-y|}{t} + \frac{|y-z|}{s} \geq \frac{|x-z|}{t+s},$$

which implies that,

$$e^{\frac{|x-y|}{t}+\frac{|y-z|}{s}} \geq e^{\frac{|x-z|}{t+s}},$$

so

$$\left[e^{\frac{|x-y|}{t}+\frac{|y-z|}{s}}\right]^{-1} \leq \left[e^{\frac{|x-z|}{t+s}}\right]^{-1}.$$

Thus,

$$M(x, y, t) *_P M(y, z, s) = \left[e^{\frac{|x-y|}{t}+\frac{|y-z|}{s}}\right]^{-1} \leq \left[e^{\frac{|x-z|}{t+s}}\right]^{-1} = M(x, z, t + s).$$

(v) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.
We have that the exponential function e^x is continuous on any open interval, hence $M(x, y, t) = \left[e^{\frac{|x-y|}{t}}\right]^{-1}$ is continuous on $(0, \infty)$.

□

Example 4.2.4. ([13]) Let (X, d) be a metric space. Define $a *_P b = a \cdot b$ and

$$M(x, y, t) = \frac{kt^n}{kt^n + md(x, y)},$$

where $k, m, n \in \mathbb{R}^+$. Then $(X, M, *_P)$ is a GV-fuzzy metric space.
Further, if $k = m = n = 1$, we get

$$M_d(x, y, t) = \frac{t}{t + d(x, y)},$$

we call this a fuzzy metric induced by a metric d, the standard fuzzy metric.
Definition 4.2.5. *([13]) Let $(X, M, *)$ be a GV-fuzzy metric space. Then an open ball $B_M(x, r, t)$ about $x \in X$ and radius $r \in (0, 1)$, and $t > 0$ is defined by*

$$B_M(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

Definition 4.2.6. *([13]) Let $(X, M, *)$ be a GV-fuzzy metric space. Then a closed ball $C_M(x, r, t)$ about $x \in X$ and radius $r \in (0, 1)$, and $t > 0$ is defined by*

$$C_M(x, r, t) = \{y \in X : M(x, y, t) \geq 1 - r\}.$$

Remark 4.2.2. ([38]) Every open ball is an open set and every closed ball is a closed set in a fuzzy metric space $(X, M, *)$.
Definition 4.2.7. *([13]) Let $(X, M, *)$ be a fuzzy metric space and define τ by*

$$\tau = \{A \subset X : x \in \tilde{A} \quad \textit{if and only if there exist} \quad t > 0 \quad \textit{and} \quad r \in (0, 1) \quad \textit{such that} \quad B(x, r, t) \subset A\}.$$

Then τ is a topology on X.
Lemma 4.2.8. *([3]) Let $(X, M, *)$ be a fuzzy metric space and $\alpha \in [0, 1)$. Then the collection*

$$B_\alpha = \{B_M(x, r, t) : x \in X, r \in (\alpha, 1), t > 0\}$$

is a base for a topology on X.

Proof. Let $z \in B_M(x, r_1, t) \cap B_M(y, r_2, s)$ where $r_1, r_2 \in (\alpha, 1)$. Thus $M(x, z, t) \geq 1 - r_1$ and $M(y, z, s) > 1 - r_2$. Then there exists $t_0 < t$ and $s_0 < s$ such that

$$M(x, z, t_0) > 1 - r_1 \quad \text{and} \quad M(y, z, s_0) > 1 - r_2.$$

Let $r = \min\{r_1, r_2\}$ and $p = \min\{t - t_0, s - s_0\}$. We claim that $B_M(z, 1-r, p) \subset B_M(x, r_1, t) \cap B_M(y, r_2, s)$. Let $u \in B_M(z, r, p) \subset B(z, r, t - t_0)$. Then $M(u, z, t - t_0) > 1 - r$. Therefore, $M(x, u, t) \geq M(x, z, t_0) * M(z, u, t - t_0) > (1 - r_1) * (1 - r) = 1 - r_1$. Then $u \in B_M(x, r_1, t)$ and we have $B_M(z, r, p) subet B_M(x, r_1 t)$. On the other hand let $u \in B_M(z, r, p) \subset B_M(z, r, s - s_0)$. Then $M(u, z, s - s_0) > 1 - r$. Therefore $M(y, u, s_0) \geq M(y, z, s_0) * M(z, u, s - s_0) > (1 - r_2) * (1 - r) = 1 - r_2$. Then $u \in B_M(y, r_2, s)$. We have $B_M(z, r, p) \subset B_M(y, r_2, s)$. Hence $B(z, r, p) \subset B_M(x, r_1, t) \cap B(y, r_2, s)$. □

Proposition 4.2.9. *([1]) Let (X,d) be a metric space and $M_d(x,y,t) = \frac{t}{t+d(x,y)}$ be the corresponding standard fuzzy metric on X. Then the topology τ_d induced by the metric d and the topology τ_{M_d} induced by the fuzzy metric M_d are the same. That is,*

$$\tau_d = \tau_{M_d}.$$

Proof. Suppose that $A \in \tau_d$. Then there exists $\epsilon > 0$ such that $B_d(x,\epsilon) \subset A$, for every $x \in A$, which implies that $d(x,y) < \epsilon$ for all $y \in B_d(x,\epsilon)$. Thus, for a fixed $t > 0$, we obtain that

$$M_d(x,y,t) = \frac{t}{t+d(x,y)} > \frac{t}{t+\epsilon}$$

Let $1 - r = \frac{t}{t+\epsilon}$. Then $M_d(x,y,t) > 1 - r$. Then it follows that,
$B_{M_d}(x,r,t) = \{y \in X : M_d(x,y,t) > 1-r\} \subset A$. Hence $A \in \tau_{M_d}$. This shows that $\tau_d \subseteq \tau_{M_d}$. Conversely, suppose that $A \in \tau_{M_d}$. Then there exists $\epsilon \in (0,1)$ and $t > 0$ such that $B_{M_d}(x,\epsilon,t) \subset A$ for every $x \in A$. Thus we obtain that

$$M(x,y,t) = \frac{t}{t+d(x,y)} > 1 - \epsilon,$$

which implies that

$$\frac{1}{t+d(x,y)} > \frac{1-\epsilon}{t},$$

which implies that

$$t + d(x,y) < \frac{t}{1-\epsilon},$$

which implies that

$$d(x,y) < \frac{t}{1-\epsilon} - t = \frac{t\epsilon}{1-\epsilon}.$$

Let $r = \frac{\epsilon t}{1-\epsilon}$ where $0 < \epsilon < 1$. Then $d(x,y) < r$ and therefore
$B_d(x,r) = \{y \in X : d(x,y) < r\} \subset A$. Hence $A \in \tau_d$. This implies that $\tau_{M_d} \subseteq \tau_d$. Therefore $\tau_d = \tau_{M_d}$. □

Definition 4.2.10. *([33]) Let $(X,M,*)$ be a GV-fuzzy metric space, a sequence $\{x_n\}$ in X is said to converge to a point x_0 in X if and only if for each $\epsilon > 0, t > 0$ there exists $N \in \mathbb{N}$ such that $M(x_n,x,t) > 1-\epsilon$ for all $n \geq N$.*

Definition 4.2.11. *([13]) Let $(X,M,*)$ be a GV-fuzzy metric space, a sequence $\{x_n\}$ in $(X,M,*)$ is said to be a Cauchy sequence if and only if for every given $\epsilon > 0$, $t > 0$ there exists $N \in \mathbb{N}$ such that $M(x_n,x_m,t) > 1-\epsilon$ for all $n,m \geq N$.*

Definition 4.2.12. *([13]) A GV-fuzzy metric space $(X,M,*)$ is said to be complete if and only if every Cauchy sequence in X converges to a point in X.*

4.3. The compatible metric

In this section we recall the concept of a compatible metric. This concept was introduced by Radu in [29]. later on, Castro-Company, Romaguera and Tirado used this concept to present fixed point theorems in fuzzy metric spaces. In Chapter 6, we will generalise this concept to the asymmetric setting and use this concept to present fixed point theorems in fuzzy quasi-metric spaces.

Theorem 4.3.1. *[6] Let $(X, M, *)$ be a fuzzy metric space such that $* \geq *_L$, where $a *_L b = \max\{a + b - 1, 0\}$ and for each $x, y \in X$, let*

$$d_t(x, y) = \sup\{t \geq 0 : M(x, y, t) \leq 1 - t\}.$$

Then d_t is a metric on X such that

$$d_t(x, y) < \epsilon \Longleftrightarrow M(x, y, \epsilon) > 1 - \epsilon,$$

for all $\epsilon \in (0, 1)$.

Proof. (see [6], Theorem 3) From the definition of $d_t(x, y)$, we have that $d_t(x, y) \geq 0$, for all $x, y \in X$. Secondly, we show that $d_t(x, x) = 0$, which is true since $M(x, x, t) = 1$, which implies that for all t such that $M(x, x, t) \leq 1 - t$, $t = 0$ and so
$d_t(x, x) = \sup\{t \geq 0 : M(x, x, t) \leq 1 - t\} = 0$. We now show that $d_t(x, y) = 0$ whenever $x = y$. Suppose that $x = y$, then $M(x, y, t) = 1$. Thus for any $x, y \in X$, we have that $t = 0$ and this implies that $d_t(x, y) = \sup\{t \geq 0 : M(x, y, t) \leq 1 - t\} = 0$. Next we show that $d_t(x, y) = d_t(y, x)$, which follows since $M(x, y, t) = M(y, x, t)$. Finally, we show that d_t satisfies the triangle inequality. Let $x, y, z \in X$, we know that $d_t(x, y) \leq 1$. Thus, if $1 \leq d_t(x, z) + d_t(z, y)$ then $d_t(x, y) \leq 1 \leq d_t(x, z) + d_t(z, y)$. Hence, we assume, without loss of generality, that $d_t(x, z) + d_t(z, y) < 1$. We use the relation that

$$M(x, y, a) \geq 1 - a \Longrightarrow d_t(x, y) \leq a$$

for all $a \in (0, 1)$. Before proceeding with the proof, we first show that the relation is true. Now suppose to the contrary that $d_t(x, y) > a$. Then there exists $t \in (a, 1]$ such that $M(x, y, t) \leq 1 - t$ from the definition of $d_t(x, y)$, which implies $a < t$ and so

$$1 - a \leq M(x, y, a) \leq M(x, y, t) \leq 1 - t,$$

which implies that $a \geq t$ which contradicts $a < t$. Hence, $d_t(x, y) \leq a$ whenever $M(x, y, a) \geq 1 - a$. Choose an arbitrary $\epsilon > 0$ such that $d_t(x, z) + d_t(z, y) + 2\epsilon < 1$. Then, from the definition of d_t and since $* \geq *_L$, we have that

$$\begin{aligned} M(x, y, d_t(x, z) + d_t(z, y) + 2\epsilon) &\geq M(x, z, d_t(x, z) + \epsilon) * M(z, y, d_t(z, y) + \epsilon) \\ &\geq (1 - d_t(x, z) - \epsilon) * (1 - d_t(z, y) - \epsilon) \\ &\geq (1 - d_t(x, z) - \epsilon) *_L (1 - d_t(z, y) - \epsilon) \\ &= 1 - (d_t(x, z) + d_t(z, y) + 2\epsilon). \end{aligned}$$

It follows from the relation $M(x, y, a) \geq 1 - a \Longrightarrow d_t(x, y) \leq a$ that

$$d_t(x, y) \leq d_t(x, z) + d_t(z, y) + 2\epsilon.$$

Since ϵ was arbitrary choosen, we conclude that $d_t(x,y) \leq d_t(x,z) + d_t(z,y)$. Hence, d_t is a metric on X. $d_t(x,y) < \epsilon \iff M(x,y,\epsilon) > 1-\epsilon$ follows from the definition of d_t.

□

4.4. Fuzzy hyperconvexity

In this section, we recall the concept of fuzzy hyperconvexity. This concept was introduced by Yiǵit and Efe in [38]. For more details see [38].
We point out that the results in this section are in the framework of GV-fuzzy metric spaces. We also point out that we use the term fuzzy metric to refer to GV-fuzzy metric.

Before we recall the definition of fuzzy metrically convexity, we first state and prove the following lemma.
Lemma 4.4.1. *([38]) Let $(X, M, *)$ be a fuzzy metric space and let $x_1, x_2 \in X, r_1, r_2 \in (0,1)$, also let $t_1, t_2 \in (0,\infty)$. If $C_M(x_1,r_1,t_1) \cap C_M(x_2,r_2,t_2) \neq \emptyset$, then for any points $x_1, x_2 \in X$ and for each pair r_1, t_1 and r_2, t_2 where $r_1, r_2 \in (0,1)$ and $t_1, t_2 \in (0,\infty)$, we have that*

$$M(x_1, x_2, t_1+t_2) \geq (1-r_1) * (1-r_2).$$

***Proof*.** Suppose $C_M(x_1,r_1,t_1) \cap C_M(x_2,r_2,t_2) \neq \emptyset$, then there exists $z \in X$ such that $z \in C_M(x_1,r_1,t_1) \cap C_M(x_2,r_2,t_2)$.
Now by definition,

$$C_M(x_1,r_1,t_1) = \{z \in X : M(x_1,z,t_1) \geq 1-r_1\}$$

and

$$C_M(x_2,r_2,t_2) = \{z \in X : M(x_2,z,t_2) \geq 1-r_2\}.$$

Thus, $M(x_1,z,t_1) \geq 1-r_1$ and $M(x_2,z,t_2) \geq 1-r_2$.
And from Definition 4.2.4 axiom (iv) of a fuzzy metric, we have that

$$\begin{aligned} M(x_1,x_2,t_1+t_2) &\geq M(x_1,z,t_1) * M(x_2,z,t_2) \\ &\geq (1-r_1) * (1-r_2). \end{aligned}$$

□

However, the converse of the above lemma need not be true.
Example 4.4.1. ([38]) Let $(\mathbb{N}, M, *_P)$ be a fuzzy metric space, where M is a fuzzy set on $\mathbb{N} \times \mathbb{N} \times (0,\infty)$ defined by

$$M(x,y,t) = \frac{\min\{x,y\}+t}{\max\{x,y\}+t},$$

$a *_P b = a \cdot b$ and $\mathbb{N}$ is the set of Natural Numbers.
If we choose $t_1 = 1, t_2 = 1, r_1 = 0.3, r_2 = 0.5, x_1 = 3$ and $x_2 = 10$ then the inequality

$$M(x_1,x_2,t_1+t_2) \geq (1-r_1) * (1-r_2)$$

is satisfied, but $C_M(3, 0.3, 1) \cap C_M(10, 0.5, 1) = \emptyset$, which implies that there exists no point $z \in X$ such that

$$M(x_1, z, t_1) \geq (1 - r_1) \quad \text{and} \quad M(x_2, z, t_2) \geq (1 - r_2).$$

Definition 4.4.2. *([38]) A fuzzy metric space $(X, M, *)$ is said to be fuzzy metrically convex if for any points $x, y \in X$ and for each pair r_1, t_1 and r_2, t_2 where $r_1, r_2 \in (0, 1)$ and $t_1, t_2 \in (0, \infty)$ such that $M(x_1, x_2, t_1 + t_2) \geq (1 - r_1) * (1 - r_2)$, there exists $z \in X$ such that $M(x_1, z, t_1) \geq 1 - r_1$ and $M(x_2, z, t_2) \geq 1 - r_2$ or equivalently $z \in C_M(x_1, r_1, t_1) \cap C_M(x_2, r_2, t_2)$.*

Example 4.4.2. ([38]) Let (X, d) be a metrically convex metric space. Define the t-norm $*_P$ by $a *_P b = a \cdot b$ for all $a, b \in [0, 1]$ and let M be a fuzzy set defined on $X \times X \times (0, \infty)$ as follows:

$$M(x, y, t) = e^{\frac{-d(x,y)}{t}}.$$

Then the 3-tuple $(X, M, *_P)$ is a fuzzy metric space and under these conditions $(X, M, *_P)$ is fuzzy metrically convex.

Proof. Since (X, d) is metrically convex, for any points $x_1, x_2 \in X$ and positive real numbers α and β such that $d(x_1, x_2) \leq \alpha + \beta$, there exists $z \in X$ such that $d(x_1, z) \leq \alpha$ and $d(x_2, z) \leq \beta$ or equivalently $z \in C_d(x_1, \alpha) \cap C_d(x_2, \beta)$. Take $\alpha = -t_1 \ln(1 - r_1)$ and $\beta = -t_2 \ln(1 - r_2)$. By the choice of α and β, the inequality

$$M(x_1, x_2, t_1 + t_2) \geq (1 - r_1) *_P (1 - r_2)$$

is satisfied. That is,

$$\begin{aligned} M(x_1, x_2, t_1 + t_2) &= e^{\frac{-d(x_1,x_2)}{t_1+t_2}} \\ &\geq e^{\frac{-\alpha-\beta}{t_1+t_2}} \\ &= e^{\frac{-\alpha}{t_1+t_2}} \cdot e^{\frac{-\beta}{t_1+t_2}} \\ &= e^{\frac{t_1 \ln(1-r_1)}{t_1+t_2}} \cdot e^{\frac{t_2 \ln(1-r_2)}{t_1+t_2}} \\ &\geq e^{\frac{t_1 \ln(1-r_1)}{t_1}} \cdot e^{\frac{t_2 \ln(1-r_2)}{t_2}} \\ &= (1 - r_1) \cdot (1 - r_2) \\ &= (1 - r_1) *_P (1 - r_2). \end{aligned}$$

By the metric convexity of (X, d) we have that

$$d(x_1, z) \leq \alpha = -t_1 \ln(1 - r_1) \quad \text{and} \quad d(x_2, z) \leq \beta = -t_2 \ln(1 - r_2),$$

which implies that

$$-d(x_1, z) \geq t_1 \ln(1 - r_1) \quad \text{and} \quad -d(x_2, z) \geq t_2 \ln(1 - r_2).$$

And since $M(x,y,t) = e^{\frac{-d(x,y)}{t}}$, we have that

$$M(x_1, z, t_1) = e^{\frac{-d(x_1,z)}{t_1}} \geq e^{\frac{t_1 \ln(1-r_1)}{t_1}} = 1 - r_1$$

and

$$M(x_2, z, t_2) = e^{\frac{-d(x_2,z)}{t_2}} \geq e^{\frac{t_2 \ln(1-r_2)}{t_2}} = 1 - r_2,$$

or equivalently

$$C_M(x_1, r_1, t_1) \cap C_M(x_2, r_2, t_2) \neq \emptyset.$$

Thus the fuzzy metric space $(X, M, *_P)$ is fuzzy metrically convex. □

Definition 4.4.3. *([38]) A fuzzy metric space $(X, M, *)$ is said to be fuzzy hyperconvex if for any collection of closed balls $\{C_M(x_i, r_i, t_i)\}_{i\in I}$ in X such that $M(x_i, x_j, t_i + t_j) \geq (1 - r_i) * (1 - r_j)$ for all $i, j \in I$,*

$$\bigcap_{i\in I} C_M(x_i, r_i, t_i) \neq \emptyset.$$

Definition 4.4.4. *([38]) A fuzzy metric space $(X, M, *)$ is said to have the ball intersection property if $\bigcap_{i\in I} C_M(x_i, r_i, t_i) \neq \emptyset$ for any collection of closed balls $\{C_M(x_i, r_i, t_i)\}_{i\in I}$ such that*

$$\bigcap_{i\in A} C_M(x_i, r_i, t_i) \neq \emptyset$$

for any finite subset $A \subset I$.

Theorem 4.4.5. *([38]) Let $\mathbb{R}$ be equipped with the usual metric d, that is, $d = |x - y|$. Consider the standard fuzzy metric M defined by*

$$M(x, y, t) = \frac{t}{t + d(x, y)}$$

with $a * b = \min\{a, b\}$ for all $a, b \in (0, 1)$. Then $(X, M, *)$ is fuzzy hyperconvex.

***Proof*.** Since $(\mathbb{R}, d)$ is hyperconvex, then by definition for any collection of closed balls $C_d(x_i, r_i)$ such that $d(x_i, x_j) \leq r_i + r_j$ for any $i, j \in I$, we have that

$$\bigcap_{i\in I} C_d(x_i, r_i) \neq \emptyset.$$

Choose $R_i = \frac{r_i}{r_i+t_i}$ and $R_j = \frac{r_j}{r_j+t_j}$. Clearly $R_i, R_j \in (0, 1)$. And by the choice of R_i and R_j, and assuming without loss of generality that $R_i \geq R_j$, the inequality

$$M(x_i, x_j, t_i + t_j) \geq (1 - R_i) * (1 - R_j) = 1 - R_i$$

is satisfied. That is, by the hyperconvexity of $(\mathbb{R}, d)$, there exists $z \in \bigcap_{i\in I} C_d(x_i, r_i)$ such that $d(x_i, z) \leq r_i$ and $d(z, x_j) \leq r_j$ for any $i, j \in I$ and so

$$\begin{aligned} M(x_i, x_j, t_i + t_j) &\geq M(x_i, z, t_i) * M(z, x_j, t_j) \\ &= \frac{t_i}{t_i + d(x_i, z)} * \frac{t_j}{t_j + d(z, x_j)} \\ &\geq \frac{t_i}{t_i + r_i} * \frac{t_j}{t_j + r_j} \\ &= \left(1 - \frac{r_i}{t_i + r_i}\right) * \left(1 - \frac{r_j}{t_j + r_j}\right) \\ &= (1 - R_i) * (1 - R_j) \\ &= \min\{(1 - R_i), (1 - R_j)\} \\ &= 1 - R_i \end{aligned}$$

Also $z \in \bigcap_{i \in I} C_d(x_i, r_i)$, implies that for all $i \in I$, $d(x_i, z) \leq r_i$, which implies that $t_i + d(x_i, z) \leq t_i + r_i$ which also implies that $\frac{t_i}{t_i + d(x_i, z)} \geq \frac{t_i}{t_i + r_i}$. And this means $M(x_i, z, t_i) \geq 1 - R_i$, which implies that $z \in C_M(x_i, r_i, t_i)$ for all $i \in I$.
Hence, $(\mathbb{R}, M, *)$ is fuzzy hyperconvex. □

Proposition 4.4.6. *([38]) Every fuzzy hyperconvex space $(X, M, *)$ has the Ball Intersection Property.*

***Proof*.** This follows from Definition 4.4.3. □

Remark 4.4.1. ([38]) A nonempty subset F of a fuzzy metric space $(X, M, *)$ has fuzzy diameter zero if and only if F is a singleton set.
Theorem 4.4.7. *([38]) Every fuzzy metric space $(X, M, *)$ with the ball intersection property is complete. Particularly any fuzzy hyperconvex metric space is complete.*

***Proof*.** Let $(X, M, *)$ be a fuzzy metric space which has ball intersection property and let $\{x_n\}$ be a Cauchy sequence in X. For any $n \geq 1$, take the set

$$r_n = \sup_{t_n > 0} \left\{ \inf_{m \geq n} \left\{ \sup_{s < t_n} \{M(x_n, x_m, s)\} \right\} \right\}.$$

Consider the collection of closed balls $\{C_M(x_n, r_n, t_n)\}_{n \geq 1}$. Since $\{x_n\}$ is Cauchy and by the choice of r_n, for $m \geq n$, we have that $M(x_n, x_m, t_n) \geq 1 - r_n$, that is, $\{r_n\}$ has fuzzy diameter zero. Now we examine this situation for any finite index $n_1 < n_2 < ... < n_k$. For $n_1 < n_2 < ... < n_k$, we have

$$M(x_{n_1}, x_{n_k}, t_{n_1}) \geq 1 - r_{n_1}, M(x_{n_2}, x_{n_k}, t_{n_2}) \geq 1 - r_{n_2}, ..., M(x_{n_k}, x_{n_k}, t_{n_k}) \geq 1 - r_{n_k}$$

which means that

$$\begin{aligned}
x_{n_1}, x_{n_2}, ..., x_{n_k} \in &\quad C_M(x_{n_1}, r_{n_1}, t_{n_1}) \\
x_{n_2}, ..., x_{n_k} \in &\quad C_M(x_{n_2}, r_{n_2}, t_{n_2}) \\
&\quad ... \\
x_{n_k} \in &\quad C_M(x_{n_k}, r_{n_k}, t_{n_k})
\end{aligned}$$

therefore,

$$x_{n_k} \in C_M(x_{n_1}, r_{n_1}, t_{n_1}) \cap C_M(x_{n_2}, r_{n_2}, t_{n_2}) \cap ... \cap C_M(x_{n_k}, r_{n_k}, t_{n_k}).$$

Since X has the ball intersection property, then we may conclude that

$$\bigcap_{n \geq 1} C_M(x_n, r_n, t_n) \neq \emptyset$$

for any $n \in \mathbb{N}$. Since x_n is a Cauchy sequence and $\{r_n\}$ has fuzzy diameter zero, the intersection $\bigcap\limits_{n \geq 1} C_M(x_n, r_n, t_n)$ is reduced to one point z which is the limit of the sequence $\{x_n\}$. So indeed, the point

$$z \in \bigcap_{n \geq 1} C_M(x_n, r_n, t_n)$$

then for each pair of $r_n, t_n > 0$ there exists $n_1 \in \mathbb{N}$ such that $M(x_n, z, t_n) > 1 - r_n$ for all $n \geq n_1$. Therefore, $M(x_n, z, t_n)$ converges to 1 when $n \to \infty$, for each $t_n > 0$ and $(X, M, *)$ is complete. □

Proposition 4.4.8. *([38]) A fuzzy metric space $(X, M, *)$ is fuzzy hyperconvex if and only if it has the ball intersection property and is fuzzy metrically convex.*

Proof. Suppose $(X, M, *)$ is fuzzy hyperconvex, then by Proposition 4.4.8, $(X, M, *)$ has ball intersection property, that is

$$\bigcap_{i \in I} C_M(x_i, r_i, t_i) \neq \emptyset,$$

for any collection of closed balls $\{C_M(x_i, r_i, t_i)\}_{i \in I}$ such that

$$\bigcap_{i \in A} C_M(x_i, r_i, t_i) \neq \emptyset.$$

And this implies there exists $z \in C_M(x_i, r_i, t_i)$ for any $i \in A$. Since A is an arbitrary subset of I, we have that for any $i, j \in I$

$$M(x_i, z, t_i) \geq 1 - r_i \quad \text{and} \quad M(x_j, z, t_j) \geq 1 - r_j,$$

which implies that $M(x_i, x_j, t_i + t_j) \geq (1 - r_i) * (1 - r_j)$ and

$$z \in C_M(x_i, r_i, t_i) \cap C_M(x_j, r_j, t_j) \neq \emptyset.$$

Thus, $(X, M, *)$ is fuzzy metrically convex.
Conversely, if two closed balls $C_M(x_i, r_i, t_i)$ and $M(x_j, r_j, t_j)$ satisfy the relation $M(x_i, xj, t_i + t_j) \geq (1 - r_i) * (1 - r_j)$; they must intersect since $(X, M, *)$ has ball intersection property. □

CHAPTER 5

FUZZY ISBELL CONVEXITY

In this chapter, we present the concept of Isbell convexity in fuzzy quasi-metric spaces. This concept generalises fuzzy hyperconvexity, introduced by Yiǵit and Efe to the framework of fuzzy quasi-metric spaces (see [38]). We begin the chapter by recalling the concept of fuzzy quasi-metric spaces, a concept that generalises fuzzy metric spaces to the asymmetric setting.

5.1. Fuzzy quasi-metric spaces

The concept of fuzzy quasi-metric spaces was introduced by Romaguera and Gregory in [36]. In [36], Ramoguera and Gregory generalised KM (Kramosil and Michalek) and GV (George and Veeramani) fuzzy metric spaces to the asymmetric setting by removing the symmetry axiom in the definition of KM and GV fuzzy metric spaces respectively. In this section we recall KM and GV fuzzy quasi-metric spaces.

Definition 5.1.1. *[36] Let X be a non-empty set. A KM (Kramosil and Michalek)-fuzzy quasi-pseudometric on a set X is a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ such that for all $x, y, z \in X$, the following axioms are satisfied:*

(i) $M(x, y, 0) = 0$

(ii) $M(x, x, t) = 1$ *for all* $t > 0$

(iii) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$

(iv) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ *is left continuous.*

Thus $(X, M, *)$ is called a KM-fuzzy quasi-pseudometric space. If further $M(x, y, t) = M(y, x, t) = 1$ for all $t > 0$ if and only if $x = y$, then $(X, M, *)$ is called a KM fuzzy quasi-metric space.

Definition 5.1.2. *[36] Let $(M, *)$ be a KM-fuzzy quasi-metric on a non-empty set X, then the conjugate $(M^{-1}, *)$ of $(M, *)$ is a KM fuzzy quasi-metric where M^{-1} is a fuzzy set defined by*

$$M^{-1}(x, y, t) = M(y, x, t).$$

Example 5.1.1. ([36]) Let $(X, M, *)$ be a KM-fuzzy quasi-metric space and let $M^s(x, y, t)$ be a fuzzy set defined by

$$M^s(x, y, t) = \min\{M(x, y, t), M^{-1}(x, y, t)\},$$

then $(X, M^s, *)$ is called a KM-fuzzy metric space.

Example 5.1.2. [31] Let (X, q) be a quasi-metric space, let $*_P$ be the multiplication continuous t-norm on X and let M_q be the function defined on $X^2 \times (0, \infty)$ by

$$M_q(x, y, t) = \frac{t}{t + q(x, y)}.$$

Then $(X, M_q, *_P)$ is a KM-fuzzy quasi-metric space.

Definition 5.1.3. *[36] Let $(X, M, *)$ be a KM-fuzzy quasi-metric space, then an open ball $B_M(x, r, t)$ about $x \in X$, for $0 < r < 1, t > 0$ is defined as*

$$B_M(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

Definition 5.1.4. *[36] Let $(X, M, *)$ be a KM-fuzzy quasi-metric space. Then a set $A \subset X$ is called an open set if for every $x \in A$ there exists $0 < r < 1$ such that $B_M(x, r, t) \subset A$ for $t > 0$.*

Definition 5.1.5. *[36] Let $(X, M, *)$ be a KM-fuzzy quasi-metric space, then we define a topology τ_M on X as*

$$\tau_M = \{A \subset X : \quad \text{for each} \quad x \in A, \quad \text{there are} \quad r \in (0, 1), t > 0 \quad \text{with} \quad B_M(x, r, t) \subset A\}.$$

Definition 5.1.6. *[36] A KM-fuzzy quasi-metric space $(X, M, *)$ is called bicomplete if and only if $(X, M^s, *)$ is a complete fuzzy quasi-metric space. Thus we say that $(M, *)$ is a bicomplete KM-fuzzy quasi-metric on X.*

Definition 5.1.7. *[36] Let X be a non-empty set. A GV (George and Veeramani)-fuzzy quasi-pseudometric on X is a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times (0, \infty)$ such that for all $x, y, z \in X$, the following axioms are satisfied:*

(i) $M(x, y, t) > 0$

(ii) $M(x, x, t) = 1$

(iii) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$

(iv) $M(x, y, \cdot) : (0, \infty) \to (0, 1]$ *is continuous.*

Therefore, $(X, M, *)$ is called a GV-fuzzy quasi-pseudometric space. Furthermore, if $M(x, y, t) = M(y, x, t) = 1$ for all $t > 0$ if and only if $x = y$, then $(X, M, *)$ is a GV-fuzzy quasi-metric space.

We note that the KM-fuzzy quasi-metric and the GV-fuzzy quasi-metric are similar in their properties, the only major difference is that while the KM-fuzzy quasi-metric is defined on $X^2 \times [0, \infty)$, the GV-fuzzy quasi-metric is defined on $X^2 \times (0, \infty)$. Thus, while the GV-fuzzy quasi-metric has the axiom that $M(x, y, t) > 0$, the KM-fuzzy quasi-metric includes $M(x, y, 0) = 0$. The GV-fuzzy quasi-metric is continuous on $(0, \infty)$, while the KM-fuzzy quasi-metric is continuous on $[0, \infty)$.

Definition 5.1.8. *([36]) Let $(X, *)$ be a GV-fuzzy quasi-metric on a non-empty set X, then the conjugate $(M^{-1}, *)$ of $(M, *)$ is a GV-fuzzy quasi-metric, where M^{-1} is a fuzzy set defined by*

$$M^{-1}(x, y, t) = M(y, x, t).$$

Example 5.1.3. ([36]) Let $(X, M, *)$ be a GV-fuzzy quasi-metric space and let $M^s(x, y, t)$ be a fuzzy set defined by

$$M^s(x, y, t) = \min\{M(x, y, t), M^{-1}(x, y, t)\},$$

then $(X, M^s, *)$ is called a GV-fuzzy metric space.

Example 5.1.4. ([36]) Let (X, q) be a quasi-metric space, let $*_P$ be the multiplication continuous t-norm on X and let M_q be the function defined on $X^2 \times (0, \infty)$ by

$$M_q(x, y, t) = \frac{t}{t + q(x, y)}.$$

Then $(X, M_q, *_P)$ is a GV-fuzzy quasi-metric space.

Definition 5.1.9. *([36]) Let $(X, M, *)$ be a GV-fuzzy quasi-metric space, then an open ball $B_M(x, r, t)$ about $x \in X$, for $0 < r < 1, t > 0$ is defined as*

$$B_M(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

Definition 5.1.10. *([30]) Let $(X, M, *)$ be a GV-fuzzy quasi-metric space, then a closed ball $C_M(x, r, t)$ about $x \in X$, for $0 < r < 1, t > 0$ is defined as*

$$C_M(x, r, t) = \{y \in X : M(x, y, t) \geq 1 - r\}.$$

Definition 5.1.11. *([36]) Let $(X, M, *)$ be a GV-fuzzy quasi-metric space. Then a set $A \subset X$ is called an open set if for every $x \in A$ there exists $0 < r < 1$ such that $B_M(x, r, t) \subset A$ for $t > 0$.*

Definition 5.1.12. *([36]) Let $(X, M, *)$ be a GV-fuzzy quasi-metric space, then we define a topology τ_M on X as*

$$\tau_M = \{A \subset X : \quad \textit{for each} \quad x \in A, \quad \textit{there are} \quad r \in (0, 1), t > 0 \quad \textit{with} \quad B_M(x, r, t) \subset A\}.$$

Definition 5.1.13. *[36] A GV-fuzzy quasi-metric space $(X, M, *)$ is called bicomplete if and only if $(X, M^s, *)$ is a complete fuzzy metric space. Thus we say that $(M, *)$ is a bicomplete GV-fuzzy quasi-metric on X.*

5.2. Fuzzy Isbell convexity

This concept is a generalisation of fuzzy hyperconvexity to the setting of fuzzy quasi-metric spaces. Before we present this concept, we first look at two related concepts, namely, fuzzy metric convexity and fuzzy Isbell completeness.

We point out that the results in this section are in the framework of GV-fuzzy quasi-metric spaces. We also point out that we use the term fuzzy quasi-metric to refer to GV-fuzzy quasi-metric.

In order to understand the definition of fuzzy metric convexity in fuzzy quasi-metric spaces, we need the following lemma.

Lemma 5.2.1. *([30]) Let $(X, M, *)$ be a fuzzy quasi-metric space and let $x, y \in X$, $r_1, s_2 \in (0,1)$ and $t_1, t_2 \in (0, \infty)$. If $C_M(x, r_1, t_1) \cap C_{M^{-1}}(y, s_2, t_2) \neq \emptyset$, then $M(x, y, t_1 + t_2) \geq (1 - r_1) * (1 - s_2)$.*

Proof. Suppose $C_M(x_1, r_1, t_1) \cap C_{M^{-1}}(x_2, s_2, t_2) \neq \emptyset$. Then there exists $z \in X$ such that $z \in C_M(x_1, r_1, t_1) \cap C_{M^{-1}}(x_2, s_2, t_2)$. Thus, $M(x_1, z, t_1) \geq (1 - r_1)$ and $M^{-1}(x_2, z, t_2) = M(z, x_2, t_2) \geq (1 - s_2)$ and this implies that

$$M(x_1, x_2, t_1 + t_2) \geq M(x_1, z, t_1) * M(z, x_2, t_2) \geq (1 - r_1) * (1 - s_2).$$

□

Definition 5.2.2. *([30]) A fuzzy quasi-metric space $(X, M, *)$ is said to be fuzzy-metrically convex if for any points $x_1, x_2 \in X$ and for each pair $r_1, t_1 > 0$ and $s_2, t_2 > 0$ $\quad (r_1, s_2 \in (0,1)$ and $t_1, t_2 \in (0, \infty))$ such that $M(x_1, x_2, t_1 + t_2) \geq (1 - r_1) * (1 - s_2)$, there exists $z \in X$ such that $M(x_1, z, t_1) \geq (1 - r_1)$ and $M^{-1}(x_2, z, t_2) = M(z, x_2, t_2) \geq (1 - s_2)$ or equivalently $z \in C_M(x_1, r_1, t_1) \cap C_{M^{-1}}(x_2, s_2, t_2)$.*

Example 5.2.1. ([30]) Let (X, q) be a metrically convex quasi-metric space and let $a *_P b = a \cdot b$ for all $a, b \in [0, 1]$ be a continuous t-norm. Let M be the fuzzy set on $X \times X \times (0, \infty)$ defined as follows:

$$M(x, y, t) = e^{\frac{-q(x,y)}{t}}.$$

Then $(X, M, *)$ is a fuzzy quasi-metric space.

Proof. Since (X, q) is metrically convex, then for any $x_1, x_2 \in X$ and $\alpha, \beta \in (0, \infty)$ such that $q(x_1, x_2) \leq \alpha + \beta$, there exists $z \in X$ such that $q(x_1, z) \leq \alpha$ and $q(z, x_2) \leq \beta$.
Now take $\alpha = -t_1 \ln(1 - r_1) \quad$ and $\quad \beta = t_2 \ln(1 - s_2)$.
Then the inequality;

$$\begin{aligned} M(x_1, x_2, t_1 + t_2) &\geq M(x_1, z, t_1) * M(z, x_2, t_2) \\ &= e^{\frac{-q(x_1,z)}{t_1}} * e^{\frac{-q(z,x_2)}{t_2}} \\ &\geq e^{\frac{-\alpha}{t_1}} * e^{\frac{-\beta}{t_2}} \\ &= e^{\frac{t_1 \ln(1-r_1)}{t_1}} * e^{\frac{t_2 \ln(1-s_2)}{t_2}} \\ &= e^{\ln(1-r_1)} * e^{\ln(1-s_2)} \\ &= (1 - r_1) * (1 - s_2) \end{aligned}$$

is satisfied for $x_1, x_2 \in X$, $r_1, s_2 \in (0,1)$ and $t_1, t_2 \in (0, \infty)$.
Thus, since (X, q) is metrically convex, we have that

$$q(x_1, z) \leq \alpha \quad \text{and} \quad q(z, x_2) \leq \beta,$$

which implies that

$$q(x_1, z) \leq -t_1 \ln(1 - r_1) \quad \text{and} \quad q(z, x_2) \leq -t_2 \ln(1 - s_2),$$

which implies that

$$\frac{-q(x_1,z)}{t_1} \geq \ln(1-r_1) \quad \text{and} \quad \frac{-q(z,x_2)}{t_2} \geq \ln(1-s_2),$$

which also implies that

$$e^{\frac{-q(x_1,z)}{t_1}} \geq e^{\ln(1-r_1)} \quad \text{and} \quad e^{\frac{-q(z,x_2)}{t_2}} \geq e^{\ln(1-s_2)},$$

which implies that

$$M(x_1,z,t_1) \geq (1-r_1) \quad \text{and} \quad M(z,x_2,t_2) \geq (1-s_2).$$

And this implies that

$$z \in C_M(x_1,r_1,t_1) \cap C_{M^{-1}}(x_2,s_2,t_2).$$

Hence, $(X,M,*)$ is a fuzzy quasi-metrically convex space. □

Proposition 5.2.3. *([30]) Let $(X,M,*)$ be a fuzzy quasi-metric space. If $(X,M,*)$ is fuzzy metrically convex, then $(X,M^{-1},*)$ is fuzzy metrically convex.*

Proof. Assume that $(X,M,*)$ is fuzzy metrically convex. Let $x_1,x_2 \in X$, $r_1,s_2 \in (0,1)$ and $t_1,t_2 \in (0,\infty)$ be such that $M^{-1}(x_1,x_2,t_1+t_2) \geq (1-r_1)*(1-s_2)$. Now $M^{-1}(x_1,x_2,t_1+t_2) = M(x_2,x_1,t_1+t_2) \geq (1-r_1)*(1-s_2)$, and since $(X,M,*)$ is fuzzy metrically convex, there exists $z \in X$ such that $M(x_2,z,t_2) \geq 1-s_2$ and $M^{-1}(z,x_1,t_1) \geq 1-r_1$. And this implies that $z \in C_M(x_2,s_2,t_2) \cap C_{M^{-1}}(x_1,r_1,t_1)$. Hence $(X,M^{-1},*)$ is fuzzy metrically convex. □

Definition 5.2.4. *([30]) Let $(X,M,*)$ be a fuzzy quasi-metric space. A collection of double balls $\{C_M(x_i,r_i,t_i); C_{M^{-1}}(x_i,s_i,t_i)\}_{i\in I}$, where $x_i \in X$, $r_i,s_i \in (0,1)$ and $t_i \in (0,\infty)$ whenever $i \in I$, is said to have a mixed binary intersection property if for all indices $i,j \in I$, $C_M(x_i,r_i,t_i) \cap C_{M^{-1}}(x_j,s_j,t_j) \neq \emptyset$.*

Definition 5.2.5. *([30]) A fuzzy quasi-metric space $(X,M,*)$ is said to be fuzzy-Isbell hypercomplete if for every collection $\{C_M(x_i,r_i,t_i); C_{M^{-1}}(x_i,s_i,t_i)\}_{i\in I}$ of double balls, where $x_i \in X$, $r_i,s_i \in (0,1)$ and $t_i \in (0,\infty)$ whenever $i \in I$, with the mixed binary intersection property satisfies*

$$\bigcap_{i\in I} C_M(x_i,r_i,t_i) \cap C_{M^{-1}}(x_i,s_i,t_i) \neq \emptyset.$$

Proposition 5.2.6. *([30]) Let $(X,M,*)$ be a fuzzy quasi-metric space. If $(X,M,*)$ is fuzzy Isbell hypercomplete, then $(X,M^{-1},*)$ is fuzzy Isbell hypercomplete and $(X,M^s,*)$ is fuzzy hypercomplete.*

Proof. Let $(X,M,*)$ be fuzzy-Isbell hypercomplete, then by definition for every collection $\{C_M(x_i,r_i,t_i); C_{M^{-1}}(x_i,s_i,t_i)\}_{i\in I}$ of double balls, where $x_i \in X$, $r_i,s_i \in (0,1)$ and $t_i \in (0,\infty)$ whenever $i \in I$, with the mixed binary intersection property,

$$\bigcap_{i\in I} C_M(x_i,r_i,t_i) \cap C_{M^{-1}}(x_i,s_i,t_i) \neq \emptyset.$$

Thus, for a collection $\{C_M(x_i, s_i, t_i); C_{M^{-1}}(x_i, r_i, t_i)\}_{i\in I}$ of double balls, we have by the fuzzy Isbell hypercompleteness of $(X, M, *)$ that

$$\bigcap_{i\in I} C_M(x_i, s_i, t_i) \cap C_{M^{-1}}(x_i, r_i, t_i) \neq \emptyset.$$

Hence, $(X, M^{-1}, *)$ is fuzzy Isbell hypercomplete.

Also, suppose that $(X, M, *)$ is fuzzy Isbell hypecomplete. And let the collection $\{C_{M^s}(x_j, r_j, t_j)\}_{j\in I}$ of closed balls have the binary intersection property. Then $\{C_M(x_j, r_j, t_j); C_{M^{-1}}(x_j, r_j, t_j)\}_{j\in I}$ has the mixed binary intersection property. Therefore,

$$\bigcap_{j\in I} C_M(x_j, r_j, t_j) \cap C_{M^{-1}}(x_j, r_j, t_j) = \bigcap_{j\in I} C_{M^s}(x_j, r_j, t_j) \neq \emptyset.$$

Therefore, $(X, M^s, *)$ is fuzzy hypercomplete. □

We now present the concept of fuzzy Isbell convexity.

Definition 5.2.7. *([30]) A fuzzy quasi-metric space $(X, M, *)$ is said to be fuzzy Isbell convex, if for any collection $\{x_i\}_{i\in I}$ of points in X, collections $\{r_i\}_{i\in I}$ and $\{s_i\}_{i\in I}$ of reals in $(0, 1)$ and collection of reals $\{t_i\}_{i\in I}$ in $(0, \infty)$ satisfying $M(x_i, x_j, t_i + t_j) \geq (1 - r_i) * (1 - s_j)$ whenever $i, j \in I$, we have that*

$$\bigcap_{i\in I} C_M(x_i, r_i, t_i) \cap C_{M^{-1}}(x_i, s_i, t_i) \neq \emptyset.$$

Proposition 5.2.8. *([30]) Let $(X, M, *)$ be a fuzzy quasi-metric space. If $(X, M, *)$ is fuzzy-Isbell convex, then $(X, M^{-1}, *)$ is fuzzy Isbell convex and $(X, M^s, *)$ is fuzzy hyperconvex.*

Proof. Let $(X, M, *)$ be fuzzy Isbell convex, then by definition any collection $\{x_i\}_{i\in I}$ of points in X, collections $\{r_i\}_{i\in I}$ and $\{s_i\}_{i\in I}$ of reals in $(0, 1)$ and collection of reals $\{t_i\}_{i\in I}$ in $(0, \infty)$ satisfying $M(x_i, x_j, t_i + t_j) \geq (1 - r_i) * (1 - s_j)$ whenever $i, j \in I$, we have that

$$\bigcap_{i\in I} C_M(x_i, r_i, t_i) \cap C_{M^{-1}}(x_i, s_i, t_i) \neq \emptyset.$$

Thus, for a collection $\{x_i\}_{i\in I}$ of points in X, collections $\{r_i\}_{i\in I}$ and $\{s_i\}_{i\in I}$ of reals in $(0, 1)$ and collection $\{t_i\}_{i\in I}$ of reals in $(0, \infty)$ such that

$$M^{-1}(x_i, x_j, t_i + t_j) = M(x_j, x_i, t_i + t_j) \geq (1 - r_i) * (1 - s_j),$$

we have by the fuzzy Isbell convexity of $(X, M, *)$ that

$$\bigcap_{i\in I} C_M(x_i, s_i, t_i) \cap C_{M^{-1}}(x_i, r_i, t_i) \neq \emptyset.$$

Hence $(X, M^{-1}, *)$ is fuzzy Isbell convex.

Also, suppose that $(X, M, *)$ is fuzzy Isbell convex. And let $\{x_i\}_{i\in I}$ be a collection of points in X, $\{r_i\}_{i\in I}$ and $\{s_i\}_{i\in I}$ be collections of reals in $(0,1)$ and $\{t_i\}_{i\in I}$ be a collection of reals in $(0,\infty)$ such that $M^s(x_i,x_j,t_i+t_j)\geq(1-r_i)*(1-s_j)$ whenever $i,j\in I$. By fuzzy Isbell convexity of $(X,M,*)$, we have that

$$\bigcap_{i\in I}C_M(x_i,r_i,t_i)\cap C_{M^{-1}}(x_i,r_i,t_i)=\bigcap_{i\in I}C_{M^s}(x_i,r_i,t_i)\neq\emptyset.$$

Hence, $(X,M^s,*)$ is fuzzy hyperconvex. □

Corollary 5.2.9. *([30]) Every fuzzy Isbell-convex quasi-metric space $(X,M,*)$ is bicomplete.*

Proof. By Proposition 5.2.8, $(X,M^s,*)$ is fuzzy hyperconvex. Since fuzzy hyperconvex spaces are complete by Theorem 4.4.7, $(X,M,*)$ is bicomplete. □

In the following result, we show the relationship among fuzzy Isbell convexity, fuzzy metric convexity and fuzzy Isbell hypercompleteness.

Lemma 5.2.10. *([30]) A fuzzy quasi-metric space $(X,M,*)$ is fuzzy Isbell convex if and only if it is fuzzy metrically convex and fuzzy Isbell hypercomplete.*

Proof. Suppose that $(X,M,*)$ is fuzzy Isbell convex. Let $x_1,x_2\in X$, $r_1,s_2\in(0,1)$ and $t_1,t_2\in(0,\infty)$ such that $M(x_1,x_2,t_1+t_2)\geq(1-r_1)*(1-s_2)$. By fuzzy Isbell convexity of $(X,M,*)$, we have that

$$C_M(x_1,r_1,t_1)\cap C_{M^{-1}}(x_2,s_2,t_2)\neq\emptyset.$$

Thus, there exists $z\in X$ such that

$$z\in C_M(x_1,r_1,t_1)\cap C_{M^{-1}}(x_2,s_2,t_2),$$

so that

$$M(x_1,z,t_1)\geq 1-r_1\quad\text{and}\quad M^{-1}(x_2,z,t_2)\geq 1-s_2.$$

Hence $(X,M,*)$ is fuzzy metrically convex.
Also let $\{C_M(x_i,r_i,t_i);C_{M^{-1}}(x_i,s_i,t_i)\}_{i\in I}$ have a mixed binary intersection property, that is

$$C_M(x_i,r_i,t_i)\cap C_{M^{-1}}(x_i,s_i,t_i)\neq\emptyset.$$

Then $M(x_i,x_j,t_i+t_j)\geq(1-r_i)*(1-s_j)$ whenever $i,j\in I$. And by fuzzy Isbell convexity of $(X,M,*)$, we have that

$$\bigcap_{i\in I}C_M(x_i,r_i,t_i)\cap C_{M^{-1}}(x_i,s_i,t_i)\neq\emptyset.$$

Hence $(X,M,*)$ is fuzzy Isbell hypercomplete.
Conversely, suppose $(X,M,*)$ is fuzzy metrically convex and fuzzy Isbell hypercomplete and let $\{x_i\}_{i\in I}$ is a collection of points in X, $\{r_i\}_{i\in I}$ and $\{s_i\}_{i\in I}$ are collection of reals in $(0,1)$ and $\{t_i\}_{i\in I}$ is a collection of reals in $(0,\infty)$ such that

$$M(x_i,x_j,t_i+t_j)\geq(1-r_i)*(1-s_j)$$

whenever $i, j \in I$. Then $\{C_M(x_i, r_i, t_i); C_{M^{-1}}(x_i, s_i, t_i)\}_{i \in I}$ has a mixed binary intersection property by fuzzy metric convexity of $(X, M, *)$. Therefore, by fuzzy Isbell completeness of $(X, M, *)$, we have that

$$\bigcap_{i \in I} C_M(x_i, r_i, t_i) \cap C_{M^{-1}}(x_i, s_i, t_i) \neq \emptyset.$$

Hence $(X, M, *)$ is fuzzy Isbell convex. □

Theorem 5.2.11. *([30]) Let $(\mathbb{R}, q)$ be a quasi-metric space with the usual quasi-metric $q(x, y) = \max\{x - y, 0\}$ for any $x, y \in \mathbb{R}$. Let $(\mathbb{R}, M, *)$ be a fuzzy quasi-metric space, where $*$ is a continuous t-norm defined by $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$ and M is a fuzzy set in $\mathbb{R} \times \mathbb{R} \times (0, \infty)$ defined by*

$$M(x, y, t) = \frac{t}{t + q(x, y)}$$

whenever $x, y \in \mathbb{R}$ and $t \in (0, \infty)$. Then $(\mathbb{R}, M, *)$ is fuzzy Isbell convex.

Proof. It is known that $\mathbb{R}$ equipped with the quasi-metric $q(x, y) = \max\{x - y, 0\}$ is Isbell convex. Thus for any collection $\{x_i\}_{i \in I}$ of points in X and collections of non-negative real numbers $\{r_i\}_{i \in I}$ and $\{s_i\}_{i \in I}$ satisfying $q(x_i, x_j) \leq r_i + s_j$ whenever $i, j \in I$, we have that

$$\bigcap_{i \in I} C_q(x_i, r_i) \cap C_{q^{-1}}(x_i, s_i) \neq \emptyset.$$

Let $R_i = \frac{r_i}{t_i + r_i}$ and $S_i = \frac{s_i}{t_i + s_i}$, where $\{t_i\}_{i \in I}$ is a collection of reals in $(0, \infty)$. Then $\{R_i\}_{i \in I}$ and $\{S_i\}_{i \in I}$ are collections of points in $(0, 1)$. Also, using the metric convexity of (X, q), a calculation shows that

$$\begin{aligned}
M(x_i, x_j, t_i + t_j) &\geq M(x_i, z, t_i) * M(z, x_j, t_j) \\
&= \frac{t_i}{t_i + q(x_i, z)} * \frac{t_j}{t_j + q(z, x_j)} \\
&\geq \frac{t_i}{t_i + r_i} * \frac{t_j}{t_j + s_j} \\
&= (1 - R_i) * (1 - S_j)
\end{aligned}$$

whenever $i, j \in I$ is satisfied, where $z \in X$. Also, by the Isbell convexity of (X, q), we have that

$$\bigcap_{i \in I} C_q(x_i, r_i) \cap C_{q^{-1}}(x_i, s_i) \neq \emptyset.$$

Thus, there exists

$$z \in \bigcap_{i \in I} C_q(x_i, r_i) \cap C_{q^{-1}}(x_i, s_i)$$

such that $q(x_i, z) \leq r_i$ and $q(z, x_i) \leq s_i$ for all $i \in I$. And this implies that

$$t_i + q(x_i, z) \leq t_i + r_i \quad \text{and} \quad t_i + q(z, x_i) \leq t_i + s_i,$$

which implies that

$$\frac{t_i}{t_i + q(x_i, z)} \geq \frac{t_i}{t_i + r_i} \quad \text{and} \quad \frac{t_i}{t_i + q(z, x_i)} \geq \frac{t_i}{t_i + s_i},$$

which also implies that

$$M(x_i, z, t_i) \geq 1 - R_i \quad \text{and} \quad M(z, x_i, t_i) \geq 1 - S_i.$$

Thus, $z \in C_M(x_i, R_i, t_i) \cap C_{M^{-1}}(x_i, S_i, t_i)$ for all $i \in I$ and so

$$\bigcap_{i \in I} C_M(x_i, R_i, t_i) \cap C_{M^{-1}}(x_i, S_i, t_i) \neq \emptyset.$$

Therefore, $(X, M, *)$ is fuzzy Isbell convex. □

Example 5.2.2. ([30]) Let $(\mathbb{R}, q)$ be a quasi-metric space, where $q(x, y) = x \dot{-} y = \max\{x - y, 0\}$ for all $x, y \in \mathbb{R}$. Then by Example 3.2.2, $(\mathbb{R}, q^s)$ is not Isbell convex, where $q^s(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. Now let $(\mathbb{R}, M, *_P)$ be a fuzzy quasi-metric space, where $*_P$ is a continuous t-norm defined by $a *_P b = a \cdot b$ for all $a, b \in [0, 1]$ and M is a fuzzy set in $\mathbb{R} \times \mathbb{R} \times (0, \infty)$ defined by

$$M(x, y, t) = \frac{t}{t + q(x, y)},$$

whenever $x, y \in \mathbb{R}$ and $t \in (0, \infty)$. Then $(\mathbb{R}, M^s, *_P)$, where

$$M^s(x, y, t) = \frac{t}{t + q^s(x, y)}$$

is a fuzzy set defined on $\mathbb{R} \times \mathbb{R} \times (0, \infty)$ is a fuzzy metric space and $(\mathbb{R}, M^s, *_P)$ is not Isbell convex.

***Proof*.** To show that $(\mathbb{R}, M^s, *_P)$ is not fuzzy Isbell convex, there must exist a sub-collection $\{x_j\}_{j \in I}$ of points in $\mathbb{R}$, sub-collections $\{r_j\}_{j \in I}$ and $\{s_j\}_{j \in I}$ of reals in $(0, 1)$ and sub-collection $\{t_j\}_{j \in I}$ of reals in $(0, \infty)$ such that

$$M^s(x_j, x_k, t_j + t_k) \geq (1 - r_j) *_P (1 - s_k) \quad \text{whenever} \quad i, j \in I,$$

but

$$\bigcap_{j \in I} C_{M^s}(x_j, r_j, t_j) \cap C_{M^i}(x_j, s_j, t_j) = \emptyset.$$

Now for any $x_j \in [0, 1]$, set $r_j = \frac{1}{4}$, $s_j = \frac{3}{4}$ and $t_j = 1$, we define R_i and S_i by $R_i = \frac{\frac{1}{4}}{1+\frac{1}{4}} = \frac{1}{5}$ and $S_i = \frac{\frac{3}{4}}{1+\frac{3}{4}} = \frac{3}{7}$. Thus, for any $x_j, x_k \in [0, 1]$, we have that

$$M^s(x_j, x_k, t_j + t_k) = \frac{2}{2 + q^s(x_j, x_k)} \geq \frac{2}{2 + 1} = \frac{2}{3} > (1 - \tfrac{1}{5}) \cdot (1 - \tfrac{3}{7}) = \frac{4}{5} \cdot \frac{4}{7} = \frac{16}{35}.$$

But

$$\bigcap_{j\in I} C_{M^s}(x_j, R_j, t_j) \cap C_{M^s}(x_j, S_j, t_j) \subseteq C_{M^s}(0, \tfrac{1}{5}, 1) \cap C_{M^s}(1, \tfrac{1}{5}, 1)$$
$$= \left[\frac{-1}{4}, \frac{1}{4}\right] \cap \left[\frac{3}{4}, \frac{5}{4}\right] = \emptyset.$$

Therefore, $(\mathbb{R}, M^s, *_P)$ is not fuzzy Isbell convex. □

Remark 5.2.1. ([30]) Note that $(\mathbb{R}, M^s, *_P)$, where $*_P$ is a continuous t-norm defined by $a *_P b = a \cdot b$ for all $a, b \in [0, 1]$ and M be a fuzzy set in $\mathbb{R} \times \mathbb{R} \times (0, \infty)$ defined by $M^s(x, y, t) = \frac{t}{t+|x-y|}$ whenever $x, y \in \mathbb{R}$ and $t \in (0, \infty)$, is fuzzy hyperconvex by Theorem 4.4.5. Therefore, $(\mathbb{R}, M, *_P)$ is an example of a space for which $(\mathbb{R}, M^s, *_P)$ is fuzzy hyperconvex but not fuzzy Isbell convex.

5.3. Fuzzy admissible subsets

In this section, we introduce the concept of fuzzy admissible subsets and show that every fuzzy admissible subset of a fuzzy Isbell convex quasi-metric space is fuzzy Isbell convex.

Definition 5.3.1. *([30]) A subset A of a fuzzy quasi-metric space $(X, M, *)$ is said to be F-bounded if for every $x, y \in A$ and $t \in (0, \infty)$ there exists a real number $r \in (0, 1)$ such that $M(x, y, t) > 1 - r$.*

Definition 5.3.2. *([30]) Let A be an F-bounded subset of a fuzzy quasi-metric space $(X, M, *)$. Then*

$$cov(A)_M = \bigcap\{C_M(x, r, t) : A \subseteq C_M(x, r, t), x \in X, r \in (0, 1) \quad and \quad t \in (0, \infty)\} \quad and$$
$$cov(A)_{M^{-1}} = \bigcap\{C_{M^{-1}}(x, s, t) : A \subseteq C_{M^{-1}}(x, s, t), x \in X, s \in (0, 1) \quad and \quad t \in (0, \infty)\}.$$

Also, we define the bicover of A by $bicov(A) = cov(A)_M \cap cov(A)_{M^{-1}}$.

Definition 5.3.3. *([30]) An F-bounded subset B of a fuzzy quasi-metric space $(X, M, *)$ is said to be fuzzy admissible if $B = bicov(B)$.*

By $\mathcal{A}_M(X)$, we will denote the set of fuzzy admissible subsets of $(X, M, *)$.

Remark 5.3.1. ([30]) A subset of a fuzzy quasi-metric space $(X, M, *)$ is fuzzy admissible if and only if it can be written as the intersection of a family of sets of the form

$$C_M(x, r, t) \cap C_{M^{-1}}(x, s, t),$$

where $r, s \in (0, 1)$, $t \in (0, \infty)$ and $x \in X$. For this reason, the family $\mathcal{A}_M$ is closed under nonempty intersections.

The following proposition is a generation of Proposition 3.3.4 in quasi-metric spaces.

Proposition 5.3.4. *([30]) Let $(X, M, *)$ be a fuzzy convex quasi-metric space. Then a fuzzy admissible subset $D \in \mathcal{A}_M(X)$ is fuzzy Isbell convex.*

Proof. Since $D \in \mathcal{A}_M(X)$, then $D = \text{bicov}(D)$, which implies that there exists a collection $\{x_i\}_{i\in I}$ of points in X, collections $\{r_i\}_{i\in I}$ and $\{s_i\}_{i\in I}$ of reals in $(0,1)$ and collection $\{t_i\}_{i\in I}$ of reals in $(0,\infty)$ such that

$$\bigcap_{i\in I} C_M(x_i, r_i, t_i) \cap C_{M^{-1}}(x_, s_i, t_i) \neq \emptyset.$$

Now let $\{C_M(x_\alpha, r_\alpha, t_\alpha); C_{M^{-1}}(x_\alpha, s_\alpha, t_\alpha)\}_{\alpha\in\Gamma}$ be a collection of double balls, where $x_\alpha \in D$, $r_\alpha, s_\alpha \in (0,1)$ and $t_\alpha \in (0,\infty)$ whenever $\alpha \in \Gamma$ such that

$$M(x_\alpha, x_\beta, t_\alpha + t_\beta) \geq (1 - r_\alpha) * (1 - s_\beta)$$

whenever $\alpha, \beta \in \Gamma$. Then since $(X, M, *)$ is fuzzy Isbell convex, we have that

$$\bigcap_{\alpha\in\Gamma} C_M(x_\alpha, r_\alpha, t_\alpha) \cap C_{M^{-1}}(x_\alpha, s_\alpha, t_\alpha) \neq \emptyset.$$

Now consider the collection
$[\{C_M(x_\alpha, r_\alpha, t_\alpha)\}_{\alpha\in\Gamma}, \{C_{M^{-1}}(x_\alpha, s_\alpha, t_\alpha)\}_{\alpha\in\Gamma}, \{C_M(x_i, r_i, t_i)\}_{i\in I}, \{C_{M^{-1}}(x_i, s_i, t_i)\}_{i\in I}]$ of closed balls. We have for each $\alpha \in \Gamma$ and $i \in I$,

$$\begin{aligned} M(x_\alpha, x_i, t_\alpha + t_i) &\geq M(x_\alpha, z, t_\alpha) * M(z, x_i, t_i) \\ &\geq (1 - r_\alpha) * (1 - s_i) \end{aligned}$$

and

$$\begin{aligned} M(x_i, x_\alpha, t_\alpha + t_i) &\geq M(x_i, z, t_\alpha) * M(z, x_\alpha, t_i) \\ &\geq (1 - r_i) * (1 - s_\alpha) \end{aligned}$$

for some $z \in D$. Furthermore, for all $i, j \in I$, $t_i, t_j \in (0,\infty)$, we have that

$$\begin{aligned} M(x_i, x_j, t_\alpha + t_i) &\geq M(x_i, x_\alpha, t_\alpha) * M(x_\alpha, x_j t_j) \\ &\geq (1 - r_i) * (1 - s_j). \end{aligned}$$

It follows from fuzzy Isbell convexity of $(X, M, *)$ that

$$\begin{aligned} &\left(\bigcap_{\alpha\in\Gamma} C_M(x_\alpha, r_\alpha, t_\alpha) \cap C_{M^{-1}}(x_\alpha, s_\alpha, t_\alpha)\right) \cap \left(\bigcap_{i\in I} C_M(x_i, r_i, t_i) \cap C_{M^{-1}}(x_, s_i, t_i)\right) \\ = &\left(\bigcap_{\alpha\in\Gamma} C_M(x_\alpha, r_\alpha, t_\alpha) \cap C_{M^{-1}}(x_\alpha, s_\alpha, t_\alpha)\right) \cap D \neq \emptyset. \end{aligned}$$

Thus, the fuzzy admissible subspace D of $(X, M, *)$ is fuzzy Isbell convex. □

Definition 5.3.5. *([30]) Let $(X, M, *)$ be a fuzzy quasi-metric space. For a fuzzy quasi-metric subspace A of X, we define for $\epsilon_1, \epsilon_2 \in (0,1)$ and $t \in (0,\infty)$ the ϵ_1, ϵ_2-parallel set of A as*

$$N_{\epsilon_1,\epsilon_2}(A) = \bigcup_{a\in A} C_M(a, \epsilon_2, t) \cap C_{M^{-1}}(a, \epsilon_1, t).$$

The following is a generalization of Lemma 3.3.11.

Lemma 5.3.6. *([30]) Let $(X, M, *)$ be a fuzzy Isbell-convex fuzzy quasi-metric space, where $* \geq *_L$. Let A be a fuzzy admissible subset of $(X, M, *)$, that is there exists a collection $\{x_i\}_{i\in I}$ of points in X, collections $\{r_i\}_{i\in I}$ and $\{s_i\}_{i\in I}$ of reals in $(0,1)$ and collection $\{t_i\}_{i\in I}$ of reals in $(0,\infty)$ such that*

$$\bigcap_{i\in I} C_M(x_i, r_i, t_i) \cap C_{M^{-1}}(x_i, s_i, t_i) \neq \emptyset$$

whenever $i \in I$. Then for each $\epsilon_1, \epsilon_2 \in (0,1)$ and $t \in (0,\infty)$, we have that

$$N_{\epsilon_1,\epsilon_2}(A) = \bigcap_{i\in I} C_M(x_i, r_i + \epsilon_2, t + t_i) \cap C_{M^{-1}}(x_i, s_i + \epsilon_1, t + t_i).$$

Proof. Suppose $y \in N_{\epsilon_1,\epsilon_2}(A) = \bigcup_{a\in A} C_M(a, \epsilon_2, t) \cap C_{M^{-1}}(a, \epsilon_1, t)$, where $\epsilon_1, \epsilon_2 \in (0,1)$, $t \in (0,\infty)$ for some $a \in A$. Then $y \in C_M(a, \epsilon_2, t)$ and $y \in C_{M-1}(a, \epsilon_1, t)$, which implies that

$$M(a, y, t) \geq (1 - \epsilon_2) \quad \text{and} \quad M^{-1}(a, y, t) = M(y, a, t) \geq (1 - \epsilon_1).$$

By the properties of a fuzzy quasi-metric

$$\begin{aligned} M(x_i, y, t + t_i) &\geq M(x_i, a, t_i) * M(a, y, t) \\ &\geq (1 - r_i) * (1 - \epsilon_2) \\ &\geq (1 - r_i) *_L (1 - \epsilon_2) \\ &= (1 - (r_i + \epsilon_2)) \end{aligned}$$

and

$$\begin{aligned} M(y, x_i, t + t_i) &\geq M(y, a, t) * M(a, x_i, t_i) \\ &\geq (1 - s_i) * (1 - \epsilon_1) \\ &\geq (1 - s_i) *_L (1 - \epsilon_1) \\ &= (1 - (s_i + \epsilon_1)). \end{aligned}$$

Thus, for each $i \in I$, we have that $y \in C_M(x_i, r_i + \epsilon_2, t + t_i)$ and $y \in C_{M^{-1}}(x_i, s_i + \epsilon_1, t + t_i)$, which implies that $y \in \bigcap_{i\in I} C_M(x_i, r_i + \epsilon_2, t + t_i) \cap C_{M^{-1}}(x_i, s_i + \epsilon_1, t + t_i)$.

Hence,

$$N_{\epsilon_1,\epsilon_2}(A) \subseteq \bigcap_{i\in I} C_M(x_i, r_i + \epsilon_2, t + t_i) \cap C_{M^{-1}}(x_i, s_i + \epsilon_1, t + t_i).$$

Conversely, suppose

$$y \in \bigcap_{i\in I} C_M(x_i, r_i + \epsilon_2, t + t_i) \cap C_{M^{-1}}(x_i, s_i + \epsilon_1, t + t_i).$$

Then, for each $i \in I$, we have that $y \in C_M(x_i, r_i + \epsilon_2, t + t_i)$ and $y \in C_{M^{-1}}(x_i, s_i + \epsilon_1, t + t_i)$ which implies that

$$M(x_i, y, t + t_i) \geq (1 - (r_i + \epsilon_2)) \quad \text{and} \quad M(y, x_i, t + t_i) \geq (1 - (s_i + \epsilon_1)).$$

Since A is nonempty and by the definition of A, we must have for any $i, j \in I$, $M(x_i, x_j, t_i + t_j) \geq M(x_i, a, t_i) * M(a, x_j, t_j) \geq (1 - r_i) * (1 - s_j) \geq (1 - (r_i + s_j))$. So by fuzzy Isbell convexity of X, we have that

$$\begin{aligned}
\emptyset &\neq \left(\bigcap_{i \in I} C_M(x_i, r_i, t_i) \cap C_M(y, \epsilon_1, t)\right) \cap (C_{M^{-1}}(x_i, s_i, t_i) \cap C_{M^{-1}}(y, \epsilon_2, t)) \\
&= \left(\bigcap_{i \in I} C_M(x_i, r_i, t_i) \cap C_{M^{-1}}(x_i, s_i, t_i)\right) \cap (C_M(y, \epsilon_1, t) \cap C_{M^{-1}}(y, \epsilon_2, t)) \\
&= A \cap C_M(y, \epsilon_1, t) \cap C_{M^{-1}}(y, \epsilon_2, t).
\end{aligned}$$

Therefore, $a \in A$ such that $M(y, a, t) \geq (1 - \epsilon_1)$ and $M(a, y, t) \geq (1 - \epsilon_2)$.
Hence $y \in N_{\epsilon_1, \epsilon_2}(A)$, which implies that

$$\bigcap_{i \in I} C_M(x_i, r_i + \epsilon_2, t + t_i) \cap C_{M^{-1}}(x_i, s_i + \epsilon_1, t + t_i) \subseteq N_{\epsilon_1, \epsilon_2}(A).$$

□

CHAPTER 6

SOME FIXED POINT THEOREMS IN ISBELL CONVEX FUZZY QUASI-METRIC SPACES

In this chapter, we present some fixed point theorems in Isbell convex fuzzy quasi-metric spaces. To achieve this, we introduce the concept of a compatible quasi-metric. This concept generalises compatible metrics, introduced by Radu in [29], to the asymmetric setting. We then use the concept of compatible quasi-metric to generalise some fixed point theorems from Isbell convex quasi-metric spaces to the framework of Isbell convex fuzzy quasi-metric spaces.

6.1. The compatible quasi-metric

In this section, we introduce the concept of a compatible quasi-metric whose symmetrised metric corresponds to that of Theorem 4.3.1.

Theorem 6.1.1. *Let $(X, M, *)$ be a fuzzy quasi-metric space such that $* \geq *_L$, where $a *_L b = \max\{a+b-1\}$ and for any $x, y \in X$, let*

$$d_t(x,y) = \sup\{t \geq 0 : M(x,y,t) \leq 1-t\}.$$

Then d_t is a quasi-metric on X such that

$$d_t(x,y) < \epsilon \iff M(x,y,\epsilon) > 1-\epsilon$$

for all $\epsilon \in (0,1)$.

Proof. The proof follows by removing the symmetry axiom in the proof of Theorem 4.3.1. □

Remark 6.1.1. Let $(M, *)$ be a fuzzy quasi-metric on X and let d_t be a compatible quasi-metric of $(M, *)$. The conjugate d_t^{-1} of d_t is a mapping $d_t^{-1} : X \times X \to [0,1]$ defined by

$$d_t^{-1}(x,y) = d_t(y,x) = \sup\{t \geq 0 : M(y,x,t) \leq 1-t\}$$

for all $x, y \in X$ and $t \in [0,1]$. Thus, d_t^{-1} is a quasi-metric on X. Furthermore, the mapping $d_t^s = \max\{d_t, d_t^{-1}\}$ is a metric on X which corresponds to the metric in Theorem 4.3.1.

The following result is a consequence of Theorem 6.1.1.

Proposition 6.1.2. *Let $(X, M, *)$ be a fuzzy quasi-metric space, $x \in X$, $\epsilon \in (0,1)$. Then*

(i) $B_{d_t}(x, \epsilon) = B_M(x, \epsilon, \epsilon)$.

(ii) $C_{d_t}(x, \epsilon) = C_M(x, \epsilon, \epsilon)$.

Proof. (i)

$$\begin{aligned} B_{d_t}(x, \epsilon) &= \{y \in X : d_t(x, y) < \epsilon\} \\ &= \{y \in X : M(x, y, \epsilon) > 1 - \epsilon\} \\ &= B_M(x, \epsilon, \epsilon) \end{aligned}$$

(ii)

$$\begin{aligned} C_{d_t}(x, \epsilon) &= \{y \in X : d_t(x, y) \leq \epsilon\} \\ &= \{y \in X : M(x, y, \epsilon) \geq 1 - \epsilon\} \\ &= C_M(x, \epsilon, \epsilon) \end{aligned}$$

□

6.2. Some fixed point theorems

In this section, we present some fixed point theorems in Isbell convex fuzzy quasi-metric spaces and we begin with the following lemma.

Lemma 6.2.1. *Suppose $(X, M, *)$ is a fuzzy Isbell convex fuzzy quasi-metric space, where $* \geq *_L$. Then (X, d_t) is Isbell convex.*

Proof. Suppose $(X, M, *)$ is fuzzy Isbell convex and let $\{x_i\}_{i \in I}$ be a collection of points in X and let $\{r_i\}_{i \in I}$ and $\{s_i\}_{i \in I}$ be collections of reals in $(0,1)$ satisfying $d_t(x_i, x_j) \leq r_i + s_j$ whenever $i, j \in I$. Then the collection of points $\{x_i\}_{i \in I}$ in X and the collections $\{r_i\}_{i \in I}$ and $\{s_i\}_{i \in I}$ of reals in $(0,1)$ satisfy $M(x_i, x_j, r_i + s_j) \geq (1 - r_i) * (1 - s_j)$ whenever $i, j \in I$ and by the fuzzy Isbell convexity of $(X, M, *)$ and Proposition 6.1.2, we have that

$$\bigcap_{i \in I} C_M(x_i, r_i, r_i) \cap C_{M^{-1}}(x_i, s_i, s_i) = \bigcap_{i \in I} C_{d_t}(x_i, r_i) \cap C_{d_{t^{-1}}}(x_i, s_i) \neq \emptyset.$$

□

Remark 6.2.1. If $(X, M, *)$ is F-bounded, then for any $x, y \in X$, there exists $r \in (0,1)$ such that $M(x, y, r) \geq 1 - r$ which implies that $d_t(x, y) \leq r$. Thus, (X, d_t) is bounded.

Definition 6.2.2. *Let $(X, M, *)$ be a fuzzy quasi-metric space. We say that a function $T : (X, M, *) \to (X, M, *)$ is t-nonexpansive if*

$$M(T(x), T(y), t) \geq M(x, y, t)$$

for all $x, y \in X$ and $t > 0$.

We show a fixed point theorem in a fuzzy quasi-metric space.

Proposition 6.2.3. *Let $(X, M, *)$ be a fuzzy quasi-metric space such that $* \geq *_L$. If $T : (X, M, *) \to (X, M, *)$ is t-nonexpansive then $T : (X, d_t) \to (X, d_t)$ is nonexpansive.*

Proof. Suppose that $T : (X, M, *) \to (X, M, *)$ is t-nonexpansive then for all $t > 0$,

$$M(T(x), T(y), t) \geq M(x, y, t)$$

And this implies that

$$\begin{aligned} d_t(T(x), T(y)) &= \sup\{t \geq 0 : M(T(x), T(y), t) \leq 1 - t)\} \\ &\leq \sup\{t \geq 0 : M(x, y, t) \leq 1 - t)\} \\ &= d_t(x, y) \end{aligned}$$

□

Example 6.2.1. Let (X, q) be a quasi-metric space and $T : (X, q) \to (X, q)$ be a nonexpansive map. Let $(M, *)$ be a fuzzy quasi-metric on X such that $a * b = \min\{a, b\}$ and

$$M(x, y, t) = \frac{t}{t + q(x, y)}.$$

Then $T : (X, M, *) \to (X, M, *)$ is t-nonexpansive.

Proof. $T : (X, q) \to (X, q)$ is nonexpansive so for any $x, y \in X$, we have that $q(Tx, Ty) \leq q(x, y)$, which implies that $t + q(Tx, Ty) \leq t + q(x, y)$, which also implies that $\frac{t}{t+q(Tx,Ty)} \geq \frac{t}{t+q(x,y)}$. Thus, $M(T(x), T(y), t) \geq M(x, y, t)$ and so $T : (X, M, *) \to (X, M, *)$ is t-nonexpansive. □

Remark 6.2.2. The fixed point set $Fix(T)$ in a fuzzy quasi-metric space $(X, M, *)$ is defined the same way as the fixed point set in quasi-metric space (X, q). That is,

$$Fix(T) = \{x \in X : Tx = x\}.$$

Theorem 6.2.4. *(Compare, Theorem 3.3.5). Suppose $(X, M, *)$ is an F-bounded fuzzy Isbell convex fuzzy quasi-metric space, where $* \geq *_L$ and let $T : (X, M, *) \to (X, M, *)$ be a t-nonexpansive map. Then the fixed point set $Fix(T)$ of T in $(X, M, *)$ is nonempty and fuzzy Isbell convex.*

Proof. We first show that $Fix(T) \neq \emptyset$. Since $(X, M, *)$ is fuzzy Isbell convex, (X, d_t) is Isbell convex by Lemma 6.2.1. Also since $(X, M, *)$ is F-bounded, (X, d_t) is a bounded quasi-metric space by Proposition 6.2.1. Furthermore, $T : (X, d_t) \to (X, d_t)$ is a nonexpansive map by Remark 6.2.3. Now by Theorem 3.3.5 we have that, $Fix(T) \neq \emptyset$ in (X, d_t), which implies that $Fix(T) \neq \emptyset$ in $(X, M, *)$.

We now show that $Fix(T)$ is fuzzy Isbell convex. Let

$$[C_M(x_i, r_i, t_i), C_{M^{-1}}(x_i, s_i, t_i)]_{i \in I}$$

be a nonempty collection of double balls, where $x_i \in Fix(T)$, $\{r_i\}_{i\in I}$ and $\{s_i\}_{i\in I}$ are collections of reals in $(0,1)$ and $(t_i)_{i\in I}$ is a collection of reals in $(0,\infty)$ such that

$$M(x_i, x_j, t_i + t_j) \geq (1 - r_i) * (1 - s_j)$$

whenever $i, j \in I$. Since $(X, M, *)$ is fuzzy Isbell convex,

$$X_0 = \bigcap_{i\in I} C_M(x_i, r_i, t_i) \cap C_{M^{-1}}(x_i, s_i, t_i) \neq \emptyset.$$

Let $x \in X_0$. Then $M(T(x), x_i, t) = M(T(x), T(x_i), t) \geq M(x, x_i, t) \geq (1 - s_i)$ and

$$M(x_i, T(x), t) = M(T(x_i), T(x), t) \geq M(x_i, x, t) \geq (1 - r_i)$$

for all $i \in I$ and $t > 0$. Thus $T(x) \in X_0$ and $T : (X_0, M, *) \to (X_0, M, *)$ is t-nonexpansive. Furthermore, X_0 is fuzzy Isbell convex since it is fuzzy admissible. So the first part of the proof implies that T has a fixed point in X_0, which implies that

$$Fix(T) \cap \left[\bigcap_{i\in I} C_M(x_i, r_i, t_i) \cap C_{M^{-1}}(x_i, s_i, t_i)\right] \neq \emptyset.$$

Thus, $Fix(T)$ is fuzzy Isbell convex. □

We now give the following definition.

Definition 6.2.5. *(Compare, Definition 3.3.12). Let Y be a subset of a fuzzy quasi-metric space $(X, M, *)$. A map $T : X \to Y$ is said to be a t-nonexpansive retraction if*

(i) For each $x \in Y$, $T(x) = x$ that is, T is the identity function on its image.

(ii) For any $x, y \in X$ and $t > 0$, $M(T(x), T(y), t) \geq M(x, y, t)$, that is, T is t-nonexpansive.

Lemma 6.2.6. *(compare, Lemma 3.3.13). Let A be a nonempty fuzzy admissible subset of a fuzzy Isbell convex fuzzy quasi-metric space $(X, M, *)$. Then, for each $\epsilon_1, \epsilon_2 \geq 0$ there exists a t-nonexpansive retraction R of $N_{\epsilon_1,\epsilon_2}(A)$ onto A which has the property that $M(x, R(x), \epsilon_1) \geq 1 - \epsilon_1$ and $M(R(x), x, \epsilon_2) \geq 1 - \epsilon_2$ for each $x \in N_{\epsilon_1,\epsilon_2}(A)$.*

Proof. Since $(X, M, *)$ is fuzzy Isbell convex and A is nonempty and fuzzy admissible, then (X, d_t) is Isbell convex by Lemma 6.2.1. And since A is fuzzy admissible, it can be written as

$$A = \bigcap_{i\in I} C_M(x_i, r_i, r_i) \cap C_{M^{-1}}(x_i, s_i, s_i) = \bigcap_{i\in I} C_{d_t}(x_i, r_i) \cap C_{d_t^{-1}}(x_i, s_i) \neq \emptyset,$$

where $r_i, s_i \in (0,1)$, thus A is q-admissible. By Lemma 3.3.13 there exists a non-expansive retraction R of $N_{\epsilon_1,\epsilon_2}(A)$ onto A which has the property $d_R(x, R(x)) \leq \epsilon_1$ and $d_R(R(x), x) \leq \epsilon_2$ for each $x \in N_{\epsilon_1,\epsilon_2}(A)$. Hence by the definition of d_R, we have that

$$M(x, R(x), \epsilon_1) \geq 1 - \epsilon_1 \quad \text{and} \quad M(R(x), x, \epsilon_2) \geq 1 - \epsilon_2.$$

□

Definition 6.2.7. *(compare, Definition 3.3.15). Let $(X, M, *)$ be a fuzzy quasi-metric space. For a map $T : (X, M, *) \to (X, M, *)$ and for any $\epsilon_1, \epsilon_2 \in (0, 1)$, we use $F_{\epsilon_1,\epsilon_2}(T)$ to denote the set ϵ_1, ϵ_2-fixed points of T, that is,*

$$F_{\epsilon_1,\epsilon_2}(T) = \{x \in X : M(x, T(x), \epsilon_2) \geq 1 - \epsilon_2 \quad \text{and} \quad M(T(x), x, \epsilon_1) \geq 1 - \epsilon_1\}.$$

Theorem 6.2.8. *(compare, Theorem 3.3.16). Suppose that $(X, M, *)$ is a fuzzy Isbell convex fuzzy quasi-metric space such that $* \geq *_L$, where $a *_L b = \max\{a + b - 1\}$ and suppose that the map $T : (X, M, *) \to (X, M, *)$ is t-nonexpansive. Furthermore, suppose that for some $\epsilon_1, \epsilon_2 \in (0, 1)$, $F_{\epsilon_1,\epsilon_2}(T) \neq \emptyset$. Then, the set $F_{\epsilon_1,\epsilon_2}(T)$ is fuzzy Isbell convex.*

Proof. Suppose that $F_{\epsilon_1,\epsilon_2}(T) \neq \emptyset$. For each $i \in I$, let $x_i \in F_{\epsilon_1,\epsilon_2}(T)$, and let $r_i, s_i \in (0, 1)$ be such that $M(x_i, x_j, r_i + s_i) \geq (1 - r_i) * (1 - s_i)$. We need to show that

$$\left[\bigcap_{i\in I}(C_M(x_i, r_i, r_i) \cap C_{M^{-1}}(x_i, s_i, s_i))\right] \cap F_{\epsilon_1,\epsilon_2}(T) \neq \emptyset.$$

Since $(X, M, *)$ is fuzzy Isbell convex, by Proposition 5.3.4,
$\emptyset \neq J = (C_M(x_i, r_i, r_i) \cap C_{M^{-1}}(x_i, s_i, s_i))$ is fuzzy Isbell convex. Furthermore, J is F-bounded in $(X, M, *)$ by definition of a fuzzy admissible subset. Also, if $x \in J$, then for each $i \in I$,

$$\begin{aligned} M(x_i, T(x), r_i + \epsilon_2) &\geq M(x_i, T(x_i), r_i) * M(T(x_i), T(x), \epsilon_2) \\ &\geq (1 - r_i) * (1 - \epsilon_2) \\ &\geq (1 - r_i) *_L (1 - \epsilon_2) \\ &= 1 - (r_i + \epsilon_2) \end{aligned}$$

and

$$\begin{aligned} M(T(x), x_i, \epsilon_1 + s_i) &\geq M(T(x), T(x_i), \epsilon_1) * M(T(x_i), x_i, s_i) \\ &\geq (1 - \epsilon_1) * (1 - s_i) \\ &\geq (1 - \epsilon_1) *_L (1 - s_i) \\ &= 1 - (\epsilon_1 + s_i). \end{aligned}$$

This implies that $T(x) \in C_M(x_i, r_i + \epsilon_2, r_i + \epsilon_2)$ and $T(x) \in C_M(x_i, s_i + \epsilon_1, s_i + \epsilon_1)$, which implies that $T(x) \in N_{\epsilon_1,\epsilon_2}(J)$ by Lemma 3.3.11. Now, by Lemma 6.2.6 there is a nonexpansive retraction R of $N_{\epsilon_1,\epsilon_2}(J)$ onto J for which $M(R(x), x, \epsilon_2) \geq (1 - \epsilon_2)$ and $M(x, R(x), \epsilon_1) \geq (1 - \epsilon_1)$ whenever $x \in N_{\epsilon_1,\epsilon_2}(J)$. Also since $R \circ T$ is a nonexpansive map of J into J, it must have a fixed point by Theorem 6.2.4. Suppose that $R \circ T(x_0) = x_0$ for some $x_0 \in J$. Then, $M(x_0, T(x_0), \epsilon_2) = M(R \circ T(x), T(x_0), \epsilon_2) \geq 1 - \epsilon_2$ and $M(T(x_0), x_0, \epsilon_1) = M(T(x_0), R \circ T(x)) \geq 1 - \epsilon_1$. Thus, the proof is complete, since $x_0 \in J \cap F_{\epsilon_1,\epsilon_2}(T)$.

□

CHAPTER 7

DISCUSSION

We now present a discussion of the results of this research. In chapter 5, we successfully generalised the concept of Isbell convexity in quasi-metric spaces to fuzzy quasi-metric spaces, where we successfully introduced fuzzy Isbell convexity in Definition 5.2.7 and fuzzy Isbell hypercompleteness in Definition 5.2.5 and also proved in Lemma 5.2.10 that a fuzzy quasi-metric space $(X, M, *)$ is fuzzy Isbell convex if and only if it is fuzzy metrically convex and fuzzy Isbell hypercomplete. We also proved in Theorem 5.2.11 that if $(\mathbb{R}, q)$ is a quasi-metric space with the usual metric $q(x, y) = \max\{x - y, 0\}$ for any $x, y \in \mathbb{R}$, then the fuzzy quasi-metric space $(\mathbb{R}, M, *)$, where $*$ is a continuous t-norm defined by $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$ and M is a fuzzy set in $\mathbb{R} \times \mathbb{R} \times (0, \infty)$ defined by $M(x, y, t) = \frac{t}{t+q(x,y)}$ is fuzzy Isbell convex. We also introduced the concept of fuzzy admissible subsets and proved in Proposition 5.3.4 that every subset of a fuzzy convex quasi-metric space is fuzzy Isbell convex.

In chapter 6, we successfully presented some fixed point theorems in fuzzy quasi-metric spaces. We first presented the concept of a compatible quasi-metric which is related to the compatible metric presented by Francisco Castro-Company, Salvador Romaguera and Pedro Tirado in [6]. With this compatible quasi-metric we successfully generalised fixed point theorems to the fuzzy asymmetric setting. For instance Olela Otafudu in [27] proved that if (X, q) is an Isbell convex quasi-metric space and that the map $T : (X, q) \to (X, q)$ is nonexpansive and if further for some $\epsilon_1, \epsilon_2 \geq 0$, $F_{\epsilon_1,\epsilon_2}(T) \neq \emptyset$. Then, the set $F_{\epsilon_1,\epsilon_2}(T)$ is Isbell convex (see Theorem 3.3.16), we generalised as follows; Suppose that $(X, M, *)$ is a fuzzy Isbell convex fuzzy quasi-metric space such that $* \geq *_L$ and suppose that the map $T : (X, M, *) \to (X, M, *)$ is t nonexpansive. Furthermore, suppose that for some $\epsilon_1, \epsilon_2 \in (0, 1)$, $F_{\epsilon_1,\epsilon_2}(T) \neq \emptyset$. Then, the set $F_{\epsilon_1,\epsilon_2}(T)$ is fuzzy Isbell convex (see Theorem 6.2.8).

CHAPTER 8

CONCLUSION

In this thesis, we have successfully generalised the concept of Isbell convexity in quasi-metric spaces to fuzzy quasi-metric spaces. We have successfully introduced fuzzy Isbell convexity and fuzzy Isbell hypercompleteness and also proved that a fuzzy quasi-metric space $(X, M, *)$ is fuzzy Isbell convex if and only if it is fuzzy metrically convex and fuzzy Isbell hypercomplete. We then introduced the concept of fuzzy admissible subsets and proved that every subset of a fuzzy convex quasi-metric space is fuzzy Isbell convex. Finally in chapter six, we successfully generalised some fixed point theorems presented by Otafudu in quasi-metric spaces (see [27]) to fuzzy quasi-metric spaces, and this was done after the introduction of the concept of a compatible quasi-metric and we proved that if $(X, M, *)$ is a fuzzy Isbell convex fuzzy quasi-metric space such that $* \geq *_L$ and if the map $T : (X, M, *) \to (X, M, *)$ is t nonexpansive. Furthermore, if for some $\epsilon_1, \epsilon_2 \in (0, 1)$, $F_{\epsilon_1,\epsilon_2}(T) \neq \emptyset$. Then, the set $F_{\epsilon_1,\epsilon_2}(T)$ is fuzzy Isbell convex. Our conclusion leads us to list some open problems encountered throughout the present investigations. We hope to study these problems in future work.

FUTURE PROBLEMS

For the following problem we give the following background information.
The injective hull by Isbell of A, denoted $\epsilon(A)$, for any subset A of a metric space (X, d) is set $\epsilon(A)$ of extremal functions defined on A. The function $f : A \to [0, \infty)$ is extremal if $d(x, y) \leq f(x) + f(y)$ for all $x, y \in A$ and is pointwise minimal, i.e. if $g : A \to [0, \infty)$ such that $d(x, y) \leq g(x) + g(y)$ for all $x, y \in A$ and $g(x) \leq f(x)$ for all $x \in A$, then we must have $f = g$ (see [12]).
Problem 1. Is it possible to construct the Isbell hull of a fuzzy quasi-metric space?

For the next problem we give the following background information.
A binary operation $\Diamond : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-conorm if it satisfies the following conditions:

(i) $\Diamond$ is commutative and associative.

(ii) $\Diamond$ is continuous.

(iii) $a\Diamond 0 = a$ for all $a \in [0, 1]$.

(iv) $a\Diamond b \leq c\Diamond d$ whenever $a \leq c$ and $b \leq d$ with $a, b, c, d \in [0, 1]$.

A 5-tuple $(X, M, N, *, \Diamond)$ is said to be an intuitionistic fuzzy quasi-metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm and M and N are fuzzy sets on $X \times X \times [0, \infty)$ satisfying the following conditions:

(i) $M(x, y, t) + N(x, y, t) \leq 1$ for all $x, y \in X$ and $t > 0$.

(ii) $M(x, y, 0) = 0$ for all $x, y \in X$.

(iii) $M(x, y, t) = M(y, x, t) = 1$ if and only if $x = y$ for all $t > 0$.

(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$.

(v) for all $x, y \in X$, $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous.

(vi) $N(x, y, 0) = 1$ for all $x, y \in X$.

(vii) $N(x, y, t) = N(y, x, t) = 0$ if and only if $x = y$ for all $t > 0$.

(viii) $N(x, y, t) \Diamond N(y, z, s) \geq N(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$.

(ix) for all $x, y \in X$, $N(x, y, \cdot) : [0, \infty) \to [0, 1]$ is right continuous.

In this case, we say that $(X, M, N, *, \Diamond)$ is an intuitionistic fuzzy quasi-metric space (see [9]).
Problem 2. Is it possible to extend the concept of Isbell convexity to intuitionistic fuzzy quasi-metric spaces?

For the next problem we give the following background information.
An ultra-quasi-metric on a nonempty set X is a function mapping $u : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$, the axioms $u(x, x) = 0$, $u(x, y) = 0 = u(y, x)$ implies that $x = y$ and $u(x, y) \leq \max\{u(x, z), u(z, y)\}$ are satisfied. The pair (X, u) is called an ultra-quasi-metric space. The closed ball $C_u(x, r)$ of an ultra-quasi-metric space (X, u) is the set defined by

$$C_u(x, r) = \{y \in X : u(x, y) \leq r\},$$

where $x \in X$ and $r \in [0, \infty)$. Thus, if (X, u) is an ultra-quasi-metric space, and if $\{x_i\}_{i \in I}$ is a collection of points in X also if $\{r_i\}_{i \in I}$ and $\{s_i\}_{i \in I}$ are collections of nonnegative reals. We say that $\{C_u(x_i, r_i), C_{u^{-1}}(x_i, s_i)\}_{i \in I}$ has the strong mixed binary intersection property provided that $u(x_i, x_j) \leq \max\{r_i, s_j\}$ whenever $i, j \in I$. We say that (X, u) is q-spherically complete provided that each collection $\{C_u(x_i, r_i), C_{u^{-1}}(x_i, s_i)\}_{i \in I}$ possessing the strong mixed binary intersection property satisfies

$$\bigcap_{i \in I} (C_u(x_i, r_i) \cap C_{u^{-1}}(x_i, s_i)) \neq \emptyset \quad \text{(see [24]).}$$

Problem 3. Is it possible to study the concept of spherical completeness in ultra fuzzy quasi-metric spaces?

For the next problem we give the following background information.
Let (X, d) be a quasi-metric space. A subspace E of X is said to be externally Isbell convex

relative to X if given any collection $\{x_i\}_{i\in I}$ of points in X and collections of nonnegative real numbers $\{r_i\}_{i\in I}$ and $\{s_i\}_{i\in I}$ the following condition holds: if $d(x_i, x_j) \leq r_i + s_j$ whenever $i, j \in I$ $\operatorname{dist}(x_i, E) \leq r_i$ and $\operatorname{dist}(E, x_i) \leq s_i$ whenever $i \in I$, then

$$\bigcap_{i\in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)) \cap E \neq \emptyset \quad \text{(see [25])}.$$

Problem 4. Is it possible to extend the concept of externally hyperconvex subsets to fuzzy quasi-metric spaces?

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