ON PROJECTIVE REPRESENTATIONS OF FINITE ABELIAN GROUPS.

by

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I declare that this dissertation represents my own work and that it has not previously been submitted for a degree at this or another University.

Signature

TO

my

entire family

and

Charity Mwananyau.

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CONTENTS

ABSTRACT

Saeed [11] has considered Schur multipliers of some of the finite abelian groups.The study of the schur multipliers of abelian groups is the first step in the studying of the projective representations of such groups. Our objective here is to determine the Inequivalent Irreducible projective representations of these groups which correspond to certain classes of factor sets.

Let $C_m^{(n)}$ denote the direct product of n cyclic groups C_m of order m. Then in [9] and [10] the α-regular classes have been determined; these being the classes at which non trivial projective representations with factor set α take on non zero character values. Here we review these results, and determine the Inequivalent Irreducible characters corresponding to these α-regular classes. In particular, a complete set of irreducible inequivalent projective characters is obtained for these classes.

The following is a brief description of how the work in the sequel has been organised. Chapter one gives the basic facts about factor sets and projective representations of finite groups together with some of their properties. The concepts of schur multipliers and twisted group algebras are also considered. The central

and stem extensions of finite groups are discussed in chapter two; while chapter three is concerned with projective character theory. Here the interest is in reviewing those properties of projective characters which are analogous to those of ordinary characters. Finally the work in the previous chapters is applied in chapter four to obtain the irreducible projective characters of certain finite abelian groups; and the results follow the works of Morris and Saeed (c.f [8], [9], [10] and [11].)

CHAPTER ONE

1 PROJECTIVE REPRESENTATIONS OF FINITE GROUPS.

In this chapter we give the required basic facts on factor sets and projective representations, and review the essential results. This work is more completely treated in Morris [8] and Karpilovsky [7].

1.1 Factor Sets

In what follows K denotes an algebraically closed field and K^* a field of non zero elements.

1.1.1 Definition

Let G be a finite group. The mapping α defined by

$\alpha : G X G \rightarrow K^*$ such that for all $g,h,k \varepsilon G$

$$\alpha(g,h)\alpha(gh,k) = \alpha(g,hk)\alpha(h,k)$$

and $\alpha(g,e) = \alpha(e,g) = 1$, where e is the identity element in G, is known as a factor set of G.

1.1.2 Definition

The set of all the factor sets of G forms a group denoted by $Z^2(G,K^*)$ known as the group of 2 cocycles.

1.1.3 Remark

Factor sets are also known as 2-cocycles.

1.1.4 Definition

Let α and β be two factor sets of G, then α and β are said to be equivalent if there exists a function

$\delta : G \rightarrow K^*$ such that for all $g,h \varepsilon G$,

$\delta(gh)\alpha(g,h) = \delta(g)\delta(h)\beta(g,h)$

Equivalent factor sets are also known as cohomologous factor sets.

It is an easy matter to show that equivalence of factor sets is an equivalence relation on $Z^2(G,K^*)$.

1.1.5 Definition

A factor set α of a group G is said to be normalized in G if

$\alpha(h,h^{-1}) = 1$ for all $h \varepsilon G$

Infact, for any factor set α, there exists a factor set β of G equivalent to α which can be normalized.(c.f.[8])

1.1.6 Lemma

Let α be a normalized factor set of G where α^{-1} denotes the inverse of α, then

$\alpha^{-1}(g,h) = \alpha(h^{-1},g^{-1})$ for all $g,h \varepsilon G$

Proof

$\alpha(g,h)\alpha(gh,h^{-1}) = \alpha(g,hh^{-1})\alpha(h,h^{-1})$

(by definition 1.1.1)

$= \alpha(g,e)\alpha(h,h^{-1})$

$= 1.\alpha(h,h^{-1})$

(by definition 1.1.1)

$= 1$ (by definition 1.1.5)

Therefore $\alpha(g,h)\alpha(gh,h^{-1}) = 1$ for all $g,h \varepsilon G$

$$\alpha(gh,h^{-1}) = (\alpha(g,h))^{-1} \qquad \text{(i)}$$

Also by definition (1.1.5),

$$\alpha(gh,h^{-1}) = \alpha(gh,h^{-1})\alpha(g,g^{-1})$$

$$= \alpha((gh),(gh)^{-1})\alpha(h^{-1},g^{-1})$$

$$= \alpha(h^{-1},g^{-1}) \qquad \text{(ii)}$$

From (i) and (ii) it follows that

$$\alpha(gh,h^{-1}) = (\alpha(g,h))^{-1} = \alpha(h^{-1},g^{-1})$$

so that $\alpha^{-1}(g,h) = \alpha(h^{-1},g^{-1})$

1.1.7 Definition

Let α be a factor set of a group G. An element $g\varepsilon G$ is said to be an α-regular element of G if

$\alpha(g,h) = \alpha(h,g)$ for all h in $C_G(g)$, the centralizer of g in G.

1.1.8 Theorem.

If g is an α-regular element of a group G, then all the elements conjugate to g are also α-regular.

Proof

Since $f_\alpha(k,h) = \alpha(k,h)\alpha^{-1}(khk^{-1},k)$ for all $k \in G$,then

if $k \in C_G(h)$, $f_\alpha(k,h) = 1$ (see theorem 1.1.10)

Given h conjugate to g,it implies that $x^{-1}hx = g$ for some $x \in G$ Therefore $f_\alpha(a,x^{-1}hx)=\alpha(a,x^{-1}hx)\alpha^{-1}(a(x^{-1}hx)a^{-1},a)$

$$=\alpha(a,x^{-1}hx)\alpha^{-1}(x^{-1}hx,a)$$

$$= 1 \text{ if all } a \in C_G(x^{-1}hx)$$

Therefore $\alpha(a,x^{-1}hx)\alpha^{-1}(x^{-1}hx,a) = 1$ for all $a\in C_G(x^{-1}hx)$

1.1.9 Remark

It is now a consequence of theorem 1.1.8 that being

α-regular is a class function. From now on a conjugacy class containing an α-regular element g shall be referred to as an α-regular class.

If for every factor set α and every α-regular element $h\varepsilon G$, we define

$$f_\alpha(k,h) = \alpha(k,h)\alpha^{-1}(khk^{-1},k) \text{ for all } k\varepsilon G$$

then we can prove the following

1.1.10 Theorem

(i) If k is an element in the cetralizer $C_G(h)$ of h in G then $f_\alpha(k,h) = 1$

(ii) If α is a normalized factor set, then $f_\alpha(k,h) = \alpha(k,h)\alpha(kh,k^{-1})$

Proof

(i) We note that since $f_\alpha(k,h) = \alpha(k,h)\alpha^{-1}(khk^{-1},k)$ and that if $k\varepsilon C_G(h)$, then $khk^{-1} = h$.

Hence,

$$\begin{aligned} f_\alpha(k,h) &= \alpha(k,h)\alpha^{-1}(khk^{-1},k) \\ &= \alpha(k,h)\alpha^{-1}(h,k) \\ &= \alpha(k,h)\alpha(k^{-1},h^{-1}) \\ &= 1 \text{ for all } k\varepsilon G \end{aligned}$$

(ii) this follows from definition 1.1.5 and theorem 1.1.10 (i)

The following result is useful.

1.1.11 Lemma

For any factor set α, define a map

$\alpha' : GXG \to \mathbb{C}^*$ by

$\alpha'(x,y) = \alpha(x,y)\alpha(y,x)^{-1}$. Then we have

$\alpha'(x,yz) = \alpha'(x,y)\alpha'(x,z)$ for all $x \varepsilon G$ and $y,z \varepsilon C_G(x)$.

Proof

$$\alpha'(x,yz) = \alpha(x,yz)\alpha(yz,x)^{-1})$$
$$= \frac{\alpha(x,yz)}{\alpha(yz,x)}$$
$$= \frac{\frac{\alpha(x,y)\alpha(xy,z)}{\alpha(y,z)}}{\frac{\alpha(y,zx)\alpha(z,x)}{\alpha(y,z)}}$$
$$= \frac{\alpha(x,y)\alpha(xy,z)}{\alpha(y,zx)\alpha(z,x)}$$
$$= \frac{\alpha(x,y)\alpha(yx,z)}{\alpha(z,x)\alpha(y,xz)}$$
$$= \frac{\frac{\alpha(x,y)\alpha(y,xz)\alpha(x,z)}{\alpha(y,x)}}{\alpha(z,x)\alpha(y,xz)}$$
$$= \frac{\alpha(x,y)\alpha(y,x)^{-1}\alpha(x,z)}{\alpha(z,x)}$$
$$= \alpha'(x,y)\alpha'(x,z)$$

1.1.12 Lemma

If α is a normalized factor set of G and an element $h \varepsilon G$ is α-regular, then

$$f_\alpha(k,h) = f_\alpha^{-1}(k,h^{-1}) \text{ for all } k \in G$$

Proof

$$f_\alpha(k,h)f_\alpha(k,h^{-1}) = \alpha(k,h)\alpha(kh,k^{-1})\alpha(k,h^{-1})\alpha(kh^{-1},k^{-1})$$
$$= \alpha(k,h)\alpha(kh,k^{-1})\alpha(k,h^{-1}k^{-1})\alpha(h^{-1},k^{-1})$$
$$= \alpha(k,h)\alpha^{-1}(k,h)\alpha(kh,k^{-1})\alpha^{-1}(kh,k^{-1})$$
$$= 1 \text{ for all } k \varepsilon G$$

i.e $f_\alpha(k,h) = (f_\alpha(k,h^{-1}))^{-1} = f_\alpha^{-1}(k,h^{-1})$

for all $k \varepsilon G$

1.1.13 Theorem

Let α be a normalized factor set and let h be an

α-regular element of G. Then h^{-1} is also α-regular

Proof

Since $C_G(h) = C_G(h^{-1})$ and by the earlier observation, $f_\alpha(k,h) = 1$ for $k \varepsilon C_G(h) = C_G(h^{-1})$

Hence $f_\alpha(k,h^{-1}) = \alpha(k,h^{-1})\alpha^{-1}(kh^{-1}k^{-1},k)$

$= \alpha(k,h^{-1})\ \alpha^{-1}(h^{-1},k)$

$= 1$ for all $k \varepsilon C_G(h^{-1})$

i.e $\alpha(k,h^{-1})\alpha^{-1}(h^{-1}k) = 1$ for all $k \varepsilon C_G$

$\alpha(k,h^{-1}) = \alpha(h^{-1},k)$ for all $k \varepsilon C_G(h^{-1})$

showing that h^{-1} is α-regular by definition.

We prove the following result which is due to conlon (c.f. Haggarty and Humphreys [5]):

1.1.14 Theorem

If α is any factor set of G, then there exists a factor set β equivalent to α such that

(i) $\beta(h,h^{-1}) = 1$ for all $h \varepsilon G$

(ii) $f_\beta(k,h) = 1$ for all β-regular elements h, in G

Proof

We define $\delta(h) = (\alpha(h,h^{-1}))^{-1/2}$ for all $h \varepsilon G$

set $\beta(h,k) = \delta(h)\delta(k)(\delta(hk))^{-1}\alpha(h,k)$

i.e α and β are equivalent

Then $\beta(h,h^{-1}) = (\alpha(h,h^{-1}))^{-1/2}(\alpha(h^{-1},h))^{-1/2}\alpha(e.e)\alpha(h,h^{-1})$

$= 1$

That is β is a normalized factor set.

To prove (ii), first we let $\{a_1,a_2,\ldots,a_n\}$ to be an arbitrary β-regular class and let

$$a_i = k_i a_1 k_i^{-1}, \quad i=1,2,....,n$$

Then $G = \bigcup_{i=1}^{n} C_G(g_i)K_i$.

Now define $\delta(a_i) = f_\beta(k_i,a_1)$; $i = 1,2,...,n$ and let $\delta(a)$ be similarly constructed, where 'a' is an aribitrary β-regular element of G. If 'a' is not β-regular, then set $\delta(a) = 1$.

$$\begin{aligned} \text{Let } \gamma(a_i,a_i^{-1}) &= \delta(a_i)\delta(a_i^{-1})(\delta(e))^{-1}\beta(a_i,a_i^{-1}) \\ &= f_\beta(k_i,a_1)f_\beta(k_i,a_1^{-1}) \\ &= 1 \end{aligned}$$

That is the factor set γ satisfies condition (i) implying that γ is normalized.

Further more,

$$\begin{aligned} f_\beta(k,a_i) &= \beta(k,a_i)\beta^{-1}(ka_ik^{-1},k) \\ &= \delta(a_i)(\delta(ka_ik^{-1}))^{-1}\gamma(k,a_i)\ \gamma^{-1}(ka_ik^{-1},k)) \\ &= \gamma(k,k_i)\gamma(kk_i,k^{-1}) \\ &= 1 \end{aligned}$$

which proves the theorem.

1.1.15 Definition

Factor sets which satisfy theorem (1.1.13) are said to be simple factor sets.

1.2 Projective Representations and Twisted Group Algebras

In what follows, we let G be a finite group, K an algebraically closed field, V a finite dimensional vector space of the field K and let GL(V) be the general linear group of V over K.

$P(g)u\varepsilon U$ for all $u\varepsilon U$.

If P is not reducible, then it's said to be irreducible

1.2.4 Definition

Let α be a factor set of G. A twisted group algebra, denoted by $(KG)\alpha$ is the formal sum

$$(KG)_\alpha = \{\sum_{g\epsilon G} \varepsilon_g \gamma(g) : \varepsilon_g \epsilon K\},$$

where addition and scalar multiplication are defined in the usual way; and multiplication is defined by

$$(\sum_{g\epsilon G} \varepsilon_g \gamma(g))(\sum_{h\epsilon G} \varepsilon_h \gamma(h)) = \sum_{g,h\epsilon G} \varepsilon_g \varepsilon_h (\alpha(g,h)\gamma(gh))$$

and the relation $\alpha(g,h)\alpha(gh,k) = \alpha(g,hk)\alpha(h,k)$ implies now that as an algebra, $(KG)_\alpha$ is associative with $\gamma(e)$ as the identity element.

1.2.5 Definition

Let M and M′ be $(KG)_\alpha$-modules. Then M and M′ are said to be isomorphic as modules if there exists a vector space isomorphism τ of M onto M′ such that

$$\gamma(g)\tau m = \tau(\gamma(g)m) \text{ for all } g\varepsilon G,\ m\varepsilon M$$

The following underlines the importance of twisted group algebras in projective representation theory.

1.2.6 Lemma.

There is a one to one correspondence between the projective representations of G with factor set α and the representations of $(KG)_\alpha$.

Proof

Let T be a representation of $(KG)_\alpha$ as an algebra, then a projective representation of G can be obtained by setting

$$P(g) = T(\gamma(g)) \quad \text{for all } g\varepsilon G.$$

Therefore

$$\begin{aligned} P(g)P(h) &= T(\gamma(g))T(\gamma(h)) \text{ for all } g,h\varepsilon G \\ &= T(\gamma(g)\gamma(h)) \\ &= T(\alpha(g,h)\gamma(g,h)) \\ &= \alpha(g,h)T(\gamma(gh)) \\ &= \alpha(g,h)P(gh) \end{aligned}$$

so that P as defined is a projective representation of G with factor set α.

1.2.7 Remark

Classfying projective representations of G with factor set α is equivalent to the problem of classfying $(KG)_\alpha$-modules which are finite dimensional.

1.2.8 Definition

Let V be a $(KG)_\alpha$-module. Then V is said to be completely reducible if for every submodule N of V, there exists a proper submodule u such that

$$V = N \oplus U$$

otherwise V is said to be indecomposable .

We now prove the following result:

1.2.9 Theorem

Every $(kG)_\alpha$-module is completely reducible.

Proof

Let V be a $(KG)_\alpha$-module and let N_1 be its non trivial submodule. Hence N_1 is a subspace of V as a K-space, and there exists a subspace u of v such that

$$v = N_1 \oplus u$$

We can now find a homomophism $\phi \ \varepsilon \ \mathrm{Hom}(V,N_1)$ such that if $V = n_1 + u$, $v\varepsilon V$, $n_1 \varepsilon N_1$, $u \varepsilon U$ then $\phi(v) = n_1$

Let P:V V be defined by

$$Pv = \frac{1}{|G|} \sum_{g\varepsilon G} \gamma(g)\phi\gamma(g^{-1})v;\ v\varepsilon V$$

since for all $v\varepsilon V$ we have

$$P\gamma(y)v = \frac{1}{|G|} \sum_{g\varepsilon G} \gamma(g)\phi\gamma(g^{-1})\gamma(y)v$$

$$= \frac{1}{|G|} \gamma(y) \sum_{h\varepsilon G} \frac{\alpha(h^{-1}y^{-1},y)\alpha(h,h^{-1})\gamma(h)\phi\gamma(h^{-1})v}{\alpha(yh,(yh)^{-1})\alpha(y,h^{-1})}$$

But $\alpha(h^{-1}y^{-1},y)\alpha(h^{-1},h) = \alpha(h^{-1}y^{-1},yh)\alpha(y^{-1},h)$

Thus $P\gamma(y)v = \gamma(y)Pv$ for all $y\varepsilon G$, $v\varepsilon V$. Furthermore,

$Pv \subseteq N_1$ since

$$\frac{1}{|G|} \sum_{g\varepsilon G} \gamma(g^{-1})v \in \frac{1}{|G|} \sum_{g\varepsilon G} \gamma(g)\phi v \in N_1$$

Also, for each $n_1 \varepsilon N_1$, then

$$Pn_1 = \frac{1}{|G|} \sum_{g\varepsilon G} \gamma(g)\phi\gamma(g^{-1})n_1 \in N_1$$

i.e $Pv = N_1$, $v\varepsilon V$

Now let $N_2 = \{v - Pv : v\varepsilon V\}$. Then N_2 is a submodule of V and $V = N_1 \oplus N_2$ so that V is completely reducible.

We can now count the number of inequivalent irreducible projective representations of G with factor set α and to do this, we require the following (-c.f. Curtis and Reiner [3]).

1.2 10 Remark

$(KG)_\alpha$ is a semi-simple algebra and if we let

$M_{n_i}(K)$ (i= 1 ,2,...,n) be a full matrix algebra of $n_i x n_i$ matrices over K, then

$$(KG)_\alpha \cong M_{n_1}(K) + M_{n_2}(K) + ... + M_{n_n}(K)$$

1.2.11 Lemma.

Let $Z((KG)_\alpha)$ denote the centre of $(KG)_\alpha$ and let $y \varepsilon Z((KG)_\alpha)$ That is $y = \sum_{g \varepsilon G} \xi_g \gamma(g)$

where $\xi_g \varepsilon$ K. If $\xi_g \neq 0$ then g is an α-regular element in G.

Proof

We need only show that $f_\alpha(h,g) = 1$ for all elements h in the centralizer of g in G. Suppose α is a simple factor set of G. Since $y \varepsilon Z((KG)_\alpha)$, then

$$\gamma(h)^{-1} y \gamma(h) = y \text{ for all } h \varepsilon G$$

$$\text{Hence, } \xi_g \gamma(h)^{-1} \gamma(g) \gamma(h) + \sum_{g' \neq g \varepsilon G} \xi_{g'} \gamma(g') \gamma(h)$$
$$= \xi_g \gamma(g) + \sum_{g' \neq g} \xi_{g'} \gamma(g')$$

This implies that

$$\xi_g f_\alpha(h,g) \gamma(g) + \sum_{g' \neq g \varepsilon G} \xi_{g'} f_\alpha(h,g') \gamma(h^{-1} g' h)$$
$$= \xi_g \gamma(g) + \sum_{g' \neq g \varepsilon G} \xi_g \gamma(g')$$

since ξ_g is finite and $\neq 0$, we have $f_\alpha(h,g) = 1$ by comparing coefficients on both sides.

Now $h^{-1}g'h = g$ would contradict the fact that $g' \neq g$. Therefore, g is an α-regular element in G as α is assumed to be simple

1.2.12 Theorem

Let $c_1, c_2, \ldots, c_s$ be a complete set of α-regular classes in G and define

$$C_i = \sum_{x_i \in c_i} \gamma(x_i) \quad i = 1,2,\ldots,s$$

Then $\{C_1, C_2, \ldots, C_s\}$ forms a K-basis for $Z((KG)_\alpha)$

Proof

For $h \in G$

$$\gamma(h)^{-1} C_j \gamma(h) = \sum_{x \in c_j} \gamma(h)^{-1} \gamma(x) \gamma(h)$$

$$= \sum_{x \in c_j} \gamma(h^{-1}xh)$$

$$= C_j$$

Thus, $C_j \in Z((KG)_\alpha)$ and hence each C_i lies in $Z((KG)_\alpha)$. Furthermore, $\{C_1, C_2, \ldots, C_s\}$ is a set of linearly independent elements since the el e ments are sums of elements from distinct classes in G.

Let $y = \sum_{g \in G} \xi_g \gamma(g) \in Z((KG)_\alpha)$

where $\xi_g \in K$.

If $\xi_g \neq 0$ then g is an α-regular element in G by lemma 1.2.11.

That is, y has precisely the α-regular elements in G.

Hence the elements $y = \sum_{g \in G} \xi_g \gamma(g)$ in the centre $Z((KG)_\alpha)$ are precisely those sums of elements which are

1.2.12)

1.2.14 Corollary.

The number of inequivalent irreducible projective representations of a group G with factor set α is equal to the number of α-regular classes in G.

Proof

Inequivalent irreducible projective representations of G are in a one-one correspondence with the finite dimensional $(KG)_\alpha$-modules by lemma 1.2.6.

Also, the number of non isomorphic irreducible $(KG)_\alpha$-modules is the same as the number of α-regular classes of G by theorem 1.2.13.

1.2.15 Lemma

Let α and β be equivalent factor sets of G. Then the number of inequivalent irreducible projective representations of G with factor set α is the same as the number of inequivalent irreducible projective representations of G with factor set β.

Proof

Let $\{\tau_1,\tau_2,\ldots,\tau_n\}$ be a complete set of inequivalent irreducible projective representations of G with factor set α. For each i=1,2,...,n define

$S_i : G \to GL(V)$ by

$S_i(g) = \delta(g)\tau_i(g)$ where $\delta : G \to K^*$

as α and β are equivalent.

Therefore,

$$S_i(g)S_i(h) = \delta(g)\delta(h)\tau_i(g)\tau_i(h)$$

$$= \delta(g)\delta(h)\alpha(g,h)\tau_i(gh)$$

$$= \delta(gh)\beta(g,h)\tau_i(gh)$$

by equivalence of α and β

$$= \beta(g,h)\delta(gh)\tau_i(gh)$$

$$= \beta(g,h)S_i(gh)$$

from definition above.

Hence S_i is a projective representation with factor set β.

Also, $\{S_i,S_2,\ldots,S_n\}$ is a set of inequivalent representations of G, since if suppose that S_i is equivalent to S_j for some $i \neq j$, then there exists a non singular matrix T such that for all $g\varepsilon G$;

$$T^{-1}S_i(g)T = S_j(g)$$

i.e $T^{-1}\delta(g)\tau_i(g)T = \delta(g)\tau_j(g)$

(from definition)

Hence τ_i is equivalent ot τ_j for $i \neq j$ which is a contradiction since $\{\tau_1,\tau_2,\ldots,\tau_n\}$ is a set of inequivalent representations.

As a direct consequence of lemma 1.2.15, we now have

1.2.16 Corollary

Let α and β be equivalent factor sets of G. Then the number of α-regular classes in G is the same as the number of β-regular classes of G. Furthermore, the number of indecomposable non isomorphic $(KG)_\alpha$-modules equals the number of indecomposable non isomorphic $(KG)_\beta$-modules.

1.3 SCHUR MULTIPLIERS OF FINITE GROUPS.

Let α be a factor set of G and let $[\alpha]$ denote the equivalence class containing α via the equivalence relation in 1.1.4. Define the inverse of $[\alpha]$ by $[\alpha^{-1}]$ and multiplication by $[\alpha]\ [\beta] = [\alpha\beta]$

Let $\xi = \{[\alpha] : [\alpha]$ is an equivalence class in $Z^2(G,H^*)\}$ be the collection of all the distinct equivalence classes arising in this way.

(i) For any $[\alpha]$ and $[\beta]$ in ξ,

$[\alpha][\beta] = [\alpha\beta] = [\gamma]\varepsilon\xi$ where $\gamma = \alpha\beta$.

That is multiplication is closed.

(ii)
$$
\begin{aligned}
[\alpha\beta][\gamma] &= ([\alpha][\beta])[\gamma] \\
&= [\alpha][\beta][\gamma] \\
&= [\alpha]([\beta][\gamma]) \\
&= [\alpha][\beta\gamma]
\end{aligned}
$$

That is associativity law holds.

(iii) Every element in ξ has its own inverse defined by $[\alpha]^{-1} = [\alpha^{-1}]$

(iv) $[\alpha][\alpha]^{-1} = [\alpha][\alpha^{-1}] = [\alpha\alpha^{-1}] = [1]\varepsilon\xi$.

The identity element lies in ξ.

Thus ξ forms a group.

Further, for every $[\alpha]$, $[\beta]$ in ξ, $[\alpha][\beta] = [\alpha\beta] = [\beta\alpha]=[\beta][\alpha]$.

Hence ξ forms an abelian group under this composition.

1.3.1 Definition

The group of distinct equivalence classes $[\alpha]$ considered earlier is known as the Schur multiplier of G, denoted by $H^2(G,K^*)$ or simply M(G).

We now consider an alternative way in which schur multipliers arise. Let α and β be equivalent factor sets of G. Then there exists a function $\delta: G \rightarrow K^*$ such that for all $g,h \varepsilon G$, we have

$$\alpha(g,h) = \delta(g)\delta(h)(\delta(gh))^{-1}\beta(g,h)$$

Now define $\mu(\delta) : GXG \rightarrow K^*$ by $\mu(\delta)(g,h) = \delta(g)\delta(h)\delta(gh)^{-1}$

μ is a factor set and a homomorphism and $im\mu$ is a subgroup of $Z^2(G,K^*)$

1.3.2 Definition

$Im\mu$ is called the group of 2-coboundaries dentoed by $B^2(G,K^*)$.

We now have the following

1.3.3 Definition

The quotient group $\dfrac{Z^2(G,K^*)}{B^2(G,K^*)}$ is known as the second cohomology group of G denoted by $H^2(G,K^*)$ also known as the Schur multiplier of G.

1.3.4 Lemma

Let P be a projective representation of G with factor set α. The projective representations of G equivalent to P are those representations whose factor sets belong to the coset $\alpha\beta^2(G,K^*)$. In particular, representations with trivial factor sets are those whose factor sets lie in the coset $B^2(G,K^*)$.

Proof

Let P' be equivalent to P and α' be its associated factor set. Then there exists a non singular matrix T and a map $\varphi: G \rightarrow K^*$ such that for all $g \varepsilon G$,

$$P'(g) = \varphi(g)T^{-1}P(g)T$$

For all $h \varepsilon G$, $\alpha'(g,h)P'(gh) = P'(g)P'(h)$

$$= \varphi(g)\varphi(h)T^{-1}P(g)P(h)T$$

$$= \varphi(g)\varphi(h)\alpha(g,h)\varphi(gh)^{-1}P'(gh)$$

Hence $\alpha'(g,h) = \varphi(g)\varphi(h)\alpha(g,h)(\varphi(gh))^{-1}$

$\Rightarrow$ α' lies in $\alpha\ B^2(G,K^*)$

1.3.5 Remark

By the above lemma, it's clear that equivalent factor sets of G, give rise to the same coset $\alpha B^2(G,K^*)$ of $B^2(G,K^*)$ in $Z^2(G,K^*)$.

1.3.6 Lemma

Let β be the factor set of G. Then every class $[\beta]$ in $H^2(G,K^*)$ of order q contains a representative β' whose values are qth roots of unity. Further, β' is a normalized factor set of G.

Proof

Let K be a field of char.m $\geq$ 0. We can write $q = m^d n$ where $d \geq 0$ and $m \not| n$ as $[\beta]$ is of order q.

There exists a function $\mu: G \to K^*$ such that

$$\beta(g,h)^q = \frac{\mu(g)\mu(h)}{\mu(gh)}.$$

Thus, $\beta(g,h)^n = \dfrac{\mu(g)^{1/m^d}\ \mu(h)^{1/m^d}}{\mu(gh)^{1/m^d}}$ which implies that $\beta(g,h)^n$ is a coboundary contradicting the fact that $[\beta]$ is of order q, unless $m^d = 1$. Hence $m \not| n$

From $\beta(g,h)^q = \dfrac{\mu(g)\mu(h)}{\mu(gh)}$, we can find $v(g) \varepsilon K^*$ such that $v(g) = \mu(g)^{-1}$. Define $\beta'(g,h) = \dfrac{v(g)v(h)}{v(gh)}\ \beta(gh)$

Then $\beta'(g,h)^q = \dfrac{\mu(g)^{-1}\ \mu(h)^{-1}}{\mu(gh)^{-1}}\ \beta(g,h) = 1$

Hence $\beta'(g,h)$ are qth roots of unity.

To show that β' is normalized, set

$$V(g) = (\beta(g,g^{-1}))^{-1/2}$$

Then $\beta'(g,g^{-1}) = 1$ for all $g \varepsilon G$

1.3.7 Theorem

The schur multiplier is a finite abelian group and the order of every element in it is a factor of the order of G

Proof

The schur multiplier is an abelian group as earlier shown. Let $[\alpha]$ be an arbitrary element in $H^2(G,K^*)$ of order e, $\Rightarrow \alpha^e = 1$. For each $g \varepsilon G$, define $\mu(g) = \prod_{y \varepsilon G} \alpha(g,h)$ Then for fixed $g,h \varepsilon G$ and taking products over $g \varepsilon G$, we have,

$$\frac{\mu(g)\mu(h)}{\mu(gh)} = \frac{\prod_{y \varepsilon G} \alpha(g,y) \prod_{y \varepsilon G} \alpha(h,y)}{\prod_{y \varepsilon G} \alpha(gh,y)} = \alpha(g,h)^{|G|}$$

From $\alpha(g,h)^{|G|} = \frac{\mu(g)\mu(h)}{\mu(gh)}$, we conclude that the order of α divides the order of G. Furthermore, since for each $[\beta] \in H^2(G,K^*)$ there exists a representative β' which is a qth root of unity (where q is the order or $[\beta]$) and q divides $|G|$. Then there are atmost a finite number of classes in $H^2(G,K^*)$.That is, it is finite.

CHAPTER TWO

REPRESENTATION GROUPS.

2.1 Central Extensions.

We first define the following:

2.1.1 Definition

A central extension(H,ϕ) of a group G is a group H together with a homomorphism ϕ such that $\ker\phi \subseteq Z(H)$ and $H/\ker\phi \cong G$ where $Z(H)$ denotes the centre of H.

Let (H,ϕ) be a central extension of G with $N=\ker\phi$. Let $\{\gamma(g):g\varepsilon G\}$ be a set of coset representatives of N in H, and suppose that $H = \bigcup_{g \varepsilon G} N\gamma(g)$. Define elements $n(g,h)\varepsilon N$ by $\gamma(g)\gamma(h) = n(g,h)\gamma(gh)$ for all $g,h\varepsilon G$, so that $n(g,h) = \gamma(g)\gamma(h)(\gamma(gh)^{-1})$ It now easily follows from associativity law in H that for all $g,h,k\varepsilon G$

$$n(g,h)n(gh,k) = n(g,hk)n(h,k) \qquad (2.1)$$

Consider ψ to be an ordinary linear character of N and let $\alpha(g,h) = \psi\ n(g,h)$ for all $g,h \in G$

Then it follows from (2.1) that α is a factor set of G

The following is due to Haggarty and Humphreys [5].

2.1.2 Definition

The factor set α arising from an ordinary linear character ψ as described above is known as a special factor set.

Let T be an ordinary representation of H. Since (H,ϕ) is a central extension of G, then $N=\mathrm{Ker}\phi \subseteq Z(H)$. Define a map $P:G \to GL(V)$ by

$$P(g) = T(\gamma(g)) \text{ for all } g\varepsilon G$$

Then since T is a homomorphism on H and $\gamma(g)\gamma(h) = n(g,h)\gamma(gh)$, we have

$$T(\gamma(g))T(\gamma(h)) = T((n(g,h))(\gamma(gh)))$$

that is

$$T(\gamma(g))T(\gamma(h)) = \alpha(g,h)T(\gamma(gh))$$

and $P(g)P(h) = \alpha(g,h)P(gh)$ for all $g,h\varepsilon G$. That is P is a projective representation of G with special factor set α.

2.1.3 Definition

The projective representation P of G obtained above via the ordinary representation T of H is said to be a projeective representation of G linearized by the representation T of H.

We prove the following result.

2.1.4 Theorem

Let G be a finite group and H an arbitrary group with an abelian normal subgroup N such that $H/N \cong G$ Let $\psi \;\varepsilon\; \mathrm{Hom}(N,K^*)$ and let α' be a special factor set of G given by $\alpha'(g,h) = \psi(n(g,h))$ where $n(g,h)\varepsilon N$ is as defined above. Then the map

$\xi : \mathrm{Hom}\,(N,K^*) \to H^2(G,K^*)$ defined by $\xi(\psi) = \alpha' B^2(G,K^*)$ is a homomorphism with Kernel $(N\cap H')^{\perp}$,where H' is the derived group of H. Furthermore, ξ is an isomorphism if and only if $N \subseteq H'$

Proof

Let $H = \bigcup_{g \varepsilon G} N\gamma(g)$ and define $\gamma(g)\gamma(h) = n(g,h)\gamma(gh)^{-1}$. Suposse that $\alpha' \varepsilon B^2(G,K^*)$. Then $\alpha'(g,k) = \mu(g)\mu(h)\mu(gh)^{-1}$ for some $\mu \varepsilon B^2(G,K^*)$ with $\mu(e) = 1$. Let $\tau: H \to K^*$ be defined by

$$\tau(n(\gamma(g))) = \psi(n)\mu(g).$$

Then

$$\begin{aligned}\tau(\gamma(g))\tau(\gamma(h)) &= \mu(g)\mu(h)\\ &= \alpha'(g,h)\mu(gh)\\ &= \psi(n(g,h))\mu(gh)\\ &= \tau(n(g,h))\gamma(gh)\end{aligned}$$

and

$\tau(\gamma(e)) = \mu(e) = 1$, so that τ is a homomorphism. Since $\tau(g^{-1}h^{-1}gh) = 1$, the restriction of τ to H' is $\tau/H' = 1$

Also

$$\begin{aligned}1 = \tau(\gamma(e)\gamma(e)^{-1}) &= \psi(n(e,e^{-1})\mu(e)\\ &= \psi(e)\mu(e).\end{aligned}$$

Therefore $\tau(n\gamma(e)\gamma(e)^{-1}) = \tau(n) = \psi(n)$ and

$\tau/N = \psi(n)$, $n \varepsilon N$. Thus, if $n \ \varepsilon \ N \cap H'$, then $\tau(n) = 1 = \psi(n)$.

Now, let $(N \cap H')^{\perp}$ be defined as follows:

$$(N \cap H')^{\perp} = \{\lambda \varepsilon \ \mathrm{Hom}(N,K^*)/\lambda(n)=1,\ n \varepsilon N \cap H'\}.$$

Then $\psi \ \varepsilon (N \cap H')$

Conversely, suppose that $\psi \ \varepsilon \ (N \cap H')^{\perp}$. Let $ł: NH' \to K^*$ be defined by $ł(ng) = \psi(n)$. Then $ł$ is a homomorphism which can be extended to a homomorphism $\phi: H \longrightarrow K^*$ defined by

$$\begin{aligned}\phi(\gamma(g))\phi(\gamma(h)) &= \phi(\gamma(g)\gamma(h))\\ &= \phi(n(g,h)\gamma(gh))\end{aligned}$$

$$= \psi(n(g,h))\phi(\gamma(gh)).$$

Now if we let $\mu(g) = \phi(\gamma(g))$, then

$$\phi(\gamma(g))\phi(\gamma(h)) = \psi(n(g,h))\psi(\gamma(gh))$$
$$= \alpha'(g,h)\mu(gh)$$

or $\mu(g)\mu(h) = \alpha'(g,h)\mu(gh)$ so that $\alpha' \varepsilon B^2(G,K^*)$. Therefore, given a linear character ψ of N, the special factor set α' determined by ψ lies in $B^2(G,K^*)$ if and only if $\psi\ \varepsilon (N \cap H')^{\perp}$. Now defining $\xi:\text{Hom}(N,K^*) \to H^2(G,K^*)$ by $\xi(\psi) = \alpha' B^2(G,K^*)$, then ξ will be a homorphism with

$$\text{Ker}\ \xi = \{\psi\ \varepsilon\ \text{Hom}(N,K^*):\psi(n)=1\} = (N \cap H')^{\perp}$$

In particular, ξ is an isomorphism if and only if $N \subseteq H'$, that is

$N = \text{Ker}\xi$ is trivial.

2.1.5 Definition (See Haggarty and Humphreys [5])

Let ξ be as above, and for a linear character w of N, let τ be such that $\tau(g,h) = w(n(g,h))$. Then $\xi:\text{Hom}(N,K^*) \to H^2(G,K^*)$ defined by $\xi(w) = \tau\ B^2(G,K^*)$ is known as the standard map.

The following gives a necessary and sufficient condition for projective representations of G to be linearized.

2.1.6 Theorem

Let (H,ϕ) be the central extension of a group G with $N = \text{Ker}\ \phi$ and let ε be the associated standard map. Then the projective representation of G are linearized in H if and only if $\alpha B^2(G,K^*) \subseteq \text{im}\ \xi$.

Proof

Consider $\{\gamma(g):g\varepsilon G\}$ to be the right transversal of N in H and $\phi(\gamma(g)) = g$ in G. For $\lambda \ \varepsilon \ \mathrm{Hom}(N,K^*)$, set $\xi(\lambda) = [\alpha]\varepsilon H^2(G,K^*)$. Then the factor set defined by $\alpha'(g,h)=\lambda(n(g,h))$ lies in $[\alpha]$ since by theorem, there exists a representative $\alpha'\varepsilon[\alpha]$ which implies that α' and α are equivalent. Thus there exists a function $\mu:G\to K^*$ such that

$$\alpha'(g,h) = \frac{\mu(g)\mu(h)}{\mu(g,h)}\alpha(g,h)$$

Define $T:H \to GL(V)$ by

$$T(a\gamma(g)) = \lambda(a)P(g)\mu(g); \ a\varepsilon N, \ g\varepsilon G$$

Clearly, T is a representation of H linearizing P. Conversely, let P be linearized in H by T. Then $T(\gamma(g))= P(g)\mu(\gamma(g))$

$P(1) = \mu(1)^{-1}T(1)$ and for any $a \ \varepsilon \ N$,

$T(a) = P(1)\mu(a) = \mu(a)\mu(1)^{-1}T(1)$

implying that $\lambda(a) = \mu(a)\mu(1)^{-1}$ is a linear character of N. Define $\eta(g) = \mu(\gamma(g))$

Then $\alpha'(g,h)T(\gamma(gh))= T(\gamma(g))T(\gamma(h))$

$= P(g)P(h)\eta(g)\eta(h)$

$\alpha'(g,h)\eta(gh)P(gh) = \alpha(g,h)\eta(g)\eta(h)P(gh)$

Thus $\alpha'(g,h) = \eta(g)\eta(h)\eta(gh)^{-1}\alpha(g,h)$

so that α' lies in $[\alpha] \ \varepsilon \ H^2(G,K^*)$

That is $\xi(\lambda) \ \varepsilon \ [\alpha]$

It is important to characterise $N = \mathrm{Ker}\ \phi$ given a central extension (H,ϕ). More specifically, we have

2.1.7 Lemma (c.f Haggarty and Humphreys [5])

Let (H,ϕ) be a central extension of a group G with $N = \text{Ker}\,\phi$, and let $n(g,h) \varepsilon N$ be as defined above. Then

$N = gp\{n(g,h):g,h\varepsilon G\}$

The following result is due to Haggarty and Humphrey [6].

2.1.8 Lemma

Let (H,ϕ) be a central extension of G, with $N = \text{Ker}\phi$. Then a transversal $\{\delta(g):g\varepsilon G\}$ for N may always be chosen to be cojugacy preserving.

That is $\delta(g_1)$ will be conjugate to $\delta(g_2)$ whenever g_1 is conjugate to g_2 in G.

2.2 STEM EXTENSIONS AND REPRESENTATION GROUPS

2.2.1 Definition

A stem extension of a group G is a pair (H,ϕ) such that $1 \to \text{Ker}\ \phi = N \xrightarrow{i} H \xrightarrow{\phi} G \to 1$ is a short exact sequence and $N \subseteq Z(H) \cap H'$.

2.2.2 Definition

A representation group H of a group G is a finite group of lowest possible order which is a central extension of G such that every projective representation of G is linearizable in H.

2.2.3 Corollary

Let (H,ϕ) be a finite central extension of G with $\text{Ker}\phi = N$, and let ε be the associated standard map.

Then (i) If $N \subseteq H'$, then N is isomorphic to a subgroup of $H^2(G,K^*)$

(ii) Assume $|N| = |H^2(G,K^*)|$. Then $N \subseteq H'$ if and only if every projective representation of G is linearized by a representation of H.

Proof

Let ξ:Hom(N,K*) $\rightarrow H^2(G,K^*)$ be the standard map defined by $\xi(\lambda) = \alpha\beta^2(G,K^*)$ where $\lambda \in$ Hom(N,K*). By Theorem 2.1.4, Ker $\xi = (N \cap H')^{\perp}$ and Im $\xi \subseteq H^2(G,K^*)$. Thus $\text{Hom}\,(N \cap H',K^*) \cong \dfrac{\text{Hom}(N,K^*)}{(N\cap H')^{\perp}} \cong \text{Im}\ \xi \cong H^2(G,K^*)$

Since $H^2(G,K^*)$ is a finite group, so is Hom($N\cap H'$,K*). But Hom($N\cap H'$,K*) $\cong N \cap H'$. Therefore $N \cap H'$ is finite and isomorphic to a subgroup of $H^2(G,K^*)$

If $N \cap H' = \{e\}$ then Im $\xi = \{e\}$ which corresponds to a trivial factor set in $H^2(G,K^*)$. This yields ordinary representations of G. Suppose $N\cap H' \neq \{e\}$, then α is a non trivial factor set, giving rise to non trivial projective representations of G. In particular, projective representations of G are linearizable in H if and only if $N \subseteq H'$. In this case $N \cong H^2(G,K^*)$.

2.2.4 Theorem

Let K* be a complex field. Then a finite group G has at least one representation group H of order $|H^2(G,K^*)|\,|G|$. Furthermore, the Kernel, Kerϕ of the homomorphism $\phi:H \rightarrow G$ is isomorphic to $H^2(G,K^*)$.

Proof

As a finite and abelian group, we can express $H^2(G,K^*)$ as a finite product of cyclic groups $<[\alpha^{(i)}]>$ where each generator $[\alpha^{(i)}]$ is of order d_i. Since the $\alpha^{(i)}(x,y)$ have the property that

$$\alpha(x,y)\alpha(xy,z) = \alpha(x,yz)\alpha(y,z), \text{ we have}$$

$$a^{(i)}_{x,y} + a^{(i)}_{xy,z} \equiv a^{(i)}_{x,yz} + a^{(i)}_{y,z} \pmod{d_i} \qquad (2.2)$$

If $[\alpha]$ ε $H^2(G,K^*)$, then α is equivalent to β where

$$\beta(x,y)=(\alpha^{(1)}(x,y))^{\ell_1} x (\alpha^{(2)}(x,y))^{\ell_2} x \ldots x (\alpha^{(r)}(x,y))^{\ell_r}$$

$$= (\xi_1^{\ell_1})^{a^{(1)}_{x,y}} x (\xi_2^{\ell_2})^{a^{(2)}_{x,y}} x \ldots x (\xi_r^{\ell_r})^{a^{(r)}_{x,y}}$$

$$0 \leq \ell_i \leq d_i - 1 \qquad (2.3)$$

Now let Ker ϕ = A, and let $a_1, a \ldots, a_n$ be generators of A corresponding to the generators $[\alpha^{(i)}]$ of $H^2(G,K^*)$. For each x,yεG, let a(x,y)εA be defined by

$$a(x,y) = \prod_{i=1}^{n} a_i \, a^{(i)}_{x,y} . \quad \text{Then}$$

$$a(x,yz)a(y,z) = a(x,y)a(xy,z) \qquad (2.4)$$

By (2.2) above.

If for χε Hom(A,K*), we define

$$\psi_\chi(x,y) = \xi(a(x,y)), \text{ then}$$

$$\psi_\chi = \prod_{i=1}^{n} \chi(a_i)^{a^{(i)}_{x,y}}$$

That is ψ_χ ε $H^2(G,K^*)$; that is as χ runs through all the linear characters of A, ψ_χ runs through the elements of $H^2(G,K^*)$. Now, set H = {(x,a) : xεG, aεA} with a composition on H being defined as follows: for x,yεG and a,bεA,

$$(x,a)(y,b) = (xy,a(x,y)ab).$$

Then H is a group with {(1,a): aεA} ≅ A contained in

$Z(H)$, the centre of H. Furthermore, if we let $v(x) = (x,1)$, for all $x \varepsilon G$, $1 \varepsilon A$, then $\{v(x):x\varepsilon G\}$ is a set of coset representatives of H mod A. Therefore $H/A \cong G$ and H is a central extension of G.

Now, to show that every projective representation of G can be linearized by a linear representation of H, let $P:G \to GL(V)$ be a projective representation of G with factor set α. Then as earlier seen, there exists a linear character ψ of A such that $\psi\ (a(x,y)) = \alpha(x,y)$, for all $x,y \varepsilon G$. Now, let $T:H \to GL(V)$ be defined by

$$T(v(x)a) = P(x)\psi(a).$$

Then

$$\begin{aligned} P(x)P(y) &= T(v(x))T(v(y)) = T(v(x)v(y)) \\ &= T(a(x,y)v(xy)) = \psi(a(x,y))P(xy) \\ &= \alpha(x,y)P(xy) \end{aligned}$$

so that P is a projective representation of G with factor set α, which is linearized by T.

That $|H| = |H^2(G,K^*)||G|$ follows from the fact that $H/A \cong G$ and that $A \cong H^2(G,K^*)$. Thus, H is a representation group of G.

2.2.5 Theorem

Let φ be an irreducible projective representation of G with factor set α. Then there exists a projective representation φ' equivalent to φ which can be linearized in a stem extension of G.

Proof

Let (H,δ) be a stem extension of G. Then we can find $\psi \varepsilon$ Hom(N,K*) where N = Ker δ, which is such that $\xi(\psi) = \alpha$ $B^2(G,K^*)$ and ξ is the standard map. Also, there exists a special factor set α' with $\alpha'(g,h) = \psi$ n(g,h). Then α' is equivalent to α via ξ. Let φ' be a projective representation correspond i ng to α'. Then φ' is projectively equivalent to φ. F u rth e r, φ' is irreducible since φ is.

φ' can be linearized in H by an ordinary representation D as $\varphi'(g) = D(\gamma(g))$ $\forall g \varepsilon G$. Hence φ is equivalent to the one that can be linearized in H.

The following result can be proved.(see e.g [8])

2.2.6 Lemma

Let G be a finite group with arbitrary factor set α. Then there exists a stem extension of G which linearizes projective representations of G with factor set α.

We now prove

2.2.7 Lemma

A finite group G has atmost a finite number of inequivalent irreducible projective representations over an algebraically closed field K.

Proof

Since G is finite, by the above lemma 2.2.5, There exists a stem extension which linearizes P. Let (H,ϕ) be a stem extension which linearizes a projective represenation P of G. i.e there exists an ordinary representation T of H such that $T(\gamma(g)) = P(g)$ for all $g \varepsilon G$. If P' is a projective representation of G equivalent to P and T' is the ordinary representation of H linearizing P' then T is equivalent to T'. Hence the number of inequivalent irreducible projective representations is less or equal to the number of inequivalent irreducible ordinary representations of H, which is known to be finite

2.2.8 Remark

Recall that the number of inequivalent irreducible projective representations of G with factor set α is equal to the number of α-regular classes in G. In addition, for finite G, $H^2(G,K^*)$ is a finite group and can be expressed as a product of cyclic groups. Hence the number of inequivalent irreducible projective representations of a finite group corresponding to an arbitrary factor set α is necessarily finite.

2.2.9 Theorem

The degrees of the irreducible projective representations of a finite group G divides the order of G.

Proof

By 2.2.5 lemma, since G is finite, there exists a stem extension (H,ϕ) linearizing the projective representation P of G. Let P be an arbitrary irreducible projective representation of G and T an ordinary representation of H which linearizes P. Then deg P = deg T. Furthermore, by Huppert,

deg T divides [H:Z(H)].

Since (H,ϕ) is a stem extension of G,

$N \subseteq Z(H)$ and $H/N \cong G$ where $N = \text{Ker } \phi$

Hence deg T divides $|H/N| = |G|$

so that deg P = deg T divides $|G|$

Given the importance of representation groups, It is instructive to provide the following characterization of representation groups. The characterization is due to I. Schur.

<u>2.2.10 Theorem</u>

Let G be a finite group. Then over the complex field $\mathbb{C}$, there exists a representation group H of G such that

$N \subseteq Z(H)$ with

(i) $N \subseteq (H')$

(ii) $H/N \cong G$

(iii) $|N| = |H^2(G,K^*)|$

CHAPTER THREE

CHARACTER THEORY

In this chapter the main aim is to review those properties of projective characters of G that are analogous of those of linear characters. We do this by considering projective characters of G as linear characters of its corresponding representation group H, via the linearization process discussed earlier.

3.1 Projective Characters

3.1.1 Definition

Let (H,ϕ) be a central extension of G which linearizes the representations of G. Let P be a projective representation with a special factor set α. Let D be an ordinary representation of H which linearizes P. Then the projective character of P denoted by χ shall be defined to be

$$\chi(g) = \text{Trace}(D(\gamma(g))$$

Note that the definition makes use of the fact that P has a special factor set. If α is an arbitrary factor set, then it is possible to choose a factor set α' equivalent to α which is a special factor set. Thus, if P' is a projective representation corresp o nding to α' and D' linearizes P' then the character of P shall be defined to be that of P'.

That is, $\chi(g) = \text{Trace}(D'(\gamma(g)))$

3.1.2 Remark

By lemma (2.1.8), since the transversal $\{\delta(g):g\varepsilon G\}$ has been chosen to be conjugacy preserving, the character of a projective representation as defined above is a class function on G.

3.1.3 Theorem

Let P_1 and P_2 be projective representations of G with special factor set β and respective characters χ_1 and χ_2. Then P_1 is projectively equivalent to P_2 if and only if there exists a one dimensional linear character λ of G such that $\chi_1(g) = \lambda(g)\chi_2(g)$

Proof

P_1 equivalent P_2 implies that both are linearly equivalent to a direct sum of irreducible projective representations with factor set α. Hence there exists ordinary representations D_1 and D_2 of H which linearizes P_1 and P_2 respectively.

That is $D_1(\gamma(g)) = P_1(g)$ and $D_2(\gamma(g)) = P_2(g)$where $D_1(\gamma(g))$ and $D_2(\gamma(g))$ have the same meaning as before.

Since α is a special factor set, there exists a one-dimensional character ψ of N such that

$$P_i(g)P_i(h) = \psi(n(g,h))P_i(gh) \quad i=1,2$$

But $D_i(\gamma(g))D_i(\gamma(h)) = D_i(\gamma(g)\gamma(h))$

$$= D_i(n(g,h)\gamma(gh)$$

$$= D_i(n(g,h))D_i(\gamma(gh)$$

Which implies that $D_i(n(g,h)) = \psi(n(g,h))$

$N = gp\{n(g,h):g,h\varepsilon G\}$ by Haggarty and Humphrey

so that $D_i(a) = \psi(a)I \;\; \forall a\varepsilon N$.

P_1 and P_2 are projectively equivalent and therefore, there exists an invertible matrix R and a map $c:G \to K^*$ such that

$$R^{-1}P_1(g)R = c(g)P_2(g) \quad \forall g\varepsilon G.$$

setting $R^{-1}D_1(\gamma(g))R = D_1'(\gamma(g))$ we get,

$$D_1' ((\gamma(g)) = c(g)D_2(\gamma(g))$$

Hence; $c(g)D_2(\gamma(g))c(h)D_2(\gamma(h)) = D_1' (\gamma(g))D_1' (\gamma(h)$

$$= D_1' (\gamma(g)\gamma(h))$$

$$= D_1' (n(g,h)\gamma(gh))$$

$$= \psi(n(g,h))D_1' (\gamma(gh))$$

$$= \psi(n(g,h))c(gh)D_2(\gamma(gh))$$

$$= c(gh)D_2(\gamma(g)\gamma(h))$$

Hence $c(g)c(h)D_2(\gamma(g))D_2(\gamma(h)) = c(gh)D_2(\gamma(g)\gamma(h))$ so that $c:G \to K^*$ is a representation of G. By $D_1'(\gamma(g) = c(g)D_2(\gamma(g))$ and the definition of character, it follows that

$$\chi_1(g) = c(g)\chi_2(g)$$

<u>Conversely</u>, assume $\chi_1 = \lambda\chi_2$ and let $L:H \to K^*$ be defined by $L(a\gamma(g)) = \lambda(g) \; \forall a\varepsilon N,\ g\varepsilon G$

Then λ is a one dimensional character of H. Let ϕ_1 and ϕ_2 be the characters of D_1 and D_2 respectively. Then since $\forall a\varepsilon N,\ g\varepsilon G$ we have

$$D_1(a\gamma(g)) = \psi(a)D_1(\gamma(g))$$

Then $\phi_1(a\gamma(g)) = \psi(a)\phi_1(\gamma(g))$

$$= \psi(a)\phi_2(\gamma(g))\lambda(g)$$

$$= \phi_2(a\gamma(g))L(\gamma(g))$$

That is $T^{-1}D_1(a\gamma(g))T = L(\gamma(g))D_2(a\gamma(g))$

where T is an invertible matrix which is allowed to be the identity.

Therefore D_1 is equivalent to $L \oplus D_2$ implying that P_1 is projectively equivalent to P_2.

From the above, we have the following:

3.1.4 Corollary

Let P_1 and P_2 be projective representations of G with respective characters ξ_1 and ξ_2 together with the factor set α. Then P_1 and P_2 are linearly equivalent if and only if

$$\xi_1 = \xi_2$$

In such a case, $\lambda = c(g) = 1$

3.2 Inner Products of Projective Characters

3.2.1 Definition

Let X_1 and X_2 be class functions in G, we define an inner product for X_1 and X_2 on G with values in $\mathbb{C}$ by

$$<X_1,X_2>_G = \frac{1}{|G|} \sum_{g \in G} X_1(g)\, \overline{X_2(g)}$$

where $\overline{X_2}$ is the complex conjugate of X_2.

3.2.2 Remark

Let τ_1 and τ_2 be irreducible projective characters of G and ξ_1 and ξ_2 be irreducible linear characters of H such that $\tau_i(g) = \xi_i(g)$ for all $g \in G$ i=1,2.Suppose that ξ_i determines the character ψ_i of N and D_i are ordinary representations of H.Then since each of τ_i is a class function and

$$D_i((\gamma(g))^{-1} = D_i(n(g^{-1},g)\gamma(g^{-1}))$$

$$= \psi_i(n(g^{-1},g))D_i(\gamma(g^{-1}))$$

where ξ_i = trace of D_i

Then, $\overline{\tau_2(g)} = \psi_2(n(g^{-1},g))\tau_2(g^{-1})$

Thus for the projective characters τ_i, i =1,2 the inner product is as follows:

$$<\tau_1,\tau_2>_G = \frac{1}{|G|} \sum_{g \in G} x_2(n(g^{-1},g))\tau_1(g)\tau_2(g^{-1})$$

We now prove

3.2.3 Corollary

Let P_1 and P_2 be irreducible projective representations of G with factor set $\alpha(g,h) = \psi(n(g,h))$. Then if ξ_i is the character of P_i(i=1,2.),

$$<\xi_1,\xi_2>_G = \begin{cases} 1 & \text{if } P_1 \text{ and } P_2 \text{ are linearly equivalent} \\ 0 & \text{otherwise} \end{cases}$$

Proof

If P_1 and P_2 are linearly equivalent, then $\xi_1 = \xi_2$ and the result follows, since in this case $X_1 = X_2$ so that

$$<X_1,X_2>_H = <\xi_1,\xi_2>_G = 1$$

That is, by the orthogonality relations in H.

If P_1 and P_2 are inequivalent, then from the orthogonality relations in H, we have $<\xi_1,\xi_2>_G = 0$.

3.2.4 Corollary

A projective representation P of G with character ξ is irreducible if and only if $<\xi,\xi>_G = 1$

Proof

If P is irreducible, then P is linearly equivalent to itself and hence, by corollary (3.2.3)

$$<\xi,\xi>_G = 1$$

Conversely, suppose that $<\xi,\xi>_G = 1$, where ξ is the character of P. P is linearly equivalent to the following projective representation

$$
\begin{matrix} a_1\left\{\begin{matrix}\\ \\ \end{matrix}\right. \\ \vdots \\ a_n\left\{\begin{matrix}\\ \\ \end{matrix}\right. \end{matrix}
\begin{pmatrix}
P_1(g) & & & & & & \\
 & \ddots & & & & \text{ZERO} & \\
 & & P_1(g) & & & & \\
 & & & \ddots & & & \\
 & & & & P_n(g) & & \\
 & \text{ZERO} & & & & \ddots & \\
 & & & & & & P_n(g)
\end{pmatrix}
$$

where $P_i (i=1,2,...n)$ are irreducible projective representations. Consider ξ_i to be the character of P_i, then

$$\xi(g) = \sum_{i=1} a_i\xi_i(g)$$

Hence $$<\xi,\xi>_G = \frac{1}{|G|} \sum_{g\in G} \xi(g)\xi(g^{-1})$$

$$= \sum_{g\in G} \frac{1}{|G|} \left(\sum_1^n a_i\xi_i(g)\right)\left(\sum_1^n a_j\xi_j(g^{-1})\right)$$

$$= \sum_{i=1}^n \sum_{j=1}^n a_i a_j \frac{1}{|G|} \sum_{g\in G} \xi_i(g)\xi_j(g^{-1})$$

$$= \sum_1^n a_i^2 \quad \text{for } \underline{i=j}$$

since the a_i are positive integers, $\sum_{i=1}^n a_i^2 = 1$ for some $1 \le i \le n$ and $a_j = 0$ for $i \neq j$. That is P has only one irreducible constituent and hence it is irreducible.

3.3 First Orthogonality Relations

Recall that for the set of n orthogonal idempotents $\{ e_1, e_2, \ldots, e_n \}$, $e_i e_j = e_i\delta_{ij}$ and let

$$e_i = |G|^{-1} \sum \chi_i(1)\chi_i(g^{-1})g$$

We now prove

3.3.1 Theorem

Let $\chi_1, \chi_2, \ldots, \chi_n$ be all the irreducible projective characters of G corresponding to a factor set α which we assume to be normalized and let g_i $(i = 1,2,\ldots,n)$

be a complete set of representatives for α regular classes of G,then

$$\sum_{g\in G} \chi_i(g)\chi_j(g^{-1}) = |G|\delta_{ij}$$

<u>Proof</u>

Since $e_i e_j = e_i\delta_{ij}$ and $e_i = |G|^{-1} \sum \chi_i(1)\chi_i(g^{-1})$

then

$$(|G|^{-1} \sum \chi_i(1)\chi_i(g^{-1})g)(|G|^{-1}\sum \chi_j(1)\chi_j(g^{-1})g)$$
$$= |G|^{-1}\sum \chi_i(1)\chi_i(g^{-1})g.\delta_{ij}$$

$$\frac{1}{|G|^2} \sum \chi_i(1)\chi_j(1)\chi_i(g^{-1})\chi_j(g^{-1})g^2 = \frac{1}{|G|} \sum \chi_i(1)\chi_i(g^{-1})g\cdot\delta_{ij}$$

$$\frac{\chi_i(1)\chi_j(1)}{|G|^2} \sum \chi_i(g)\chi_j(g^{-1}) = \frac{\chi_i(1)\chi_i(1)}{|G|}\delta_{ij}$$

Then by comparing the coefficients of the identity element,we have

$$\sum \chi_i(g)\chi_j(g^{-1}) = |G|\delta_{ij}$$

<u>3.4 Second Orthogonality Relations</u>

With $h_k = |c_k|$, the order of the conjugacy class c_k, we may write the first orthogonality relations as;

$$|G|^{-1}\sum_{k=1}^{r} h_k\chi_i(g_k)\chi_j(g_k^{-1}) = \delta_{ij} \qquad (1)$$

Also, since $|C_G(g_k)|\,|C_k| = |G|$ then

$$|C_G(g_k)|h_k = |G|$$

That is $\dfrac{h_k}{|G|} = |C_G(g_k)|^{-1}$ and (1) can now

be written as $\sum \chi_i(g_k)\chi_j(g_k^{-1}) \dfrac{h_k}{|G|} = \delta_{ij}$

This implies that

$$\sum \chi_i(g_k)(|C_G(g_k)|^{-1})\chi_j(g_k^{-1}) = \delta_{ij} \qquad (2)$$

from where the second orthogonality relations follow by letting A be an rxr matrix whose (i,k)-entry a_{ik} is $\chi_i(g_k)$ and B an rxr matrix whose (k,j)-entry b_{kj} is $|C_G(g_k^{-1})|^{-1}\chi_j(g_k^{-1})$.

That is,for all i,j

$$\sum_{k=1}^{r} b_{jk}a_{ki} = \delta_{ij}$$

which can also be written as

$$\sum_{k=1}^{r} |C_G(g_j)|^{-1}\chi_k(g_j^{-1})\chi_k(g_i) = \delta_{ij}$$

Thus we have the following lemma

3.4.1 lemma

Let χ_i (i=1,2,...,n) be the projective characters of G with the assumed normalized factor set α and let c_i (i=1,2,...,n) be the α-regular classes of G with g_i as the representative element of c_i, then

$$\sum_{i=1} \chi_i(g_j)\chi_i(g_k^{-1}) = |C_G(g_j)|\delta_{jk}$$

We now show the following important result which will be used later.

3.4.2 Lemma

Let $\alpha(g,h) = \psi(n(g,h))$ be a special factor set of G and let g be any α-regular element of G. Then,

$$(\alpha(h^{-1},h))^{-1}\alpha(h^{-1},gh)\alpha(g,h) = 1 \quad \text{for all } h \in G$$

Proof

Let $\delta_\alpha(g,h) = (\alpha(h^{-1},h))^{-1}\alpha(h^{-1},gh)\alpha(g,h)$. As g is α-regular, there exists an irreducible projective character χ of G with factor set α such that $\chi(g) \neq 0$.

Let P be the projective representation of G with character χ and D an ordinary representation linearizing P.

That is, $P(g) = D(\gamma(g))$ for all $g \in G$.

Then for $h \varepsilon G$,

$D(\gamma(h^{-1})\gamma(g)\gamma(h))$

$$= D((n(h^{-1},h)^{-1}n(h^{-1},gh)n(g,h)\gamma(h^{-1}gh))$$

$$= \psi(n(h^{-1},h)^{-1})\alpha(h^{-1},gh)\alpha(g,h)P(h^{-1}gh)$$

$$= \delta_{\alpha}(g,h)P(h^{-1}gh)$$

That is $\chi(g) = \delta_{\alpha}(g,h)\chi(h^{-1}gh) = \delta_{\alpha}(g,h)\chi(g)$

That is $\chi\ (g) = \delta_{\alpha}(g,h)\chi(g)$

since $\chi(g)$ is finite and $\chi(g) \neq 0$ then

$$\delta_{\alpha}(g,h) = 1$$

That is $(\alpha(h^{-1},h))^{-1}\alpha(h^{-1},gh)\alpha(g,h) = 1$

3.4.3 Corollary

An element $g \varepsilon G$ is α-regular if and only if there exists an irreducible projective character χ of G with factor set α such that $\chi(g) \neq 0$

Proof

If g is α-regular, then $\chi(g) \neq 0$ for some χ conversely, suppose $\chi(g) \neq 0$ where χ is the irreducible character corresponding to a special factor set α. Then by lemma 3. 4.2. $\delta_{\alpha}(g,h) = 1$ for all$h \varepsilon G$.

In particular, choose $h \varepsilon C_G(g)$ so that

$$\delta_{\alpha}(g,h) = 1 \text{ for all} h \varepsilon C_G(g)$$

That is $(\alpha(h^{-1},h))^{-1}\alpha(h^{-1},gh)\alpha(g,h) = \alpha(g,h)(\alpha(h,g))^{-1} = 1$

That is $\alpha(g,h)(\alpha(h,g))^{-1} = 1$ for all $h \varepsilon C_G(g)$

That is $\alpha(g,h) = \alpha(h,g)$ for all$h \varepsilon C_G(g)$.

3.5 Induced Characters and Representations

Consider a subgroup M of G. Then a left $(KG)_{\alpha}$-module V can be regarded as a left $(KM)_{\alpha}$-module by restriction. Let V_M denote the left $(KM)_\alpha$-module of V when restricted to M. Let V afford a representations P and V_M afford a representation P_M. If P corresponds to a factor set α, then α determines a factor set α_M of P_M by restriction. Let χ and χ_M be the respective characters of P and P_M. If χ is irreducible, then χ_M is in general reducible and elements which are α_M regular in M are not in general α-regular in G.

We now describe a construction which associates with each $(KM)_\alpha$-module W, a left $(KG)_\alpha$-module W^G.

3.5.1 Definition

An induced left $(KG)_\alpha$ -module W^G from M is one which is such that $W^G = (KG)_\alpha \oplus_{(KM)_\alpha} W$ where M is a subgroup of G and W is a left $(KM)_\alpha$-module.

The representation afforded by W^G is said to be an induced representation of G.

When given a projective representation $P(g) = (S_{ij}(g))$ afforded by W, we can obtain an induced representation of G. We now briefly explain how to get this

Let $(W:K) = r$ and $\{w_1,w_2,...,w_r\}$ be a K-basis for W and let $G = \bigcup_{i=1}^{n} g_iM$ be a coset decomposition of G modulo M. Then every element of G can be uniquely expressed in the form g_im $(1\le i \le n)$ and $m\varepsilon M$. Hence every element of $(KG)_\alpha$ is uniquely expressed in the form $\sum_{i=1}^{n}\gamma(g_i)b_i$, $b_i\varepsilon(KM)_\alpha$.

Thus we have,

$$(KG)_\alpha = \gamma(g_1)(KM)_\alpha \oplus \gamma(g_2)(KM)_\alpha \oplus \dots \oplus \gamma(g_n)(KM)_\alpha$$

That is $(KG)_\alpha$ is a free right $(KM)_\alpha$-module with basis $\{\gamma(g_1), \gamma(g_2), \dots, \gamma(g_n)\}$. Therefore,

$$W^G = (\gamma(g_1)(KM)_\alpha \otimes_{(KM)_\alpha} W) \otimes (\gamma(g_2)(KM)_\alpha \otimes_{(KM)_\alpha} W)$$
$$\otimes \dots \otimes (\gamma(g_n)(KM)_\alpha \otimes_{(KM)_\alpha} W)$$

Since $\gamma(g_i)b_i \otimes w = \gamma(g_i) \otimes bw \quad \forall b \varepsilon (KM)_\alpha\ w \varepsilon W$, we may write

$$G = \gamma(g_1) \otimes W \oplus \gamma(g_2) \otimes W \oplus \dots \oplus \gamma(g_n) \otimes w$$

From the isomorphism, $\gamma(g_i)b \to b$ between $\gamma(g_i)(KM)_\alpha$ and

$(KM)_\alpha$ we get $\gamma(g_i)(KM)_\alpha \otimes_{(KM)_\alpha} W \cong (KM)_\alpha \otimes_{(KM)_\alpha} W \cong W$

Thus $\gamma(g_i)(KM)_\alpha \otimes_{(KM)_\alpha} W \cong W$

so that $(W^G:K) = [G:M](W:K)$

Thus the elements of W^G have a unique expression of the form $\Sigma\gamma(g_i) \otimes u_i$ where the u_i are uniquely determined in W. Hence a K-basis for W^G is $\{\gamma(g_i) \otimes w_j / i=1,2,\dots,n.\ \ j=1,2,\dots,r\}$. Express $\gamma(g)(\gamma(g_i) \oplus w_j)$ as a K -linear combination of basis elements. Rearranging the basis elements in order, we get

$$\gamma(g_1)\otimes w_1, \dots, \gamma(g_1)\otimes w_r, \dots, \gamma(g_2)\otimes w_1, \dots, \gamma(g_2)\otimes w_r, \dots, \gamma(g_n)\otimes w_1, \dots, \gamma(g_n) \oplus w_r$$

Then the above implies that $\forall\ g \varepsilon G$,

$$P^G(g) = \left|\begin{array}{c|ccc|c} * & (i,1) & & (i,r) & * \\ \hline * & & \alpha(g,g_i)\alpha^{-1}(g_k, g_k^{-1}gg_i)P(g_k^{-1}gg_i) & & * \\ \hline * & & & & * \end{array}\right| \begin{array}{c}(k,1)\\ \vdots \\ (k,r)\end{array}$$

Where P is extended to the whole group by setting $P(g) = 0$. for all $g \notin M$. Infact, as obtained, P^G is the induced projective representation. Also let X be the character of P^G. Then,

$$\varepsilon^G(g) = \sum_1^n \alpha(g,g_i)\alpha^{-1}(g_i,g_i^{-1}gg_i)\varepsilon(g_i^{-1}gg_i)$$

$$= \sum_{i=1}^n (\alpha(g_i^{-1},g_i))^{-1}\alpha(g_i^{-1},gg_i)\alpha(g,g_i)\varepsilon(g_i^{-1}gg_i)$$

Which gives the formula for the induced character for which the F r oben i us recip r ocit y theorem holds.

CHAPTER FOUR

PROJECTIVE REPRESENTATIONS OF FINITE ABELIAN GROUPS.

4.1 ABELIAN GROUPS

We start the chapter by discussing the structure of arbitary finite abelian groups. Abelian groups mer i t attention because they are fundamental in all of group theory as well as in many other branches of mathematics.

The following result is useful.

4.1.1 Theorem [2]

Any abelian group whose order is not a power of a prime number is a direct product of all its sylow subgroups.

We now prove

4.1.2 Theorem

Every finite abelian group is a direct product of cyclic groups.

Proof

We assume that the abelian group G is not a cyclic group and that S is an element of highest order P^a in G where P is a prime. Consider the quotient group $\frac{G}{<S>}$. Let $H<S>$ be an element of highest order P^r in this quotient group for the same prime P. Then $H^{pr} = S^{npt}$ $t \leq a$, where n is not divisible by P. In the coset $H<S>$ it is possible to select an element of order P^r; since the order of H is not higher than the order of S, $r \leq t$.

If $H^{p^r} \not\equiv 1$, take the element $HS^{-np^{t-r}}$ instead of H and obtain

$$(HS^{-np^{t-r}})^{p^r} = 1$$

Assume that H has been properly selected so that the groups <S> and <H> are relatively prime and their direct product <S>x<H> is contained in G. If G is not exhausted by this product, then let us again take an element F<S>x<H> of highest order P^{β} in the quotient group $\frac{G}{<S> x <H>}$. Then

$$F^{p^{\beta}} = S^{np^{\alpha}} H^{mp^{\sigma}}$$

$$\beta \leq \alpha, \beta \leq \sigma$$

If $FP^{\beta} \not\equiv 1$ then instead of F it would be possible to select $FS^{-np^{\alpha-\beta}} H^{-mp^{\sigma-\beta}}$, which would be of order P^{β}. Selecting F in such a manner, we get the direct product

<S> x <H> x <F>

If this does not exhaust G, we continue in the same way until we get a direct product

<S> x <H> x <F> x...x <K>

which is equal to the group G.

<u>4.1.3 Definition</u>

Let G be an arbitrary finite abelian group and r an integer, then

$$G(r) = \{x \varepsilon G : x^r = e\}$$

clearly, G(r) is a subgroup of G as G is an abelian group. The following can now be proved.

4.1.4 Lemma

If G and G′ are isomorphic abelian groups, then for every integer S, G(S) and G′(S) are isomorphic.

Proof

Since G and G′ are isomorphic then there exists an isomorphism $\varphi: G \to G'$. If $x \varepsilon G(s)$ then $x^s = e$ and $\varphi(x^s) = \varphi(e) = e'$. Hence $\varphi(x)^S = e'$ and so $\varphi(x)$ lies in $G'(s)$. Thus $\varphi(G(s)) \subset G'(s)$. Again, if $u' \varepsilon G'(s)$ then $(u')^s = e'$. But since φ is onto, $u' = \varphi(y)$ for some $y \varepsilon G$. Therefore $e' = (u')^s = \varphi(y)^s = \varphi(y^s)$. $\varphi : G(s) \to G'(s)$ is injective since for $x,y \varepsilon G(s)$ if $\varphi(x^S) = \varphi(y^S)$ then

$\varphi(e) = \varphi(e)$ implying that $e' = e'$

i.e $e' = e' = e$. Hence

$e = e \Rightarrow x^s = y^s$

Thus we have $y^s = e$ and so $y \varepsilon G(s)$. Thus showing that G(s) and G′(s) are isomorphic.

4.1.5 Remark

It now follows from (4.1.1) and (4.1.2) that two abelian groups are isomorphic when and only when their sylow subgroups are isomorphic in some order and therefore the problem of determining all possible abstract abelian groups is readily reduced to that of determining all prime power abelian groups.

From now on we shall assume that G is a prime power abelian group.

4.1.6 Definition

Let G be an abelian group of order P^n, P a prime, so that it is a direct product of K cyclic groups each of order P^{n_i} (i=1,,2,...,k) with $n_1 \geq n_2 \geq ... \geq n_k$. Then the integers $n_1, n_2, ..., n_k$ are known as the invariants of G.

4.1.7 Definition

If G is an abelian group of order P^m, P a prime and the orders of its direct summands are p^{m_1}, p^{m_2}, ...,p^{mk}; ($m = m_1 + m_2 + ... + m_k$), then the group G is said to be of type $(m_1, m_2, ..., m_k)$

The following result is due to Heisteins.

4.1.8 Lemma [6]

Let G be an abelian group fo order P^n, p a prime. Suppose that $G = A_1 \times A_2 x... \times A_k$, where each $A_i=(a_i)$ is cyclic of order p^{ni}, and $n_1 \geq n_2 \geq n_k > 0$. If $m \varepsilon Z$ such that $n_t > m \geq n_{t+1}$ then $G(P^m) = B_1 x...xB_t x A_{t+1} x...xA_k$ where B_i is cyclic of order P^m, generated by $a_i^{p^{n_i - m}}$, for $i \leq t$. The order of $G(P^m)$ is p^u

where $u = mt + \sum_{i=t+1}^{k} n_i$

Furthermore, the order $0(G(p))$ of $G(p)$ is p^k.

We now prove

4.1.9 Theorem

Two abelian groups of order p^n are isomorphic if and only if they have the same invariants.

Proof

Suppose G and G′ are abelian groups of order P^n with same invariants $n_1,n_2,...,n_k$, then

$$G = A_1 x A_2 x ... x A_k \text{ and } G' = B'_1 \ x \ B'_2 \ x ... x B'_k$$

where $A_i = (a_i)$ and $B'_i = (b'_i)$ are cyclic groups of order P^{ni}. If we define a mapping $\varphi:G \rightarrow G'$ by $\varphi(a_i^{\alpha_i} ... a_k^{\alpha_k}) = = (b'_1)^{\alpha_1} ... (b'_k)^{\alpha_k}$ then φ is a homomorphism. φ as defined is both injective and surjective. Hence $\varphi:G \rightarrow G'$ is a bijective homomorphism and therefore, G is isomorphic to G′. Conversely, let $G = A_1 x ... x A_k$, $G' = B'_1 \ x \ B'_2 x \ ... \ x \ B'_2$; A_i, B'_i are cyclic groups of orders p^{ni}, p^{hi} respectively, where $n_1 \geq ... \geq n_k > 0$ and $h_1 \geq ... \geq h_s > 0$. If G and G′ are isomorphic, we only need to show that k=s for G and G′ to have the same invariants Since G and G′ are isomorphic then by lemma 4.1.4 $G(p^m)$ must be isomorphic for any integer $m \geq 0$, hence must have the same order. When m=1, $0(G(P)) = 0(G'(P))$. But from Lemma 4.1.8, $0(G(P)) = P^k$ and $0(G'(P)) = P^s$. Hence $P^k = P^s$ and therefore,

$$k = s$$

and G and G′ have same invarinats.

4.1.10 Remark

Theorem 4.1.9 has two immediate consequences, those being;

(i) that an abelian group of order p^n can be decomposed in only one way, as a direct product of cyclic subgroups and the invariants of G completely determine G.

(ii) that two different partitions of n give rise to non isomorphic abelian groups of order p^n.

4.1.11 Corollary [7]

The number of non isomorphic abelian groups of order $P_1^{\alpha_1} \ldots P_r^{\alpha_r}$ where the P_i are distinct primes and each $\alpha_i > 0$, is $P(\alpha_1)P(\alpha_2) \ldots P(\alpha_r)$, where P(u) denotes the number of partitions of u.

4.2 SCHUR MULTIPLIERS OF FINITE ABELIAN GROUPS.

In this section, we consider the schur multipliers of some of the finite abelian groups. It follows trivially from the definition of a projective representation that all the representations of cyclic groups are ordinary representations. Their schur multipliers are therefore isomorphic to {1}.

NOTATION

The following notation shall be used throughout. Let k_1 and k_2 be any positive integers. Then $C_{(k_1,k_2)}$ shall denote a cyclic group of order (k_1,k_2) where (k_1,k_2) is the greatest common divisor of K_1 and k_2.

We first consider the abelian groups which can be written as a direct product of two cyclic groups. Let $C_{k,2}$ denote the direct product of two cyclic groups of orders k_1 and k_2 respectively, so that

$$C_{k,2} = \{S_1, S_2 : S_1^{k_1} = S_2^{k_2} = e;\ S_1S_2 = S_2S_1\}$$

then we prove the following

4.2.1 Theorem [13]

Keeping the above notation, we have

$$H^2(C_{k,2}, C^*) \cong C_{(k_1,k_2)}$$

Proof

For any factor set α of $C_{k,2}$ it is possible to choose a cohomologous factor set μ which is such that

$$\mu(S_1^{w_1}, S_2^{w_2}) = \alpha(S_1^{w_1}, S_2^{w_2})$$

for all $w_1 \varepsilon \{0,...k_1-1\}$ and $w_2 \varepsilon \{0,...,k_2-1\}$ (i)

Consider P to be the projective representation of $C_{k,2}$ with th e factor set satisfying (i) and put $P_i=P(S_i)$; i=1,2. From the definition of a projective representation, it follows that

$$P_1^{k_1} = aI,\ P_2^{k_2} = bI;\ P_1P_2 = \lambda P_2P_1$$

where $a,b,\lambda\ \varepsilon\ C^*$

We may obtain a projective representation Q of $C_{k,2}$ by letting

$$\mu(S_1^{w_1}, S_2^{w_2}) = a^{w_1/k_1} b^{w_2/k_2} \quad \text{where}$$

$$w_1 \varepsilon \{0,....,k_1-1\},\ w_2 \varepsilon \{0,...,k_2-1\}$$

and defining Q by

$Q(s) = \mu(s)P(S)$ for all $S\varepsilon C_{k,2}$. If we now set $Q_i = Q(S_i)$ i=1.2. then we have

$$Q_1^{k_1} = I = Q_2^{k_2};\ Q_1Q_2 = \lambda Q_2Q_1 \qquad \text{(ii)}$$

Thus our new representation Q is such that

$$Q(S_1^{w_1} S_2^{w_2}) = Q_1^{w_1} Q_2^{w_2}$$

Therefore the class $[\alpha]$ depends entirely on λ. The proof to show that $H^2(C_{k,2}, \mathbb{C}^*)$ is isomorphic to a subgroup of $C_{(k_1,k_2)}$ is now completed by multiplying (ii) through by Q_1^{-1} from the right and raising to the power k_2 i.e $(Q_1 Q_2 Q_1^{-1})^{k_2} = (\lambda Q_2 Q_1 Q_1^{-1})^{k_2}$ thereby getting $1 = \lambda^{k_2}$. Also multiplying Q_2^{-1} throughout (ii) from the left and raising to the power k_1

That is $(Q_2^{-1} Q_1 Q_2)^{k_1} = (Q_2^{-1} \lambda Q_2 Q_1)^{k_1}$ gives $1 = \lambda^{k_1}$ so that $\lambda^{(k_1,k_2)} = 1$.

$H^2(C_{k,2}, \mathbb{C}^*)$ is infact equal to $C_{(k_1,k_2)}$ for if we construct a pair of non singular matrices Q_1 and Q_2 as follows

$$Q_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & \lambda & 0 & \cdots & 1 \\ 0 & 0 & \lambda^2 & \cdots & 0 \\ \cdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Then,

$$Q_1^2 = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdots & & & & & \\ 0 & 0 & 0 & \cdots & & 1 \\ 1 & 0 & 0 & \cdots & & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \cdots & & & & & \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

so that

$$Q_1^{k_1-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ \cdots & & & & \\ 0 & 0 & \cdots 1 & 0 & 0 \\ 0 & 0 & \cdots 0 & 1 & 0 \end{pmatrix}$$

Hence,

$$Q_1 Q_1^{k_1-1} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \cdots & & & & & \\ 0 & 0 & 0 & & \cdots & 1 \\ 1 & 0 & 0 & & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ \cdots & & & & \\ 0 & 0 & \cdots 1 & 0 & 0 \\ 0 & 0 & \cdots 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots 1 & 0 \\ 0 & 0 & 0 & \cdots 0 & 1 \end{pmatrix}$$

$$= I$$

That is $Q_1^{k_1} = Q_1 Q^{k_1 - 1} = I$

$$Q_2 = \begin{pmatrix} 0 & \lambda & 0 & \cdots & 1 \\ 0 & 0 & \lambda^2 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

In a similar manner

$$Q_2^{k_2} = I$$

Hence Q_1 and Q_2 satisfy the equation in (ii) for all values of λ such that $\lambda^{(k_1,k_2)} = 1$ and generate a projective represenation of $C_{k,2}$.

We now consider the general finite abelian group G. That is G is a direct product of r cyclic groups. The following shall be required in the sequel:

4.2.2 Definition

Let $P(G,H,K^*) = \{f:GxH \to K^*: f(g_1g_2,h)$
$= f(g_1,h)f(g_2,h)$ and $f(g,h_1h_2) = f(g,h_1)f(g.h_2)$
for all $g,g_1,g_2 \varepsilon G$; $h,h_1,h_2 \varepsilon H\}$

The set $P(G,H,K^*)$ is known as a set of pairings.

The following result is due to Yamazaki [13].

4.2.3 Lemma

Let $G \cong G_1 \times G_2 \times \ldots \times G_r$, then

$$H^2(G,K^*) \cong \prod_{i=1}^{r} H^2(G_i,K^*) \times \prod_{1 \leq j, i \leq r} P(G_i,G_j;K^*)$$

We prove

4.2.4 Corollary [13]

$$P(C_m \times C_m; K^*) \cong C(m_i,m_j)$$

Proof

Since $H^2(C_{m_i},K^*) = 1$ for each $i=1,2,\ldots,r$ then $H^2(C_{m_i} \times C_{m_j}; K^*) \cong P(C_{m_i} \times C_{m_j}; K^*)$ by Lemma 4.2.3.

Also, $H^2(C_{m_i} \times C_{m_j};K^*) \cong C(m_i,m_j)$ by theorem 4.2.1 so that $P(C_{m_i} \times C_{m_j};K^*) \cong C(m_i,m_j)$

As a consequence of the above, we have the following:

4.2.5 Remark

Let $C_{m,k}$ denote the direct product of k cyclic groups $C_{m_1} \ldots C_{m_k}$ of orders $m_1,\ldots,m_k$ respectively. Then $H^2(C_{m,k}; \mathbb{C}^*)$ is equal to the direct product of the cyclic groups $C(m_i,m_j)$ of order (m_i,m_j), $1 \leq i<j\leq k$.

4.3 α-REGULAR CLASSES OF FINITE ABELIAN GROUPS

As earlier seen, the number of irreducible projective representations of a group G with factor set α is the same as the number of α-regular classes. Since for abelian groups, conjugacy classes consist of single elements, it will suffice to consider α-regular elements in the sequel.

Let us consider an abelian group $C_m^{(n)} = C_m \times \ldots \times C_m$ (n-times) which is such that $|C_m^{(n)}| = m^n$ and it is generated by $s_1,s_2,\ldots,s_n$ each of order m.

We may now obtain α-regular elements of $C_m^{(n)}$.

and our work follows that of Morris and Saeed [10].

4.3.1 Theorem

Let $C_m^{(n)}$ have the same meaning as before and let α be the factor set of $C_m^{(n)}$ satisfying the property that $\alpha'(s_i, s_j) = \varphi$ $(1 \leq i < j \leq n)$ where φ is a promitive k^{th} root of unity, where k divides m. If m=kd then

(i) If n is even, then $C_m^{(n)}$ has d^n α-regular elements.

(ii) If n is odd, then $C_m^{(n)}$ has kd^n α-regular elements.

Proof

Since for any $s \varepsilon C_m^{(n)}$, s can be expressed in the form $s = s_1^{a_1} \ldots s_n^{a_n}$, $a_i \varepsilon \{0,\ldots,m-1\}$ for all i=1,2,...,n. Then $\alpha'(s_i, s) = 1$ for all i=1,2,..,n if s has to be α-regular. But $\alpha'(s_i, s_1^{a_1} \ldots, s_n^{a_n})$

$$= \prod_{j=1}^{n} \alpha'(s_i, s_j^{a_j})$$

$$= \left(\prod_{i.j} \alpha'(s_i, s_j^{a_i}) \right) \left(\alpha'(s_i, s_i^{a_i}) \left(\prod_{j>i} \alpha'(s_i, s_j^{a_j}) \right) \right)$$

$$= \varphi^{(-a_1 \ldots . a_{i-1} + a_{i+1} \ldots + a_n)}$$

$$= \varphi^{-(a_1 + \ldots + a_{i-1} - a_{i+1} \ldots - a_n)}$$

$= 1$ *if and only if*

$$a_1 + \ldots + a_{i-1} - a_{i+1} \ldots a_n \equiv 0 \pmod{k} \quad i=1,2,\ldots,n$$

and this system of congruences is equivalent to $AX = 0 \pmod{k}$ where A is an nxn matrix given by

$$A = \begin{pmatrix} 1 & 1 & \cdots\cdots\cdots & 1 & 0 \\ 1 & 1 & \cdots\cdots\cdots & 0 & -1 \\ 1 & \cdots\cdots\cdots & & -1 & -1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 0 & & -1 & -1 \\ 0 & -1 & & -1 & -1 \end{pmatrix}$$

and $\chi^t = (a_1,\ldots,a_n)$

Thus, the determinant of A will give 1 or -1 whenever n is an even positive integer and zero otherwise.

$$\text{i.e} \quad \det A = \begin{cases} (-1)^{\frac{1}{2}n} & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

<u>case (i)</u>

If n is even, the only solution to the system $a_1+\ldots+ a_{i-1} - a_{i+1} \ldots -a_n \equiv 0 \pmod{k}$ is $a_1 = a_2 = \ldots = a_n = 0 \pmod{k}$ which is equivalent to $a_i \ \varepsilon \ (0,k,2k,\ldots,(d-1)k\}$ $i = 1,2,\ldots,n$

Therefore s is α-regular only if

$$s = s_1^{kt_1} \ldots s_n^{kt_n} \tag{1}$$

where $t_i \ \varepsilon \ \{0,1,\ldots, d-1\}$

Also for $h = 0,\ldots, m-1$; $i=1,\ldots,n$

we have $\alpha'(s_i^h, s) = \alpha'(s_i,s)^h$

$$= \left(\prod_{j=1}^{n} \alpha'(s_i,s_j^{kt_j})\right)^h$$

$$= \varphi^{(kt_1 +\ldots kt_{i-1} -kt_{i+1} \ldots -kt_n)}$$

$$= \varphi^{h(kt_1 + \ldots kt_{i-1} - kt_{i+1} \ldots -kt_n)}$$

$$= \varphi^{hk(t_1 + \ldots t_{i-1} - t_{i+1} \ldots -t_n)}$$

$$= (\varphi^k)^{h(t_1 +\ldots t_{i-1} - t_{i+1} \ldots - t_n)}$$

$$1^{h(t_1 + \dots t_{i-1} - t_{i+1} \dots -t_n)}$$

$= 1$ showing that all such elements in (1) are α-regular. Therefore, for any $s' = s_1^{a_1} \dots s_n^{a_n}$ in $C_m^{(n)}$,

$$\alpha'(s',s) = \alpha'(s_1^{a_1},s) \dots \alpha'(s_n^{a_n},s)$$
$$= 1.1.1\dots1$$
$$= 1$$

Hence the number of elements as in (1) above is d^n, confirming case (i)

<u>Case (ii)</u>

Now suppose n is odd, then the above system reduces to

$a_1 = -2_2 = a_3 = -2_4 = -2_5 = \dots = a_n (\mathrm{mod}\ k)$ if a_i satisfies the above system and α′ has the same meaning as above, then

$$\alpha'(s_i,s) = \varphi^{(a_1+\dots+a_{i-1} - a_{i+1} \dots -a_n)}$$

for all $i=1,2,\dots,n$

Then $\alpha'(s_1^{t_1} \dots s_n^{t_n},s) = \prod_{i-1}^{n} \alpha'(s_i^{t_i},s)$

$$= \prod_{i=1}^{n} \alpha'(s_i, s)^{t_i}$$

$= 1$ for all $s \varepsilon C_m^{(n)}$

so that s is α-regular.

We now note that by fixing $a_1 = h$, $0 \leq h \leq m-1$ a_2 and a_4 may be allowed to take any value from $\{-h + \delta_k : \delta = 0,\dots,(d-1)\}$ and $a_3,a_5,\dots$ may be allowed to take values from $\{h + \delta_k : \delta = 0,\dots, (d-1)\}$ and the number of α-regular elements equaling $md^{n-1} = kd^n$ consequently completing the proof.

4.3.2 Corollary

When m is a prime integer and n is odd, then the number of α-regular elements in $C_m^{(n)}$ is equal to m, the order of the generators s_i(i=1,2,...,n) of $C_m^{(n)}$.

Proof

Since $C_m^{(n)} = \underbrace{C_m \times C_m \times \ldots \times C_m}_{n\text{-times}}$ and

any $s \varepsilon C_m^{(n)}$ has the form $s = s_1 \ldots s_n$ each of order m, then if m is a prime integer, in 4.3.1 above, we have m = m.1 and the rest now follows from theorem 4.3.2 (ii).

4.4 THE NUMBER AND DEGREES OF THE IRREDU PROJECTIVE REPRESENTATIONS OF FINITE ABELIAN GI

Here we consider the number and degrees of the irreducible inequivalent projective representations of finite abelian groups. In particular, we obtain the number and degrees of irreducible representations of abelian groups $C_m^{(n)}$ of type $(a_1,\ldots,a_n)$.

The following result follows directly from theorem 4.3.1

4.4.1 Theorem

Let α be a factor set of $C_m^{(n)}$ and α′ have the same meaning as before. Let m=kd then

(i) If n is even, $C_m^{(n)}$ has d^n number of inequivalent irreducible projective representations each of degree $k^{n/2}$.

(ii) If n is odd, $c_m^{(n)}$ has kd^n number of inequivalent irreducible projective representations each of degree $k^{(n-1)/2}$.

Following from the work of Morris [9] we can determine the number of inequivalent irreducible projective representations with factor set α over $\mathbb{C}$ in the following two cases,

(a) when $\beta(i,j) = w$, where w is a primitive m^{th} root of unity and

(b) when m is even and $\beta(i,j) = -1$ $(1< i<j<n)$

When α satisfies condition (a) or (b) then a particular case of 4.4.1 gives the following

4.4.2 Theorem

We keep the above notation. Let G be an abelian group of order m^n generated by $g_1, \ldots, g_n$ and let α be a factor set of G such that $\mu(i) = 1$ $(i = 1,2,\ldots,n)$ and $\beta(i,j) = w$ $(1\leq i < j\ n)$ where w is a primitive m^{th} root of unity. Then if $n = 2\mu$ is even, G has only one irreducible inequivalent projective representation of degree m^{μ} and if $n=2\mu + 1$ is odd, G has m inequivalent irreducible projective representations of degree m^{μ}.

Proof

Let T be a projective representation of G with factor set α and let $T(g_i) = e_i (i=1,2,\ldots,n)$. We can now determine the number of elements $a = e_{k_r}^{\alpha_1} \ldots e_{k_r}^{\alpha_r}$ where

$1\leq k_1 < k_2 < \ldots < k_r \leq n;\ 0 \leq \alpha_i \leq m-1$

such that $e_i^{-1} ae_i = a$ (i=1,2,...,n). In particular, we have $e_{k_i}^{-1} ae_{k_i} = a$ (i=1,2,...,n)

$e_{k_i}^{-1} ae_{k_i} = \omega^{(\alpha_1+...\alpha_{i-1}-\alpha_{i+1}...-\alpha_r)}.a$ since

$w^{(\alpha_1+...\alpha_{i-1}-\alpha_{i+1}...-\alpha_r)} = 1$ from which we get AX = 0(mod m)

where A is the rxr matrix given in the proof of theorem 4.3.2 and

$$X^t = (\alpha_1,...,\alpha_r)$$

$$\text{Hence det } A = \begin{cases} 0 & \text{if } r \text{ is odd} \\ (-1)^{\lambda} & \text{if } r = 2\lambda \text{ is even} \end{cases}$$

If r is even, the only solution is a trivial one i.e $\alpha_1 = \alpha_2 = ... = \alpha_r = 0$.

Also if $r = 2\lambda + 1$ is odd, the above linear system reduces to $\alpha_1 \equiv -\alpha_2 \equiv \alpha_3 \equiv ... \equiv -\alpha_{2\lambda} \equiv \alpha_{2\lambda+1}$ (mod m).

Thus if $g_{k_1}^{\alpha_1} ... g_{k_r}^{\alpha_r}$ is an α-regular element it can only take the form

$$g_{k_1}^{i}\, g_{k_2}^{-i} \ldots g_{k_{r-1}}^{-i}\, g_{k_r}^{i} \quad (i = 0,1,...,m-1)$$

Also when $K \not\equiv K_i$, $e_k^{-1} ae_k = a$. If $k \not\equiv k_i$ (i = 1,2,...,r) and put $k_o = 1$, $k_{r+1}=n$ then $k > k_j$ and $k < k_{j+1}$ for some $0 \leq j \leq r + 1$ and

$$e_k^{-1} ae_k = W^{(\alpha_1+...+\alpha_j - \alpha_{j+1}... - \alpha_r)}.a=w^{\mp}a$$

for som $0 \leq i \leq m-1$

i.e if r<m, i=0. Thus if $n=2\mu$ is even, 1 is the only α-regular element and if $n=2\mu+1$ is odd, the α-regular elements are given by

$g_1^i g_2^{-i} \ldots g_{2\mu+1}^{-i}$ (i = 0,1,...,m-1)

Hence when n=2μ is even, G has only one inequivalent irreducible projective representation whose degree is m^{μ}. When n=2μ+1 is odd, the m inequivalent projective representations have same degree m^{μ}.

case (b)

Let G = $\langle g_1\rangle \times \langle g_2\rangle \times \ldots \times \langle g_m\rangle$ as before, n=2μ (even) and α satisfies

$$\mu(i) = 1 \quad \text{for all } i\varepsilon\{1,2,\ldots,m\}$$

$$\text{and } \beta(i,j) = -1 \quad (1\leq i < j \leq m)$$

consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; $e = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$;

$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and for i = 1,2,...,t, let

$M_{2i-1} = D \otimes D \otimes \ldots \otimes D \otimes B \otimes A \otimes \ldots \otimes A$

$M_{2i} = D \otimes D \otimes \ldots \otimes D \otimes C \otimes A \otimes \ldots \otimes A$

$M_{2t+1} = D \otimes D \otimes \ldots \otimes D \otimes \ldots \otimes D$

be tensor products, where B and C are in the ith position and each product has t factors.

If m=2t and w is a primitive 2k-th root of 1 for i=1,2,...,m and 0≤ 2i<k, define

$$P_{(\lambda_1,\ldots,\lambda_m)}(g_i) = \omega^{\lambda_i} M_i$$

In Morris [9], the projective representations are given by the following result

(i) If m = 2t is even, then G has k^m inequivalent irreducible projective representations with factor set α

(ii) If m = 2t + 1 is odd, then G has $2k^m$ inequivalent irreducible projective representations with factor set α.

The following fundamental result is due to Yamazaki.

4.4.3 Lemma

All the irreducible projective representations of an abelian group have the same degrees.

This follows easily from Theorem 4.4.1

We now prove the following result,

4.4.4 Lemma [13]

The sum of the squares of the degrees of the irreducible projective representations of a group with a fixed factor set is equal to the order of G.

Proof

Let $\{T_i\}$ be a complete set of projective representations of the group G with character $\{x_i : i=1,2,\ldots,S\}$ with degrees f_i respectively then

$|G| = r_G(e)$ where r_G is the regular character of G

$$= \sum_{i=1}^{s} f_i x_i(e)$$

$$= \sum_{i=1}^{s} f_i f_i$$

$$= \sum_{i=1}^{s} f_i^2$$

4.5 SOME IRREDUCIBLE PROJECTIVE REPRESENTATIO[NS] OF FINITE ABELIAN GROUPS.

Here we determine some irreducible projective representations of abelian groups corresponding to certain types of factor sets.

Let G be an abelian group of order m^n generated by n-elements $g_i(i=1,2,..,n)$ each of order m. That is,

$G \cong c_m \times c_m \times \ldots \times c_m$ (n-copies) where c_m is a cyclic group of order m. Furthermore, let

$$\mu(i) = \prod_{j=1}^{m-1} \alpha(g_i^i, g_j) \quad (1 \leq i, \leq n)$$

and $\beta(i,j) = \alpha(g_i, g_j)\alpha^{-1}(g_j, g_i)$ $(1 \leq i \leq n)$

$(1 \leq j \leq m-1)$

Then the factor set α can be chosen such that

$\mu(i) = 1$ $(i=1,2,\ldots,n)$ and $\beta(i,j)$ $(1 \leq i \leq m-1)$ is

$(1 \leq j \leq m-1)$

an mth root of unity then the factor set α considered earlier on, may be chosen, upto equivalence, such that

$\alpha'(s_i, s_j) = \varphi_{ij}$ satisfy the relations:

$$\varphi_{ij}^{a_i} = 1, \ \varphi_{ii} = 1, \ \varphi_{ji} = \varphi_{ij}^{-1} \qquad (1)$$

$1 \leq i,j \leq m, \ i \neq j$

$$\text{and} \ \prod_{j=1}^{a_i} \alpha(s_i^j, s_i) = 1 \quad 1 \leq i \leq m \qquad (2)$$

We shall call (φ_{ij}) the matrix associated with α and write $\alpha\varepsilon(\varphi_{ij})$. Now we determine irreducible projective representations corresponding to the factor sets $\alpha\varepsilon(\varphi_{ij})$ belonging to the following special classes such that each φ_{st} is a primitive a_s^{th} root of unity, for a fixed pair of indices (s,t), s<t and

$\varphi_{ij} = 1$ for $(i,j) \neq (s,t)$.

Consider a set of indices $1 \leq s_2 < \ldots < s_{2r} \leq m$ with

$\alpha\varepsilon(\varphi_{ij})$ where $\varphi_{i\, s_i s_{i+1}}$ is a primitive $a s_i^{th}$ root of unity for i=1,3,5,..., 2r-1 and φ_{ij} = 1 otherwise. Let $\alpha\varepsilon(\varphi_{ij})$ such that φ_{ij} is a primitive a_i^{th} root of unity and let $T:G \to GL(n,\mathbb{C})$ be a projective representation of G with the factor set satisfying (1) and (2). If

$T_i = T(s_i)$ i = 1,2,...,m then

$T_1,\ldots, T_m$ satisfy the following relations

$$\left.\begin{array}{l} T_i^{a_i} = I \quad i = 1,2, \ldots,m \\ T_iT_j = \varphi_{ij}T_jT_i, \; i,j = 1,2,\ldots,m \end{array}\right\} \quad (3)$$

Let us now consider an abelian group

$$G = \langle s_j, j = 1,2,\ldots,m : s_i^{a_i} = 1;\ 1 \leq i \leq m,$$
$$a_i \mid a_{i+1} \quad 1 \leq i \leq m\text{-}1\rangle$$

that is an abelian group of type $(a_1,\ldots,a_m)$. Let w_i be a primitive a^{th} root of unity for i=1,2,...,m. Then a complete set of inequivalent irreducible ordinary representations of G is given by (see eg [10]

$$\{\lambda(x_1,\ldots,x_m) : x_i\varepsilon(0,1,\ldots,a_{i-1}\}, i = 1,2,\ldots,m\}$$

where $\lambda(x_1,\ldots,x_m)(s_1^{\alpha_1}\ldots s_m^{\alpha_m}) = \prod_{i=1}^{m} w_i^{x_i\alpha_i}$

for all $\alpha_i\varepsilon$ $\{0,1,\ldots, a_{i-1}\}$ i = 1,2,...,m. Let $\mathbb{C}^*$ be the non zero elements of the complex field and let α' have the same meaning as in lemma 4.3.1,

Also, suppose $T_1,\ldots, T_m$ are non singular nxn matrices satisfying equations (3) and (φ_{ij}) is an nxn matrix whose entries satisfy (1) then these matrices define a projective representation T of G with factor set $\alpha\varepsilon(\varphi_{ij})$ defined by

$$T(s_1^{\alpha_1} \dots s_m^{\alpha_m}) = T_1^{\alpha_1} \dots T_m^{\alpha_m}$$

since G is a finite abelian group then this number of inequivalent irreducible projective representations is the same as the number of α-regular elements of a group.

Let α(G) denote the set of α-regular elements of G, $n_\alpha(G)$ the number of α-regular elements in α(G) and $d_\alpha(G)$ the degree of the irreducible projective representation of G. In view of lemma 4.4.3 we have

$$n_\alpha(G)(d_\alpha(G))^2 = |G|.$$

For $1 \leq i < j \leq m$ let

$$\varphi_{ij} = \begin{cases} w_{a_i}, \text{ a primitive } a_i^{th} \text{ root of unity} \\ 1, \text{ otherwise} \end{cases}$$

then with the above notation we prove

4.5.1 Theorem

The number of inequivalent irreducible, projecitve representations of a finite abelian group G with factor set $\alpha \varepsilon (\varphi_{ij})$ is $\dfrac{|G|}{a_i^2}$

where $d_\alpha(G) = a_i$

Proof

We only require to find the number of α-regular elements of G with th factor set α.

$s \varepsilon G$ is α-regular if and only if

$$\alpha'(s_i,s) = \alpha'(s_j,s) = 1 \quad 1 \leq i < j \leq m$$

But $s \varepsilon G$ has the form $s = s_1^{\alpha_1} \dots s_m^{\alpha_m}$, Therefore $s \varepsilon G$ is α-regular if and only if

$$\alpha'(s_i, s_1^{\alpha_1} \dots s_m^{\alpha_m}) = \alpha'(s_j, s_1^{\alpha_1} \dots s_m^{\alpha_m}) = 1$$

since $\alpha'(s_p, s_m^{\alpha_m}) = 1$ for all $p \neq i,j$ then s is α-regular if $\alpha'(s_i, s_j^{\alpha_j}) = 1$ and $\alpha'(s_j, s_i^{\alpha_i}) = 1$ which implies that

$$w_{a_i}^{\alpha_j} = w_{a_i}^{\alpha_i} = 1$$

Thus, $\alpha_j \equiv 0 \pmod{a_i}$ implying that

$$\alpha_i = 0 \text{ and } \alpha_j = ba_i, \quad b = 0,1,\dots,(a_j/a_i) - 1$$

Therefore, the total number of α-regular elements of G is

$$a_1 a_2 \dots a_{i-1} a_{i+1} \dots (a_j/a_i) a_{j+1} \dots a_r = \frac{|G|}{a_i^2}$$

For any fixed pair of indices (s,t), let

$$\Lambda(S,T) = \{(x_1, \dots x_m) : x_s = 0, \ a_s / x_t, \ 0 \leq x_k < a_k; 1 \leq k \leq m\}$$

and $\lambda_{(x_1,\dots,x_m)}$ be an irreducible ordinary representation of G assoiciated with the sequence $(x_1,\dots x_m) \ \varepsilon \ \Lambda(s,t)$, with $B_{st} = (\varphi_{ij})$ being defined as before. Then we prove the following result.

4.5.2 Theorem [6]

Let G be a finite abelian group of type$(a_1,\dots,a_m)$. Then $\{\lambda_{(x_1,\dots x_m)} \oplus T_{st} : (x_1,\dots,x_m) \ \varepsilon \ \Lambda \ (s,t)\}$ is a complete set of inequivalent irreducible projective representations of G with factor set $\alpha \ \varepsilon \ B_{st}$.

Proof

To give an explicit construction of the inequivalent irreducible projective representations we construct a set of non singular kxk matrices. Let w_k be a primitive k^{th} root of unit and $\delta_k^2 = w_k$. Then, if k is odd, let P_k and $Q_k(w_k)$ be the non singular kxk matrices defined by

$$P_k = \begin{bmatrix} 0 & 1 & 0 \cdots 0 \\ 0 & 0 & 1 \ldots 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0 \ldots 1 \\ 1 & 0 & 0 \ldots 0 \end{bmatrix}, \quad Q_k(w_k) = \begin{bmatrix} 0 & \omega_k & 0 \cdots 0 \\ 0 & 0 & w_k^2 \ldots 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0 \ldots w_k^{k-1} \\ 1 & 0 & 0 \ldots 0 \end{bmatrix}$$

as in [6]. If k is even, let P_k be defined as above and $Q_k(w_k)$ as

$$\begin{bmatrix} 0 & \delta_k & 0 \ldots 0 \\ 0 & 0 & \delta_k^3 \ldots 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0 \ldots \delta_k^{2k-3} \\ \delta_k^{2k-1} & 0 & 0 \ldots 0 \end{bmatrix}$$

In the same way as in the proof of theorem 4.2.1, we obtain

$$P_k^k = I_k \quad \text{and} \quad (Q_k(w_k))^k = I_k$$

Furthermore,

$$P_k Q_k(w_k) = \begin{pmatrix} 0 & 1 & 0 \ldots 0 \\ 0 & 0 & 1 \ldots 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0 \ldots 1 \\ 1 & 0 & 0 \ldots 0 \end{pmatrix} \begin{pmatrix} 0 & w_k & 0 \ldots 0 \\ 0 & 0 & w_k^2 \ldots 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0 \ldots w_k^{k-1} \\ 1 & 0 & 0 \ldots 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & w_k^2 & 0 \ldots 0 \\ 0 & 0 & 0 & w_k^3 \ldots 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & w_k & 0 \ldots & \ldots 0 \end{pmatrix}$$

$$= w_k \begin{pmatrix} 0 & 0 & w_k & 0 \ldots 0 \\ 0 & 0 & 0 & w_k^2 \ldots 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & w_k & 0 \ldots & \ldots 0 \end{pmatrix}$$

$$= w_k \begin{pmatrix} 0 & w_k & 0 & \ldots\ldots 0 \\ 0 & 0 & w_k^2 & \ldots\ldots 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots\ldots w_k^{k-1} \\ 1 & 0 & 0 & \ldots\ldots 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & \ldots 0 \\ 0 & 0 & 1 & \ldots\ldots 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots\ldots 1 \\ 1 & 0 & 0 & \ldots\ldots 0 \end{pmatrix}.$$

$$= W_k Q_k(W_k) P_k$$

Hence $P_k^k = (Q_k(W_k))^k = I_k$; $P_k Q_k(W_k) = W_k Q_k(W_k) P_k$ where I_k is the kxk identity matrix. If we let $T_s = P_{a_s}$, $T_t = Q_{a_s}(w_{a_s})$ and $T_i = I_{a_s}$ for all $i \neq s,t$ then $T_i (i = 1,2,...,m)$ satisfies $T_i^{a_i} = I$ and $T_i T_j = \varphi_{ij} T_j T_i$ $i,j = 1,2,...,m$. Thus the T_i $i = 1,...m$ generate a projective representation T_{st} Of G with factor set $\alpha \varepsilon B_{st} = (\varphi_{ij})$. Since T_{st} is of degree $d_\alpha(G) = a_s$, it is irreducible by theorem 4.5.1. Now let $\lambda(x_1,...,x_m)$ be as above. Then

$$\lambda(x_1,...,x_m) \neq \lambda(x_1',...,x_m') \text{ on } \alpha(G)$$

If and only if $(x_1,...,x_m) \neq (x_1',...,x_m')$ for $(x_1,...,x_m)$ and $(x_1',...,x_1',...,x_m')$ in Λ (S,t)

If T_{st} is the irreducible projective representation of G with factor set $\alpha\varepsilon$ B_{st} as defined above and S$\oplus$T denotes the tensor products of representations S and T, then

$\lambda_{(x_1, \ldots, x_m)} \oplus T_{st}$ is also an irreducible projective representation of G with factor set $\alpha\varepsilon B_{st}$ since $\lambda_{(x_1,...,x_m)}$ and T_{st} are irreducible. Infact the set

$\{\lambda_{(x_1 \ldots x_m)} \oplus T_{st} : (x_1,...,x_m)\ \varepsilon\ \Lambda(s,t)\}$ is a complete set of inequivalent irredicible Projective representations of G with factor set α (see e.g [10, p200])

Define an irreducible projective representation $T_{s_i s_{i+1}}$ of G with factor set lying in the class $B_{s_i s_{i+1}}$ for i = 1,3,...,2r-1 and let

$$T_s = T(s_1 \ldots s_{2r}) = T_{s_1 s_2} \oplus T_{s_3 s_4} \oplus \ldots \oplus T_{s_{2r-1} s_{2r}}.$$

Then T_s is an irreducible projective representation of G with the needed factor set as it is a direct product of irreducible representations. Furthermore, if $X_{s_i} = 0$, $a_{s_i}/x_{s_{i+1}}$, i = 1,3,..., 2r-1 and $1 \leq k \leq m$, then

$F_{(x_1, \ldots, x_m)} = \lambda_{(x_1, \ldots, x_m)} \oplus T_s$ is an irreducible projective representation of G with factor set α and

$$\{F_{(x_1, \ldots, x_m)} : X_{s_i} = 0;\ a_{s_i}/X_{s_{i+1}}\ ;\ i = 1,3,\ldots,2r-1$$

and $1 \leq x_k < a_k$ for $1 \leq k \leq m\}$ gives a complete set of inequivalent irreducible projective representations of G with factor set α because $\lambda_{(x_1, \ldots, x_m)}$ and the projective character of $F_{(x_1, \ldots, x_m)}$ are distinct when restricted to $\alpha(G)$ and the number of sequences $(X_1, X_2, \ldots, X_m)$ is equal to the number of irreducible projective representations of G with the determined factor set α.

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