Differential Spaces, Diffeological Spaces and Frölicher Spaces: A Comparative Smootheology

Hamilton Zaninge Chirwa
512801771

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Supervisor: Professor Batubenge T.A.

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ABSTRACT

The contemporary smooth structures that describe the motion of particles on underlying sets through the related specified differential geometry need to be assessed in the sense of stabilising the advantage of working with one or another. Such a study is what we call a comparative smootheology.

A diffeological structure,differential structure and a Frölicher structure are each a generalisation of a smooth manifold structure. However, it is known that a smooth manifold is a Frölicher space, a Frölicher space is a subcategory of a diffeology and a Frölicher space is a subcategory of a differential space.In this study we will carry out a comparative study on the three spaces and see to which extent the diffeologies, the differential structure in the sense of Sikorski or the Frölicher structure will be more suited in describing any field of application that require the tools of differential geometry.

The method of comparison will be based on comparing, their structures, their topologies and their tangent structures.

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DECLARATION

I Hamilton Zaninge Chirwa, declare that this dissertation titled **Differential Spaces, Diffeological Spaces and Frölicher Spaces: A Comparative Smootheology**, and the work presented in it are my own. The work described in this Master of Science (MSc) dissertation was carried out under the supervision of the Department of Mathematics and statistics, University of Zambia, Lusaka.
This MSc dissertation represents original work by the author and has not otherwise been submitted in any form for any degree or diploma to any other University. Where use has been made of the work of others it is duly acknowledged in the text.

Signed:

Hamilton Zaninge Chirwa (Student)

Date:

Signed:

Professor Augustine Tshidibi Batubenge. (Supervisor)

Date:

DEDICATION

I dedicate this work to my beautiful wife Racheal Musonda Chirwa and to my kids Malumbo, Nyali, Joy, Racheal, Brian and Memory for their understanding and patience. To my parents Rev and Mrs Chirwa whose love,insistence on fearing and puting God the almighty first,strictness and discipline, enabled me to reach this far.Also my brothers Mumbake, Augustine and my sister clara for their invaluable encouragement,financial and moral support.

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Finally I am grateful to management of Mukuba University for the financial support they rendered to me during this study.

APPROVAL

This dissertation of Hamilton Zaninge Chirwa is a fulfilling requirement of the award of the degree of master of science in mathematics of the University of Zambia.

Signed:

Hamilton Zaninge Chirwa (Student)

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Professor Augustine Tshidibi Batubenge. (Supervisor)

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Examiner 1

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Examiner 2

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Examiner 3

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Contents

KEY TERMS AND DEFINITIONS

- comparative smootheology
- diffeological space
- manifold
- homomeophism
- chart
- atlas
- diffeomophism
- topological manifold
- differentiable manifold
- Sikorski differential space
- Sikorski structure
- topology on a Sikorski differential space
- tangent space on a Sikorski differential space
- tangent cone on a Sikorski differential space
- Parametrization of a set
- diffeology
- diffeological space
- D-topology
- plot derivation
- tangent space on a diffeological space.

- tangent space on a diffeological space
- tangent cone on a diffeological space
- Frölicher space
- Frölicher structure
- curvaceous topology on the Frölicher space
- functional topology on the Frölicher space
- operational Tangent Vector
- kinematic Tangent Vector
- operational Tangent space
- kinematic Tangent space
- tangent cone space
- pre-Frölicher space

INDEX OF NOTATION

Below is a list of some important symbols to be used and a brief indication of their meaning.

$\mathbb{R}$	The set of real numbers
$\mathbb{R}^n$	n-dimension Euclidean space
$\mathbb{Q}$	a set of rational numbers
C^∞	continuous and infinitely differentiable, all partial derivatives exist
C^k	continuous and differentiable, partial derivatives exist up to k^{th} order.
$\mathcal{F}$	a non empty family of real valued functions into $\mathbb{R}$ for a Sikorski differential space
$\tau_{\mathcal{F}}$	initial topology on $\mathcal{F}$.
$\mathcal{D}$	a diffeology
$\mathcal{D}_{\mathcal{M}}$	a diffeology on a set $\mathcal{M}$
$\mathcal{D}^o$	indiscrete diffeology
$\mathcal{D}^\star$	discrete diffeology
$\mathcal{F}_{\mathcal{M}}$	collection of real valued functions f from $\mathcal{M}$ to $\mathbb{R}$
$\mathcal{C}_{\mathcal{M}}$	a collection of curves c from $\mathbb{R}$ to $\mathcal{M}$
$\mathbb{R}^{\mathcal{M}}$	from $\mathcal{M}$ to $\mathbb{R}$
$C^\infty(\mathbb{R})$	Infinitely differentiable on $\mathbb{R}$
$C^\infty(\mathbb{R},\mathbb{R})$	Infinitely differentiable from $\mathbb{R}$ to $\mathbb{R}$
$C^\infty(\mathcal{U},\mathbb{R})$	Infinitely differentiable from $\mathcal{U}$ to $\mathbb{R}$
$\Phi\mathcal{D}$	a set of those real-valued functions whose precomposition with each element of $\mathcal{D}$ is infinitely-differentiable
$\Pi\mathcal{F}$	a set of those parametrisations whose composition with each element of $\mathcal{F}$ is infinitely differentiable
$T_p\mathcal{M}$	a tangent space at a point p $\in \mathcal{M}$ on a given space.
$i_{\mathcal{A}}$	inclusion map from a subset $\mathcal{A}$ of a given set.
$Hom_{set}(\mathbb{Q},\mathbb{R})$	a set functions from $\mathbb{Q}$ to $\mathbb{R}$
$id_{\mathbb{R}}$	identity on $\mathbb{R}$

1

INTRODUCTION :

1.1. Background

Differential geometry is the study of geometric objects by means of differential calculus. During the era before 20th and 21st century, everything on differential geometry was based on curves and surfaces. However, the differential geometry of 20th and 21st century lies on differentiable manifolds developed mostly from the studies done by Johann Carl Friedrich Gauss (1777-1855) and Bernhard Riemann (1826-1866).
Classical differentiation in linear spaces of arbitrary dimension use Banach spaces structure, but infinite dimension spaces appear in particular as function spaces, and these are not essentialy Banach spaces. Any attempts to develop a theory of differentiation covering non-normable linear spaces have always involved arbitrary conditions. It was then important to think of reducing the differentiability of general maps to that of functions on the real numbers that could consist of replacing the property of "continuosly differentiable" by that of "lipschitz differentiable" then further on think of using J Boman's result (see [12]) and reduce this property by a more natural theory of conceptual simplicity that leads to the same categories of linear spaces, but in a more general setting.

The other aspect is that most spaces that are useful in various applications of analysis to other sciences are not even linear spaces, but these spaces are to be used as the modelling spaces for mechanical systems so that a kind of differential calculus on them is needed.
To this effect the theory of differentiable manifold was developed in 18th century and constantly adjusted to finally have a modern presentation. The concept of a manifold originates from the Euclidean space. For an n-dimensional real manifold can be viewed as the result of gluing together the blocks of $\mathbb{R}^n$ (the real n-dimensional Euclidean space). The most essential character of an n-dimensional real manifold is that there is an n-dimensional system of local

coordinates in an open neighborhood of each point of the underlying set.(see [4]). Basically, a Manifold is a space that locally looks like some Euclidean space $\mathbb{R}^n$ and on which we can do calculus (see [28]) or roughly speaking we view a manifold as a topological space for which one can locally make charts which piece together in a consistent way (see [25]). Theories were developed and more work was done based on the differentiable manifold. However,there were also some problems encountered, such as how to deal with objects having corners, which meant coming up or rather defining charts at the corners. Also the local modeling makes it difficult to endow a large class of geometric objects with a smooth structure. In the mid sixties, the need of working out a theory of differentiation without using norms appeared through several works. Some more weaker structures were introduced in the literature, some of these defining differential spaces, diffeological spaces and Frölicher spaces. We can cite some contributions in this direction, begining with the paper by Jan Boman (1967) ([12]). Differential spaces were introduced by Roman Sikorski, diffeologies or diffeological spaces by Jean-Marie Souriau and Frölicher spaces result from the work of Alfred Frölicher, Andreas Kriegl and Peter Micher. The structures are a generalisation of a smooth manifold. Each of them comes equipped with its own topology and tangent structure. In this dissertation we will gather and present the three spaces, we will compare their structures, their induced topologies and their tangent structures so as to find out which ones are well behaved.This is what we will call a comparative smootheology.

1.2. Organisation of the dissertation

This dissertation is organized as described below.

Chapter 1.
In this chapter we shall start by giving a brief background on the differential geometry of smooth manifolds and the motivation for generalisation to new structures. We will introduce the three structures and the individuals who introduced them.

Chapter 2.
This chapter is devoted to the study of smooth manifolds. We will give preliminary definitions such as topology on a set, topological space, open cover, a chart, an atlas, a differential structure and a differentiable manifold. We will give example and counter example of a differentiable manifold. An example of a geometric set which is not a differentiable manifold will be given. Smooth mapping between differentiable manifolds will be explained and examples given. In this chapter we will show that composition of smooth manifolds is a smooth manifold and the sum smooth functions on a smooth manifold is a smooth function. We will

give the definition of a smooth function on a manifold and the pullback of smooth function. We will end the chapter by giving the definition of a diffeomorphism with an example.

Chapter 3.

In chapter three we shall start by giving a brief background on the Sikorski differential structure. Then we will give some preliminary definitions and concepts on the Sikorski differential space giving some examples as well. An example of a Sikorski differential space which is not a differentiable manifold will be given. Smooth mapping between Sikorski differential spaces will be explained and example given. We will show that composition of two Sikorski differential spaces is a Sikorski differential space. We will end the chapter by giving the definition of functional diffeomorphism of two Sikorski differential spaces.

Chapter 4.

Chapter four looks at diffeological spaces. We begin this chapter with a brief background on diffeolofies. Then we will give some preliminary definitions and concepts on diffeologies with some examples. We will give the definition of indiscrete and discrete diffeological space. An example of a diffeological space which is not a smooth manifold will be given. Smooth mapping between diffeological spaces will be defined and example given. We will show that composition of diffeological smooth maps is a diffeological smooth map. We will give the definition of a difeological smooth function then we will end the chapter by giving the definition of diffeomorphism on diffeological spaces.

Chapter 5.

In this chapter we introduce Frölicher spaces. Then we will give some preliminary definitions and concepts on Frölicher spaces with some examples. An example of a Frölicher space which is not a smooth manifold will be given. Smooth mapping between Frölicher spaces will be defined and example given. We will show that composition of Frölicher smooth maps is a Frölicher smooth map. We will give the definition of a difeological smooth function then we will end the chapter by giving the definition of Frölicher diffeomorphism.

Chapter 6.

In chapter six we will compare the structures of the four spaces.

Chapter 7.

In this chapter the topology of each space will be discussed, thereafter we will do a comparative study on the spaces based on their topologies.

Chapter 8.

In chapter eight we will do a comparative study of tangent structures on the spaces.

Chapter 9.

A chapter for the conclusion.

2

MANIFOLD SPACE :

We will need to consider and recall some useful definitions which will lead us to a formal or rather more mathematical definition of a smooth Manifold.

2.1. Preliminary concepts and definitions :

definition 2.1.1. ***(A Topology on a set $\mathcal{M}$)***
Let $\mathcal{M}$ be a set, a topology on $\mathcal{M}$ is a collection τ of open subsets of $\mathcal{M}$ satisfying the following properties:

(1) $\mathcal{M}$, $\emptyset \in \tau$ that is $\mathcal{M}$, $\emptyset$ are open.

(2) $\bigcup_{i \in j} O_i \in \tau$; τ is closed under arbitrary unions. That is the union of any family of open sets is open.

(3) $\bigcap_{i=1}^{n} O_i \in \tau$; τ is closed under finite intersections. That is the finite intersection of any collection of open sets is open.

definition 2.1.2. ***(A Topological space)***
A topological space is a pair ($\mathcal{M}$,τ) where $\mathcal{M}$ is the underlying set and τ a collection of open subsets of $\mathcal{M}$.

definition 2.1.3. ***(Open cover)***
Let $\mathcal{M}$ be a topological space.A collection $\{U_i\}_{i\in j}$ of open sets in the topology of $\mathcal{M}$ is called an open cover of $\mathcal{M}$ if $\bigcup_{i\in j} U_i = \mathcal{M}$,that is $\forall\ x \in \mathcal{M}$, $\exists\ U_i \in \{U_i\}_{i\in j}$ such that $x \in U_i$.

definition 2.1.4. ***(Chart)***
Let $\mathcal{M}$ be a topological space.A n-dimensional chart at $x \in \mathcal{M}$ is a pair (U,φ) where U is an open neighbourhood of x called the domain of chart and
$\varphi : U \longrightarrow \varphi(U) \subseteq \mathbb{R}^n$ is a homeomorphism.

definition 2.1.5. ***(Topological manifold)***
A topological manifold $\mathcal{M}$ of dimension n, is a topological space satisfying the following properties

(1) $\mathcal{M}$ is Hausdorff .

(2) $\mathcal{M}$ is second countable .

(3) $\mathcal{M}$ is locally Euclidean, that is there is a local chart at each point $x \in \mathcal{M}$.

Example 2.1.1. ***(A cusp)***
The graph of $y = x^{\frac{2}{3}}$ in $\mathbb{R}^2$ is a topological manifold (Figure 1). Since it is a subspace of $\mathbb{R}^2$, it is Hausdorff and second countable. It is locally Euclidean, because it is homeomorphic to $\mathbb{R}$ through the chart $(x,x^{\frac{2}{3}}) \mapsto x$. (see figure 1 the below)

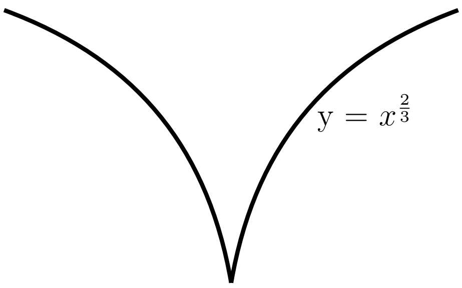


figure 1 (A cusp)

definition 2.1.6. ***(Compatible charts)***

Two charts (U,φ) and (V,ψ) are C^k-compatible if $U \cap V \neq \emptyset$ and the transition functions $\psi \circ \varphi^{-1}$: $\varphi(U \cap V) \longrightarrow \psi(U \cap V)$ and
$\varphi \circ \psi^{-1}$: $\psi(U \cap V) \to \varphi(U \cap V)$ are C^k-diffeomorphisms of $\mathbb{R}^n$.

definition 2.1.7. ***(C^k-atlas)***

A C^k-atlas is a collection $\mathcal{A} = \{(U_i,\varphi_i)\}_{i\in j}$, of charts such that;

(1) $\mathcal{M} = \bigcup_{i\in j} U_i$.

(2) *The charts are C^k-compatible.*

(3) *Any chart,say (V_i,ψ) compatible with any of the $\{(U_i,\varphi_i)\}_{i\in j}$, with $V_i \cap U_i \neq \emptyset$ and the transition functions $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ are C^k-diffeomorphisms of $\mathbb{R}^n$, then $(V_i,\psi) \in \mathcal{A}$. That is $\mathcal{A}$ is a maximal atlas.*

definition 2.1.8. ***(Differentiable structure on a manifold)***

A n-dimensional differentiable structure on a topological manifold $\mathcal{M}$ is a maximal atlas of compatible charts on $\mathcal{M}$.

definition 2.1.9. ***(Differentiable manifold)***

A differentiable manifold is a pair ($\mathcal{M}$,$\mathcal{A}$) where $\mathcal{M}$ is a topological manifold and $\mathcal{A}$ is a maximal atlas of compatible charts on $\mathcal{M}$.

The figure below shows a trnsition map.

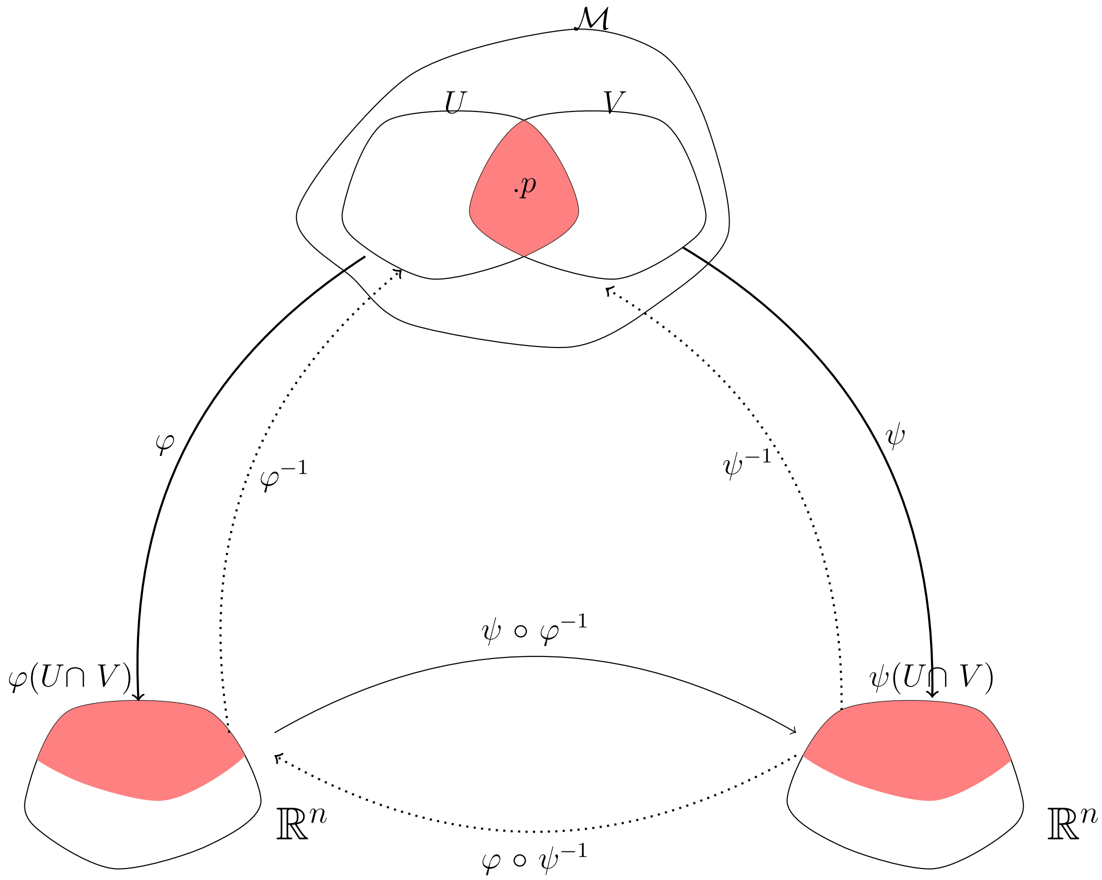


figure 2 (A Transition map)

Example 2.1.2. *([4] [25])*
The Euclidean space $\mathbb{R}^n$ ***:***
The Euclidean Space $\mathbb{R}^n$ *is a n-dimensional differentiable manifold.A single chart that covers the whole of* $\mathbb{R}^n$ *is* $(\mathbb{R}^n, id_{\mathbb{R}^n})$ *which is the unique atlas where* $id_{\mathbb{R}^n}$ *is the identity map* $id_{\mathbb{R}^n}(x) = x$*. Clearly* $\mathbb{R}^n$ *is domain of chart since it is an open set of* $\mathbb{R}^n$ *and it covers* $\mathbb{R}^n$*. Also the usual Euclidean topology of* $\mathbb{R}^n$ *is Hausdorff and has countable basis of open sets. Next,* $id_{\mathbb{R}^n}$ *is a homeomorphism onto* $\mathbb{R}^n$*. So* $\mathbb{R}^n$ *is a topological manifold of dimension n. Finally, transition maps (functions) are exactly* $id_{\mathbb{R}^n}$*, which is a diffeomorphism of class* C^∞*.*

Example 2.1.3. *(Counter example)*
Let $\{(\mathbb{R}, x), (\mathbb{R}, x^3)\}$ *be an atlas.*
Let $\varphi(x) = x$ *and* $\psi(x) = x^3$ *. We have* $\varphi^{-1}(x) = x$ *and* $\psi^{-1}(x) = x^{\frac{1}{3}}$*. Now the transition functions will be,* $\varphi \circ \psi^{-1}(x) = \varphi\ (\psi^{-1}(x)) = \varphi(x^{\frac{1}{3}}) = x^{\frac{1}{3}}$ *and this is not* C^1*-differentiable at* $x = 0$ *. Although,* $\psi \circ \varphi^{-1}(x) = \psi\ (\varphi^{-1}(x)) = \psi(x) = x^3$ *and this is* C^∞ *on* $\mathbb{R}$*. That is, the charts* $(\mathbb{R}, x)$ *and* $(\mathbb{R}, x^3)$ *are not compatible. Therefore, the atlas* $\{(\mathbb{R}, x), (\mathbb{R}, x^3)\}$ *is not a* C^k *structure on* $\mathbb{R}$*.*

Example 2.1.4. *;*

A geometric set which is not a differentiable manifold :

Let Set $\mathcal{M} = \{ (x, y) : x^2 = y^2 \}$. We have that x^2-y^2= 0, $(x-y)(x+y)$= 0 , and this is equivalent to x = y or x = -y . Thus it is the union of two lines of $\mathbb{R}^2$ which intersects at (0,0). It is not a differentiable manifold because the local Euclidean property fails at a point (0,0), that is if say U is an arbitrary neighborhood of the origin in $\mathbb{R}^2$, there does not exist an open set V in $\mathbb{R}$ and a smooth map $\varphi : U \longrightarrow V \subseteq \mathbb{R}$ whose derivative has rank one everywhere.

2.2. Smooth mappings between differentiable manifolds :

definition 2.2.1. ***(Smooth or C^∞-mapping between differentiable manifolds)***
Let ($\mathcal{M}$,$\mathcal{A}$) and ($\mathcal{N}$,$\mathcal{B}$) be C^k-manifolds of dimension n and m respectively. A continuous map $h : \mathcal{M} \longrightarrow \mathcal{N}$ is said to be C^k- differentiable at $p \in \mathcal{M}$, if for every chart (U,φ) at p and (V,ψ) at $h(p)$,the local representation $\psi \circ h \circ \varphi^{-1}$ of h is a differentiable function from $\mathbb{R}^n$ to $\mathbb{R}^m$.

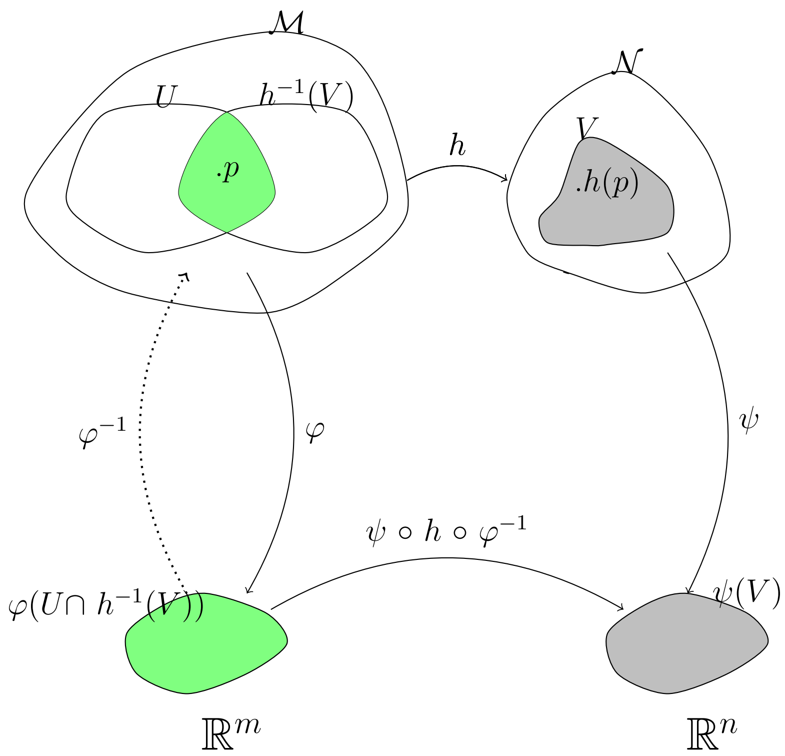


figure 3 (Smooth or C^∞ mapping between manifolds)

Example 2.2.1. *;*

Let $\mathcal{M} = \mathbb{R}^n$, $\mathcal{N} = \mathbb{R}^n$ and $h = id_{\mathbb{R}^n}$. Then we have the map $id_{\mathbb{R}^n}\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Clearly h is continuous since $(id_{\mathbb{R}^n})^{-1}(U) = U'$, U' is open in $\mathbb{R}^n$ for all U open in $\mathbb{R}^n$. We have the local representation as $id_{\mathbb{R}^n} \circ id_{\mathbb{R}^n} \circ (id_{\mathbb{R}^n})^{-1} = id_{\mathbb{R}^n} \circ id_{\mathbb{R}^n} \circ id_{\mathbb{R}^n} = id_{\mathbb{R}^n} \in C^\infty$. Therefore the map $id_{\mathbb{R}^n}\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is smooth.

proposition 2.2.1. *([28]);*

Composition of smooth maps between smooth manifolds is smooth:

Suppose $F : \mathcal{M} \longrightarrow \mathcal{N}$ and $G : \mathcal{N} \longrightarrow \mathcal{S}$ are smooth maps. Then the composition $F \circ G : \mathcal{M} \longrightarrow \mathcal{S}$ is smooth.

Proof.

Given p $\in \mathcal{M}$, choose charts (U,φ) around p , (V,ψ) around F(p), and (W,ϕ) around G(F(p)), with G(V) $\subseteq W$ and F(U) $\subseteq$ V. Then G(F(V)) $\subseteq W$, and we have: $\phi \circ (G \circ F) \circ \varphi^{-1} = (\phi \circ G \circ \psi^{-1}) \circ (\psi \circ F \circ \varphi^{-1})$; which is a composition of smooth maps between open subsets of Euclidean spaces. ☐

definition 2.2.2. ***(C^k-differentiable function)***

Let $\mathcal{M}$ be a C^k-manifold of dimension n , a continuous map f :$\mathcal{M} \longrightarrow \mathbb{R}$ is C^k-differentiable at p $\in \mathcal{M}$ if for any chart (U,φ) at p ,

the local representation $f \circ \varphi^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}$ is C^k-differentiable.

proposition 2.2.2. *;*

Suppose $\{(U_\alpha,\ \varphi_\alpha)\}_{\alpha \in j}$ is a smooth atlas for $\mathcal{M}$. If f : $\mathcal{M} \longrightarrow \mathbb{R}^k$ is a function such that f $\circ\ \varphi_\alpha^{-1}$ is smooth for each α , then f is smooth.

Proof.

We just need to check that $f \circ \varphi^{-1}$ is smooth for any smooth chart (U,φ) on $\mathcal{M}$. It suffices to show it is smooth in a neighbourhood of each point x = φ(p) $\in \varphi$(U). For any p $\in$ U, there is a chart (U_α,φ_α) in the atlas whose domain contains p. Since (U,φ) is smoothly compatible with (U_α,φ_α), the transition map $\varphi_\alpha \circ \varphi^{-1}$ is smooth on its domain of definition,which includes x. Thus $f \circ \varphi^{-1}$ = f $\circ\ \varphi_\alpha^{-1} \circ \varphi_\alpha \circ \varphi^{-1}$ is smooth in a neighborhood of x. ☐

Theorem 2.2.1. *;*
Let $\mathcal{M}$ be a smooth manifold. The sum of smooth functions on $\mathcal{M}$ is a smooth function.

Proof.

Let $f, g \in C^{\infty}(\mathcal{M})$. We need to check that $f+g \in C^{\infty}(\mathcal{M})$. By definition it means that for any chart $\varphi : \mathcal{V} \longrightarrow \mathcal{M}$ (where $\mathcal{V} \subset \mathbb{R}^n$), the composition $(f+g) \circ \varphi \in C^{\infty}(\mathcal{M})$. We have that $((f+g) \circ \varphi)(x) = (f+g)(\varphi(x)) = f(\varphi(x)) + g(\varphi(x)) = ((f \circ \varphi) + (g \circ \varphi))(x) = (f \circ \varphi)(x) + (g \circ \varphi)(x)$ for all $x \in \mathcal{V}$. Therefore $(f+g) \circ \varphi = (f \circ \varphi) + (g \circ \varphi) \in C^{\infty}(\mathcal{M})$. $\square$

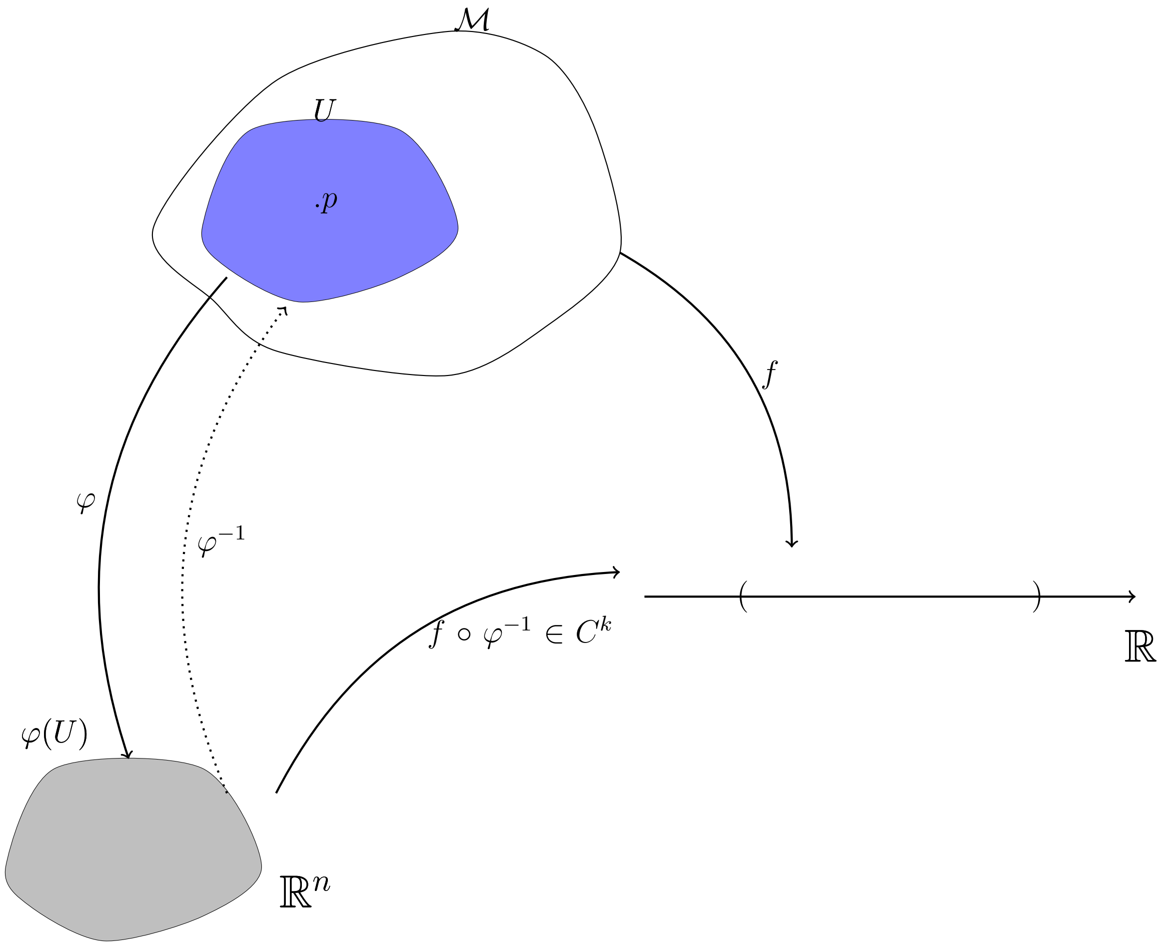


figure 4 (C^k-differentiable function)

definition 2.2.3. ***(Diffeomorphism)***
A C^{∞} mapping $F : \mathcal{M} \to \mathcal{N}$ between C^{∞} manifolds is a diffeomorphism if it is a homeomorphism and F^{-1} is C^{∞}.

Remark 2.2.1.

- *Two manifolds are said to be diffeomorphic if there is a diffeomorphism between them.*
- *A C^∞ homeomorphism is not necessarily a diffeomorphism.*

Example 2.2.2. *;*
We consider the maps $F : S^n \to R^n$ and $G : R^n \to S^n$ given by $F(x) = \frac{x}{\sqrt{1-|x|^2}}$ and $G(y) = \frac{y}{\sqrt{1+|y|^2}}$, respectively.
Cleary the two maps are smooth and it is easy to compute that they are inverses of each other. Therefore they are both diffeomorphisms, and thus S^n is diffeomorphic to $\mathbb{R}^n$.
Example 2.2.3. *;*
Let $F : \mathbb{R}^n. \to \mathbb{R}^n.$ be defined by F(t)= t^3. Then F is C^∞ and a homeomorphism, but it is not a diffeomorphism since $F^{-1}(t)= t^{\frac{1}{3}}$ and this is not even of class C- it is not C^∞ , since it is not continuous and differentiable at $t = 0$.
definition 2.2.4. ***(Pullback**)[4]*
Let $\mathcal{M}$ and $\mathcal{N}$ be smooth manifolds. Let $h : \mathcal{M} \to \mathcal{N}$ be a smooth map and f a smooth function on $\mathcal{N}$ ($f \in \mathcal{F}_\mathcal{N}$). The pullback of f under h denoted h^*f, is the function defined on $\mathcal{M}$ by $h^*f = f \circ h$.

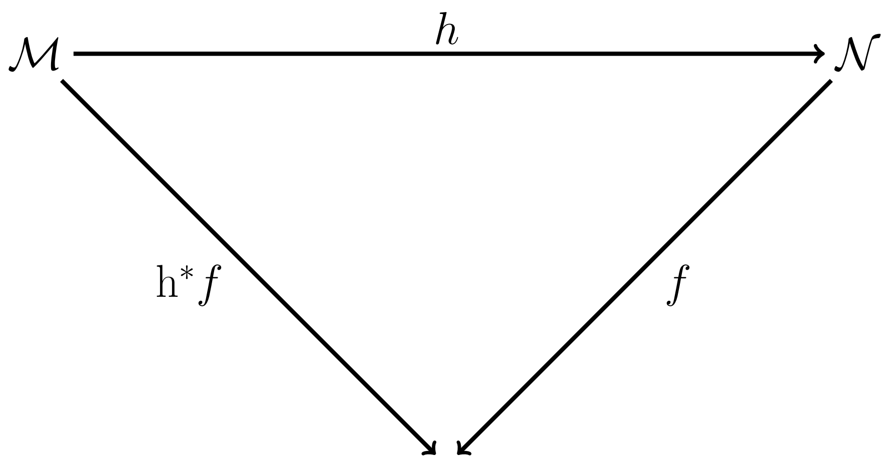


figure 5 (pullback of a function under a smooth map)

3

DIFFERENTIAL SPACES (in the sense of Sikorski) :

3.1. Background

The notion on differential spaces traces back to the work of N.Aronszajn ([1]) in the late 1960's who expressed the need for a theory of smooth structures on arbitrary subsets of $\mathbb{R}^n$, and together with Marshall ([30]), they developed the theory of the so called sub-cartesian spaces, which essentially are manifolds with singularities.Later Aronszajn and Szeptycki ([2]) further developed the theory and applied it in the study of the Bessel potential in functional analysis.The theory on differential spaces was further developed by R.Sikorski (Polish mathematician) ([33]) in "Differential Modules" (1972),where he introduced differentiable structures, without referring to the locally Euclidean setting,hence the space known as differential or Sikorski space.The concepts on a differential space allows us to explore or rather investigate problems in differential geometry, which can not be investigated on differentiable manifolds.

3.2. Preliminary concepts and definitions :

definition 3.2.1. ***(Sikorski differential structure)***
Let $\mathcal{M}$ be a non empty set. A Sikorski differential structure on $\mathcal{M}$ is a non empty family $\mathcal{F}$ of real valued functions into $\mathbb{R}$, along with its initial topology $\tau_{\mathcal{F}}$ in which the f_i , $i \in J$ are continuous, such that the following conditions are satisfied ;

(1) (Smooth composition) For any positive integer $n \in \mathbb{N}$, functions f_1 , ... , $f_n \in \mathcal{F}$ and for all smooth real valued function $\omega \in C^\infty(\mathbb{R}^n,\mathbb{R})$ the composition $\omega \circ (f_1 , \ldots , f_n) \in \mathcal{F}$.

(2) (Locality) Given a function $g : \mathcal{M} \longrightarrow \mathbb{R}$. If for every $x \in \mathcal{M}$, there exist an open

neighborhood $\mathcal{V} \subseteq \mathcal{M}$ of x such that $g\mid_{\mathcal{V}} = f\mid_{\mathcal{V}}$, $f \in \mathcal{F}$. Then $g \in \mathcal{F}$.

definition 3.2.2. ***(Sikorski differential space)***

A Sikorski differential space is a triple $(\mathcal{M},\tau_{\mathcal{F}},\mathcal{F})$ where $\mathcal{M}$ is a nonempty set , $\tau_{\mathcal{F}}$ is the topology in which the f_i , $i \in J$ are continuous and $\mathcal{F}$ is a Sikorski differential structure on $\mathcal{M}$.

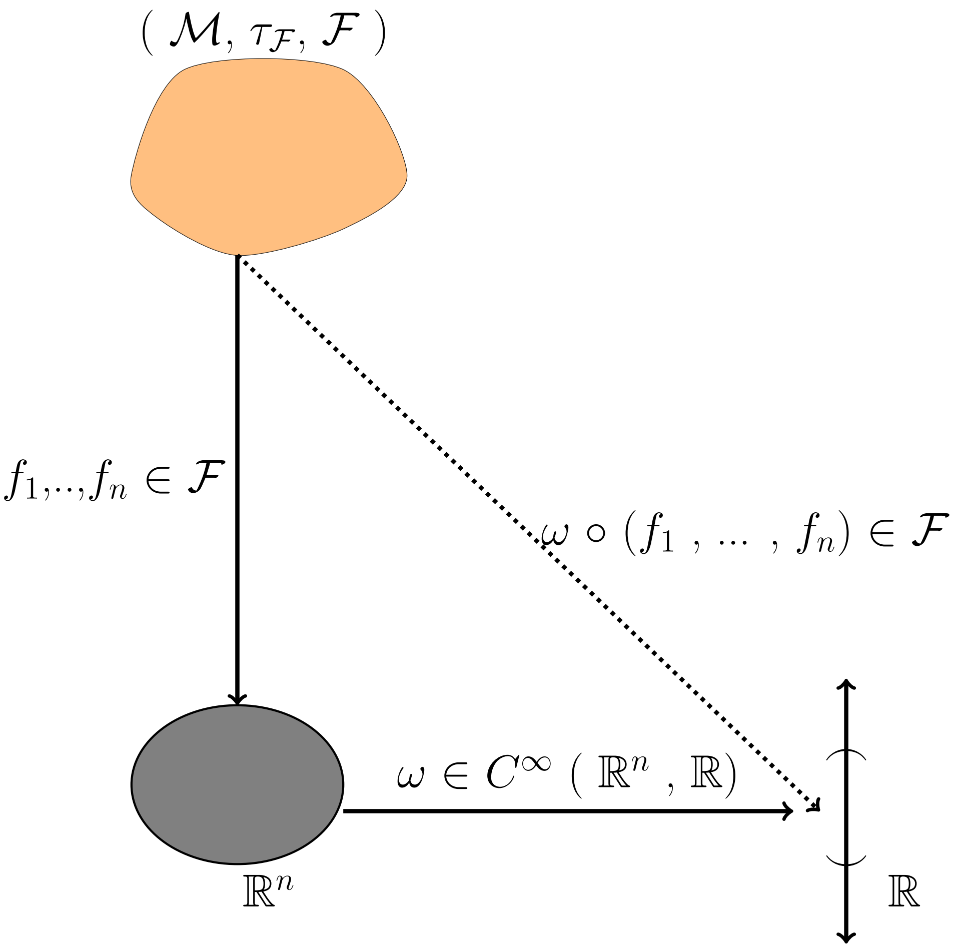


figure 6 (Sikorski structure)

Example 3.2.1. *;*

(1) *The set of all infinitely differentiable functions on* $\mathbb{R}^n$*, represented by the pair* $(\mathbb{R}^n, \mathcal{A}_n)$ *for* $n \in \mathbb{N}$ *where* $\mathcal{A}_n = C^\infty(\mathbb{R}^n, \mathbb{R})$ *is a differential space.*

(2) *A* C^∞ *n-dimensional manifold* $\mathcal{M}$ *is a differential space, since all smooth functions on* $\mathcal{M}$ *determine a differential structure on* $\mathcal{M}$*.*

(3) *Let* $\mathcal{M} = \mathbb{R}$ *and* p *,* $q \in \mathcal{M}$ *,* $p \neq q$ *. Let* $\mathcal{F} = \{f \in C^\infty(\mathbb{R}) | f(p) = f(q)\}$*.* $(\mathcal{M}, \mathcal{F})$ *is a differential space.*

Example 3.2.2. *;*
Let $\mathcal{M}$ *be the open interval* $(0,1) \subset \mathbb{R}$ *with the usual Euclidean topology* τ *and a function* $f : x \mapsto \frac{1}{x}$*. Since* $f : x \mapsto \frac{1}{x}$ *is not bounded as it approaches* $x = 0$*, it is not a* $C^\infty(\mathbb{R})$ *function, therefore any set of a collection say* $\mathcal{B}$ *of* $C^\infty(\mathbb{R})$ *functions together with any function or functions of the form* $f : x \mapsto \frac{1}{x}$ *is not a sikorski differential space.*

Example 3.2.3. *;*
A geometric set which is not a smooth manifold but is a Sikorski space :
The graph of function $|x|$ *:* $[-1,1] \longrightarrow R$ *is not a smooth manifold, but it is a differential space, that is, let* $\mathcal{M}=\{(x,|x|) \subset \mathbb{R}^n \mid x \in \mathbb{R}\}$*. Then* $(\mathcal{M}, C^\infty(\mathbb{R}^2))$ *is a Sikorski differential space.*

3.3. Smooth mappings between Sikorski differential spaces :

definition 3.3.1. ***(Smooth mappings between Sikorski differential spaces)***
Let $(\mathcal{M}, \mathcal{F})$ *and* $(\mathcal{N}, \mathcal{G})$ *be Sikorski differential spaces. A map* $F : \mathcal{M} \longrightarrow \mathcal{N}$ *is said to be smooth if for all* $h \in \mathcal{G}$ *the composition* $h \circ F \in \mathcal{F}$*.*

This is illustrated in figure 7 below.

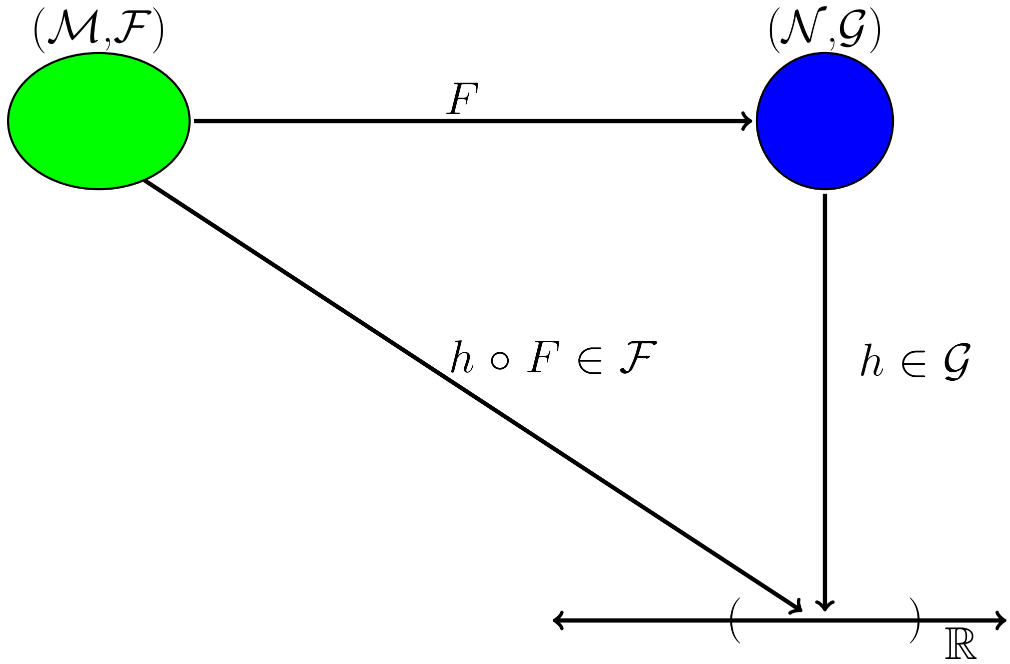


figure 7 (Sikorski smooth maps)

definition 3.3.2. *([43]);*
Let $\mathcal{M}$ and $\mathcal{N}$ be sikorski differential spaces, a map $\varphi : \mathcal{M} \longrightarrow \mathcal{N}$ is smooth if for every $f \in C^{\infty}(\mathcal{N})$, $\varphi^{*}f := f \circ \varphi \in C^{\infty}(\mathcal{M})$.

definition 3.3.3. *([43]);*
A map $\varphi : \mathcal{M} \longrightarrow \mathcal{N}$ is a diffeomorphism if it is a smooth homeomorphism with a smooth inverse.

Example 3.3.1. *;*
*(**Smooth Maps Between Manifolds**). Given two manifolds $\mathcal{M}$ and $\mathcal{N}$, the smooth maps between $\mathcal{M}$ and $\mathcal{N}$ are exactly the usual smooth maps $C^{\infty}(\mathcal{M}, \mathcal{N})$.*

proposition 3.3.1. *;*
The composition of two Sikorski smooth maps is a Sikorski smooth map:

Proof. (see also figure 8.)
Let $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{P}$ be sikorski differential spaces and let $\varphi : \mathcal{M} \longrightarrow \mathcal{N}$ and $\psi : \mathcal{N} \longrightarrow \mathcal{P}$ be smooth maps. Then for any $f \in C^{\infty}(\mathcal{P})$, $(\psi \circ \varphi)^{*}f = \varphi^{*}(\psi^{*}f) \in C^{\infty}(\mathcal{M})$ □

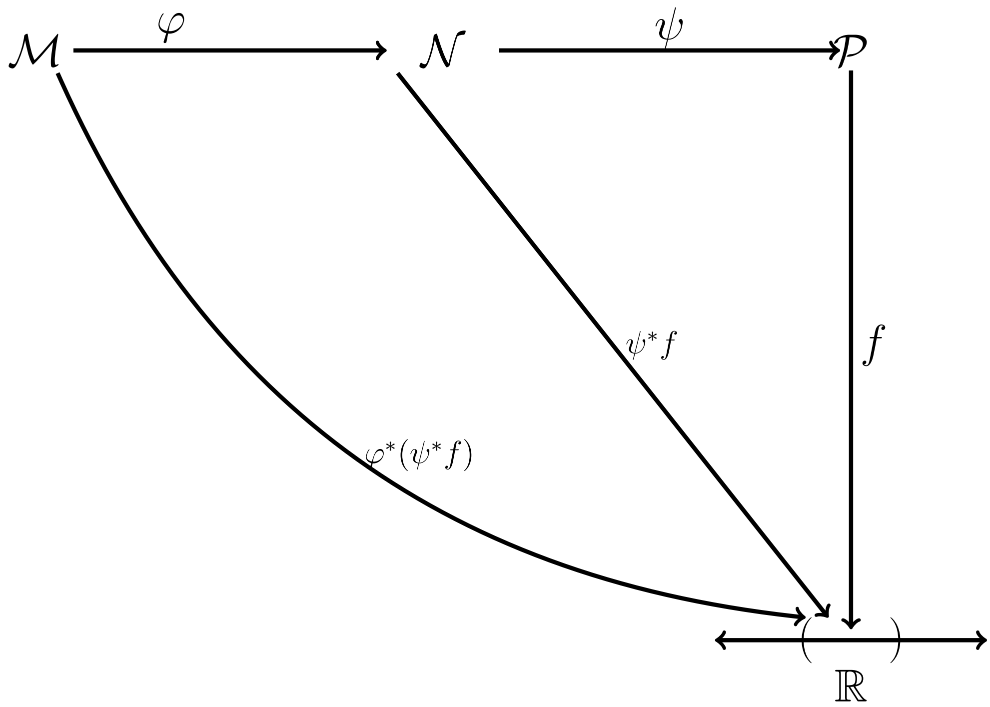


figure 8 ([43])

definition 3.3.4. ***(Functional Diffeomorphism)***
Let $\mathcal{M}$ and $\mathcal{N}$ be two Sikorski differential spaces. A map $F : \mathcal{M} \longrightarrow \mathcal{N}$ is a functional diffeomorphism if it is functionally smooth and has a functionally smooth inverse,that is, if $F : \mathcal{M} \longrightarrow \mathcal{N}$ is a bijection and $F^{-1} : \mathcal{N} \longrightarrow \mathcal{M}$ is smooth.

4

DIFFEOLOGICAL SPACES :

4.1. Background

The work on diffeologies goes back to J.M . Souriau ([34]) in the early 1980s, whose work in physics prompted him to introduce and work on the theory of diffeological spaces. Earier in the 1970, K . T . Chen ([14]) had developed a theory on a smooth or differentiable space which was later referred to as a "Chen Space ". Souriau generalised the notion of smooth spaces in the sense of Chen and obtained the smooth structure defined on an arbitrary set by using plots from open domains of R^n . However, an earlier theory by P. Donato and P. Iglesias - Zemmour ([19]) on the irrational torus ; helped to motivate further development on diffeologies. This structure was then intensively investigated by Souriau's Phd student P. Iglesias - Zemmour ([44],[45]) whose notion will be our main reference on the theory of diffeologies. The theory of diffeological spaces tries to capture the essence of smooth spaces .It generalizes smooth manifolds.

4.2. Preliminary concepts and definitions :

definition 4.2.1. ***(Parametrisation of a set)***
Let $\mathcal{M}$ be a nonempty set. A parametrisation of $\mathcal{M}$ is a function $p : U \longrightarrow \mathcal{M}$, where U is an open set of $\mathbb{R}^n$ for some $n \in \mathbb{N}$.
definition 4.2.2. ***(Diffeology)***
The diffeology $\mathcal{D}$ of a nonempty set $\mathcal{M}$ is a set of parametrisations of $\mathcal{M}$ satisfying the following three conditions;

(1) *(Covering) For every $x \in \mathcal{M}$ and every nonnegative integer n, the constant function $p : \mathbb{R}^n \longrightarrow \{x\} \subseteq \mathcal{M}$ is in $\mathcal{D}$.*

(2) *(Locality) Let $p : U \longrightarrow \mathcal{M}$ be a parametrisation such $\forall\ x \in U$, $\exists$ an open neighbourhood $\mathcal{V} \subset U$ of x satisfying $p\,|_{\mathcal{V}} \in \mathcal{D}$ then $p \in \mathcal{D}$.*

(3) *(Smooth Composition) Let $p : U \subset \mathbb{R}^n \longrightarrow \mathcal{M}$ be a plot. For every $n \in \mathbb{N}$, for every open subset $\mathcal{V} \subseteq \mathbb{R}^n$ and for every smooth map $F : \mathcal{V} \longrightarrow U$, the composition $P \circ F \in \mathcal{D}$.*

definition 4.2.3. ***(Diffeological space)***
A diffeological space is a pair $(\mathcal{M},\mathcal{D})$ where $\mathcal{M}$ is the nonempty set and $\mathcal{D}$ is the diffeology. on $\mathcal{M}$.

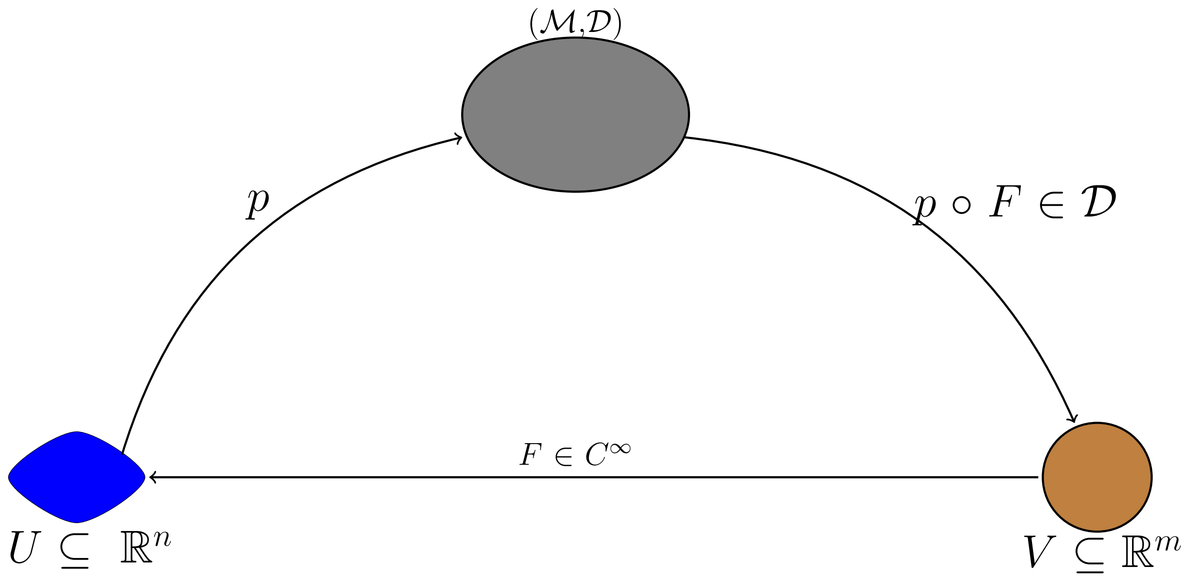


figure 9 (Diffeological structure)

Example 4.2.1. *([26])([44]);*

(1) ***(Standard diffeology on a Manifold)*** *The set of all smooth maps from open subsets of* $\mathbb{R}^n$*,* $n \in \mathbb{N}$*, to a manifold* $\mathcal{M}$ *is a standard diffeology on a manifold.*
Let $\mathcal{M}$ *be a smooth manifold, then the smooth map* $\alpha : U \longrightarrow \mathcal{M}$ *for an open set* $U \subseteq \mathbb{R}^n$ *is a diffeology.*
Proof.
we check for the three conditions in definition 4.2.2.

(i) Cleary, constant maps are smooth.

(ii) Let α be the smallest extension of the α_i, then given x $\in U_\alpha$, there is an i $\in$ I such that x $\in U_i$ and hence $\alpha|_{U_i} = \alpha_i$ is smooth. Now smoothness is a local condition, so α is smooth on all of U_α.

(iii) Composition of smooth maps are smooth.

□

(2) **A diffeology for square ;**
If we consider a square given by [0,1] $\times$ {0,1} $\cup$ {0,1}$\times$[0,1] $\subset \mathbb{R}^2$. The set of the parametrisations of a square which regarded as parametrisations of $\mathbb{R}^2$,are smooth, is a diffeology.

Proof.
using definition 4.2.2.

(i) Every constant parametrisation regarded as a parametrisation in $\mathbb{R}^2$ is smooth.

(ii) A parametrisation in the square regarded as a parametrisation in $\mathbb{R}^2$, locally smooth at each point of its domain, is smooth.

(iii) The composite of a plot of the square with any smooth parametrisation in the source of the plot does not change its set of values regarded as a parametrisation in $\mathbb{R}^2$, is smooth.

□

(3) **Subsets of a diffeological space ;**
Let ($\mathcal{M}$,$\mathcal{D}_\mathcal{M}$) be a diffeological space then the subsets ($\mathcal{N}$,$\mathcal{D}_\mathcal{N}$) of a diffeological space ($\mathcal{M}$,$\mathcal{D}_\mathcal{M}$) are also diffeological spaces.The plots of the subsets ($\mathcal{N}$,$\mathcal{D}_\mathcal{N}$) are those plots of ($\mathcal{M}$,$\mathcal{D}_\mathcal{M}$) whose image is contained in ($\mathcal{N}$,$\mathcal{D}_\mathcal{N}$).

(4) ***The discrete and indiscrete diffeology ;***

Let $\mathcal{M}$ be a fixed set, $\mathcal{D}$ a set of diffeologies on $\mathcal{M}$ and any open set U in $\mathbb{R}^n$ for each $n \in \mathbb{N}$, we have the following definitions;

definition 4.2.4. ***(discrete Diffeology)***

The discrete diffeology denoted by $\mathcal{D}^*$ is the finest diffeology on $\mathcal{M}$, that is the smallest element in $\mathcal{D}$ which is the collection of all locally constant functions $U \longrightarrow \mathcal{M}$.

definition 4.2.5. ***(Indiscrete diffeology)***

The indiscrete diffeology denoted by $\mathcal{D}^o$ is the coarsest diffeology on $\mathcal{M}$, that is the largest element in $\mathcal{D}$ that contains any other diffeology and is the collection of all functions $U \longrightarrow \mathcal{M}$. It is a diffeology in which every function is a plot.

Example 4.2.2. *;*

Diffeological space which is not a smooth manifold:

The cross in $\mathbb{R}^2$ equipped with the subspace diffeology is a diffeological space, and with the subspace topology it is not locally Euclidean at the intersection, and so cannot be a topological manifold.

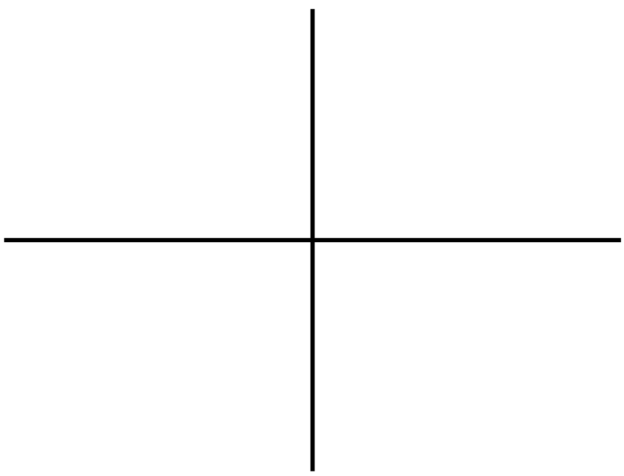


figure 10 (Cross)

lemma 4.2.1. ***(Generating families)***

Given a family $\mathcal{A}_i \mid i \in I$ of diffeologies on a set $\mathcal{M}$, the intersection $\bigcap_i \mathcal{A}_i$ is again a diffeology.

Proof.

Here, we verify the three axioms of definition 4.2.2; for $\mathcal{A} := \bigcap_i \mathcal{A}_\rangle$. By definition, each of the diffeologies $\mathcal{A}_i$ contains all constant maps, thus the same is true for their intersection, (**covering**). every compatible family of n-plots in $\mathcal{A}$ is also compatible in each of the $\mathcal{A}_i$, therefore the smallest and common diffeology is the element of each $\mathcal{A}_i$, hence it is also of $\mathcal{A}$ since $\mathcal{A}_i \subset \mathcal{A}$, (**locality**). Now, if p $\in \mathcal{A}$ and f (smooth) are composable then p $\circ$ f $\in \mathcal{A}_i$

for all $i \in I$, therefore also p ∘ f $\in \mathcal{A}$. (**smooth composition**). □

4.3. Smooth mappings between diffeological spaces:

definition 4.3.1. ***(Smooth mappings between diffeological spaces)***

Let $(\mathcal{M},\mathcal{D}_{\mathcal{M}})$ *and* $(\mathcal{N},\mathcal{D}_{\mathcal{N}})$ *be two diffeological spaces. A map* $F : \mathcal{M} \longrightarrow \mathcal{N}$ *is said to be diffeologically smooth if for any plot* $P \in \mathcal{D}_{\mathcal{M}}$*, the composition* $F \circ P \in \mathcal{D}_{\mathcal{N}}$*. (see figure 11)*

Example 4.3.1. *;*

Any Smooth map between $\mathbb{R}^n$ *and a differentiable manifold.*

proposition 4.3.1. *;*

The composition of two diffeological smooth maps is a diffeological smooth map:

Proof.

Let $(\mathcal{M},\mathcal{D}_{\mathcal{M}})$, $(\mathcal{N},\mathcal{D}_{\mathcal{N}})$ and $(\mathcal{P},\mathcal{D}_{\mathcal{P}})$ be three diffeological spaces . Let $f : \mathcal{M} \longrightarrow \mathcal{N}$ and $g : \mathcal{N} \longrightarrow \mathcal{P}$ be two smooth maps that is $f \circ \mathcal{D}_{\mathcal{M}} \subset \mathcal{D}_{\mathcal{N}}$ and $g \circ \mathcal{D}_{\mathcal{N}} \subset \mathcal{D}_{\mathcal{P}}$. Then we have that $(g \circ f) \circ \mathcal{D}_{\mathcal{M}} = g \circ f \circ \mathcal{D}_{\mathcal{M}} \subset g \circ \mathcal{D}_{\mathcal{N}} \subset \mathcal{D}_{\mathcal{P}}$. □

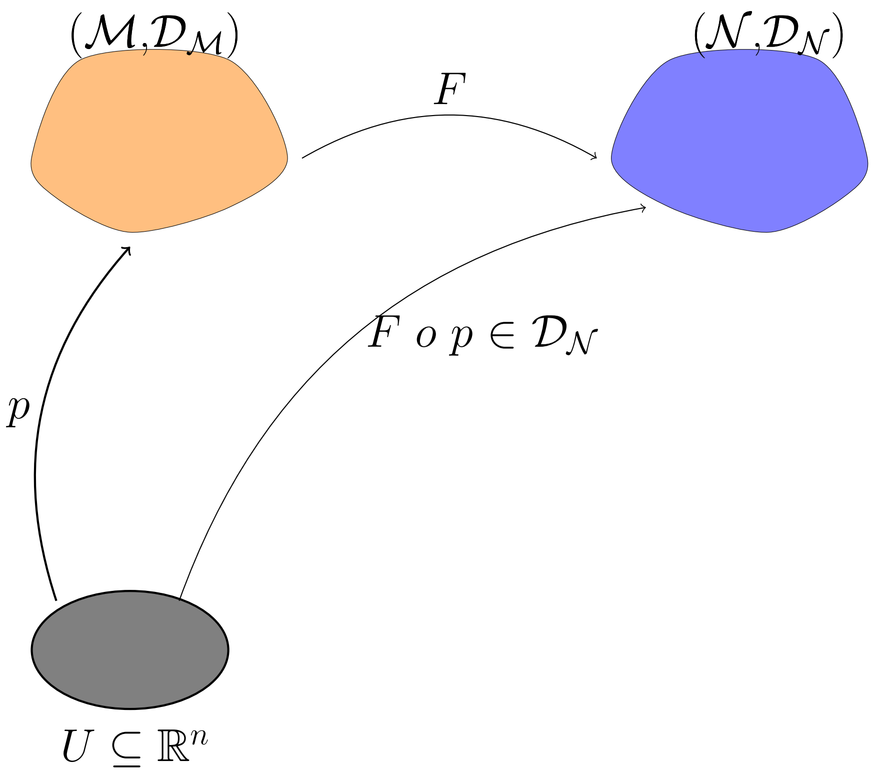


figure 11 (diffeological smooth map)

definition 4.3.2. ***(Diffeomorphism)***

Let Let $(\mathcal{M},\mathcal{D}_{\mathcal{M}})$ *and* $(\mathcal{N},\mathcal{D}_{\mathcal{N}})$ *be two diffeological spaces . A map* $f : \mathcal{M} \longrightarrow \mathcal{N}$ *is called a diffeomorphism if f is bijective and if both f and* f^{-1} *are smooth.*

definition 4.3.3. ***(Diffeological smooth function)***

Let $(\mathcal{M},\mathcal{D})$ *be a diffeological space. A (real-valued) diffeologically smooth function on* $\mathcal{M}$ *is a diffeologically smooth map* $f : \mathcal{M} \longrightarrow \mathbb{R}$ *where* $\mathbb{R}$ *is equipped with the standard diffeology. Hence , for any plot* $p : U \longrightarrow \mathcal{M}$*,*

$f \circ p \in C^\infty(U,\mathbb{R})$*. (see figure the diagram below.)*

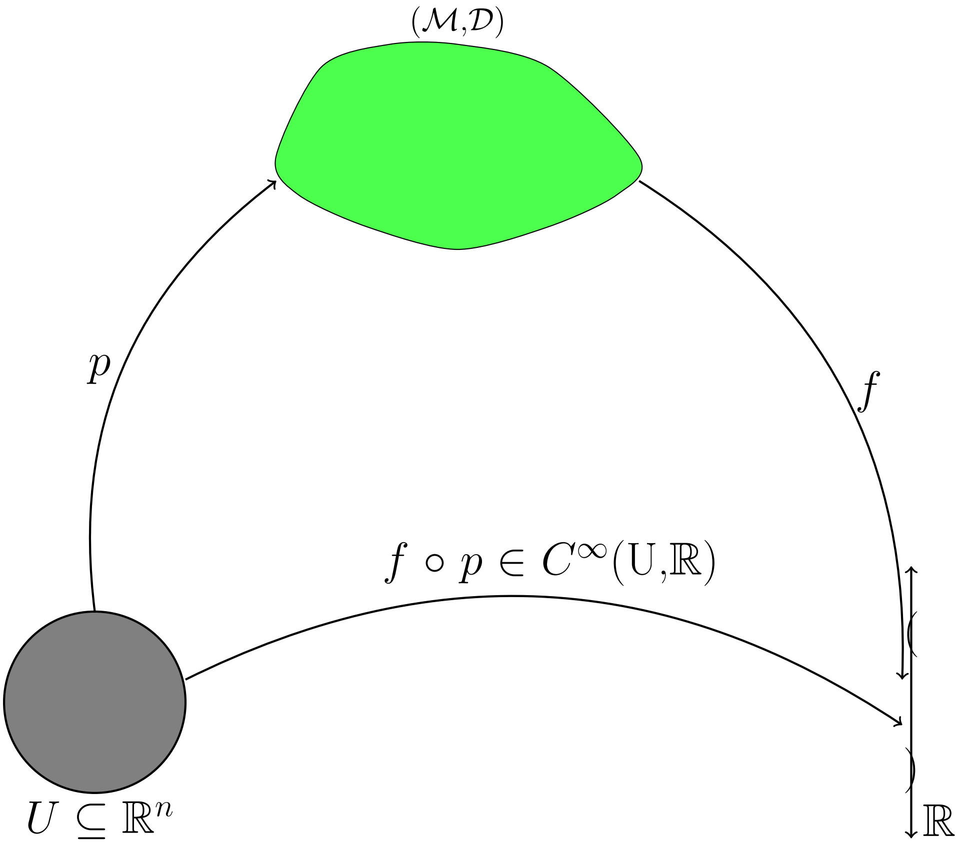


figure 12 (diffeological smooth function)

Remark 4.3.1.

(i) A function $f : \mathcal{M} \longrightarrow \mathcal{N}$ *between diffeological spaces is smooth if for every plot* $p : U \longrightarrow \mathcal{M}$ *of* $\mathcal{M}$ *, the composite* $f \circ p$ *is a plot of* $\mathcal{N}$ *.*

(ii) In a case where $n = 1$*, then we will have 1-plots into* $\mathcal{M}$ *and this gives a Frölicher structure.*

5

FRÖLICHER SPACES:

5.1. Background :

Frölicher spaces where introduced by Alfred Frölicher who called them "smooth spaces" , then later on they where named after him by Paul Cherenack , Andreas Kriegl and Peter Michor . Later more studies and publications have been done on Frölicher spaces by A .T Batubenge et al . ([3][7] [9][8][39][10]). This is a smooth structure generated on a set $\mathcal{M}$ by a set of maps from $\mathbb{R}$ into $\mathcal{M}$, called curves (or contours) and/or the set of maps from $\mathcal{M}$ into $\mathbb{R}$ called functions . Frölicher spaces are a generalisation to an arbitrary set by J Boman's theorem ([12]); that is "A function
f : $\mathbb{R}^n \longrightarrow \mathbb{R}$ is smooth if and only if it is smooth along Eucleadean smooth Curves " ; otherwise a Frölicher space is a generalization of a smooth manifold .

5.2. Preliminary concepts and definitions :

definition 5.2.1. ***(Frölicher Space*** *)*
Let $\mathcal{M}$ be a nonempty set . A triple $(\mathcal{M},\mathcal{C}_{\mathcal{M}},\mathcal{F}_{\mathcal{M}})$ is said to be a Frölicher space if the following property called compatibility condition is satisfied ;
$\Phi\mathcal{C}_{\mathcal{M}} := \{f : \mathcal{M} \longrightarrow \mathbb{R} \mid f\ o\ c \in C^{\infty}(\mathbb{R},\mathbb{R})$ *for all* $c \in \mathcal{C}_{\mathcal{M}}\} = \mathcal{F}_{\mathcal{M}}$ *and*
$\Gamma\mathcal{F}_{\mathcal{M}} := \{c : \mathbb{R} \longrightarrow \mathcal{M} \mid f\ o\ c \in C^{\infty}(\mathbb{R},\mathbb{R})$ *for all* $f \in \mathcal{F}_{\mathcal{M}}\} = \mathcal{C}_{\mathcal{M}}$.

definition 5.2.2. ***(Frölicher Structure)***
A pair ($\mathcal{C}_{\mathcal{M}}$,$\mathcal{F}_{\mathcal{M}}$) is called a Frölicher structure on $\mathcal{M}$ with the elements of $\mathcal{C}_{\mathcal{M}}$ called structure curves on $\mathcal{M}$ and the elements of $\mathcal{F}_{\mathcal{M}}$ called structure functions on $\mathcal{M}$.

From the two definitions , $\mathcal{M}$ is a set, $\mathcal{F}_{\mathcal{M}}$ a collection of real-valued functions f from $\mathcal{M}$ to $\mathbb{R}$ and $C_{\mathcal{M}}$ a collection of curves c from $\mathbb{R}$ to $\mathcal{M}$. We thus have the following diagram.

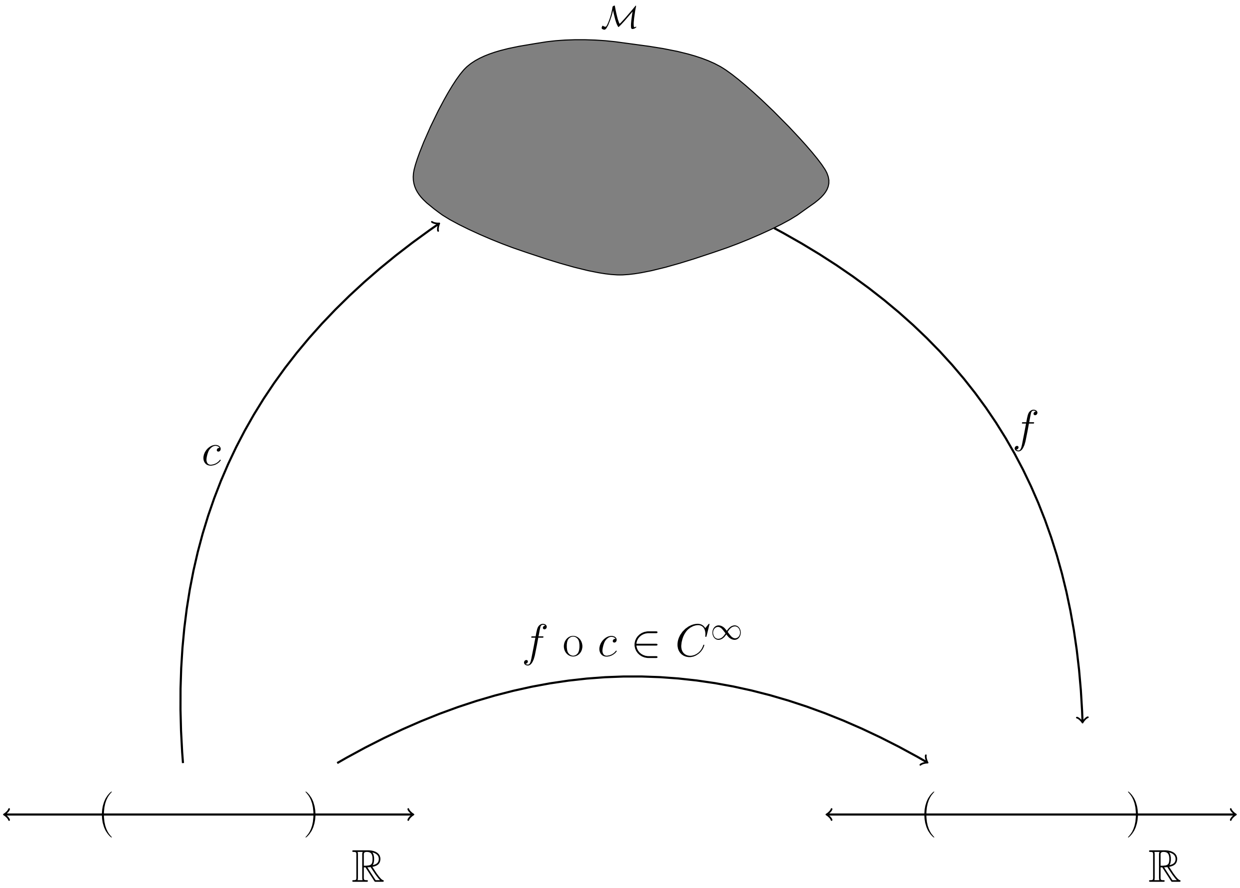


figure 13 (Frölicher Structure)

Theorem 5.2.1. *(Boman's Theorem)([27]) ;*
Let $f \in \mathrm{map}(\mathbb{R}^{\mathrm{n}}, \mathbb{R})$ be such that $f \circ c$ is C^{∞} whenever $c : \mathbb{R} \to \mathbb{R}^{n}$ is C^{∞}, then f is C^{∞}.

Example 5.2.1. *;*
We show that the canonical structure of $\mathbb{R}$ *is generated by* $\mathcal{F}_0 = \{id_{i\mathbb{R}}\}$

$$\mathcal{F}_0 \xrightarrow{\Gamma} \Gamma\mathcal{F}_0 \xrightarrow{\Phi} \Phi\Gamma\mathcal{F}_0$$

$$\begin{aligned}
\Gamma\mathcal{F}_{\prime} &= \{c : \mathbb{R} \to \mathbb{R} : f \circ c \in C^{\infty} \quad \textit{for all} \;\; f \in \mathcal{F}_0\} \\
&= \{c : \mathbb{R} \to \mathbb{R} : c \in C^{\infty}(\mathbb{R}, \mathbb{R})\} \\
\Phi\Gamma\mathcal{F}_0 &= \{f : \mathbb{R} \to \mathbb{R} : f \circ c \in C^{\infty}(\mathbb{R}, \mathbb{R}) \quad \textit{for all} \;\; c \in C^{\infty}(\mathbb{R}, \mathbb{R})\} \\
&= C^{\infty}(\mathbb{R}, \mathbb{R})
\end{aligned}$$

For, if $f \notin C^{\infty}(\mathbb{R}, \mathbb{R})$*, then particular choice* $c = id_{\mathbb{R}}$ *will yield a contradiction.*

This first example above appears then as an immediate consequence of Boman's theorem. That is:
Corollary 5.2.2. *If* M *is a smooth finite-demensional manifold, then*

$$(C^{\infty}(\mathbb{R}, M), C^{\infty}(M, \mathbb{R}))$$

is a Frölicher structure on M*.*

Γ and Φ are order reversing , hence called contravariant functors . We have the following lemma.
lemma 5.2.1. *;*
Let $\mathcal{C}_1$,$\mathcal{C}_2 \subset \mathcal{M}^{\mathbb{R}}$ ***and*** $\mathcal{F}_1$,$\mathcal{F}_2 \subset \mathbb{R}^{\mathcal{M}}$***, where*** $\mathcal{M}$ ***is a non empty set. We have that:***

(1) $\mathcal{C}_1 \subseteq \mathcal{C}_2 \Rightarrow \Phi\mathcal{C}_1 \supseteq \Phi\mathcal{C}_2$

(2) $\mathcal{F}_1 \subset \mathcal{F}_2 \Rightarrow \Gamma\mathcal{F}_1 \supset \Gamma\mathcal{F}_2$

Proof.

(1) let c $\in \mathcal{C}_1$ and $f \in \Phi\ \mathcal{C}_2$. Then $f \circ c \in C^{\infty}(\mathbb{R}, \mathbb{R})$ for all c $\in \mathcal{C}_2$, implying that $\mathcal{C}_1 \subseteq \mathcal{C}_2$. In particular, $f \circ c \in C^{\infty}(\mathbb{R}, \mathbb{R})$ for all c $\in \mathcal{C}_1$, since by assumption $\mathcal{C}_1 \subseteq \mathcal{C}_1$. Hence $f \in \Phi\mathcal{C}_1$ Thus $\Phi\mathcal{C}_2 \subset \Phi\mathcal{C}_1$.

(2) let $f \in \mathcal{F}_1$ and c $\in \Gamma\mathcal{F}_2$. Then $f \in \mathcal{F}_2$ and therefore ,$f \circ c \in C^\infty(\mathbb{R},\mathbb{R})$. Thus , c $\in \Gamma\mathcal{F}_1$ since c satisfies the compatibility condition with $f \in \mathcal{F}_1$. Thus $\Gamma\mathcal{F}_2 \subset \Gamma\mathcal{F}_1$. □

proposition 5.2.1. *;*

Let $\mathcal{M}$ be a non empty set . Let $\mathcal{C}_1$, $\mathcal{C}_2 \subset \mathcal{M}^{\mathbb{R}}$ and $\mathcal{F}_1$, $\mathcal{F}_2 \subset \mathbb{R}^{\mathcal{M}}$. Then

(i) $\mathcal{C} \subseteq \Gamma\Phi\mathcal{C}$

(ii) $\mathcal{F} \subseteq \Phi\Gamma\mathcal{F}$

Proof.

(i) let c $\in \mathcal{C}$ and assume that c $\notin \Gamma\ \Phi\ \mathcal{C}$ hence there is a function f $\in \Phi\mathcal{C}$ such that $f \circ c \notin C^\infty(\mathbb{R},\mathbb{R})$. This is a contradiction with the definition of $\Phi\mathcal{C}$. Therefore c $\in \Gamma\ \Phi\ \mathcal{C}$ and thus $\mathcal{C} \subseteq \Gamma\ \Phi\ \mathcal{C}$.

(ii) let f $\in \mathcal{F}$ and assume that f $\notin \Phi\ \Gamma\mathcal{F}$ hence there is a curve c $\in \Gamma\mathcal{F}$ such that $f \circ c \notin C^\infty(\mathbb{R},\mathbb{R})$. This is a contradiction since $\Gamma\mathcal{F}$ is by definition the set of curves $c : \mathbb{R} \mapsto$ M | $f \circ c \in C^\infty(\mathbb{R},\mathbb{R})$ for all
f $\in \mathcal{F}$. Therefore f $\in \Phi\ \Gamma\mathcal{F}$, and $\mathcal{F} \subseteq \Phi\Gamma\mathcal{F}$.

□

proposition 5.2.2. *;*

The following identities hold for the functors Φ and Γ:

(a) $\Gamma\ = \Gamma\ \Phi\ \Gamma$

(b) $\Phi\ = \Phi\ \Gamma\ \Phi$

Proof.

(a) By proposition5.2.1(ii) above, $\mathcal{F} \subseteq \Phi\Gamma\mathcal{F}$ always holds. Then applying lemma 5.2.1(2) we have that $\Gamma\mathcal{F} \supseteq \Gamma\Phi\Gamma\mathcal{F}$ which gives $\Gamma \supseteq \Gamma\Phi\Gamma$. But in proposition 5.2.1(ii) again we have that $\Gamma\mathcal{F} \subseteq \Gamma\Phi(\Gamma\mathcal{F})$ for any set of curves (here we take $\Gamma\mathcal{F} \equiv \mathcal{C}$) thus $\Gamma\Phi\Gamma \subseteq \Gamma$ and the equality follows.

(b) By proposition 5.2.1(i) above $\mathcal{C} \subseteq \Gamma\Phi\ \mathcal{C}$ holds, applying lemma 5.2.1(1) we have $\Phi\ \mathcal{C} \supseteq \Phi\Gamma\Phi\ \mathcal{C}$ which gives $\Phi \supseteq \Phi\Gamma\Phi$ but by proposition 5.2.1(i) we have that $\Phi\ \mathcal{C} \subseteq \Phi\Gamma(\Phi\ \mathcal{C})$ for any set of functions where we take $(\Phi\ \mathcal{C} \equiv \mathcal{F})$,
Thus $\Phi \subseteq \Phi\Gamma\Phi$ and the equality follows.

□

Example 5.2.2. *;*

(1) Let $\mathcal{M} = \mathbb{R}^n$ be the n-dimensional Euclidian space and $\mathcal{F}_o$ the set of all smooth functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$. By Boman's result ([12]), $f \circ c \in C^\infty(\mathbb{R},\mathbb{R})$ for each smooth curve $c : \mathbb{R} \longrightarrow \mathbb{R}^n$. Hence,
$\Gamma\mathcal{F}_o := \{c : \mathbb{R} \longrightarrow \mathbb{R}^n \mid f \circ c \in C^\infty(\mathbb{R},\mathbb{R})$ for all $f \in \mathcal{F}_o\} = C^\infty(\mathbb{R},\mathbb{R}^n)$ and $\Phi\Gamma\mathcal{F}_o := f : \mathbb{R}^n \longrightarrow \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R},\mathbb{R})$ for all $c \in \Gamma\mathcal{F}_o = C^\infty(\mathbb{R}^n,\mathbb{R})$. Thus we have the Frölicher structure $(C^\infty(\mathbb{R},\mathbb{R}^n)$, $C^\infty(\mathbb{R}^n,\mathbb{R}))$.

(2) Let $\mathcal{M} = \mathbb{R}$ be the real line and we let the set $\mathcal{C}_o = id_{\mathbb{R}}$, where $id_{\mathbb{R}}$ is the identity map on $\mathbb{R}$. By definition 4.1 the structure functions are given by the set
$\Phi\mathcal{C}_o := \{f : \mathcal{M} \longrightarrow \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R},\mathbb{R})$ for all $c \in \mathcal{C}_o\}$. We have that $f \circ c = f \circ id_{\mathbb{R}} = f$. Thus ;
$\Phi\mathcal{C}_o = \{f : \mathcal{M} \longrightarrow \mathbb{R} \mid f \in C^\infty(\mathbb{R},\mathbb{R})\} = C^\infty(\mathbb{R},\mathbb{R})$.
The structure curves are given by ;
$\Gamma\Phi\mathcal{C}_o := \{c : \mathcal{M} \longrightarrow \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R},\mathbb{R})$ for all $f \in \Phi\mathcal{C}_o\}$. Here we want to show that $f \circ c \in C^\infty(\mathbb{R},\mathbb{R})$ implies $c \in C^\infty(\mathbb{R},\mathbb{R})$, for any
$f \in \Phi\mathcal{C}_o = C^\infty(\mathbb{R},\mathbb{R})$. assuming that $c \notin C^\infty(\mathbb{R},\mathbb{R})$ then, by taking $f = id_{\mathbb{R}} \in C^\infty(\mathbb{R},\mathbb{R})$, we have a contradiction $f \circ c = id_{\mathbb{R}} \circ c =$
$c \notin C^\infty(\mathbb{R},\mathbb{R}))$. Thus,
$\Gamma\Phi\mathcal{C}_o = \{c : \mathbb{R} \longrightarrow \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R},\mathbb{R})$ for all $f \in C^\infty(\mathbb{R},\mathbb{R})\} = C^\infty(\mathbb{R},\mathbb{R})$.
Therefore we have the Frölicher structure given as
$(C^\infty(\mathbb{R},\mathbb{R}),C^\infty(\mathbb{R},\mathbb{R}))$.

(3) *Let* $\mathcal{M} = \mathbb{R}$ *be the real line and we let the set* $\mathcal{F}_o = \{(x\ ,\ |x|) \mid x \in \mathbb{R}\}$ *the graph of the absolute value function in* $\mathbb{R}$ *,* $x \mapsto |x|$ *is a frölicher subspace of* $\mathbb{R}^2$ *which is not a smooth manifold, since* $|x|$ *fails to hold at* $x = 0$*.*
structure curves are given by ;
$\Gamma\mathcal{F}_o := \{c : \mathbb{R} \longrightarrow \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R},\mathbb{R})$ *for all* $f \in \mathcal{F}_o\}$*. Since* $f(x) = |x| \mid x \in \mathbb{R})$*, let* $\lambda \in \mathbb{R}$ *then by the compatibility condition , we have that,* $(f \circ c)(\lambda) = f(c(\lambda)) = c(\lambda) \in C^\infty(\mathbb{R},\mathbb{R})$ *and* $(f \circ c)(\lambda) = f(c(\lambda)) = |c(\lambda)| \in C^\infty(\mathbb{R},\mathbb{R})$*. Therefore,*
$\Gamma\mathcal{F}_o := \{c : \mathbb{R} \longrightarrow \mathbb{R} \mid c(\lambda) \in C^\infty(\mathbb{R},\mathbb{R})$ *and* $|c(\lambda)| \in C^\infty(\mathbb{R},\mathbb{R})\}$
and
$\Phi\Gamma\mathcal{F}_o := \{f : \mathbb{R} \longrightarrow \mathbb{R} \mid f \circ c \in C^\infty(\mathbb{R},\mathbb{R})$ *for all* $c \in C^\infty(\mathbb{R},\mathbb{R})$ *such that* $|c| \in C^\infty(\mathbb{R},\mathbb{R})\}$*. Hence the Frölicher structure is given as* $(\Gamma\mathcal{F}_o\ ,\ \Phi\Gamma\mathcal{F}_o)$*.*

(4) $K = \{(x,y) \in \mathbb{R}^2 \mid xy = 0\} = \{(x,0)\ x \neq 0\} \cup \{(0,y) \mid y \neq 0\}$ *is a frölicher Space as a subspace of the frölicher Space* $\mathbb{R}^2$ *but it is not a smooth manifold.*

(5) *The cone* $c = \{(x^1, x^2, x^3 \in \mathbb{R}^3 \mid (x^1)^2 - (x^2)^2 - (x^3)^2 = 0\}$ *is a frölicher Space since it is a subset of* $\mathbb{R}^3$ *but it is not a submanifold unless (0 , 0 , 0) is removed. since* $(x^1)^2 - (x^2)^2 - (x^3)^2 = 0$ *has rank o at (0 , 0 , 0) and* $\neq 0$ *otherwise , so it is not a smooth submanifold of* $\mathbb{R}^3$*.*

Remark 5.2.1.

(3),(4) and(5) are examples of Frölicher spaces which are not smooth manifolds.

definition 5.2.3. ***(Finer Frölicher structure)***
A Frölicher structure $(\mathcal{C},\mathcal{F})$ *is said to be finer than another* $(\mathcal{C}_o,\mathcal{F}_o)$ *on the same underlying set if* $\mathcal{C} \subset \mathcal{C}_o$ *or equivalentry if* $\mathcal{F}_o \subset \mathcal{F}$ *.*

definition 5.2.4. ***(Coarser Frölicher structure)***
A frölicher structure $(\mathcal{C},\mathcal{F})$ *is said to be coarser than another* $(\mathcal{C}_o,\mathcal{F}_o)$ *on the same underlying set if* $\mathcal{C}_o \subset \mathcal{C}$ *or equivalentry if* $\mathcal{F} \subset \mathcal{F}_o$*.*

lemma 5.2.2. *[38];*

(1) Let $\mathcal{C}_o$ be a generating set of the Frölicher structure ($\Gamma\Phi\mathcal{C}_o$,$\Phi\mathcal{C}_o$) on a set $\mathcal{M}$. Let ($\mathcal{C}_\mathcal{M}$,$\mathcal{F}_\mathcal{M}$) be another Frölicher structure on the same set $\mathcal{M}$ whose generating set $\mathcal{C}_1$ contains $\mathcal{C}_o$. Then ($\Gamma\Phi\mathcal{F}_o$,$\Phi\mathcal{C}_o$) is finer than ($\mathcal{C}_\mathcal{M}$,$\mathcal{F}_\mathcal{M}$).

(2) Let $\mathcal{F}_o$ be a generating set of the Frölicher structure ($\Gamma\mathcal{F}_o$,$\Phi\Gamma\mathcal{F}_o$) on a set $\mathcal{M}$. Let ($\mathcal{C}_\mathcal{M}$,$\mathcal{F}_\mathcal{M}$) be another Frölicher structure on the same set $\mathcal{M}$ whose generating set $\mathcal{F}_1$ contains $\mathcal{F}_o$. Then ($\Gamma\mathcal{F}_o$,$\Phi\Gamma\mathcal{F}_o$) is coarser than ($\mathcal{C}_\mathcal{M}$,$\mathcal{F}_\mathcal{M}$).

Proof.
By making use of lemma 5.2.1 we have that;

(1) $\mathcal{C}_1 \supseteq \mathcal{C}_o$ implies $\mathcal{F}_\mathcal{M} = \Phi\mathcal{C}_1 \subseteq \Phi\mathcal{C}_o$

(2) $\mathcal{F}_1 \supseteq \mathcal{F}_o$ implies $\mathcal{C}_\mathcal{M} = \Gamma\mathcal{F}_1 \subseteq \Gamma\mathcal{F}_o$. □

5.3. Smooth mappings between Frölicher spaces :

definition 5.3.1. ***(Frölicher smooth map)***
Let ($\mathcal{M}$,$\mathcal{C}_\mathcal{M}$,$\mathcal{F}_\mathcal{M}$) and ($\mathcal{N}$,$\mathcal{C}_\mathcal{N}$,$\mathcal{F}_\mathcal{N}$) be two Frölicher spaces .
A map $\varphi : \mathcal{M} \longrightarrow \mathcal{N}$ is said to be Frölicher smooth if one of the following equivalent conditions is satisfied ;

(1) For each $c \in \mathcal{C}_\mathcal{M}$, φ o $c \in \mathcal{C}_\mathcal{N}$.

(2) For each $f \in \mathcal{F}_\mathcal{N}$, f o $\varphi \in \mathcal{F}_\mathcal{M}$.

(3) For each $c \in \mathcal{C}_\mathcal{M}$, For each $f \in \mathcal{F}_\mathcal{N}$, f o φ o $c \in C^\infty(\mathbb{R},\mathbb{R})$.

As illustrated in the figure below.

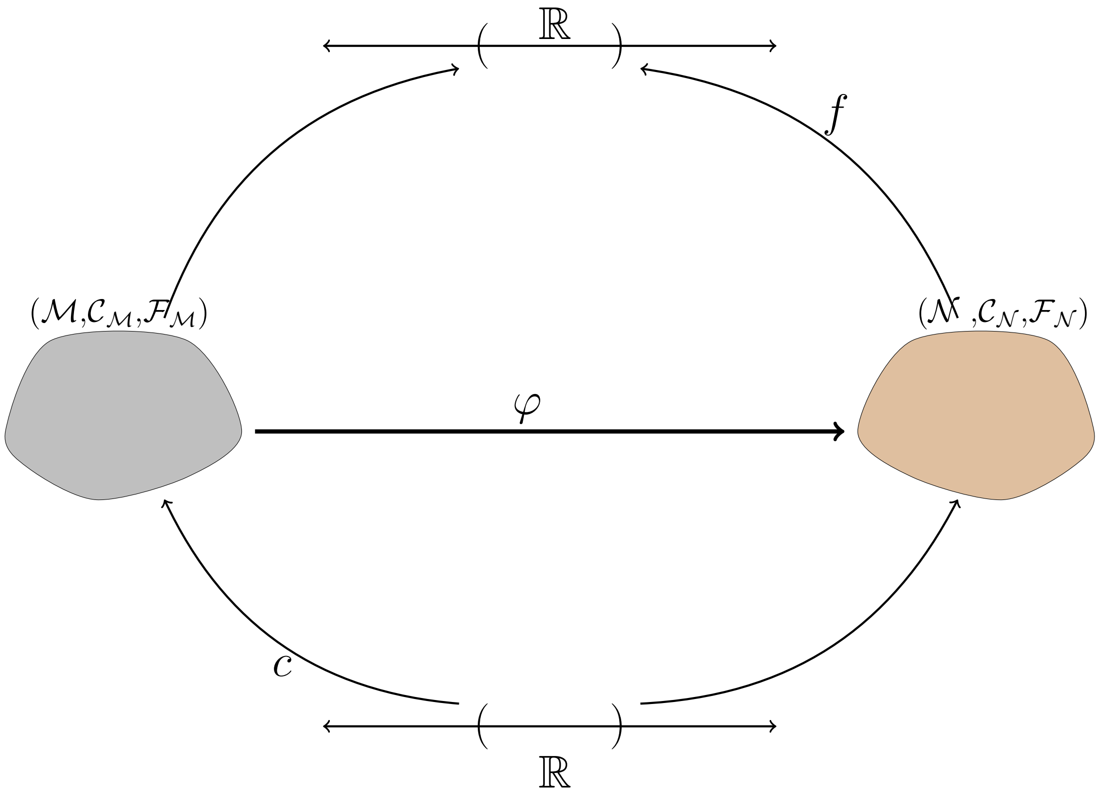


figure 14 (Frölicher smooth map)

Example 5.3.1. *;*
Let $\mathcal{M}$ be a Frölicher space , if we consider $\mathcal{N} = \mathcal{M}$ and $\varphi = id_{\mathcal{M}}$ in definition 5.3.1 then identity map $id_{\mathcal{M}}$ on $\mathcal{M}$ is a Frölicher smooth map.

proposition 5.3.1. *;*
The composition of two Frölicher smooth maps is a Frölicher smooth map :

Proof.
Let $(\mathcal{M},\mathcal{C}_{\mathcal{M}},\mathcal{F}_{\mathcal{M}})$,$(\mathcal{N},\mathcal{C}_{\mathcal{N}},\mathcal{F}_{\mathcal{N}})$ and $(\mathcal{P},\mathcal{C}_{\mathcal{P}},\mathcal{F}_{\mathcal{P}})$ be Frölicher spaces and let $\varphi : \mathcal{M} \longrightarrow \mathcal{N}$ and $\psi : \mathcal{N} \longrightarrow \mathcal{P}$ be Frölicher smooth maps ; since $f \circ \psi \in \mathcal{F}_{\mathcal{N}}$ and $\varphi \circ c \in \mathcal{C}_{\mathcal{N}}$, we have that the composite $f \circ (\psi \circ \varphi) \circ \mathrm{c} = (f \circ \psi) \circ (\varphi \circ \mathrm{c}) \in C^{\infty}(\mathbb{R},\mathbb{R})$ for some $f \in \mathcal{F}_{\mathcal{P}}$ and for some c $\in \mathcal{C}_{\mathcal{M}}$. Hence, $(\psi \circ \varphi) : \mathcal{M} \longrightarrow \mathcal{P}$ is a Frölicher smooth map. □

Frölicher spaces along with Frölicher smooth maps , form a category.

definition 5.3.2. ***(Frölicher diffeomorphism)***

A Frölicher diffeomorphism is a Frölicher map with a Frölicher inverse. i.e $\varphi : \mathcal{M} \longrightarrow \mathcal{N}$, where φ is smooth and φ^{-1} exists.

6

COMPARATIVE STUDY OF STRUCTURES :

6.1. Manifold Structure vs Sikorski structure

Differential spaces in the sense of Sikorski are generalization of the manifold concept by omitting the condition of existence of local diffeomorphisms to $\mathbb{R}^n$. The generalization admit to considerations larger class of object than manifolds. The set of real-valued functions f :$\mathcal{M} \longrightarrow \mathbb{R}$ that are infinitely-differentiable is a differential structure, moreover a differential space ($\mathcal{M}$,$\mathcal{F}$) is a manifold if every point of ($\mathcal{M}$,$\mathcal{F}$) has a neighbourhood diffeomorphic to an open subset of $\mathbb{R}^n$.

6.2. Manifold Structure vs diffeological structure

Diffeological spaces are a generalization of smooth manifolds to a category that is more stable. Every smooth manifold $\mathcal{M}$ is a diffeological space, in particular, every open subset of $\mathbb{R}^n$ has a diffeology. That is the one where the plots are taken to be all smooth maps from open subsets U of Euclidean spaces to the manifold $\mathcal{M}$, which is called the standard diffeology on $\mathcal{M}$. It is therefore easy to see that smooth maps in the usual sense between manifolds coincide with smooth maps between them with the standard diffeology. The smooth manifolds with smooth maps can then be seen as a full subcategory of the category of diffeological spaces.[45]. In fact on a smooth manifold $\mathcal{M}$, the set of parametrisations U $\longrightarrow \mathcal{M}$ that are infinitely-differentiable is a diffeology. Stacey A in his comparative smootheology research paper [35] pointed out that there are some similarlities between the Manifold structure and diffeological structure. In each there is an underlying category of objects to which one might wish to give a smooth structure. For manifolds, this is the category of topological manifolds and for diffeological spaces, this is the category of sets. In each there is a category which he called test spaces and a smooth structure consists of a family of morphisms to or from these test spaces and the object in question. In both structures, this is the category of open subsets of Euclidean spaces. He further points out that in each of the two structures, these families are not completely arbitrary. There is a condition which must be met , which he called forcing condition. For manifolds, this condition is that the maps from test spaces are diffeomorphisms on the overlaps. For diffeological spaces, the condition is that a map which is locally a plot is again a plot.

proposition 6.2.1. *;*

Given two smooth manifolds $\mathcal{M}$ and $\mathcal{N}$, then $C^\infty(\mathcal{M},\mathcal{N}) = [\mathcal{M},\mathcal{N}]$ that is the usual smooth maps coincide with the smooth maps in the diffeological sense :

Proof.

see [26] p 157 □

6.3. Manifold Structure vs Frölicher structure

A Frölicher structure is a generalisation of a manifold structure and it is known that every smooth manifold is a Frölicher space. However, some Frölicher spaces are not smooth manifolds. (see remark 5.2.1). Furthermore, we note that for the Frölicher space, the smooth structure is formed by pairs of both curves c $\in \mathcal{C}_{\mathcal{M}}$ and functions $f \in \mathcal{F}_{\mathcal{M}}$, whilst for the manifold, curves are not part of the defining axioms of its smooth structure. see [9]

Theorem 6.3.1. *;*

Let $(\mathcal{M}, \mathcal{A})$ be a smooth manifold then $(\mathcal{M}, \mathcal{A})$ is a Frölicher space.

Proof.

By $f \circ c = f \circ \psi^{-1} \circ \psi \circ c$ through local charts and boman's theorem 5.2.1)(see [12]) the manifold structure induces a frölicher structure on $\mathcal{M}$, so through local charts at each point of $\mathcal{M}$, $f \in C^\infty(M)$ satisfy $f \circ \psi^{-1} : \psi(u) \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ is C^∞ (definition of smooth function on a manifold). By Boman's theorem $f \circ \psi^{-1}$ being smooth implies that $f \circ \psi^{-1} \circ (\psi \circ c)$ is C^∞ as $(\psi \circ c)$ is a C^∞ -Euclidean curve by definition of smoothness for c on the manifold $\mathcal{M}$, and so $f\ o\ \psi^{-1} \circ \psi\ o\ c = f \circ c \in C^\infty$, which shows that smooth functions on $\mathcal{M}$ generate a frölicher structure . Thus a smooth manifold $\mathcal{M}$ is a Frölicher Space. □

As illustrated in the figure 15.

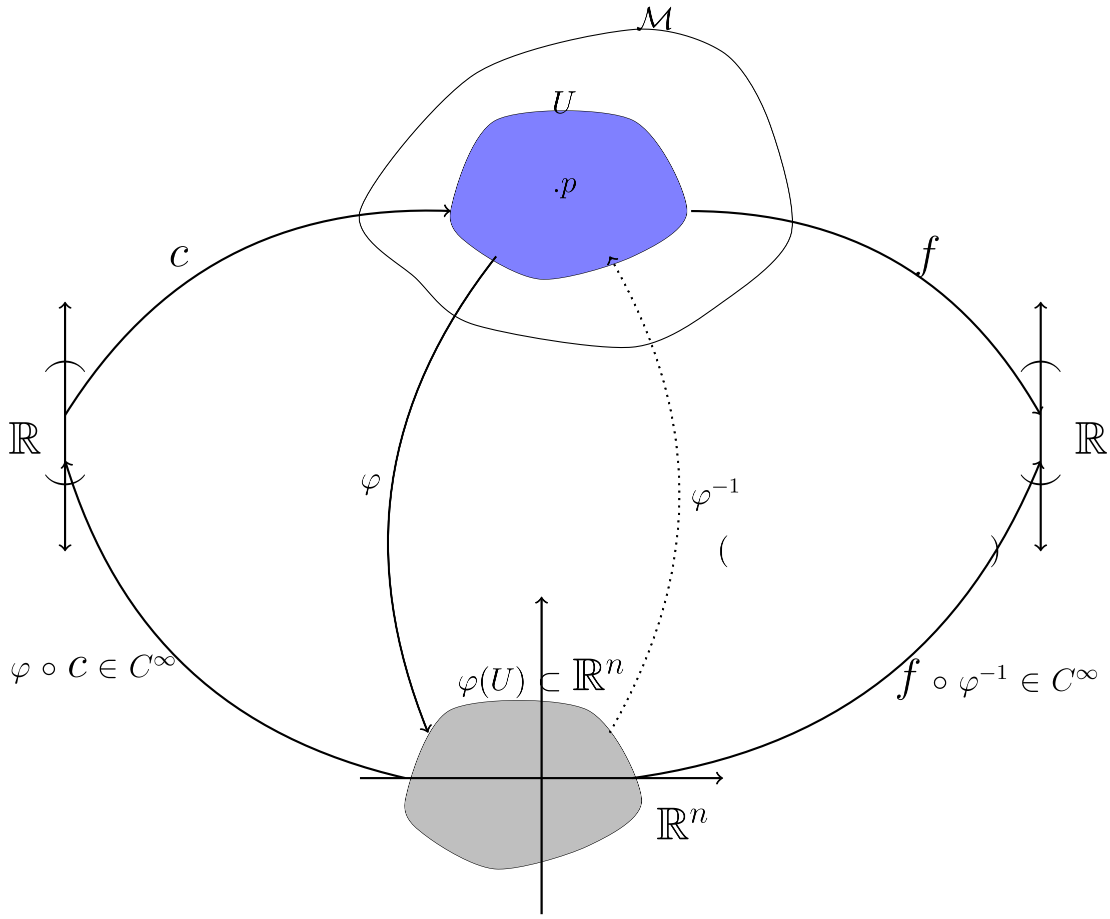


figure 15 (Smooth manifold generates a Frölicher space)

Most of the notes in the following three sections are with reference to a joint project of Batubenge A., Patrick Iglesias Z., Watts J and Yael k. (see [10], [42], [43]).

6.4. Sikorski Structure vs Frölicher structure

Frölicher spaces and differential spaces are both a generalisation of smooth manifold, however it is known from the works of Paul Cherenack [15] that Frölicher spaces constitute a full subcategory of the category of differential spaces. We let ($\mathcal{M}$,$\mathcal{C}_{\mathcal{M}}$,$\mathcal{F}_{\mathcal{M}}$) be a Frölicher space and we very much know in real sense that a Frölicher structure can not exist without the curves. Now suppose we 'ommit' the smooth curves in the structure $(\Gamma\mathcal{F}, \Phi\Gamma\mathcal{F})$ then $\mathcal{F}$ is a Sikorski differential structure. It is clearly true by the compatibility condition satisfied by $\mathcal{C}$ and $\mathcal{F}$ on M that $\Phi\Gamma\mathcal{F}$ is the generating set of $\mathcal{F}$. Now, suppose we let $(M, \mathcal{F})$ be a differential space. the differential structure induces a Frölicher structure. However, it is a property of Frölicher spaces that this Sikorski structure $\mathcal{F}$ considered as generating set will be a subset of the set of all Frölicher structure functions. That is, we have the inclusion $\mathcal{F} \subseteq \Phi\Gamma\mathcal{F}$. We show the above explanation in the following theorems.

Theorem 6.4.1. *;*
Let ($\mathcal{M}$, $\mathcal{C}_{\mathcal{M}}$, $\mathcal{F}_{\mathcal{M}}$) be a Frölicher space, then ($\mathcal{M}$, $\mathcal{F}_{\mathcal{M}}$) is a Sikorski differential space.

Proof.
We need to show that the two axioms in definition 3.2.1 are satisfied.

(i) Since the space is Frölicher by assumption,then for all c $\in \mathcal{C}_{\mathcal{M}}$ we have $(f_i \circ$ c) $\in$ $C^\infty(\mathbb{R},\mathbb{R})$, (i = 1,...,n). That is $(f_1 \circ$ c ,...,$f_n \circ$ c) $\in C^\infty(\mathbb{R},\mathbb{R}^n)$.For every smooth $\omega \in$ $C^\infty(\mathbb{R}^n$,$\mathbb{R})$ the composition $\omega \circ (f_1 \circ$ c ,...,$f_n \circ$ c) $\in C^\infty(\mathbb{R},\mathbb{R})$. Hence , by the compatibility condition that defines a Frölicher smooth structure on $\mathcal{M}$, $\omega \circ (f_1 \circ$ c ,...,f_n $\circ$ c) $\in \mathcal{F}_{\mathcal{M}}$,which satisfies the smooth compatibility condition of definition 3.2.1 . As illustrated in the diagram below.

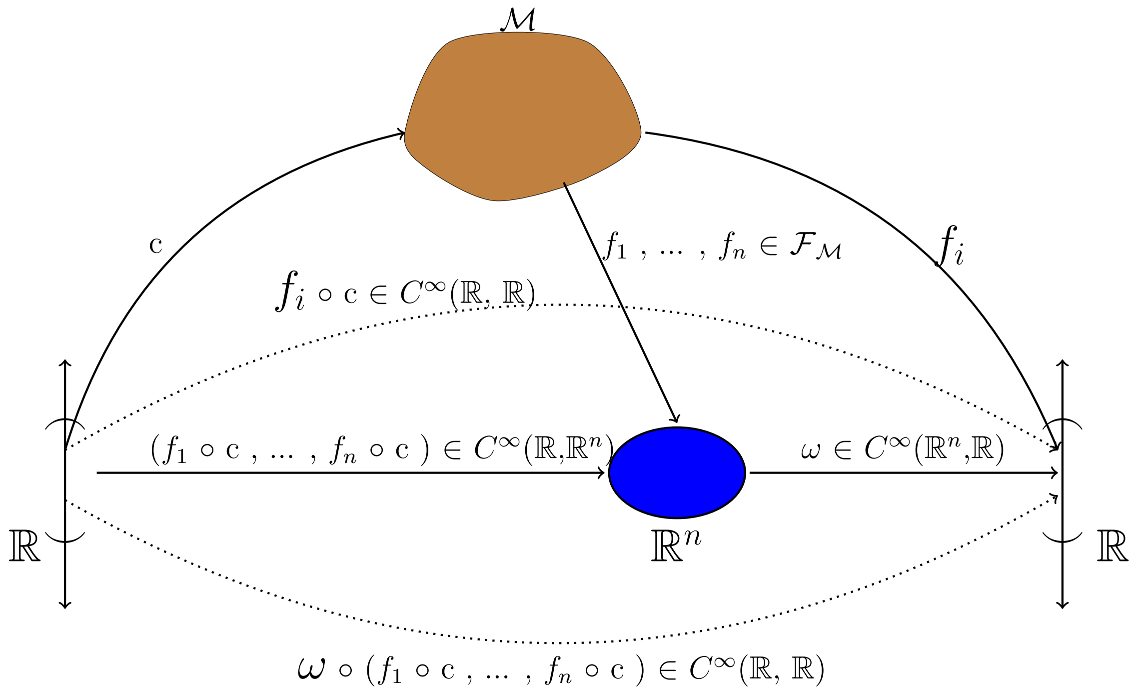


figure 16

(ii) Let c : $\mathbb{R} \longrightarrow \mathcal{M}$ be a smooth curve in $\mathcal{M}$ and $f \in \mathcal{F}_{\mathcal{M}}$ .Let x be a point in $\mathcal{M}$ and $\mathcal{A}$ an open covering of $\mathcal{M}$. There exist a neighbourhood $U_i \in \mathcal{A}$ such that $x \in U_i$ and $g_{|U_i} = f_{|U_i}$ for each i $\in$ I . $f \circ c \in C^\infty(\mathbb{R},\mathbb{R})$, therefore $g_{|U_i} \circ c_{|c^{-1}(U_i)} \in C^\infty(\mathbb{R},\mathbb{R})$ for some function g : $\mathcal{M} \longrightarrow \mathbb{R}$ defined on $\mathcal{M}$. The sets $c^{-1}(U_i)$ cover $\mathbb{R}$, thus $g \circ c \in C^\infty(\mathbb{R},\mathbb{R})$. Hence g $\in \mathcal{F}_{\mathcal{M}}$, since ($\mathcal{M}$,$\mathcal{C}_{\mathcal{M}}$,$\mathcal{F}_{\mathcal{M}}$) is a Frölicher space . □

Theorem 6.4.2. *;*
Let ($\mathcal{M}$, $\mathcal{F}$) be a Sikorski differential space. Then $\mathcal{F}$ induces a Frölicher structure on $\mathcal{M}$.

Proof.
Let $\Gamma\mathcal{F} := \{c : \mathbb{R} \longrightarrow \mathcal{M} \mid f \circ c \in C^\infty$ for all $f \in \mathcal{F}\}$ and $\Phi\Gamma\mathcal{F} := \{f : \mathcal{M} \longrightarrow \mathbb{R} \mid f \circ c \in C^\infty$ for all $c \in \Gamma\mathcal{F}\}$ We know that ($\Gamma\mathcal{F}$, $\Phi\Gamma\mathcal{F}$) is a Frölicher structure on $\mathcal{M}$, with $\mathcal{F} \subseteq \Phi\Gamma\mathcal{F}$ (see lemma) and this shows that all Sikorski functions in $\mathcal{F}$ are Frölicher structure functions in ($\Gamma\mathcal{F}$, $\Phi\Gamma\mathcal{F}$) hence the proof. □

definition 6.4.1. ***(Reflexive differential structure)***
A differential structure $\mathcal{F}$ is reflexive if $\Phi\Pi\mathcal{F} = \mathcal{F}$.

Theorem 6.4.3. *(Reflexive Theorem)([10]);*
There is a natural isomorphism of categories of Frölicher spaces to reflexive differential spaces.

Furthermore (see [42]) if we Let Ξ to be the forgetful functor from Frölicher spaces to differential spaces : $\Xi(X, C, \mathcal{F}) = (X, \mathcal{F})$, and Ξ takes maps to themselves.
We now state the following theorem:
Proposition 6.4.4 (Frölicher Stability)**.** *Let X be a set, and let $\mathcal{F}_0$ be a family of functions on X, C_0 be a family of curves into X.*

1. *Let $C = \Gamma\mathcal{F}_0$ and $\mathcal{F} = \Phi\Gamma\mathcal{F}_0$. Then X equipped with C and $\mathcal{F}$ is a Frölicher space.*
2. *$\mathcal{F} = \Phi C_0$ and $C = \Gamma\Phi C_0$ $\dot{T}$hen X equipped with C and $\mathcal{F}$ is a Frölicher space.*

Proof.
(see [42], pages 26-27). □

6.5. Sikorski Structure vs diffeological structure

Due to the fact that Diffeological spaces and Sikorski differential spaces are each a generalisation of a smooth manifold,the two structures are related in one way or the other and can be compared. We note that in their definitions, both have locality and smooth compatibility conditions.Each structure use the category of sets as the object where a smooth structure can be developed. With reference to Stacey A s paper [35] we see that the test space for each structure is the category of open subsets of Euclidean spaces. That is an open set of $\mathbb{R}^n$ for some n $\in \mathbb{N}$ for diffeological spaces and $\mathbb{R}$ for a Sikorski space , though it should be noted that for a diffeological space maps are into the set, whilst a Sikorski differential space consists of maps out of the set.
Batubenge et al [10] compares the two structures using the terms "compatible" and "determines" which are defined as follows;

definition 6.5.1. *("Compatible")*
Given a set $\mathcal{M}$ with a collection $\mathcal{D}$ of parametrisations and a collection $\mathcal{S}$ of real-valued functions, we say that $\mathcal{D}$ and $\mathcal{F}$ are compatible if $f \circ p$ is infinitely-differentiable for all $p \in \mathcal{D}$ and $f \in \mathcal{F}$.

definition 6.5.2. *("Determine")*
Given a set $\mathcal{M}$ with a collection $\mathcal{D}$ of parametrisations and a collection $\mathcal{F}$ of real-valued functions, we say that ;

(i) *$\mathcal{D}$ determines the set $\Phi\mathcal{D}$ of those real-valued functions whose precomposition with each element of $\mathcal{D}$ is infinitely-differentiable, $\Phi\mathcal{D} := \{ f : \mathcal{M} \longrightarrow \mathbb{R} \mid \forall (p\colon U \longrightarrow \mathcal{M}) \in \mathcal{D} , f \circ p \in C^\infty(U) \}$;*

(ii) *$\mathcal{F}$ determines the set $\Pi\mathcal{F}$ of those parametrisations whose composition with each element of $\mathcal{F}$ is infinitely-differentiable, $\Pi\mathcal{F} := \{$ parametrisations $p\colon U \longrightarrow \mathcal{M} \mid \forall f \in \mathcal{F} , f \circ p \in C^\infty(U) \}$.*

Remark 6.5.1.

In this way the diffeological structure and a differential structure determine each other.

lemma 6.5.1. *[10][42];*
Each of the operations $\mathcal{D} \mapsto \Phi\mathcal{D}$ and $\mathcal{F} \mapsto \Pi\mathcal{F}$ is inclusion-reversing, we have the following ;

(i) $\mathcal{D} \subseteq \prod \Phi\mathcal{D}$.

(ii) $\mathcal{F} \subseteq \Phi\Pi\mathcal{F}$

Proof.

(i) Let p $\in \mathcal{D}$. By definition of $\Phi\mathcal{D}$, for any f $\in \Phi\mathcal{D}$, $f \circ p$ is smooth. Since f is arbitrary we have that p $\in \Pi\Phi\mathcal{D}$.

(ii) Let f $\in \mathcal{F}$. By definition of $\Pi\mathcal{F}$, for all g $\in \mathcal{F}$, $g \circ p$ is smooth, then p $\in \Pi\mathcal{F}$. Thus $f \circ p$ is smooth for all such p. Since p is arbitrary we have that f $\in \Phi\Pi\mathcal{F}$.

□

proposition 6.5.1. *;*
Let $\mathcal{D}$ be a family of parametrisations, there exists a family of real valued functions that determines $\mathcal{D}$ if and only if
$\Pi\Phi\mathcal{D} = \mathcal{D}$. ***:***
proposition 6.5.2. *;*
Let $\mathcal{F}$ be a family of real-valued functions, there exists a family of parametrisations that determines $\mathcal{F}$ if and only if
$\Phi\Pi\mathcal{F} = \mathcal{F}$ ***:***
definition 6.5.3. *(**Reflexive Diffeology**)*
A diffeology $\mathcal{D}$ is reflexive if $\Pi\Phi\mathcal{D} = \mathcal{D}$.

Theorem 6.5.1. *;*
Let $(\mathcal{M}, \mathcal{D})$ be a diffeological space and $\Phi\mathcal{D} := \{ f : \mathcal{M} \longrightarrow \mathbb{R} \mid \forall (p: U \longrightarrow \mathcal{M}) \in \mathcal{D}\ ,\ f \circ p \in C^{\infty}(U,\mathbb{R}) \}$. Then the set $\Phi\mathcal{D}$ is a Sikorski differential structure on $\mathcal{M}$

Proof.
Here we need to show that the axioms in definition 3.2.1 are satisfied.

(i) Let f_1 , ... , $f_n \in \Phi\mathcal{D}$ and let $\omega \in C^{\infty}(\mathbb{R}^n,\mathbb{R})$, for any positive integer n $\in \mathbb{N}$. Let (p: U $\longrightarrow \mathcal{M}) \in \mathcal{D}$, then the composition $\omega \circ (f_1 , ... , f_n) \circ$ p $\in C^{\infty}$(U,$\mathbb{R}$), since $\omega \in C^{\infty}(\mathbb{R}^n,\mathbb{R})$ and $(f_1 , ... , f_n) \circ$ p $\in C^{\infty}$(U,$\mathbb{R}^n$) as $f_i \in \Phi\mathcal{D}$, so that each coordinate function $f_i \circ$ p $\in C^{\infty}$(U,$\mathbb{R}$), for i = 1,...,n. Now p is arbitrary therefore $\omega \circ (f_1 , ... , f_n) \in \Phi\mathcal{D}$. Therfore $\Phi\mathcal{D}$ satisfies the smooth composition property of Sikorski differential structure on $\mathcal{M}$.
As illustrated in figure 17.

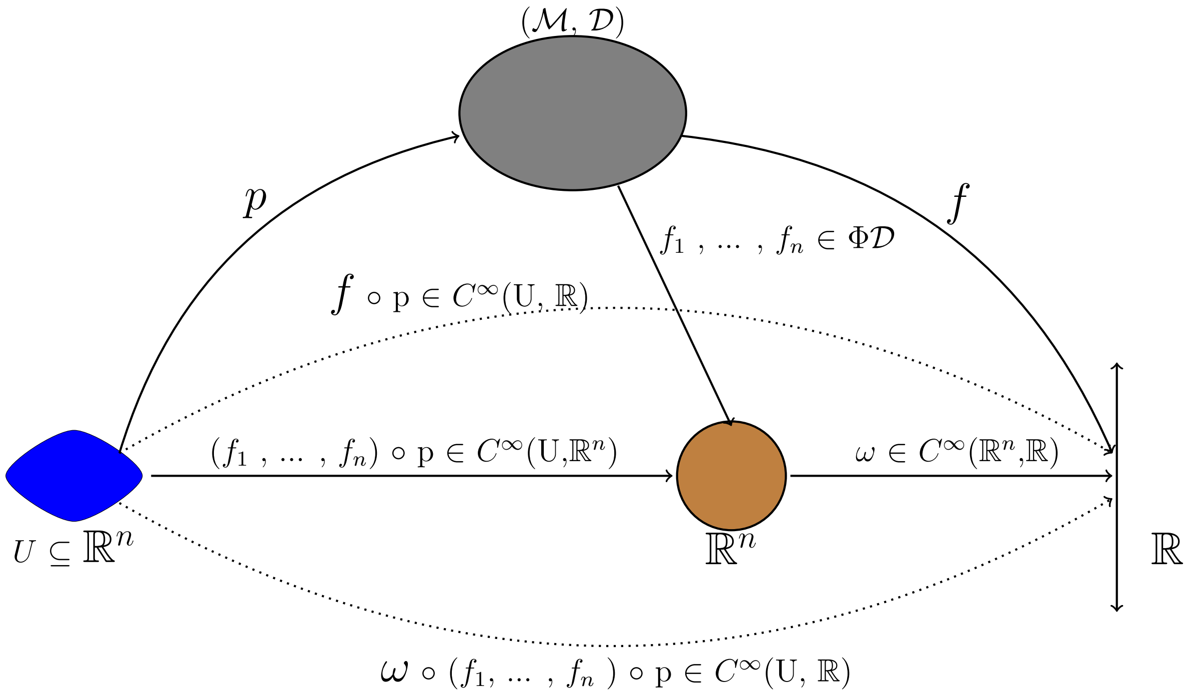


figure 17

(ii) Let $\mathcal{M}$ be equiped with the initial topology induced on $\mathcal{M}$ by the set $\Phi\mathcal{D}$. Let $f : \mathcal{M} \longrightarrow \mathbb{R}$ be a function satisfying, for every x $\in \mathcal{M}$ there is an open neighbourhood $\mathcal{V}$ of x with respect to the $\mathcal{D}$-topology on $\mathcal{M}$ and let a function g $\in \Phi\mathcal{D}$ be such that $g\mid_{\mathcal{V}} = f\mid_{\mathcal{V}}$.....$\star$. We need to show that $f \in \Phi\mathcal{D}$. Let p:U $\longrightarrow \mathcal{M} \in \mathcal{D}$,then from $\star$ we have that $g \circ p|_{p^{-1}(\mathcal{V})} = f \circ p|_{p^{-1}(\mathcal{V})}$, where the pre-image $p^{-1}(\mathcal{V})$ is an open set in U $\subseteq$

$\mathbb{R}^n$. Now each such $g \circ p$ is smooth in U and since smoothness is a local property, then $f \circ p : p^{-1}(\mathcal{V}) \subseteq$ U $\longrightarrow \mathbb{R}$ is smooth. Now the open sets $p^{-1}(\mathcal{V})$ cover U, therefore $f \circ p \in C^\infty$(U,$\mathbb{R}$) and since p $\in \mathcal{D}$ is arbitrary, we have that $f \in \Phi\mathcal{D}$. Hence proof of the locality condition.

□

Theorem 6.5.2. *;*
Let ($\mathcal{M}$, $\mathcal{F}$) be a Sikorski differential space and $\Pi\mathcal{F}$ ***:= { parametrisations p : U*** $\longrightarrow \mathcal{M} \mid \forall$ ***f*** $\in \mathcal{F}$ ***,*** $f \circ p \in C^\infty$***(U,***$\mathbb{R}$***) }.Then the set*** $\Pi\mathcal{F}$ ***is a diffeology on*** $\mathcal{M}$***.***

Proof.
we need to show that the three axioms in definition 4.2.2 are satisfied.

(i) Here we need to check that $\Pi\mathcal{F}$ contains all the constant maps into $\mathcal{M}$. Now if p: U $\longrightarrow \mathcal{M}$ is constant then for any f $\in \mathcal{F}$ the composition $f \circ p$: U $\longrightarrow \mathcal{M}$ is a constant function, hence smooth.Thus the covering axiom is shown.

(ii) Let p : U $\longrightarrow \mathcal{M}$ be a parametrisation such that for every x $\in$ U there is an open neighbourhood V $\subseteq$ U of x such that $p\mid_V \in \Pi\mathcal{F}$ .we need to show that p $\in \Pi\mathcal{F}$. Let $f \in \mathcal{F}$ then for any x $\in$ U, there is an open neighbourhood V $\subseteq$ U of x such that $f \circ p\mid_V$ is smooth. Now smoothness on U is a local condition therefore $f \circ$ p : U $\longrightarrow \mathcal{M}$ is smooth. Since $f \in \mathcal{F}$ is arbitrary we have that p $\in \Pi\mathcal{F}$, hence shows the locality axiom.

(iii) Let U and V be open subsets of Euclidean spaces, and let F : V $\longrightarrow$ U be a smooth map. Let (p: U $\longrightarrow \mathcal{M}$) $\in \Pi\mathcal{F}$ then for any $f \in \mathcal{F}$, we have that $f \circ$ p $\in C^\infty$(U,$\mathbb{R}$). Now, composition of two smooth maps is smooth therefore ($f \circ$ p) $\circ$ F = $f \circ$ (p $\circ$ F) $\in C^\infty$(V,$\mathbb{R}$). since $f \in \mathcal{F}$ is arbitrary, p $\circ$ F $\in \Pi\mathcal{F}$. Which is the smooth compatibility axiom. **See figure 18 below** . □

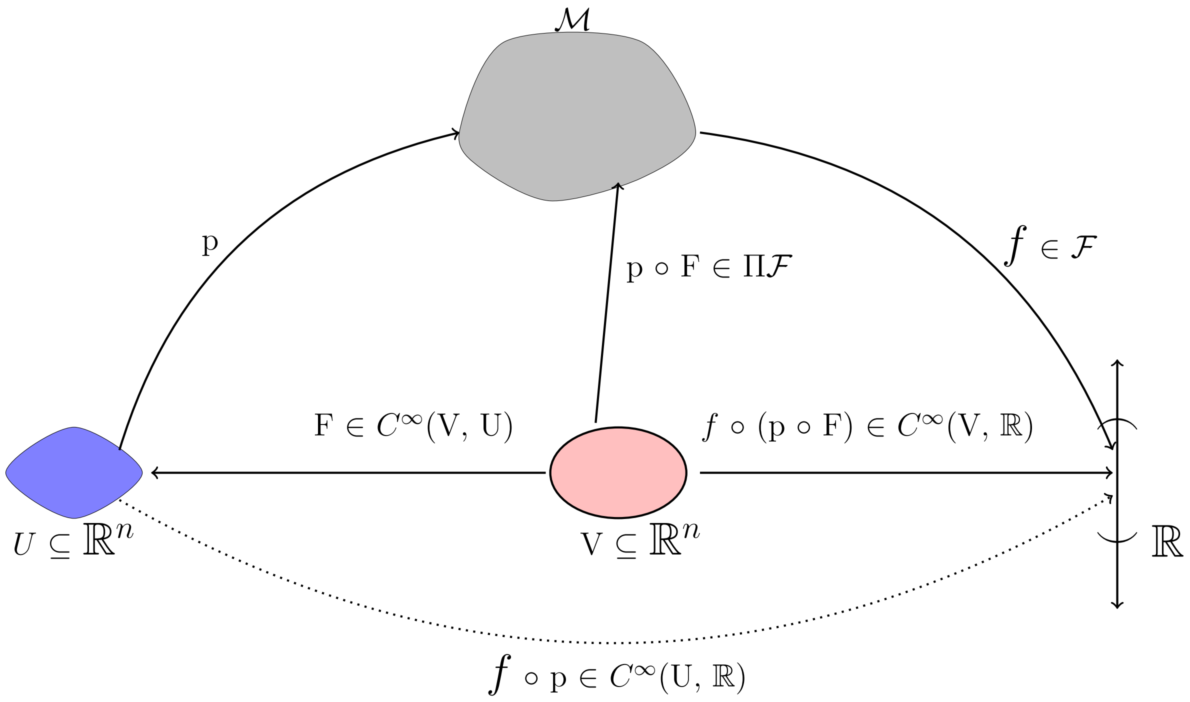


figure 18

6.6. Frölicher Structure vs Diffeological structure

Every Frölicher space is a diffeological space as noted also by Tore C A [37]. In Frölicher spaces we see that structure curves c : $\mathbb{R} \longrightarrow \mathcal{M}$ appear as plots in diffeological spaces [44],[45]. We see in the following theorem that the two structures are interrelated.

Theorem 6.6.1. *;*

Let ($\mathcal{M}$,$\mathcal{C}_{\mathcal{M}}$,$\mathcal{F}_{\mathcal{M}}$) be a Frölicher space, then ($\mathcal{M}$,$\mathcal{C}_{\mathcal{M}}$,$\mathcal{F}_{\mathcal{M}}$) is a diffeological space.

Proof.

To prove that ($\mathcal{M}$,$\mathcal{C}_{\mathcal{M}}$, $\mathcal{F}_{\mathcal{M}}$) is a diffeological space , we need to show that $\mathcal{C}_{\mathcal{M}}$ defines a diffeology on $\mathcal{M}$. That is it should satisfy the three axioms in definition 4.2.2

(i) Firstly , since constant curves are structure curves in the Frölicher structure and observing that in $\mathcal{C}_{\mathcal{M}}$ we only have maps c : $\mathbb{R} \longrightarrow \mathcal{M}$ as plots , if we let U ≡ ℝ then the axiom of covering is satisfied.

(ii) Let c : $\mathbb{R} \longrightarrow \mathcal{M}$ be a curve in $\mathcal{M}$. Let x ∈ ℝ then c(x) ∈ $\mathcal{M}$. Assume that c(x) ∈ $\mathcal{O}$, with $\mathcal{O} \in \tau_{\mathcal{C}_{\mathcal{M}}}$ the curvaceous topology of ($\mathcal{M}$,$\mathcal{C}_{\mathcal{M}}$,$\mathcal{F}_{\mathcal{M}}$). Let $\alpha \in \mathcal{C}_{\mathcal{M}}$ such that $\mathcal{V} := \alpha^{-1}(\mathcal{O}) \in \tau_{\mathbb{R}}$ and $\mathcal{V}$ is the neighbourhood of x. Now, assume that $\alpha_{|\mathcal{V}} = c_{|\mathcal{V}}$.......⋆, then for any f $\in \mathcal{F}_{\mathcal{M}}$, we have that $f \circ \alpha \in C^{\infty}(\mathbb{R},\mathbb{R})$.Therefore $(f \circ \alpha)_{|\mathcal{V}} = \mathrm{f}_{|\alpha(\mathcal{V})} \circ \alpha_{|\mathcal{V}}$, from $\star$ we have that $\mathrm{f}_{|\alpha(\mathcal{V})}$ $f \circ c_{|\mathcal{V}} \in C^{\infty}(\mathbb{R},\mathbb{R})$. Since $f \circ \alpha \in C^{\infty}(\mathbb{R},\mathbb{R})$ which is a continuous function, then $(f \circ c)_{|\mathcal{V}} \in C^{\infty}(\mathbb{R},\mathbb{R})$ which shows that c $\in \mathcal{C}_{\mathcal{M}}$ and the proof holds true for all x $\in \mathbb{R}$, which satisfies the axiom of locality. **As shown in the diagram below**.

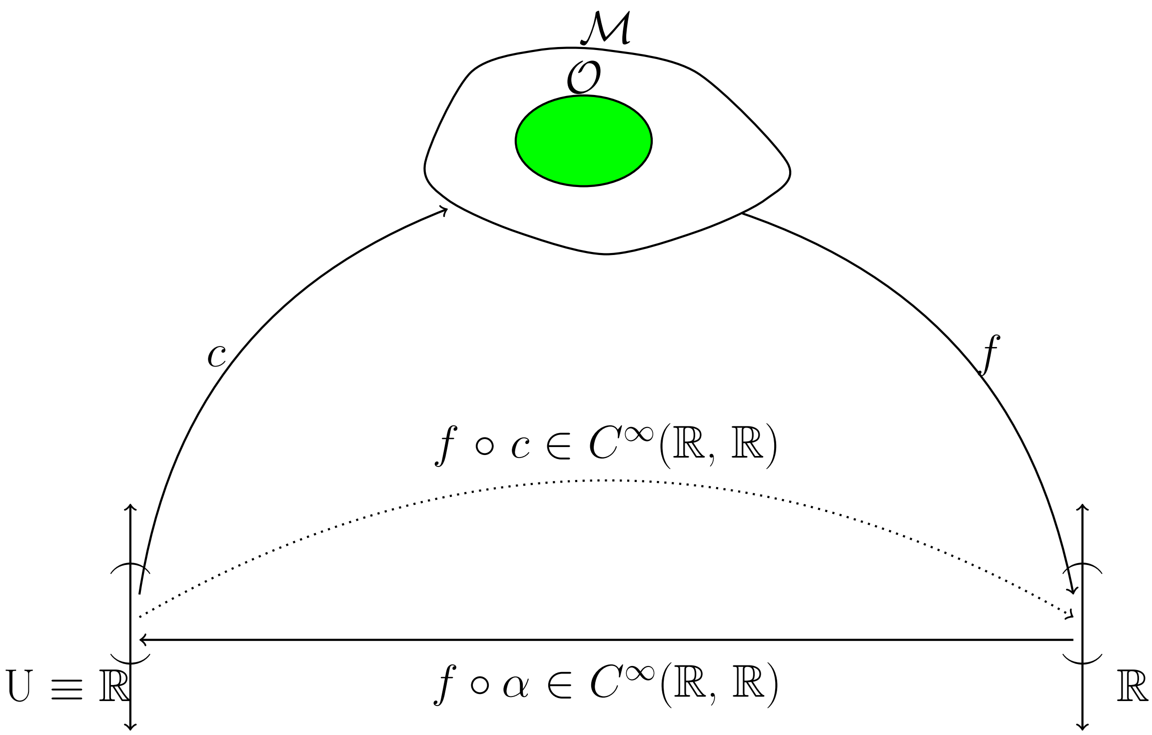


figure 19

(iii) Let $p : \mathbb{R} \longrightarrow \mathcal{M}$ be a smooth curve in $\mathcal{M}$. Then for every smooth function F : $\mathbb{R} \longrightarrow \mathbb{R}$ and $f \in \mathcal{F}_{\mathcal{M}}$, we have that $f \circ p \in C^{\infty}(\mathbb{R},\mathbb{R})$.Since F $\in C^{\infty}(\mathbb{R},\mathbb{R})$, then $(f \circ p) \circ$ F $\in C^{\infty}(\mathbb{R},\mathbb{R})$ that is $p \circ F \in \mathcal{C}_{\mathcal{M}}$, thus $p \circ F$ has same property as p, hence $p \circ F$ is a 1-plot i.e $p \circ F \in \mathcal{D}_1$. see figure 20 □

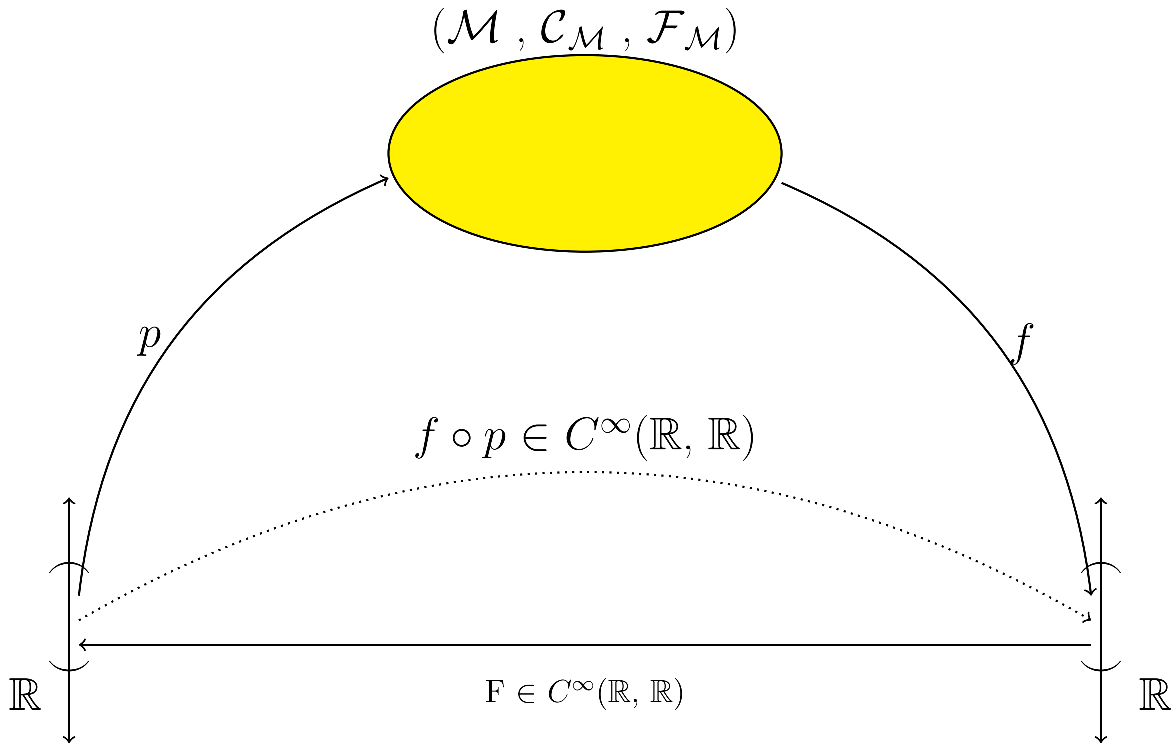
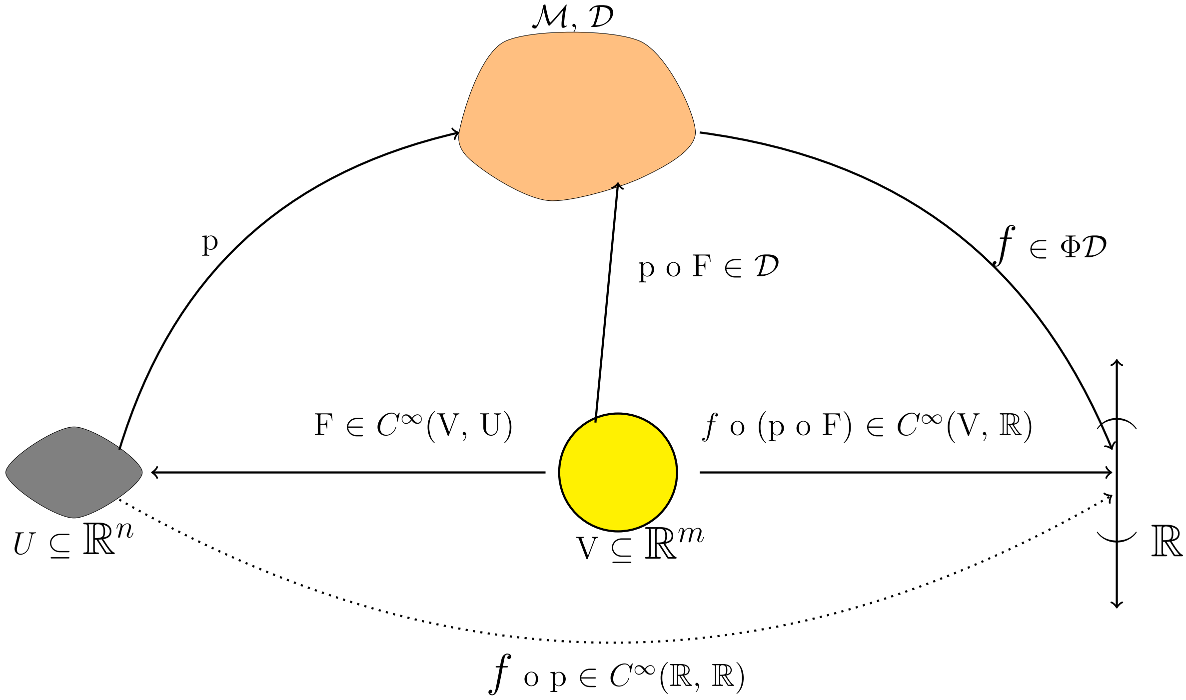


figure 20

Theorem 6.6.2. *;*
Let $(\mathcal{M}\ ,\ \mathcal{D})$ be a diffeological space then $(\mathcal{M}\ ,\ \mathcal{D})$ defines a Frölicher structure $(\mathcal{M},\mathcal{C}_{\mathcal{M}},\mathcal{F}_{\mathcal{M}}\ )$ on $\mathcal{M}$.

Proof.
Let $(\mathcal{M},\mathcal{D})$ be a diffeological space and $p : \text{U} \longrightarrow \mathcal{M} \in \mathcal{D}$, with $\text{U} \subseteq \mathbb{R}^n$. Then for all $\mathcal{V} \in \tau_{\mathbb{R}^m}$ and $\text{F} \in C^{\infty}(\mathcal{V},\mathcal{U})$, one has $(p \circ F : \mathcal{V} \longrightarrow \mathcal{M}) \in \mathcal{D}$. Now for all f $\in \Phi\mathcal{D}$, $f \circ p \in C^{\infty}(\mathbb{R},\mathbb{R})$. Therefore, $p : \mathbb{R} \longrightarrow \mathcal{M}$ and f:$\mathcal{M} \longrightarrow \mathbb{R}$ satisfy the compatibility condition for a Frölicher structure on $\mathcal{M}$. Hence, proving that a diffeology contains a Frölicher structure underlying it.This is illustrated in figure 21 below. ☐



$\mathcal{M}$, $\mathcal{D}$
p
p o F ∈ $\mathcal{D}$
f ∈ Φ$\mathcal{D}$
F ∈ C^∞(V, U)
f o (p o F) ∈ C^∞(V, ℝ)
$U \subseteq \mathbb{R}^n$
V ⊆ $\mathbb{R}^m$
ℝ
f o p ∈ C^∞(ℝ, ℝ)

figure 21

Summary of the comparative study of structures:

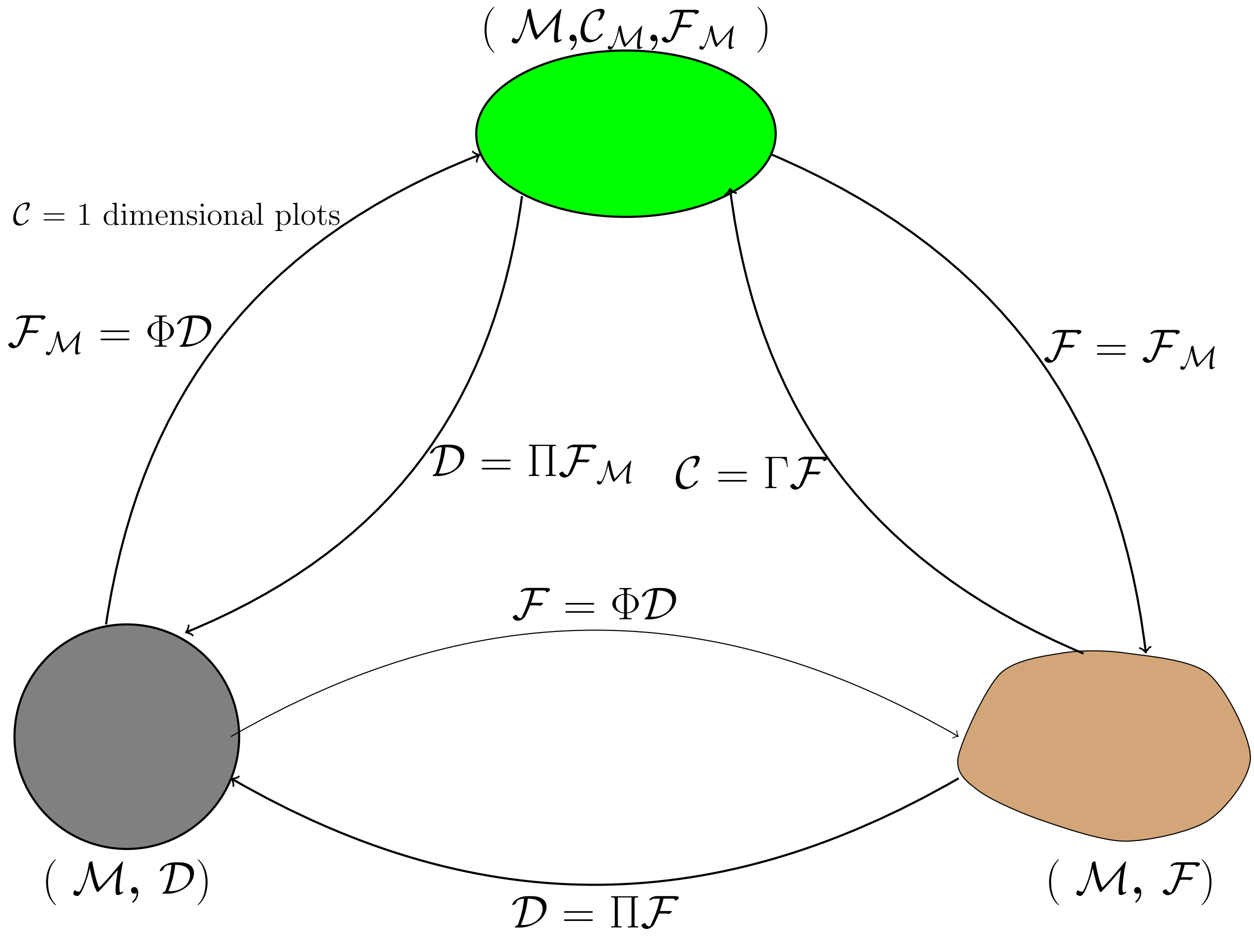


figure 22

7

COMPARATIVE STUDY OF TOPOLOGIES :

7.1. Topology of Manifold

Smooth or differentiable manifolds are modelled on Euclidean spaces, whose topological space satisfy the Hausdorff, second countable and locally Euclidean properties.

In general the topology on a manifold ,

(1) Is Hausdorff.

(2) Has a countable basis of open sets.

see definitions 2.1.1, 2.1.2, 2.1.5 and 2.1.9.

7.2. Topology of Sikorski space

definition 7.2.1. ***(Topology on a Sikorski differential space.)***
Let $(\mathcal{M}\ ,\tau_{\mathcal{F}},\ \mathcal{F})$ *be a Sikorski differential space.* $\tau_{\mathcal{F}}$ *is the topology induced on* $\mathcal{M}$ *by the family* $\mathcal{F}$ *, that is the weakest topology such that all functions from the family* $\mathcal{F}$ *are continuous and the set* $\{f^{-1}(a\ ,\ b) \subseteq \mathcal{M} \mid f \in \mathcal{F}\ ,\ (a\ ,\ b) \subseteq \mathbb{R}\ \}$ *is a sub-basis for the topology of* $\mathcal{M}$ *.*

lemma 7.2.1. .
Let $\mathcal{G}$ be the set of all $\mathcal{F}$ - smooth functions on $\mathcal{M}$ then $\tau_{\mathcal{G}} = \tau_{\mathcal{F}}$.

Proof.
Since $\mathcal{G}$ denotes the set of all $\mathcal{F}$ - smooth functions on $\mathcal{M}$, we have that $\mathcal{F} \subset \mathcal{G}$ which implies that $\tau_{\mathcal{F}} \subset \tau_{\mathcal{G}}$. On the other hand for any $\omega \in C^{\infty}$ ($\mathbb{R}^n$, $\mathbb{R}$), the composition ω o $(f_1 , ... , f_n)$ is continuous with respect to $\tau_{\mathcal{F}}$ which gives $\tau_{\mathcal{G}} \subset \tau_{\mathcal{F}}$, therefore $\tau_{\mathcal{G}} = \tau_{\mathcal{F}}$. □

7.3. Topology of Diffeological space

Any diffeological space is equipped with a natural topology with respect to all of its plots . This topology is called the $\mathcal{D}$ **- topology** , and is not an extra structure but just a topology naturally carried by a diffeological space . That is , the final topology induced by its plots , where each domain is equipped with the standard topology .

definition 7.3.1. *($\mathcal{D}$ **- open set**)*
Let $(\mathcal{M}$, $\mathcal{D}_{\mathcal{M}})$ be a diffeological space. A subset B of $\mathcal{M}$ is $\mathcal{D}$ - open if $p^{-1}(B)$ is open in U for each plot $p : U \longrightarrow \mathcal{M}$.

definition 7.3.2. *($\mathcal{D}$ **- Topology**)*
The $\mathcal{D}$ - Topology is the the finest topology induced on $\mathcal{M}$ by a collection of $\mathcal{D}$ - open subsets of $\mathcal{M}$ such that the plots are continuous .

Example 7.3.1. *;*

(1) ***Discret diffeology*** *;*
The $\mathcal{D}$-topology of the discrete diffeology is just the discrete topology.

(2) ***Indiscrete diffeology*** *;*
The $\mathcal{D}$-topology on an indiscrete diffeological space is indiscrete. Given any set $\mathcal{M}$, the collection $\mathcal{P}(\mathcal{M})$ of all possible maps into $\mathcal{M}$ with open domain is a diffeology. The only open sets are $\mathcal{M}$ and the empty set, so the -topology is the indiscrete topology.

(3) ***Smooth Manifold*** *;*
The $\mathcal{D}$-topology on a smooth manifold with the standard diffeology coincides with the usual topology on the smooth manifold.

lemma 7.3.1. *([41]).*
Let $A \subseteq \mathcal{M}$ be a subspace (i.e A is equipped with the subspace diffeology) then every set U $\subseteq$ A open in the supspace topology is $\mathcal{D}_A$-open.

Proof.
Trivial.
proposition 7.3.1. *;*
***Smooth maps are $\mathcal{D}$ - Continuous , that is continuous for the $\mathcal{D}$ - topology** ([44]);*

Proof.
Let $\mathcal{D}_M$ and $\mathcal{D}_N$ be the diffeologies on $\mathcal{M}$ and $\mathcal{N}$. Then for every plot P in $\mathcal{M}$, $f \circ P$ is a plot for $\mathcal{N}$. That is f maps every plot P $\in \mathcal{D}_M$ into the plot $f \circ P \in \mathcal{D}_N$ (see definition 4.3.1) . Let $A_o \subset \mathcal{N}$ be $\mathcal{D}$ - Open , and let A = $f^{-1}(A_o)$. For every plot P $\in \mathcal{D}_M$, $p^{-1}(A)$ = $p^{-1}(f^{-1}(A_o)) = (f \circ P)^{-1}(A_o)$. But $f \circ P$ is a plot of $\mathcal{D}_N$ and A_o is $\mathcal{D}$ - Open , therefore (f o P)$^{-1}(A_o)$ is open . Hence , $p^{-1}(A)$ is open for every plot P $\in \mathcal{D}_M$. We have that A = $f^{-1}(A_o)$ is $\mathcal{D}$ - open , thus f is $\mathcal{D}$ - continuous . □

7.4. Topology of Frölicher space

Several studies have been done on topologies of the Frölicher space, Frölicher A and Kriegl A, [21] in their book "Linear spaces and Differentiation Theory" showed that the subbasis for $\tau_{\mathcal{F}_\mathcal{M}}$, the topology induced by all structural functions is $\{f^{-1}\ (0\ ,\ 1)\}_{f \in \mathcal{F}_\mathcal{M}}$, and Dugmore B,[18] in his thesis gives the basis as $\mathcal{B} = \{f^{-1}\ (0\ ,\ \infty)\}_{f \in \mathcal{F}_\mathcal{M}}$.
Cherenack p [15] observed that the topologies $\tau_{\mathcal{F}_\mathcal{M}}$ and $\tau_{\mathcal{C}_\mathcal{M}}$ coincide when the Frölicher space under consideration is a smooth manifold.In his PhD Thesis [13], A Cap defines a balanced space as a Frölicher space whose initial topology coincides with the final topology that is $\tau_{\mathcal{F}_\mathcal{M}} = \tau_{\mathcal{C}_\mathcal{M}}$ and that it is Hausdorff if the two topologies are Hausdorff. More studies and investigations on the topologies of a Frölicher space have been done by Batubenge A and Tshilombo H .([6],[7],[8],[9],[38])

definition 7.4.1. ***(Frölicher Topologies)***

A Frölicher space carries two natural topologies induced by functions and by curves on the Frölicher space $(\mathcal{M}\ ,\ \mathcal{C}_{\mathcal{M}}\ ,\ \mathcal{F}_{\mathcal{M}})$ *;*

(1) ***The curvaceous topology :***
This is the final topology induced by all smooth curves in $\mathcal{C}_{\mathcal{M}}$ *, given by* $\tau_{\mathcal{C}_{\mathcal{M}}} = \{\mathcal{U} \subset \mathcal{M} \mid c^{-1}(\mathcal{U}) \in \tau_{\mathbb{R}}\}$ *, were* $c \in \mathcal{C}_{\mathcal{M}}$ *.*

(2) ***The functional topology :***
This is the inital topology induced by all smooth functions in $\mathcal{F}_M$ *, that is the collection* $\tau_{\mathcal{F}_{\mathcal{M}}} = \{\mathcal{U} \subset \mathcal{M} \mid \mathcal{U} = \bigcup_{f\in\mathcal{F}_{\mathcal{M}}} f^{-1}(\mathcal{V})\}$ *were* $\mathcal{V} \in \tau_{\mathbb{R}}$ *.*

Remark 7.4.1.

The functional topology is the weakest topology in which all functions are continuous (see [5],[7], [22]) . It has subbasis $\mathcal{U} = \{f^{-1}\ (0\ ,\ 1)\}_{f\in\mathcal{F}_{\mathcal{M}}}$ *[7] and basis* $\mathcal{B} = \{f^{-1}\ (0\ ,\ \infty)\}_{f\in\mathcal{F}_{\mathcal{M}}}$ *[18] , respectively.*

lemma 7.4.1. *;*

Let $(\mathcal{M}\ ,\ \mathcal{C}_{\mathcal{M}}\ ,\ \mathcal{F}_{\mathcal{M}})$ ***be a Frölicher space with*** $\tau_{\mathcal{F}_{\mathcal{M}}}$ ***and*** $\tau_{\mathcal{C}_{\mathcal{M}}}$ ***the underlying natural topologies on*** $(\mathcal{M}\ ,\ \mathcal{C}_{\mathcal{M}}\ ,\ \mathcal{F}_{\mathcal{M}})$ ***, the property*** $\tau_{\mathcal{F}_{\mathcal{M}}} \subset \tau_{\mathcal{C}_{\mathcal{M}}}$ ***holds.***

Proof.
Let $\mathcal{U} \in \tau_{\mathcal{F}_{\mathcal{M}}}$. That is $\mathcal{U} = \bigcup_{f\in\mathcal{F}_{\mathcal{M}}} \mathrm{f}^{-1}(\mathcal{V})$ where $\mathcal{V}$ is open in $\mathbb{R}$. For an arbitrary $\mathrm{c} \in \mathcal{C}_{\mathcal{M}}$,
$\mathrm{c}^{-1}(\mathcal{U}) = \mathrm{c}^{-1}(\bigcup_{f\in\mathcal{F}_{\mathcal{M}}} \mathrm{f}^{-1}(\mathcal{V})) = \bigcup_{f\in\mathcal{F}_{\mathcal{M}}} (\mathrm{f\ o\ c})^{-1}(\mathcal{V}) \in \tau_{\mathcal{C}_{\mathcal{M}}}$ but $\mathcal{V} \in \tau_{\mathbb{R}}$ and f o c is C^{∞} ,
Hence $\mathrm{c}^{-1}(\mathcal{U})$ is open in $\mathbb{R}$ as arbitrary union of elements of $\tau_{\mathbb{R}}$, thus $\mathcal{U} \in \tau_{\mathcal{C}_{\mathcal{M}}}$ implying $\tau_{\mathcal{F}_{\mathcal{M}}} \subset \tau_{\mathcal{C}_{\mathcal{M}}}$. □

proposition 7.4.1. *;*

Let $(\mathcal{M}\ ,\ \mathcal{C}_{\mathcal{M}}\ ,\ \mathcal{F}_{\mathcal{M}})$ ***and*** $(\mathcal{N}\ ,\ \mathcal{C}_{\mathcal{N}}\ ,\ \mathcal{F}_{\mathcal{N}})$ ***be Frölicher spaces and*** $\varphi : (\mathcal{M}\ ,\ \mathcal{C}_{\mathcal{M}}\ ,\ \mathcal{F}_{\mathcal{M}}) \longrightarrow (\mathcal{N}\ ,\ \mathcal{C}_{\mathcal{N}}\ ,\ \mathcal{F}_{\mathcal{N}})$ ***be a map. If*** φ ***is smooth then it is continuous for the topologies*** $\tau_{\mathcal{F}_{\mathcal{M}}}$ ***and*** $\tau_{\mathcal{C}_{\mathcal{M}}}$

Proof.
We recall that $\tau_{\mathcal{F_M}} = \{\mathcal{U} \subseteq \mathcal{M} \mid \mathcal{U} = \mathrm{h}^{-1}(\mathrm{I}), \mathrm{I} \in \tau_{\mathbb{R}}\ , \mathrm{h} \in \mathcal{F_M}\ \}$. Now $\varphi : (\mathcal{M}\ , \mathcal{C_M}\ , \mathcal{F_M}) \longrightarrow (\mathcal{N}\ , \mathcal{C_N}\ , \mathcal{F_N})$ is smooth iff $\varphi_* \mathcal{C_M} \subseteq \mathcal{C_N} \Leftrightarrow \varphi^{-1}(\tau_{\mathcal{N}}) \subseteq \tau_{\mathcal{M}}$. Let $\mathcal{U} \in \tau_{\mathcal{N}}$ for continuity we need to show that $\varphi^{-1}(\mathcal{U}) \in \tau_{\mathcal{F_M}}$ which will suffice since $\tau_{\mathcal{F_M}} \subseteq \tau_{\mathcal{C_M}}$ so that $\varphi^{-1}(\mathcal{U}) \in \tau_{\mathcal{F_M}}$ shall lie in $\tau_{\mathcal{C_M}}$ we have that ; $\mathcal{U} \in \tau_{\mathcal{F_N}} \Leftrightarrow$ there exist $\mathrm{I} \in \tau_{\mathbb{R}}$ such that $\mathcal{U} = \mathrm{h}^{-1}(\mathrm{I})$; $\mathrm{h} \in \mathcal{F_N}$ hence $\varphi^{-1}(\mathcal{U}) = \varphi^{-1}(\mathrm{h}^{-1}(\mathrm{I})) = (\mathrm{h} \text{ o } \varphi)^{-1}(\mathrm{I})$ but $\mathrm{h} \text{ o } \varphi \in \tau_{\mathcal{M}}$ since φ is smooth by assumption, therefore $(\mathrm{h} \text{ o } \varphi)^{-1}(\mathrm{I}) \equiv \mathcal{V} \in \tau_{\mathcal{F_M}}$. Thus $\mathcal{V} \in \tau_{\mathcal{F_M}}$. □

7.5. Topology of Frölicher space vs Topology of Sikorski differential space

It is proved in chapter 6 unit 6.4 that every Frölicher space is a Sikorski differential space, a result due to P.Cherenack [15]. The proof shows that every Frölicher structure (smooth) functions has properties stated in the definition of the Sikorski differential structure on the same underlying set. We also observe to prove this, we still use structure curves failing what it would not be workable. Furthermore, we see that both the topology on the Sikorski differential space and the one on the Frölicher space are induced by the functions which form the smooth structure.
Nevertheless the above outlined similarities do not imply that the two structures are the same.
The Frölicher smooth structure on a set is determined by a pair of paths (curves) from $\mathbb{R}$ into $\mathcal{M}$ together with scalar valued functions from $\mathcal{M}$ into $\mathbb{R}$. None of the two (functions or curves) alone can form the Frölicher smooth structure. The paths and the scalar functions subject to a compatibility condition will then be called structure functions and structure curves respectively.(see definition 5.2.2). So looking at the two spaces we have that $(\mathcal{M}, \mathcal{C_M}, \mathcal{F_M})$ as a Frölicher space implies that $(\mathcal{M}, \mathcal{F_M})$ is a Sikorski differential space.(see chapter 6 unit 6.4 theorem 6.4.1). However the converse is not true because there are no curves in the Sikorski differential structure.
Now, suppose we need a Frölicher structure $(\mathcal{C}, \mathcal{F})$ associated to a differential structure $\mathcal{F}$ on a set $\mathcal{M}$, then the curves for this Frölicher structure are induced by applying the Γ function on $\mathcal{F}$ as in definition 5.2.1.That is $\Gamma\mathcal{F} := \{\mathrm{c} : \mathbb{R} \longrightarrow \mathcal{M} \mid \mathrm{f} \circ \mathrm{c} \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $\mathrm{f} \in \mathcal{F}\ \} = \mathcal{C}$ and by the compatibility condition we have that $\Phi\Gamma\mathcal{F} := \{\mathrm{f} : \mathcal{M} \longrightarrow \mathbb{R} \mid \mathrm{f} \circ \mathrm{c} \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $\mathrm{c} \in \Gamma\mathcal{F} \equiv \mathcal{C}\ \}$ and this will provide the set $\mathcal{M}$ with the Frölicher smooth (structure) functions. We recall that $\mathcal{F} \subseteq \Phi\Gamma\mathcal{F}$, that is generating a Frölicher smooth structure using a differential structure will produce more smooth functions on $\mathcal{M}$. Consequently, the difference of these structures becomes obvious on subsets and on

cartesian products.Therefore what follows is now to check the two structures based on their subsets and thereafter compare their topologies.
Let ($\mathcal{M}$, $\mathcal{F}$) and ($\mathcal{M}$, $\mathcal{C}_{\mathcal{M}}$,$\mathcal{F}_{\mathcal{M}}$) be Sikorski differential space and Frölicher space respectively, on the same underlying set $\mathcal{M}$. We recall by definition 7.2.1 that the topology of ($\mathcal{M}$, $\mathcal{F}$) is the one induced by the functions of $\mathcal{F}$ in which all functions in $\mathcal{F}$ are continuous.
A subbase for this topology is given by $\{\mathrm{f}^{-1}(\mathrm{a}\ ,\ \mathrm{b})\}_{f\in\mathcal{F}}$, where (a , b) $\subseteq \mathbb{R}$.
Recall also from definition 7.4.1 that a Frölicher topology on ($\mathcal{M}$, $\mathcal{C}_{\mathcal{M}}$,$\mathcal{F}_{\mathcal{M}}$) is the one induced by structure functions in $\mathcal{F}_{\mathcal{M}}$ such that all functions in $\mathcal{F}_{\mathcal{M}}$ and all curves in $\mathcal{C}_{\mathcal{M}}$ are continuous.
A subbase for this topology is given by $\{\mathrm{f}^{-1}(\mathrm{a}\ ,\ \mathrm{b})\}_{f\in\mathcal{F}}$, (0 , 1) $\subseteq \mathbb{R}$.
Besides this functional topology the curvaceous topology on ($\mathcal{M}$, $\mathcal{C}_{\mathcal{M}}$,$\mathcal{F}_{\mathcal{M}}$) is the one induced by curves in $\mathcal{C}_{\mathcal{M}}$ and whose members are subsets of $\mathcal{M}$, the inverse images of which along curves are open sets of $\mathbb{R}$.
definition 7.5.1. *;*
Let ($\mathcal{M}$, $\mathcal{F}$) be a Sikorski differential space. The Frölicher structure associated to $\mathcal{F}$ is defined by ($\mathcal{C}_{\mathcal{M}}$,$\mathcal{F}_{\mathcal{M}}$), where $\mathcal{C}_{\mathcal{M}} = \Gamma\mathcal{F} := \{c : \mathbb{R} \longrightarrow \mathcal{M} \mid f \circ c \in C^{\infty}(\mathbb{R},\mathbb{R})$ *for all* $f \in \mathcal{F}\ \}$ *and*
$\mathcal{F}_{\mathcal{M}} = \Phi\Gamma\mathcal{F} := \{f : \mathcal{M} \longrightarrow \mathbb{R} \mid f \circ c \in C^{\infty}(\mathbb{R},\mathbb{R})$ *for all* $c \in \mathcal{C}_{\mathcal{M}} = \Gamma\mathcal{F}\ \}$
Theorem 7.5.1. *;*
Let $\mathcal{M}$ be a non empty set. Let $\mathcal{F}$ be a Sikorski differential structure and ($\mathcal{C}_{\mathcal{M}}$,$\mathcal{F}_{\mathcal{M}}$) be the Frölicher structure associated to $\mathcal{F}$; then $\mathcal{F} \subseteq \Phi\Gamma\mathcal{F} = \mathcal{F}_{\mathcal{M}}$. That is, the set of Frölicher structure functions is finer than the Sikorski generating structure.
Proof.
Since $\mathcal{F}$ is a generating set for ($\mathcal{C}_{\mathcal{M}}$,$\mathcal{F}_{\mathcal{M}}$), one has $\mathcal{F} \subseteq \Phi\Gamma\mathcal{F}$. For let f $\in \mathcal{F}$ and assume that f $\notin \Phi\Gamma\mathcal{F}$. Hence, there exists c $\in \Gamma\mathcal{F}$ such that f $\circ$ c $\notin C^{\infty}(\mathbb{R},\mathbb{R})$. This is a contradiction with the definition of $\Gamma\mathcal{F}$, therefore f $\in \Phi\Gamma\mathcal{F}$. Thus $\mathcal{F} \subseteq \Phi\Gamma\mathcal{F}$. □
definition 7.5.2. *;*
Let ($\mathcal{M}$, $\mathcal{F}$) be a Sikorski differential space and $\mathcal{A} \subset \mathcal{M}$. A function f:$\mathcal{A} \longrightarrow \mathcal{M}$ is smooth if for every $x \in \mathcal{A}$ there exist a neighbourhood $\mathcal{V} \subseteq \mathcal{M}$ of x and a function $F \in \mathcal{F}$ such that $f_{|\mathcal{A}\cap\mathcal{V}} = F_{|\mathcal{A}\cap\mathcal{V}}$.

Remark 7.5.1.

Cleary, we can see that smooth functions on a subset of a Sikoski differential space are just local restrictions of functions on the ambient space, therefore these functions will satisfy the conditions in definition 3.2.1.

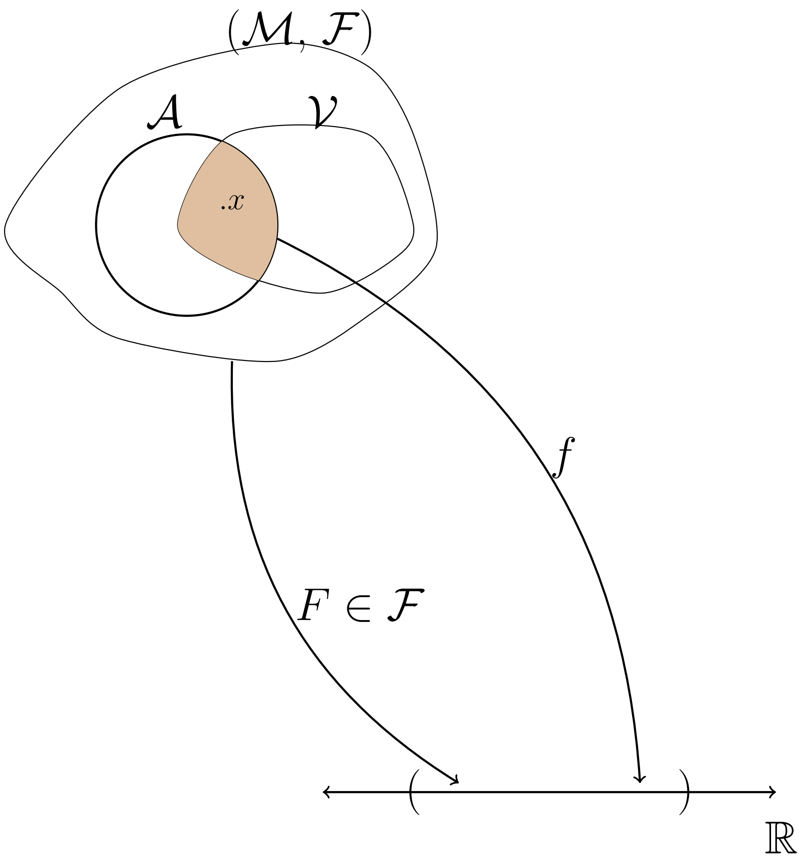


figure 23

Remark 7.5.2.

The set $\mathcal{A}$ endowed with such a differential structure is called a differential subspace of $\mathcal{M}$, and denoted by ($\mathcal{A}$, $\mathcal{F}_{\mathcal{A}}$).

Theorem 7.5.2. *;*
Let ($\mathcal{M}$, $\mathcal{C}_{\mathcal{M}}$, $\mathcal{F}_{\mathcal{M}}$) be a Frölicher space, and $\mathcal{A} \subset \mathcal{M}$. There is a Frölicher structure ($\mathcal{C}_{\mathcal{A}}$, $\mathcal{F}_{\mathcal{A}}$) on $\mathcal{A}$ induced from ($\mathcal{C}_{\mathcal{M}}$, $\mathcal{F}_{\mathcal{M}}$) such that ($\mathcal{A}$,$\mathcal{C}_{\mathcal{A}}$,$\mathcal{F}_{\mathcal{A}}$) is a Frölicher subspace of ($\mathcal{M}$,$\mathcal{C}_{\mathcal{M}}$,$\mathcal{F}_{\mathcal{M}}$).

Proof.

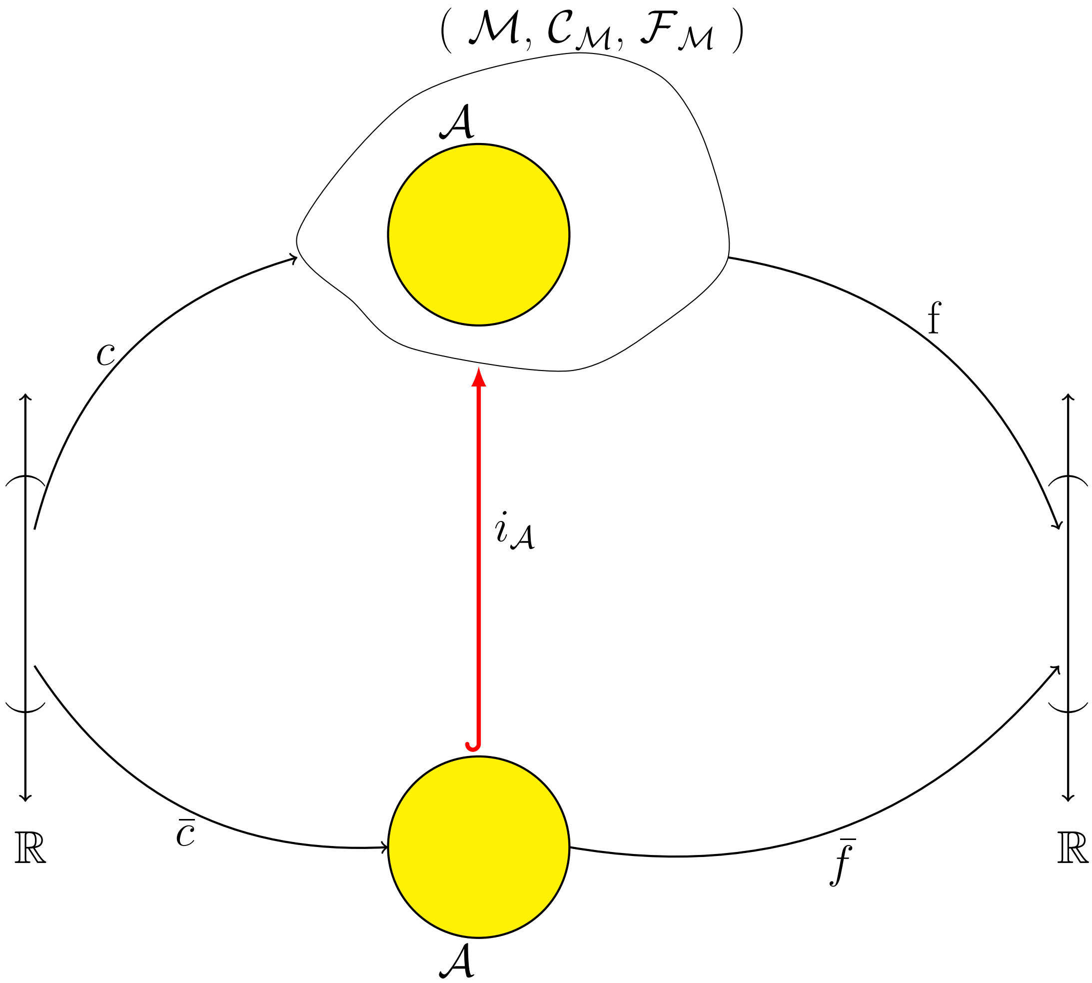


figure 24

First we consider the map $\bar{f}$:$\mathcal{A} \longrightarrow \mathbb{R}$, then $\bar{f} = f$ o $i_{\mathcal{A}}$, where $f \in \mathcal{F}_{\mathcal{M}}$. Also for the map $\bar{c}$:$\mathbb{R} \longrightarrow \mathcal{A}$, we have that $c = i_{\mathcal{A}}$ o $\bar{c}$, c $\in \mathcal{C}_{\mathcal{M}}$.

Thus $\bar{f}$ o $\bar{c} = f$ o $i_{\mathcal{A}}$ o $\bar{c}$

$= f$ o $(i_{\mathcal{A}}$ o $\bar{c})$

$= f$ o c

Now f o c $\in C^{\infty}(\mathbb{R}, \mathbb{R})$, therefore $\bar{f}$ *o* $\bar{c} \in C^{\infty}(\mathbb{R}, \mathbb{R})$

$\mathcal{F}_o := \{\bar{f} : \mathcal{A} \longrightarrow \mathbb{R} \mid \bar{f} = f \text{ o } i_{\mathcal{A}} \in \mathcal{F}_{\mathcal{A}}, f \in \mathcal{F}_{\mathcal{M}}\}$

The structure curves are given by;

$\Gamma\mathcal{F}_o(\mathcal{A}) := \{\bar{c} : \mathbb{R} \longrightarrow \mathcal{A} \mid \text{g o } \bar{c} \in C^{\infty}(\mathbb{R},\mathbb{R}), \text{ g} = f_{|\mathcal{A}}, f \in \mathcal{F}_{\mathcal{M}}\}$

The structure functions are given by

$\Phi\Gamma\mathcal{F}_o(\mathcal{A}) := \bar{f} : \mathcal{A} \longrightarrow \mathbb{R} \mid \bar{f} \text{ o } \bar{c} \in C^{\infty}(\mathbb{R},\mathbb{R})$, for all $\bar{c} \in \Gamma\mathcal{F}_o(\mathcal{A})$.

Therefore,

$(\Gamma\mathcal{F}_o(\mathcal{A}), \Phi\Gamma\mathcal{F}_o(\mathcal{A}))$ is the Frölicher structure induced on $\mathcal{A} \subset \mathbb{R}$. □

Example 7.5.1. *;*

We consider $\mathbb{Q}$ *the set of all rational numbers. Since* $\mathbb{Q} \subseteq \mathbb{R}$ *and* $\mathbb{R}$ *is a canonical Frölicher - space, we have that* $\mathbb{Q} \subseteq (\mathbb{R}, \mathcal{C}_{\mathbb{R}}, \mathcal{F}_{\mathbb{R}})$*, thus* $\mathbb{Q} \hookrightarrow \mathbb{R}$ *with* $i_{\mathbb{Q}}$ *the inclusion map.*

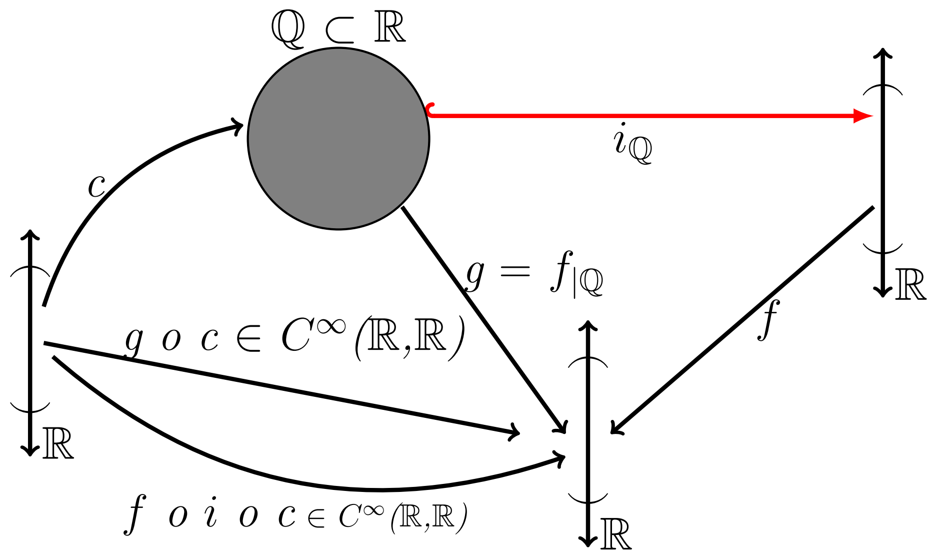


figure 25

Now, we let $\mathcal{F}_{o(\mathbb{Q})} := \{i : \mathbb{Q} \hookrightarrow \mathbb{R} \mid i = id_{\mathbb{R}|\mathbb{Q}}\} = \{i\}$ *and* $\mathcal{M} = \mathbb{Q}$,

$$\mathcal{F}_{o(\mathbb{Q})} := \{ \ f : \mathbb{Q} \longrightarrow \mathbb{R} \mid f = g_{|\mathbb{Q}}, \ g \in C^{\infty}(\mathbb{R}, \mathbb{R}) \ \}$$

The structure curves are,

$$\begin{aligned} \mathcal{C}_{\mathbb{Q}} &:= \Gamma\mathcal{F}_{o(\mathbb{Q})} \\ &:= \{c : \mathbb{R} \longrightarrow \mathbb{Q} \mid g_{|\mathbb{Q}} \ o \ c \in C^{\infty}(\mathbb{R},\mathbb{R}), \text{ for all } f \in \mathcal{F}_{o(\mathbb{Q})}\} \\ &:= \{c : \mathbb{R} \longrightarrow \mathbb{Q} \mid f \ o \ c \in C^{\infty}(\mathbb{R},\mathbb{R}), \text{ for all } f \in \mathcal{F}_{o(\mathbb{Q})}\} \\ &:= \{c : \mathbb{R} \longrightarrow \mathbb{Q} \mid i \ o \ c = id_{\mathbb{Q}} \ o \ c = c \in C^{\infty}(\mathbb{R})\} \\ &:= \{c : \mathbb{R} \longrightarrow \mathbb{Q} \mid c \in C^{\infty}(\mathbb{R}) \text{ and } c(\mathbb{R}) \subset \mathbb{Q}\} \\ &:= \mathbb{Q}^{\mathbb{R}} \cap C^{\infty}(\mathbb{R}) \text{ (which is continuous)} \end{aligned}$$

We need to characterise $\mathcal{C}_{\mathbb{Q}}$*, knowing that* $\mathbb{Q}^{\mathbb{R}} \cap C^{\infty}(\mathbb{R})$ *is the set of* C^{∞} *functions on* $\mathbb{R}$ *restricted to* $\mathbb{Q}$*;*

From the above, for a $c \in \mathcal{C}_{\mathbb{Q}}$ *such that* $c[\mathbb{R}] = \mathbb{Q}$*, we have that* $c \in C^{\infty}(\mathbb{R},\mathbb{R})$*, that is, c is continuous in the usual sense. Now we show that such a c is a constant map* $c_k : \mathbb{R} \longrightarrow \mathbb{Q}$, $k \in \mathbb{Q}$ *is constant for all* $\bar{r} \in \mathbb{R}$ *.*

suppose that for r_1, $r_2 \in \mathbb{R}$*, with* $r_1 \neq r_2$ *implying* $c(r_1) \neq c(r_2)$*. Assume without loss of generality that* $c(r_1) < c(r_2)$ *and since* $\mathbb{Q}$ *is dense in* $\mathbb{R}$*, by the intermediate value theorem, for each* $s \in [c(r_1), c(r_2)] \subset \mathbb{R}$*, there exist* $\bar{r} \in \mathrm{C}_{\mathbb{Q}}^{\mathbb{R}}$*, that is* $\bar{r} \in [r_1, r_2]$ *such that* $s = c(\bar{r})$*. This means that c will take all real values (rational and irrational) between* $c(r_1)$ *and* $c(r_2)$*. But by definition of c, we have that* $c(\mathbb{R}) \subset \mathbb{Q}$*, that is, all* $c(\bar{r})$ *are rational numbers.Therefore we have a contradiction since the range of c consists of only rational numbers. Therefore,* $c(r_1) = c(r_2)$*, for all* r_1, $r_2 \in \mathbb{R}$ *with* $r_1 \neq r_2$*. Thus, c is a constant function.consequently, the generated curves in* $\mathbb{Q}$ *are given by,*

$\mathcal{C}_{\mathbb{Q}} := \{c : \mathbb{R} \longrightarrow \mathbb{Q} \mid c(\bar{r}) = k, \text{ where } k \in \mathbb{Q}, \ k \text{ is constant for all } \bar{r} \in \mathbb{R} \ \}.$

Therefore for all $g \in C^{\infty}(\mathbb{R},\mathbb{R})$, $g_{\mathbb{Q}} \ o \ c \in C^{\infty}(\mathbb{R},\mathbb{R})$ *implying that* $g \in \mathbb{R}^{\mathbb{R}}$ *The functions for the structure are given by,*

$$\begin{aligned} \mathcal{F}_{\mathbb{Q}} &= \Phi\Gamma\mathcal{F}_{o(\mathbb{Q})} \\ &= \Phi\mathcal{C}_{\mathbb{Q}} \\ &= \{f : \mathbb{Q} \longrightarrow \mathbb{R} \mid f \ o \ c \in C^{\infty}(\mathbb{R}), \text{ for all } c \in \mathcal{C}_{\mathbb{Q}}, \\ &\quad \text{where } f = g_{|\mathbb{Q}}, \ g \in C^{\infty}(\mathbb{R}, \mathbb{R})\} \\ &= \{f : \mathbb{Q} \longrightarrow \mathbb{R} \mid f \ o \ c_k \in C^{\infty}(\mathbb{R}), \text{ such that } c_k(\bar{r}) = k, \\ &\quad \text{where } \bar{r} \in \mathbb{R} \text{ and } k \in \mathbb{Q}\} \text{ Hence,} \end{aligned}$$

$\mathcal{F}_{\mathbb{Q}} = \{f : \mathbb{Q} \longrightarrow \mathbb{R} \mid f_k \in C^{\infty}(\mathbb{R}), \ f_k(\bar{r}) = f(k), \text{ for all } \bar{r} \in \mathbb{R}.$

We now need to characterize $\mathcal{F}_{\mathbb{Q}} \subset \mathbb{R}^{\mathbb{Q}}$*, the set of real-valued functions with the source* $\mathbb{Q}$ *such that* $f \ o \ c_k \in C^{\infty}(\mathbb{R})$*. Thus for any* $f \in \mathbb{R}^{\mathbb{Q}}$*, for any* $k \in \mathbb{Q}$*,* $f(k)$ *determines a constant function,*

$f_k : \mathbb{Q} \longrightarrow \mathbb{R}, \ f_k(\bar{r}) = (f \circ c_k)(\bar{r}) = f(c_k(\bar{r})) = f(k).$

Therefore, since $f \circ c_k \in C^\infty(\mathbb{R})$ We have that $f \in \mathcal{F}_{\mathbb{Q}}$ if and only if $\mathcal{F}_{\mathbb{Q}} = \mathbb{R}^{\mathbb{Q}}$.
Finally from this example, we see that the Frölicher structure on $\mathbb{Q} \subset \mathbb{R}$ is generated by $\{i\}$ $= \{id_{\mathbb{Q}}\} = \{id_{\mathbb{R}|\mathbb{Q}}\}$ which yields only constant maps $c : \mathbb{R} \longrightarrow \mathbb{Q}$ as structure (smooth) curves. The structure (smooth) functions are given by all real-valued functions on $\mathbb{Q}$.

Remark 7.5.3. *(Important)*

Let $(\mathcal{M}, \mathcal{F})$ be a Sikorski differential space and $(\mathcal{C}_{\mathcal{M}},\mathcal{F}_{\mathcal{M}})$ be the associated Frölicher structure on the same underlying set. Let $\mathcal{A} \subset \mathcal{M}$. It follows from the above constructions that the Sikorski subspace function set $\mathcal{F}(\mathcal{A})$ is weaker than the corresponding Frölicher subspace function set $\mathcal{F}_{\mathcal{A}}$.
Since the underlying topology on $(\mathcal{A}, \mathcal{F}(\mathcal{A}))$ and $(\mathcal{A}, \mathcal{C}_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ are not the same, it lead us to a question that ; in which case does the differential structure $\mathcal{F}$ equals the set of structure functions for the Frölicher structure generated on M by $\mathcal{F}_o$? This argument motivated Batubenge [3] to introduce a class of Sikorski spaces which yield a Frölicher structure in such a way that $\mathcal{F} = \Phi\Gamma\mathcal{F}_o$, the so called Pre-Frölicher spaces.

definition 7.5.3. ***(pre-Frölicher space)***
A pre-Frölicher space is a Sikorski differential space $(\mathcal{M}, \mathcal{F})$ with the structure $\mathcal{F}$ such that $\mathcal{F} = \Phi\Gamma\mathcal{F}$ in the induced Frölicher structure $(\mathcal{M}, \Gamma\mathcal{F}, \Phi\Gamma\mathcal{F})$ on $\mathcal{M}$.

The diagram below explains the pre-Frölicher space.

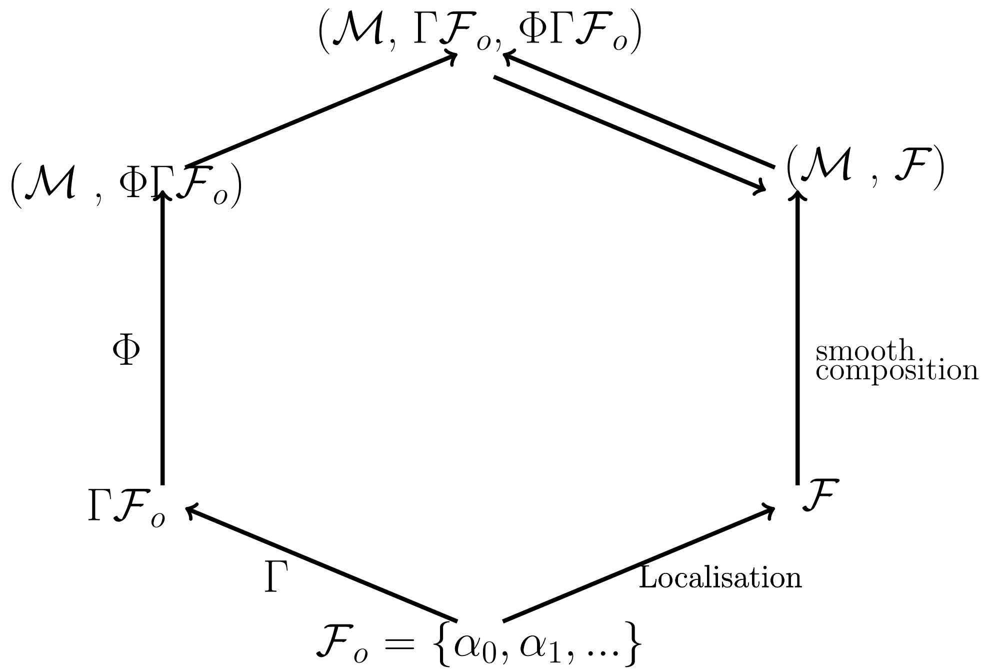


figure 26

Example 7.5.2. ***(Counter)*** *;*

As seen in example 7.5.1 $\mathbb{Q}$ as a Sikorski differential space is not a pre-Frölicher space.Infact $\mathbb{Q} \subseteq \mathbb{R}$. if $\mathcal{F}_o$ is the standard Sikorski structure on $\mathbb{R}$, C^∞-functions on $\mathbb{R}$ are the elements of $\mathcal{F}$. Therefore, $\mathcal{F}|_{\mathbb{Q}} = C^\infty(\mathbb{R},\mathbb{R}) |_{\mathbb{Q}}$ by the locality axiom.But in yielding the Frölicher associated structure on $\mathbb{Q}$ from $C^\infty(\mathbb{R},\mathbb{R})$, one has $\Gamma\mathcal{F}$ = constant paths $c : \mathbb{R} \longrightarrow \mathbb{Q}$ and $\mathcal{F} = \phi(c : \mathbb{R} \longrightarrow \mathbb{Q},\ c = constant) = Hom_{set}(\mathbb{Q},\mathbb{R})$. Clearly, $Hom_{set}(\mathbb{Q},\mathbb{R}) \neq C^\infty(\mathbb{R},\mathbb{R}) |_{\mathbb{Q}}$. Thus, $Hom_{set}(\mathbb{Q},\mathbb{R}) = \{$all functions $f : \mathbb{Q} \longrightarrow \mathbb{R}\}$ and $C^\infty(\mathbb{R},\mathbb{R}) |_{\mathbb{Q}} = \{$restrictions to $\mathbb{Q}$ of all C^∞ functions $f : \mathbb{R} \longrightarrow \mathbb{R}\}$

Theorem 7.5.3. *;*

Let $(\mathcal{M}, \mathcal{F})$ be a Pre-Frölicher space. Let $(\mathcal{M}, \Gamma\mathcal{F}, \Phi\Gamma\mathcal{F})$ be the associated Frölicher space. Then the weakest topologies on $(\mathcal{M}, \mathcal{F})$ and $(\mathcal{M}, \Gamma\mathcal{F}, \Phi\Gamma\mathcal{F})$ coincide, moreover, the tangent spaces coincide.

Proof.

The proof is straight forward since $\mathcal{F} = \Phi\Gamma\mathcal{F}$. It follows that $\tau_{\mathcal{F}}$ and $\tau_{\Phi\Gamma\mathcal{F}}$ have the same base. □

Remark 7.5.4.

Two topologies are said to coincide on the same underlying set if they have the same bases.

7.6. Topology of Sikorski space vs Topology of diffeological space

We proved in chapter 6 unit 6.5 that the diffeological and Sikorski differential spaces are compatible and determine each other. No further discussion was done based on the relationship of the topologies underlying the two structures. Batubenge et al in their paper [10] analysed the topologies basing on the subsets under the same underlying set. They pointed out that if $\mathcal{M}$ is a set and $\mathcal{A} \subseteq \mathcal{M}$ its subset then the differential structure on $\mathcal{M}$ also determines a topology on $\mathcal{A}$ without ambiguity. The initial topology corresponding to the subset differential structure on $\mathcal{A}$ coincides with the subset topology on $\mathcal{M}$ induced by the initial topology on $\mathcal{M}$. Also, the D-topology corresponding to the subset diffeology on $\mathcal{A}$ might differ from the subset topology on $\mathcal{A}$ induced by the D-topology on $\mathcal{M}$. This can occur with the subset $\mathcal{A} = \mathbb{Q}$ of $\mathcal{M} = \mathbb{R}$ (see similar example 7.5.1) .

Theorem 7.6.1. *[42];*
Let ($\mathcal{M}$,$\mathcal{D}$) be a diffeological space. Let $\tau_{\mathcal{D}}$ be the strongest topology on $\mathcal{M}$ such that all parametrisations in $\mathcal{D}$ are continuous, and let $\tau_{\Phi\mathcal{D}}$ be the weakest topology on $\mathcal{M}$ such that all functions in $\Phi\mathcal{D}$ are continuous. Then $\tau_{\Phi\mathcal{D}} \subseteq \tau_{\mathcal{D}}$.

Proof.
Let $\mathcal{V} \in \tau_{\Phi\mathcal{D}}$. Let p $\in \mathcal{D}$ be fixed, we need to show that $p^{-1}(\mathcal{V})$ is open in U := dom(p). To this end, let u $\in p^{-1}(\mathcal{V})$. Then there exists an open set $\mathcal{W}$ containing p(u) and contained in $\mathcal{V}$ of the form $\mathcal{W} := \bigcap_{i=1}^{k} f^{-1}((a_i, b_i))$

for some open intervals $(a_i, b_i) \subseteq \mathbb{R}$ and functions $f_i \in \Phi\mathcal{D}$. But then u $\in p^{-1}(\mathcal{W}) \subseteq p^{-1}(\mathcal{V})$. But

$$p^{-1}(\mathcal{W}) = \bigcap_{i=1}^{k}(f_i \circ p)^{-1}((a_i, b_i))$$

Since $f_i \circ p$ is a smooth function on U for each i, we have that $p^{-1}(\mathcal{W})$ is a finite intersection of open subsets in U, and hence itself is open. Thus $p^{-1}(\mathcal{V})$ is open in U. Hence the proof. □

7.7. Topology of Frölicher space vs Topology of diffeological space

corollary 7.7.1. *;*
Let ($\mathcal{M}$,$\mathcal{D}$) be a diffeological space. Let $\mathcal{D} = \mathcal{D}_1$, the diffeology of 1-plots, then theorem 7.6.1 gives similar result as ($\mathcal{M}$,$\mathcal{D}_1$) is considered as a Frölicher space.

8

COMPARATIVE STUDY OF TANGENT STRUCTURES :

8.1. Tangent structures on the manifold

We note that there is a linear approximation at each point of a smooth manifold. We now construct these linear spaces associated with a smooth manifold $\mathcal{M}$ at each point x $\in \mathcal{M}$, the so-called tangent spaces and their dual so-called cotangent spaces. These spaces are isomorphic to the modeling Euclidean space. Thus, the dimension of the manifold is that of each tangent space and each cotangent space. This linear structure is fundamental for several so-called exterior operators on the manifold.([4]) There are different ways of defining tangent vectors to a manifold $\mathcal{M}$.The approaches are equivalent in the sense that they end up defining the same objects and the same space.

(1) **Curves approach:**
A vector tangent to a manifold at one of its points is tangent to a curve on the manifold.

(2) **Derivation approach:**
A tangent vector at a point of a manifold is defined as a derivation of functions on the manifold at this point. That is , we think of a tangent vector as defining a directional derivative.

definition 8.1.1. *;*
A smooth curve on a smooth manifold $\mathcal{M}$ is a smooth map γ:(a,b)$\longrightarrow \mathcal{M}$, where (a,b) is a non empty interval in $\mathbb{R}$

definition 8.1.2. *;*
Let $\mathcal{M}$ be a smooth manifold and $c : I \subseteq \mathbb{R} \to \mathcal{M}$. A map c is called a smooth curve through $p \in \mathcal{M}$ if for all chart (U,φ) at p ,
φ *o* $c : I \to \varphi(U) \subseteq \mathbb{R}^n$

Example 8.1.1. *;*
We consider We will take this interval $I = [0, 1] \subset \mathbb{R}$. The curve $\gamma : I \longrightarrow \mathcal{M}$ such that $\gamma(0) = p$ and $\gamma(1) = p^{'}$ is said to join the points p and $p^{'}$ of the manifold $\mathcal{M}$.

definition 8.1.3. *;*
Let $\mathcal{M}$ be a differentiable manifold of dimension n. A smooth curve c : $\mathbb{R} \longrightarrow \mathcal{M}$ such that c(0) = $p \in \mathcal{M}$ is called a curve through p. Now let c_1 and c_2 be curves through p, c_1 is said to be tangent to c_2 at p if for all local chart (U, ϕ) at $p \in U$ on $\mathcal{M}$ one has $(\phi\ o\ c_1)^{'}(0) = (\phi\ o\ c_2)^{'}(0)$. where $\phi\ o\ c_i : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^n$.

definition 8.1.4. *;*
Given a smooth manifold $\mathcal{M}$ and a point $p \in \mathcal{M}$, the tangent vector to $\mathcal{M}$ at p is an equivalence class of differentiable curves
c:(-ε,ε)$\longrightarrow \mathcal{M}$ where $c(0) = p$ and
$c_1 \sim c_2 \Longleftrightarrow (\phi\ o\ c_1)^{'}(0) = (\phi\ o\ c_2)^{'}(0)$.

definition 8.1.5. *;*
For a point $p \in \mathcal{M}$ we let C^{∞}(p) be the set of functions such that;

(i) $f : \mathcal{U} \longrightarrow \mathbb{R}$ where $p \in \mathcal{U} \subset \mathcal{M}$ and $\mathcal{U}$ is an open set.

(ii) $f \in C^{\infty}(p)$.

Remark 8.1.1.

recall that an open subset of a manifold is a manifold.

definition 8.1.6. *(**Tangent Vector to a smooth Manifold**)*
A tangent vector v_p to a smooth manifold $\mathcal{M}$ at a point $p \in \mathcal{M}$ is a is a map $v : C^{\infty}(p) \longrightarrow \mathbb{R}$ that is a linear function from the set of functions defined and differentiable in some neighbourhood of p into $\mathbb{R}$, which satisfies,

(1) $v_p(\alpha f + \beta g) = \alpha v_p(f) + \beta v_p(g)$ (Linearity)

(2) $v_p(fg) = f(p)v_p(g) + g(p)v_p(f)$ (Leibniz rule),

(3) v_p(constant map) = 0

definition 8.1.7. *(**Differentiable function**)*
A function f : $\mathcal{M} \longrightarrow$ R is differentiable at a point $p \in \mathcal{M}$ if in a chart φ at P, the function $f \circ \varphi^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}$ is differentiable at $\varphi(p)$.

definition 8.1.8. *(**Tangent Space**)*
The set of all tangent vectors at the point $p \in \mathcal{M}$ forms the tangent space $T_p\mathcal{M}$ of a manifold $\mathcal{M}$ at the point p. That is ,the collection of all vectors obtained by differentiating

differentiable curves through p of $\mathcal{M}$ gives $T_p\mathcal{M}$.

Remark 8.1.2.

If there are n partial derivatives $\frac{\delta}{\delta x^1}$,.......,$\frac{\delta}{\delta x^n}$ then $T_p\mathcal{M}$ is a tangent space of dimension n at the point p. A tangent vector can be associated to a differential operator acting on functions on $\mathcal{M}$. That is $v \in T_p\mathcal{M}$ whenever v is a linear operator $v : \mathcal{F}(\mathcal{M}) \longrightarrow \mathbb{R}$ where $\mathcal{F}(M)$ is the ring of smooth functions on $\mathcal{M}$. ([11])

In what follows we will denote $\frac{\partial f}{\partial x^\mu} = \partial_\mu f$

definition 8.1.9. *;*

Let $(x_i, ..., x_n)$ be local coordinates about a point $p \in \mathcal{M}$. That is there exists a chart (U_i,φ_i) with $p \in U_i$ and $\varphi_i(q) = (x_1(q), x_2(q), ..., x_n(q))$ for all $q \in U_i$. We define ;

$\frac{\partial}{\partial x^\mu}|_p : C^\infty(p) \longrightarrow \mathbb{R}$ by

$$\frac{\partial}{\partial x^\mu}|_p f = \partial_\mu(f \ o \ \varphi_i^{-1})(x_1(p), x_2(p), ..., x_n(p))$$
$$= \partial_\mu(f \ o \ \varphi_i^{-1}) \ o \ \varphi_i(p)$$

Theorem 8.1.1. $\frac{\partial}{\partial x^\mu}|_p$ ***is a tangent vector to $\mathcal{M}$ at p.***

Proof.

Let f,g $\in C^\infty$(p) be defined on a common open set $\mathcal{U}$ in $\mathcal{M}$ that contains p then

$$\frac{\partial}{\partial x^\mu}|_p(f+g) = \partial_\mu((\text{f} + \text{g}) \text{ o } \varphi_i^{-1})(x_1(p), x_2(p), ..., x_n(p))$$
$$= \partial_\mu(\text{f o } \varphi_i^{-1} + \text{g o } \varphi_i^{-1})(x_1(p), x_2(p), ..., x_n(p))$$
$$= \partial_\mu (\text{f o } \varphi_i^{-1})(x_1(p), x_2(p), ..., x_n(p)) + \partial_\mu (\text{g o } \varphi_i^{-1})(x_1(p), x_2(p), ..., x_n(p))$$
$$= \frac{\partial}{\partial x^\mu}|_p (\text{f}) + \frac{\partial}{\partial x^\mu}|_p (\text{g})$$

[Linearity]

$$
\begin{aligned}
\frac{\partial}{\partial x^{\mu}}|_p(\mathrm{fg}) &= \partial_\mu((\mathrm{fg}) \text{ o } \varphi_i^{-1})(x_1(p), x_2(p), ..., x_n(p)) \\
&= \partial_\mu(\mathrm{f} \text{ o } \varphi_i^{-1} \ . \ \mathrm{g} \text{ o } \varphi_i^{-1} \ )(x_1(p), x_2(p), ..., x_n(p)) \\
&= \partial_\mu \ (\mathrm{f} \text{ o } \varphi_i^{-1})(x_1(p), x_2(p), ..., x_n(p))(\mathrm{g} \text{ o } \varphi_i^{-1})(x_1(p), x_2(p), ..., x_n(p)) \\
&\quad +(\mathrm{f} \text{ o } \varphi_i^{-1})(x_1(p), x_2(p), ..., x_n(p))\partial_\mu \ (\mathrm{g} \text{ o } \varphi_i^{-1})(x_1(p), x_2(p), ..., x_n(p)) \\
&= \frac{\partial}{\partial x^{\mu}}|_p(\mathrm{f})\mathrm{g(p)} + \mathrm{f(p)} \ \frac{\partial}{\partial x^{\mu}}|_p(\mathrm{g})
\end{aligned}
$$

[Leibniz Rule]

It is clear that ;

$\frac{\partial}{\partial x^{\mu}}|_p$ (constant map) = 0. □

definition 8.1.10. *;*

Let $T_p\mathcal{M}$ be the tangent space at a point p of a smooth manifold M of dimension n with dual space denoted by $T^*\mathcal{M}$. Then $T^*\mathcal{M}$ is called the cotangent space at $p \in \mathcal{M}$.

definition 8.1.11. *;*

Given a smooth manifold $\mathcal{M}$, a Cotangent bundle denoted $T^*\mathcal{M}$ is the disjoint union of cotangent spaces at $p \in \mathcal{M}$, that is

$$T^*\mathcal{M} = \bigcup_{p \in \mathcal{M}} T_p^*\mathcal{M}.$$

8.2. Tangent structures on Sikorski differential spaces

definition 8.2.1. ***(Tangent vector)***

Let ($\mathcal{M}$,$\mathcal{F}$) be a differential space . A tangent vector v at $p \in \mathcal{M}$ is a derivation $v : \mathcal{F} \longrightarrow \mathbb{R}$ at p, that is a linear map such that ,

$v(f.g) = v(f).g(p) + f(p).v(g)$ for all f , $g \in \mathcal{F}$.

Remark 8.2.1. *[32]*

The real number v(f) is called the directional derivative of the function $f \in \mathcal{F}$ in the direction v and is often denoted by the symbol $\partial_v f$. Thus the above definition can be rewritten as ;

$\partial_v(f.g) = \partial_v f.g(p)$ + f(p).$\partial_v g$, for all $f, g \in \mathcal{F}$.

definition 8.2.2. ***(Tangent space)****;*

Let ($\mathcal{M}$,$\mathcal{F}$) be a differential space. A tangent space $T_p\mathcal{M}$ is the set of all tangent vectors to ($\mathcal{M}$,$\mathcal{F}$) at $p \in \mathcal{M}$.

definition 8.2.3. ***(Tangent cone)*** *[15];*

Let ($\mathcal{M}$,$\mathcal{F}$) be a differential space .Suppose c(a) = p . A tangent cone $T_pC\mathcal{M}$ at $p \in \mathcal{M}$ is the set ;

$T_pC\mathcal{M} = \{V_c|\ c \in \Gamma\mathcal{F}\ ,\ c(a) = p\}$, *where* V_c *is the derivation defined by setting* $V_c(f) = \lim_{t\to a} \dfrac{foc(t) - foc(a)}{t-a}$, *and* $\Gamma\mathcal{F} := \{c : \mathbb{R} \longrightarrow \mathcal{M} \mid f\ o\ c \in C^\infty(\mathbb{R}, \mathbb{R})\}$, *for all* $f \in \mathcal{F}$.

definition 8.2.4. *(**Tangent bundle**);*

Let $(\mathcal{M},\mathcal{F})$ *be a differential space. A tangent bundle to* $(\mathcal{M},\mathcal{F})$ *is a triple* $(T\mathcal{M},\mathcal{M},\pi)$, *where* $T\mathcal{M}$ *is the disjoint union of the tangent spaces at* $p \in \mathcal{M}$, *that is* $T\mathcal{M} = \bigcup_{p\in\mathcal{M}} T_p\mathcal{M}$ *and* π *is the canonical projection* $\pi : T\mathcal{M} \longrightarrow \mathcal{M} : v_q \mapsto q$.

definition 8.2.5. *(**Differential of a** C^∞ **function**)[40];*

Let $(\mathcal{M},\mathcal{F})$ *be a differential space. The differential of a function* $f \in C^\infty$ *is a mapping* $df : T\mathcal{M} \longrightarrow \mathbb{R}$ *defined by the formula* $df(v) = \langle v, df\rangle$.

definition 8.2.6. *(**Cotangent space**);*

The cotangent space on a differential space $(\mathcal{M},\mathcal{F})$ *at* $p \in \mathcal{M}$ *is the set of all differentials of functions at the point p and will be denoted* $T_p^*\mathcal{M}$.

definition 8.2.7. *(**Cotangent bundle**)[40];*

The cotangent bundle $(T^*\mathcal{M},\mathcal{M},\tau)$ *of a differential space* $(\mathcal{M},\mathcal{F})$ *is the disjoint sum of cotangent spaces where* $T^*\mathcal{M} = \bigcup_{p\in\mathcal{M}} T_p^*X$ *, with the canonical projection* $\tau : T^*\mathcal{M} \longrightarrow \mathcal{M} : d_pf \longrightarrow p$.

8.3. Tangent structures on diffeological spaces

In this section, the construction of tangent structures on diffeologigical spaces are with reference to the work by M Laubinger ([26],[27]).For the other construction on the same, see ([41],[16]).

Given a diffeological space $(\mathcal{M}, \mathcal{D})$ and a point $x \in \mathcal{M}$, we construct a vector space $T_x\mathcal{M}$ and a linear map $d_p : T_o\mathcal{V}_p \longrightarrow T_x\mathcal{M}$ for each plot p centred at x. The maps d_p will be interpreted as differentials of the plots which satisfy the chain rule for plots of the form $p \circ g$ where g is a smooth map with $g(o) = 0$ gives a reparametrization as shown in figure 27 below.

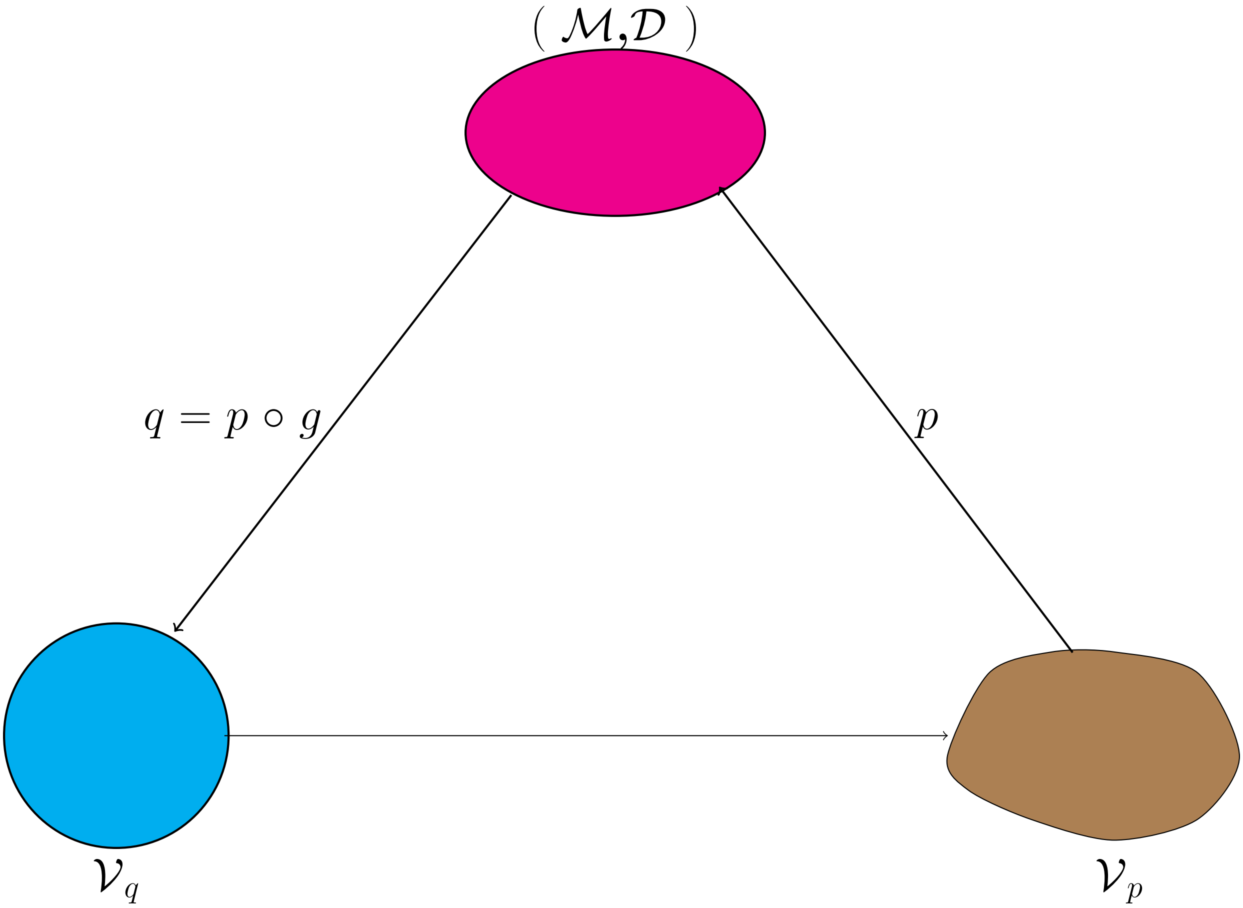


figure 27: Reparametrization

The chain rule here implies that the corresponding diagram of differentials should commute. as shown in figure 28 below.

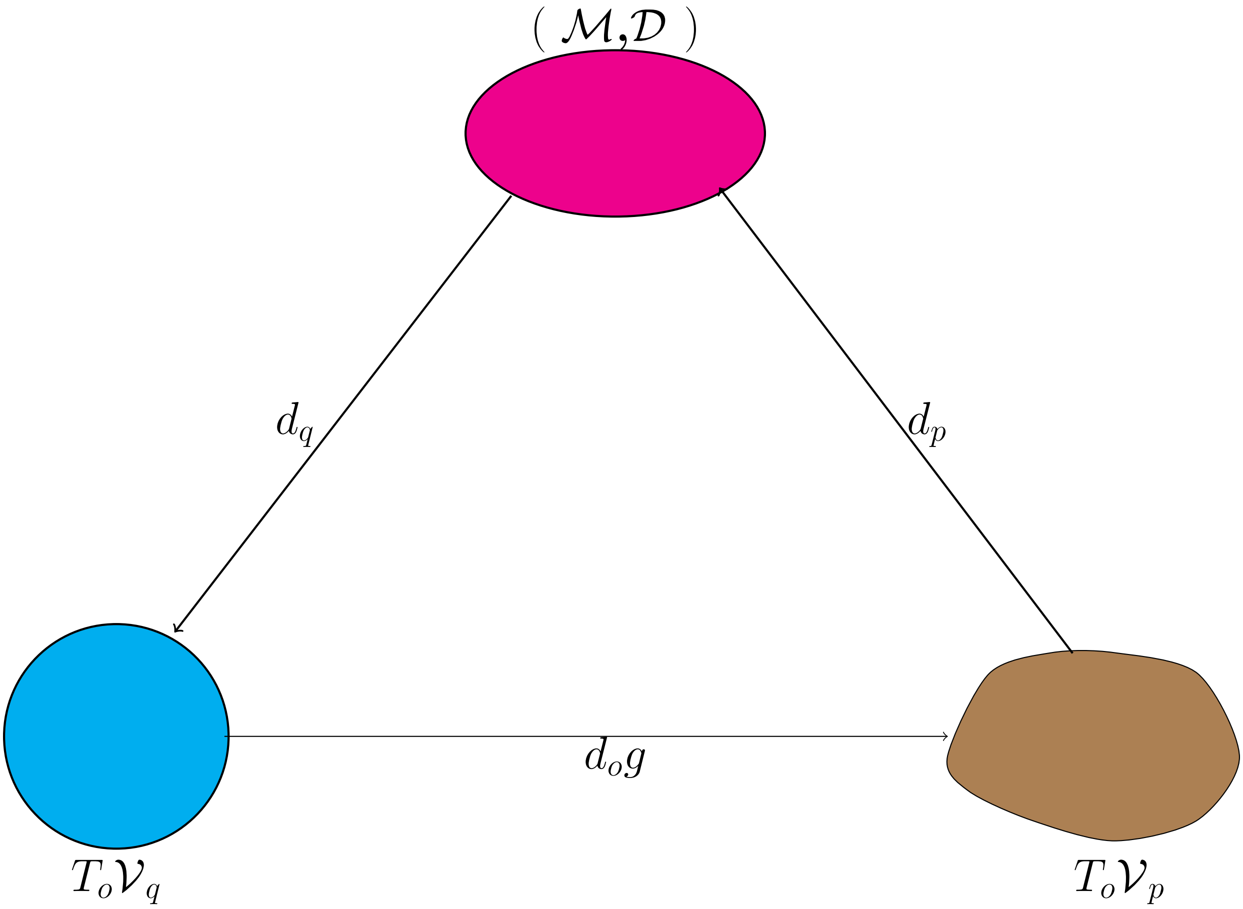


figure 28

We need $d_{p \circ g} = d_p \circ d_o g$, which gives the chain rule for the composition $p \circ g$.
Let $\mathcal{S}_p$ denote the the tangent space $T_o\mathcal{V}_p$. Let the direct sum $\mathcal{S}_x := \bigoplus_{p \in \mathcal{D}} \mathcal{S}_p$, be equipped with the final diffeology with respect to the injections $i_p : \mathcal{S}_p \longrightarrow \mathcal{S}_x$ and $u \in \mathcal{S}_p$ with its image under i_p. We define a linear subspace $\hat{\mathcal{S}}_x$ of $\mathcal{S}_x$ as

$\hat{\mathcal{S}}_x \;;= \langle\ i_q(u) - (i_p \circ d_o g)(u) \mid q = p \circ g$ and $u \in T_o\mathcal{V}_q\ \rangle$
Where $\langle\ \rangle$ denotes the linear span in $\mathcal{S}_x$. Hence the following definition.
definition 8.3.1. ***(Tangent space)***
The tangent space to a point $x \in \mathcal{M}$ is the quotient space $T_x\mathcal{M} := \mathcal{S}_x/\hat{\mathcal{S}}_x$

definition 8.3.2. ***(Tangent bundle)***

The tangent bundle denoted $T\mathcal{M}$ is the disjoint union of all tangent spaces to $\mathcal{M}$ at $x \in \mathcal{M}$. That is $T\mathcal{M} := \bigcup_{x\in\mathcal{M}} T_x\mathcal{M}$

definition 8.3.3. ***(Plot delivation)****[41]*

The p induced plot derivation is the map $d_p : C^\infty(\mathcal{M}) \longrightarrow \mathbb{R}$ given by $d_p(f) := d_o(f \circ p)$, where $p \in \mathcal{D}$ is a plot.

Example 8.3.1. *;*

Let $p \in \mathcal{D}$ be locally constant at $x \in \mathcal{M}$, then $d_p = 0$.

proposition 8.3.1. *[41];*

Let $p \in \mathcal{D}$, then the induced plot derivation $d_p : C^\infty(\mathcal{M}) \longrightarrow \mathbb{R}$ is a smooth derivation at x on the algebra $C^\infty(\mathcal{M})$.

Proof.

(i) Let $q : \mathcal{V} \longrightarrow C^\infty(\mathcal{M})$, then the map $q.p : \mathcal{V} \times \mathbb{R} \longrightarrow \mathbb{R}$ is smooth. For each $v \in \mathcal{V}$,
$d_p(q(v)) = d_o(q(v) \circ p)$
$$= \frac{\partial q.p(v,u)}{\partial u}|_{(v,0)}, \text{ for } u \in \mathcal{V}.$$
Hence the map $v \longrightarrow d_p(\mathrm{q}(v))$ is smooth, since $q.p$ is smooth and the partial derivative of a smooth map is smooth.This implies that $d_p : C^\infty(\mathcal{M}) \longrightarrow \mathbb{R}$ is smooth.

(ii) Let $f, g \in C^\infty(\mathcal{M})$
$d_o[(f+g) \circ p)] = d_o(f \circ p) + d_o(g \circ p)$ and
$d_o[(fg) \circ p)] = d_o(f \circ p)g(x) + f(x)d_o(g \circ p)$,
which satisfies the leibniz rule. Hence $dp : C^\infty(\mathcal{M}) \longrightarrow \mathbb{R}$ is a derivation on $C^\infty(\mathcal{M})$.
□

definition 8.3.4. ***(Tangent cone)****;*

Let $(\mathcal{M}, \mathcal{D})$ be a diffeological space, the tangent cone is the set given by $\mathcal{C}_x(\mathcal{M}) := \{d_p \mid p \in \mathcal{D}\}$.

Remark 8.3.1.

The tangent cone is the collection of plot derivations on $\mathcal{M}$. That is a cone in the vector space of derivations on $\mathcal{M}$.

Example 8.3.2. *;*
(Tangent space of a discrete diffeological space);
For any discrete diffeological space $(\mathcal{M}, \mathcal{D})$*, which is the collection of all locally constant functions* $U \longrightarrow \mathcal{M}$ *then by example 8.3.1, the tangent space at any point* $x \in \mathcal{M}$ *is the zero space.*

;

8.4. Tangent structures on Frölicher Spaces

Let $(\mathcal{M}, \mathcal{C}_{\mathcal{M}}, \mathcal{F}_{\mathcal{M}})$ be a Frölicher space. In this category, one defines a tangent vector at $x \in \mathcal{M}$ using either structure curves or structure functions. That is by first considering the fact that $f \circ c$ is C^{∞} for arbitrary $f \in \mathcal{F}$ and $c \in \mathcal{C}$. If we consider curves and assuming $c(0) = x$, then one defines a curvaceous tangent vector as the derivative of $f \circ c$ at $t = 0$, which clearly is a derivation at x ([9]). In this regard, Frölicher spaces have two types of tangent vectors defined on them.

(1) **Operational Tangent Vector :** A tangent vector from the structure functions.

(2) **Kinematic Tangent Vector :** A tangent vector from the strucure curves.

Operation Tangent Vectors

definition 8.4.1. ***(Operational Tangent Vector)****([3],[36],[39])*
Let $(\mathcal{M}, \mathcal{C}_{\mathcal{M}}, \mathcal{F}_{\mathcal{M}})$ *be a Frölicher space. The operation tangent vector at* $p \in \mathcal{M}$ *is a map* $v : \mathcal{F}_{\mathcal{M}} \longrightarrow \mathbb{R}$ *which satisfies the linearity condition and the Leibniz rule, that is for all* $f, g \in \mathcal{F}_{\mathcal{M}}$*,* $\alpha, \beta \in \mathbb{R}$ *we have*

(i) $v(\alpha f + \beta g) = \alpha v(f) + \beta v(g)$.

(ii) $v(fg) = f(p)v(g) + g(p)v(f)$.

Remark 8.4.1.

The map v *is also called a smooth derivation at* $p \in \mathcal{M}$.

Example 8.4.1. *[36] ;*
The cross in $\mathbb{R}^2$ *given by the set* $\mathcal{S} := \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$. $\mathcal{S}$ *as a subspace of the Frölicher space* $\mathbb{R}^2$ *has a Frölicher structure. We set* $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$, *with* $\mathcal{S}_1 = \{(x, 0) \mid x \in \mathbb{R}\}$ *and* $\mathcal{S}_2 = \{(0, y) \mid y \in \mathbb{R}\}$. *The structure functions are given by* $\mathcal{F}_o := \{f : \mathcal{S} \longrightarrow \mathbb{R} \mid f := h_{|\mathcal{S}},\ h \in \mathcal{F}_{\mathbb{R}^2}\}$.
Where

$$f(x, y) = \begin{cases} f_1(x, 0) & along \quad \mathcal{S}_1 \\ f_2(0, y) & along \quad \mathcal{S}_2 \end{cases}$$

f_1 *and* f_2 *are real functions that intersect at* $(0, 0)$. *An operation tangent vector* v *on* $\mathcal{S}$ *is a linear derivation along* $\mathcal{S}_1$ *or along* $\mathcal{S}_2$. *That is* $v = \frac{\partial f_1}{\partial x}$ *or* $v = \frac{\partial f_2}{\partial y}$.
proposition 8.4.1. *([3] ,[39]) ;*
Let ($\mathcal{M}$,$\mathcal{C}_{\mathcal{M}}$,$\mathcal{F}_{\mathcal{M}}$) *be a Frölicher space and* $p \in \mathcal{M}$. . *Let* $v : \mathcal{F}_{\mathcal{M}} \longrightarrow \mathbb{R}$ *be a linear map. Then* v *is an operational tangent vector to* $\mathcal{M}$ *at* $p \in \mathcal{M}$, *if and only if* v *satisfies the following conditions:*

(i) $v(f) = 0$ *if* f *is constant.*

(ii) $v_{|\alpha_p^2} = 0$, *where* $\alpha_p^2 := \{(f - f(p))(g - g(p)) \mid f, g \in \mathcal{F}_{\mathcal{M}}\}$

Proof.
We assume that v is an operational tangent vector. That is v is a linear derivation.

(i) Let c $\in \mathbb{R}$ be constant and $f : \mathcal{M} \longrightarrow \mathbb{R}$, a function such that $f(x) =$ c for all $x \in \mathcal{M}$. Then we have that;

$$\begin{aligned} v(f) &= v(c) \\ &= v(1.c) \\ &= cv(1) \\ \text{But } v(1) &= v(1.1) \\ &= 1v(1) + 1v(1) \\ v(1) &= 2v(1) \end{aligned}$$

Hence $v(1) = 0$
Thus $v(f) = 0$, as required.

(ii) v is a derivation at $p \in \mathcal{M}$, thus v satisfies the Leibniz property given by

$$v(f.g) = v(f)g(p) + f(p)v(g) \text{ for all f , g} \in \mathcal{F}_{\mathcal{M}}. \text{ Then}$$

$$\begin{aligned} 0 &= v(f.g) \text{ - } f(p)v(g)\text{- } g(p)v(f) \\ &= v((f.g) - f(p)(g) - g(p)(f)) \\ &= v((f.g) - f(p)(g) - g(p)(f) + f(p)g(p)) \\ &= v(g(f - f(p)) - g(p)(f - f(p)) \\ &= v((f - f(p))(g - g(p)) \ldots\ldots\ldots\ldots \star \end{aligned}$$

Conversely we assume that the linear map v vanishes on constants and on the set α_p^2 $:= \{(f - f(p))(g - g(p)) \mid f, g \in \mathcal{F}_{\mathcal{M}} \}$ we need to show that v satisfies the Leibniz property.

From $\star$
$$\begin{aligned} 0 &= v((f - f(p)).(g - g(p)) \\ &= v((f.g - f(p)g - g(p)f + f(p)g(p)) \\ &= v(f.g) - f(p)v(g) - g(p)v(f) + v(f(p)g(p)) \\ &= v(f.g) - f(p)v(g) - g(p)v(f) \end{aligned}$$

Thus $v(f.g) = g(p)v(f) + f(p)v(g)$ □

definition 8.4.2. *(**Operational Tangent space**)([3] ,[15] , [31])*
Let ($\mathcal{M}$,$\mathcal{C}_{\mathcal{M}}$,$\mathcal{F}_{\mathcal{M}}$) be a Frölicher space. The operational tangent space to $\mathcal{M}$ at $p \in \mathcal{M}$ is the set of all operational tangent vectors to $\mathcal{M}$ at $p \in \mathcal{M}$ and is denoted by $T_p\mathcal{M}$.

definition 8.4.3. *(**Operational contangent space**)([3],[36])*
Let ($\mathcal{M}$,$\mathcal{C}_{\mathcal{M}}$,$\mathcal{F}_{\mathcal{M}}$) be a Frölicher space. The operational cotangent space at $p \in \mathcal{M}$, denoted $T_p^*\mathcal{M}$, is the algebraic dual of $T_p\mathcal{M}$.

Remark 8.4.2.

The elements of $T^*\mathcal{M}$ are linear forms or linear functionals on $\mathcal{M}$ called tangent covectors or covariant vectors.

definition 8.4.4. *(**differential of smooth map**); ([3] ,[23] , [36])*
Let ($\mathcal{M}$,$\mathcal{C}_{\mathcal{M}}$,$\mathcal{F}_{\mathcal{M}}$) and ($\mathcal{N}$,$\mathcal{C}_{\mathcal{N}}$,$\mathcal{F}_{\mathcal{N}}$) be Frölicher spaces and let $\psi : \mathcal{M} \longrightarrow \mathcal{N}$ be a Frölicher smooth map. Let $p \in \mathcal{M}$ and $v \in T_p\mathcal{M}$. The Frölicher smooth linear map $\psi_{\star p}(v) : T_p\mathcal{M} \longrightarrow T_{\psi(p)}\mathcal{N}$ defined by $\psi_{\star p}(v)(g) = v(g \circ \psi)$ is called the differential of ψ at $p \in \mathcal{M}$.

Remark 8.4.3.

1. *The Frölicher smooth linear map $\psi_{\star p}(v)$ is sometimes called the tangent linear map.*
2. *The operation cotangent space $T_p^*\mathcal{M}$ is a linear Frölicher if $T_p\mathcal{M}$ is finite dimensional, then we have that dim $T_p\mathcal{M}$ = dim $T_p^*\mathcal{M}$.*

Theorem 8.4.1. *([3],[36])*
Let ψ : $(\mathcal{M},\mathcal{C}_{\mathcal{M}},\mathcal{F}_{\mathcal{M}}) \longrightarrow (\mathcal{N},\mathcal{C}_{\mathcal{N}},\mathcal{F}_{\mathcal{N}})$ be a Frölicher smooth map and $v \in T_p\mathcal{M}$. The pair (v, ψ) induces on $\mathcal{N}$ an operational tangent vector in the neighbourhood $\psi(p) \in \mathcal{N}$.

Proof.

By definition 5.3.1 (2), for each $h \in \mathcal{F}_{\mathcal{N}}$, $h \circ \psi \in \mathcal{F}_{\mathcal{M}}$, since ψ is smooth. Now for $v \in T_p\mathcal{M}$, in the neighbourhood of $\psi(p) \in \mathcal{N}$, we define the map $\psi_{\star p}(v) : \mathcal{F}_{\mathcal{N}} \longrightarrow \mathbb{R}$ by $\psi_{\star p}(v)(h) = v(h \circ \psi)$. We need to show that $\psi_{\star p}(v)$ is a linear map and satisfies the Leibniz property.
Let $\mu \in \mathbb{R}$ and $h \in \mathcal{F}_{\mathcal{N}}$ since v is linear, then by definition of $\psi_{\star p}(v)$ we have that;

$$\begin{aligned}\psi_{\star p}(v)(\mu.h) &= v((\mu.h) \circ \psi)\\ &= v(\mu.(h \circ \psi))\\ &= \mu v(h \circ \psi)\\ &= \mu\psi_{\star p}(v)(h).\end{aligned}$$

Suppose $g \in \mathcal{F}_{\mathcal{N}}$ and $h \in \mathcal{F}_{\mathcal{N}}$; Then we have ;

$$\begin{aligned}\psi_{\star p}(v)(g + h) &= v((g + h) \circ \psi)\\ &= v((g \circ \psi) + (h \circ \psi))\\ &= v(g \circ \psi) + v(h \circ \psi)\\ &= \psi_{\star p}(v)(g) + \psi_{\star p}(v)(h)\end{aligned}$$

Hence, $\psi_{\star p}(v)$ is linear.
Now let $g \in \mathcal{F}_{\mathcal{N}}$, $h \in \mathcal{F}_{\mathcal{N}}$ and $q := \psi(p) \in \mathcal{N}$. since v satisfies the Leibniz rule and by definition of $\psi_{\star p}(v)$ we have that,

$$\begin{aligned}\psi_{\star p}(v)(g.h) &= v((g.h) \circ \psi)\\ &= v((g \circ \psi).(h \circ \psi))\\ &= (h \circ \psi)(p)v(g \circ \psi) + (g \circ \psi)(p)v(h \circ \psi)\\ &= h(\psi(p))\psi_{\star p}(v)(g) + g(\psi(p))\psi_{\star p}(v)(h)\\ &= h(q)\psi_{\star p}(v)(g) + g(q)\psi_{\star p}(v)(h)\end{aligned}$$

which satisfies the Leibniz property.
Therefore (v, ψ) induces an operational tangent vector on $\mathcal{N}$ in the neighbourhood $q = \psi(p) \in \mathcal{N}$. □

Kinematic Tangent Vectors

In kinematics, the velocity vector is always tangent to the path of motion and its magnitude is determined by taking the time derivative of the path function. This motion for a body is usually described by curves. The rate of change determines the velocity and direction of the tangent at any point of the path of motion. This type of vector will be called kinematic tangent vector.

definition 8.4.5. ***(Kinematic tangent vector**([3],[36]))*

Let ($\mathcal{M}$,$\mathcal{C}_{\mathcal{M}}$,$\mathcal{F}_{\mathcal{M}}$) be a Frölicher space. We denote by $\mathcal{C}_{\mathcal{M}}^{a,p}$ the set of all structure curves $c : \mathbb{R} \longrightarrow \mathcal{M}$ such that $c(a) = p$ with $c \in \mathcal{C}_{\mathcal{M}}^{a,p}$, $a \in \mathbb{R}$ being the foot point and $p \in \mathcal{M}$ a point where the curves pass. A kinematic tangent vector to the space $\mathcal{M}$ with foot point $a \in \mathbb{R}$ is the derivation given by ;

$\bar{K}_{c,p}(f) := \dfrac{d}{dt}\, (f \text{ o } c)_{|t=a}$, *where* $f \in \mathcal{F}_{\mathcal{M}}$.

definition 8.4.6. *([3],[15],[31],[36])*

Let ($\mathcal{M}$,$\mathcal{C}_{\mathcal{M}}$,$\mathcal{F}_{\mathcal{M}}$) be a Frölicher space.

(1) ***Tangent cone space :***
The tangent cone space at a point $p \in \mathcal{M}$ denoted by $T_p\mathcal{C}\mathcal{M}$ is the set of all kinematic tangent vectors at p.

(2) ***Contangent cone space :***
The contangent cone space denoted $T_p^*\mathcal{C}\mathcal{M}$ is the algebraic dual of a linear tangent cone space.

(3) ***tangent cone bundle :***
The tangent cone bundle denoted $T\mathcal{C}\mathcal{M}$ is the direct sum of the tangent cone space.

(4) ***Contangent cone bundle :***
The contangent cone bundle denoted $T^*\mathcal{C}\mathcal{M}$ is the algebraic dual of the tangent cone bundle.

Example 8.4.2. *;*

For a set $\mathbb{Q}$ of rationals, the set $\mathcal{C}_{\mathbb{Q}}$ consists of constant maps $c : \mathbb{R} \longrightarrow \mathbb{Q}$ and $\mathcal{F}_{\mathbb{Q}}$ consists of all functions $f : \mathbb{Q} \longrightarrow \mathbb{R}$. Let $p \in \mathbb{Q}$ such that $c(a) = p$. Then the kinematic tangent vector to the set $\mathbb{Q}$ is the derivation,

$\dfrac{d(f(c(t)))}{dt}_{|t=a} = 0$, *since the curves in $\mathbb{Q}$ are constant maps.*

8.5. Tangent structure of Frölicher space vs Tangent structure of diffeological space

Any diffeology induces a Frölicher structure when we consider the diffeology as 1-plot.

Theorem 8.5.1. *;*

Let ($\mathcal{M}$,$\mathcal{D}$) be a diffeological space and $F : (\mathcal{M}, \mathcal{D}_{\mathcal{M}}) \longrightarrow (\mathcal{N}, \mathcal{D}_{\mathcal{N}})$ *a smooth map. Then ($\mathcal{M}$,$\mathcal{D}$) together with 1-plots of $\mathcal{D}$ is a Frölicher space.*

Proof.

By definition F is smooth, that is if p is a plot in $\mathcal{D}_{\mathcal{M}}$ then $F \circ p \in \mathcal{D}_{\mathcal{N}}$. Now if p is a 1-plot, then it is a curve in the Frölicher space thus one has the same characterisation. Therefore tangent structures for 1-plots are exactly tangent structures for the Frölicher space called cones. □

corollary 8.5.1. *;*

Let ($\mathcal{M}$,$\mathcal{D}$) be a diffeological space and ($\mathcal{M}$,$\mathcal{C}_{\mathcal{M}}$,$\mathcal{F}_{\mathcal{M}}$) be Frölicher space, then the tangent structures on ($\mathcal{M}$,$\mathcal{D}$) coincide with the tangent structure on ($\mathcal{M}$,$\mathcal{C}_{\mathcal{M}}$,$\mathcal{F}_{\mathcal{M}}$) when ($\mathcal{M}$,$\mathcal{D}$) is a 1-plot.

Remark 8.5.1.

We recall that plots can be multi-dimensional whilst in Frölicher spaces we only have one dimension paths or curves which are 1-plot. Therefore for Frölicher spaces there is no need to construct tangent structures starting from Euclidean spaces.

8.6. Tangent structure of Sikorski space vs Tangent structure of diffeological space

We recall that in the Sikorski differential structure,curves are not part of the structure. However, we recall that when difining tangent vectors on a Sikorski differential space we use curves. We first define curves because curves are not in the structure. That is, a path or curve from an interval I of $\mathbb{R}$ is called a differentiable curve if its composition with a structure function gives a smooth function and this coincides with the composition as in the Frölicher structure, although in the Frölicher setting the curve is globally defined on $\mathbb{R}$. If we consider tangent cones on Sikorski differential space, we see that they use curves or paths which are 1-plot in diffeologies. In this comparison one can deduce that the tangent structures on diffeologies are more wider because they use multi-dimensional paths from $\mathbb{R}^n$ into the smooth space, whilst Sikorski differential spaces use one dimensional paths from $\mathbb{R}$ into the smooth space.Therefore, for diffeological spaces the construction of tangent structures on them will only coincide with those of the Sikorski differential space only when the diffeological space has 1-plot only.

corollary 8.6.1. *;*
Let ($\mathcal{M}$,$\mathcal{D}$) be a diffeological space and ($\mathcal{M}$,$\mathcal{F}$) be Sikorski differential space, then the tangent structures on ($\mathcal{M}$,$\mathcal{D}$) coincide with the tangent structure on ($\mathcal{M}$,$\mathcal{F}$) when ($\mathcal{M}$,$\mathcal{D}$) is a 1-plot.

8.7. Tangent structure of Sikorski space vs Tangent structure of Frölicher space

We recall the definition of a pre-Frölicher space, as given in section 7 unit 7.5. Then we have the folowing theorem.

Theorem 8.7.1. *;*
Let ($\mathcal{M}$,$\mathcal{F}$) be a Sikorski differential space and ($\hat{\mathcal{M}}$, $\Gamma\mathcal{F}$, $\Phi\Gamma\mathcal{F}$) be the associated Frölicher space. Then the tangent spaces on $\mathcal{M}$ and $\hat{\mathcal{M}}$ coincide if ($\hat{\mathcal{M}}$, $\Gamma\mathcal{F}$, $\Phi\Gamma\mathcal{F}$) is a pre-Frölicher space.

Proof.
In effect $v \in T_x\mathcal{M}$ is a linear map which is a derivation on $\mathcal{F}$. Now $\mathcal{F} \subset \Phi\Gamma\mathcal{F}$ in general, and $\mathcal{F} = \Phi\Gamma\mathcal{F}$ for ($\mathcal{M}$,$\mathcal{F}$) a pre-Frölicher space. Therefore, $T_x\hat{\mathcal{M}} = T_x\mathcal{M}$ only in the case where $\mathcal{F} = \Phi\Gamma\mathcal{F}$. Otherwise the tangent spaces for Frölicher spaces and Sikorski differential spaces do not coincide. □

Example 8.7.1. *(see [15]) ;*

(1) *The rationals as a Frölicher subspace of $\mathbb{R}$ have trivial tangent spaces equal to their tangent cones: Since contours must have constant values, the tangent cones must be trivial. Let $q \in \mathbb{Q}$. The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(q) = 1$ and $f(r) = 0$ if $r \neq q$ belongs to $\mathcal{F}_{\mathbb{Q}}$. Since $f^2 = f$, one can show that, for any derivation D at q, $D(f) = 0$. Let $g \in \mathcal{F}_{\mathbb{Q}}$ and $g^{'} = fg$. Then, $D(g^{'}) = f(q)D(g)$. Since $g(q)g^{'} = (g^{'})^2$, $D(g^{'}) = 0$ and thus $D(g) = 0$. Hence, the tangent space at q is trivial.*

(2) *$\mathbb{Q}$ as a differential subspace of $\mathbb{R}$ has scalars $f : \mathbb{Q} \longrightarrow \mathbb{R}$ which are locally in the usual topology the restrictions of locally smooth functions on $\mathbb{R}$. Thus, the tangent space to a point $q \in \mathbb{Q}$ is the same as the tangent space when q is regarded as a point of $\mathbb{R}$ and one dimensional.*

9

CONCLUSION:

In this study, we have compared three spaces each a generalised manifold structure to investigate how they relate to one another in terms of their smooth structures, topologies and tangent structures. In chapter 6 we did the comparative study of the three structures and Confirmed P .Chenerack's statement that a Frölicher space is a differential space in the sense of Sikorski. An important fact is that there are some sets on which the Frölicher structure and the differential structure, although generated by the same function set, don't have same structure functions. An example is given by the set $\mathbb{Q}$ of rational numbers, with $\mathcal{F}_{\mathbb{Q}} = \{id_{\mathbb{Q}}\}$ as generating set. For instance, the differential structure on $\mathbb{Q}$ is formed by the restrictions to $\mathbb{Q}$ of all the smooth functions on $\mathbb{R}$. But, as from the definition of a Frölicher structure, we first test the curves generated by $\{id_{\mathbb{Q}}\}$, which clearly are all constant maps from $\mathbb{R}$ to $\mathbb{Q}$ as proved in example 7.5.1 using the intermediate value theorem of real analysis. Then it turns out that all real-valued functions on $\mathbb{Q}$ are made smooth in this smooth structure. The Frölicher structure on $\mathbb{Q}$ is therefore discrete in that it contains all real-valued functions. We also proved that a diffeological structure can only be a Frölicher space if it is considered as a 1-plot. In the same chapter we defined sets $\Phi\mathcal{D}$ and $\Pi\mathcal{F}$ on the spaces ($\mathcal{M}$,$\mathcal{F}$) and ($\mathcal{M}$,$\mathcal{D}$) respectively and proved that ($\mathcal{M}$,$\mathcal{D}$) is Sikorski differential space and vice versa.

In chapter 7 we see that the the initial topology $\tau_{\mathcal{F}_{\mathcal{M}}}$ induced by all smooth functions so called functional topology is the weakest topology in which all functions are continuous and that $\tau_{\mathcal{F}_{\mathcal{M}}} \subset \tau_{\mathcal{C}_{\mathcal{M}}}$ the curvaceous topology on a Frölicher space. We compare the topologies on a Frölicher space and diffeological spaces and conclude that they only coincide when the diffeological space is a 1-plot. It is shown in chapter 7 that topologies for a Sikorski differential space and a Frölicher space will only coincide in the case of a pre-Frölicher space.

In chapter 8 it is clear from theorem 8.7.1 that the tangent structures on the Sikorski space

and tangent structures on the Frölicher space do not coincide but will only coincide in the case where ($\mathcal{M}$,$\mathcal{F}$) is a pre-Frölicher space. We see in both structures that the definition of tangent vector satisfies the Leibniz property. P.Cherenack (see [15]) used the term tangent cone for the curvaceous tangent vector on a differential space where as A Batubenge (see [3]) called it Kinematic tangent vector on a Frölicher space.
When we consider the diffeological space and the Frölicher space, tangent structures for 1-plots are exactly tangent structures for the Frölicher space called cones.
All in all through this study we have illustrated Andrew Stacy's (see [35]) statement that the smooth structure in the sense of Frölicher is the best behaved, provides the underlying set with the weakest topology and defines geometric quantities in particular tangent structures in a more natural way.

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