THE UPPER BROWDER SPECTRUM IN COMMUTATIVELY ORDERED BANACH ALGEBRAS

BY

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Declaration

I, Sonick Mumba, declare that this dissertation entitled **“The upper Browder spectrum in commutatively ordered Banach algebras”**, and the work presented in it are my own. The work described in this Master of Science (MSc) dissertation was carried out under the supervision of Dr. K. Muzundu, Department of Mathematics and Statistics, University of Zambia, Lusaka and Dr. R. Benjamin, Department of Mathematical Sciences (Division Mathematics), Stellenbosch University, South Africa.

This MSc dissertation represents original work by the author and has not otherwise been submitted in any form for any degree or diploma to any other University. Where use has been made of the work of others it is duly acknowledged in the text.

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Approval

This dissertation of Sonick Mumba is approved as fulfilling part of the requirements for the award of the degree of Master of Science in Mathematics by the University of Zambia.

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Dedication

I dedicate this work to my beautiful wife Amina Mumba Zimba, my child Jamie Mphatso Sonick Mumba and my brothers Jimmy Mumba and Alex Mofya.

Abstract

A *commutatively ordered Banach algebra* ($COBA$) is a complex unital Banach algebra A containing a subset C, called an *algebra c-cone*, such that C contains the unit of A and is closed under addition, positive scalar multiplication and multiplication by commuting positive elements. If the commutativity assumption is removed, then the resulting cone is called an *algebra cone*. A Banach algebra ordered by an algebra cone is called an *ordered Banach algebra* (OBA). Evidently, every OBA is a $COBA$.
Not every $COBA$ is however an OBA. An example confirming this statement is given by the $COBA$ $(B(H), C)$, where $B(H)$ is the space of all bounded linear operators on a Hilbert space H and

$$C = \{T \in B(H) : \langle Tx, x\rangle \in \mathbb{R}^+ \text{ for all } x \in H\}.$$

Benjamin and Mouton described Fredholm theory in $OBAs$ relative to a homomorphism $T : A \to B$, where A and B are Banach algebras, and introduced an element called an upper Browder element. An upper Browder element $x \in A$ is an element of the form $y + z$, where y is invertible in A and $z \in C$ is an element of the null space of T such that $yz = zy$. We denote by $\mathcal{B}_T^+$ the set of all upper Browder elements of A, which in turn gives (in a natural way) rise to the *upper Browder spectrum*

$$\beta_T^+(x) := \{\lambda \in \mathbb{C} : \lambda e - x \text{ is not an upper Browder element}\}.$$

In an OBA setting, Benjamin examined the following natural question: given that the spectral radius of a positive element is not in the Fredholm spectrum of the element, when will it be outside the upper Browder spectrum of that element? The element satisfying this condition is said to have the *upper Browder spectrum property* (see Definition 5.1.1). They went on to show that the connected hulls of the upper Browder and the Browder spectra do not coincide in general, as well as the conditions under which the upper Browder spectrum satisfy the spectral mapping theorem.
In this study we extend these results to $COBAs$. Since every OBA is a $COBA$, it is actually concluded that some results on upper Browder spectrum of an OBA element readily extend to $COBAs$.
For further research, we recommend that the $COBAs$, rather than the $OBAs$, should be the default setting for studying Fredholm theory.

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Contents

List of Notations

Below is a list of some important symbols to be used and a brief explanation of their meaning.

Operations

$\|\|x\|\|$	norm of normed space element x
$x \circ y$	quasi-product of x and y or composition of functions x and y
$f\|_X$	restriction of a function f to a set X
$\dim B$	dimension of a vector space B
$\bigoplus_{i=1}^{n} A_{\mathrm{i}}$	direct sum of the sets A_{i}
ηK	connected hull of a compact set K
∂K	topological boundary of a set K
$\mathrm{int}(K)$	interior of a set K

Sets

A_1	unitization of a Banach algebra A
A^D	set of all almost invertible elements of a unital Banach algebra
A^{-1}	set of all invertible elements of an algebra with unity
q-A^{-1}	set of quasi-invertible elements of an algebra A
$\mathrm{QN}(A)$	set of all quasinilpotent elements of a unital Banach algebra A
$\mathrm{Rad}(A)$	radical of a unital Banach algebra A
$A^D \cap \mathcal{F}_T$	set of all almost invertible Fredholm elements relative to T
$\mathrm{N}(T)$	null space of a linear operator T
$\mathrm{R}(T)$	range of a linear operator T
$H(\Omega)$	algebra of all complex-valued functions defined and holomorphic on $\Omega \subseteq \mathbb{C}$
$\mathrm{Comm}(x)$	commutant of an algebra element x
$\mathrm{span}\ B$	linear span of a set B
$\rho(x)$	resolvent set of a unital Banach algebra element x
$\mathcal{F}_T$	set of Fredholm elements relative to T
$\mathcal{B}_T$	set of Browder elements relative to T
$\mathcal{B}_T^+$	set of upper Browder elements relative to T

Spaces

$\mathbb{R}^+$	set of all non-negative real numbers.
$\mathbb{C}^n$	set of all n-tuples of complex numbers.
$\mathbb{R}(\mathbb{C})$	set of all real (complex) numbers
$C(K)$	algebra of continuous complex-valued functions on a compact set K

$M_n(A)$	algebra of $n \times n$ matrices with entries in an algebra A
$M_n^u(A)$	algebra of upper triangular matrices in $M_n(A)$
$B(X)$	algebra of bounded linear operators on a Banach space X
$K(X)$	algebra of compact operators on a Banach space X
ℓ^∞	space of all bounded sequences of complex numbers.
$L^1(\mathbb{Z})$	discrete convolution algebra

Spectra

$\sigma(x)$	spectrum of a unital Banach algebra element x
$\sigma'(x)$	set of all non-zero elements of $\sigma(x)$
$\sigma_1(x)$	spectrum of a non-unital Banach algebra element x
iso $\sigma(x)$	set of isolated points of $\sigma(x)$
acc $\sigma(x)$	set of accumulation points of $\sigma(x)$
$\beta_T(x)$	Browder spectrum of a unital Banach algebra element x w.r.t. T
$\beta_T^+(x)$	upper Browder spectrum of a unital Banach algebra element x w.r.t. T

Elements

0_A	zero element of a Banach algebra A
e_A	unity of an algebra A
x^{-1}	inverse of an algebra element x
x^D	generalized Drazin inverse of a unital Banach algebra element x
$r(x)$	spectral radius of a unital Banach algebra element x
$p(x, \lambda)$	spectral idempotent of x corresponding to $\lambda \in$ iso $\sigma(x)$

1

Introduction

In this chapter, A will denote a complex Banach algebra with a multiplicative identity e, and we shall refer to A as a unital Banach algebra.
An *algebra cone* of a complex unital Banach algebra A is a non-empty subset C of A which is closed under addition, multiplication, positive scalar multiplication and contains the unity of A. In [23], Raubenheimer and Rode showed that an algebra cone induces a partial ordering on A in the following way: if $x, y \in A$, then $x \leq y$ if and only if $y - x \in C$. A Banach algebra ordered by an algebra cone is called an *ordered Banach algebra* (OBA). In [19], Mouton and Muzundu noted that not all Banach algebras can be ordered by an algebra cone. They then introduced and defined an *algebra c-cone* C as an algebra cone closed under multiplication of commuting elements and showed that it induces an ordering in the same way an algebra cone does. A Banach algebra ordered by an algebra c-cone is called a *commutatively ordered Banach algebra* ($COBA$). Every OBA is a $COBA$.

Fredholm theory, which is one of the main subareas of spectral theory, has its origins in the Fredholm integral equations on a Banach space of bounded linear operators. There have been some generalizations of Fredholm theory to operators and semisimple Banach algebras. In [12], Harte generalized Fredholm theory to general Banach algebras. A class of elements arising from Fredholm theory in Banach algebras are the Browder elements. An element $x \in A$ is said to be *Browder* relative to a Banach algebra homomorphism T if $x = y + z$, where y is invertible in A and z is in the null space of T such that $yz = zy$. The Browder elements give rise to the *Browder spectrum*, which is defined (for an element x in a unital Banach algebra A) as the set

$$\{\lambda \in \mathbb{C} : \lambda e - x \text{ is not a Browder element}\}.$$

In [5], Benjamin and Mouton took into account the ordering on a Banach algebra and introduced the concept of an upper Browder element in an OBA. Let $T : A \to B$ be a Banach algebra homomorphism and (A, C) be an OBA. An upper Browder element $x \in A$ is an element of the form $y + z$, where y is invertible in A

and $z \in C$ is an element of the null space of T such that $yz = zy$. We denote by $\mathcal{B}_T^+$ the set of all upper Browder elements of A, which in turn gives (in a natural way) rise to the *upper Browder spectrum*

$$\beta_T^+(x) := \{\lambda \in \mathbb{C} : \lambda e - x \text{ is not an upper Browder element}\}.$$

The purpose of this dissertation is to generalize some of the results involving the upper Browder spectrum given in [4], [5] and [6] from the OBA setting to the more general setting of $COBAs$.

This dissertation is composed of five chapters. Below we give a brief introduction to what is studied in each chapter.

The aim of Chapter 2 is to review some definitions and results and establish the terminology and notation that will be needed in the rest of this dissertation.

In Chapter 3 we discuss upper Browder elements in $COBAs$. This chapter begins with the definition of an upper Browder element of a $COBA$ relative to a Banach algebra homomorphism and an example of such an element. Additionally, a number of examples are given that demonstrate the (generally strict) inclusion properties of the upper Browder, Browder and invertible elements of a $COBA$. The second section of Chapter 3 is about the basic properties of upper Browder elements in $COBAs$. Propositions 3.2.5 and 3.2.10 state, respectively, that if the $COBA$ is commutative - in which case we are also dealing with an OBA - then the upper Browder elements (relative to a Banach algebra homomorphism with the Riesz property) remain stable under perturbation by positive null space elements and are closed under multiplication, while Theorem 3.2.12 gives conditions under which the sets of Browder and upper Browder elements coincide. As a result, these results cannot be meaningfully generalized to $COBAs$. (The disparity between the classes of Browder and upper Browder elements is evident from their algebraic properties. For example, Browder elements are known to be closed under multiplication while (in view of Example 3.2.9) upper Browder elements are generally not.)

In Chapter 4 the spectrum corresponding to the set of upper Browder elements, referred to as the upper Browder spectrum, is studied. Specifically, we focus on spectral mapping theorems and the relationship between the connected hulls of the Browder and upper Browder spectra.

The upper Browder spectrum, like the ordinary spectrum, is a non-empty and compact subset of the complex plane (see the remark following Proposition 4.1.5). However, one should be careful to substitute the element $\lambda e - x$ in the definition of the upper Browder spectrum of $COBA$ element x by $x - \lambda e$ (see Example

4.1.2). This is due to the fact that the set of upper Browder elements is not in general closed under non-zero scalar multiplication (see Example 3.2.8). From Lemma 3.2.7 we have, relative to a Banach algebra homomorphism with the Riesz property, that $\lambda e - x$ is an upper Browder element if and only if $x - \lambda e$ is an upper Browder element. In Section 4.2 we show in Proposition 4.2.1 that the upper Browder spectrum of an element x in a commutative $COBA$ (A, C) satisfies a spectral mapping theorem whenever T (defined on A) has the Riesz property and

$$p(x, \lambda) \in \text{span}(C \cap \text{N}(T)) \text{ for all } \lambda \in (\text{iso } \sigma(x)) \backslash \sigma(Tx),$$

where $p(x, \lambda)$ denotes the spectral idempotent of x corresponding to an isolated point λ of the spectrum $\sigma(x)$ of x which lies outside the Fredholm spectrum $\sigma(Tx)$ of x.
The necessity of the aforementioned assumptions is investigated further. At present we do not have examples confirming that the spectral idempotent and the commutativity assumptions cannot in general be removed. Nonetheless, as shown in Example 4.2.2, the assumption that "T has the Riesz property" is generally essential to conclude the spectral mapping theorem for an element of a commutative $COBA$.

In Section 4.3 we investigate the relationship between the connected hulls of the Browder and upper Browder spectra of a $COBA$ element. In Example 4.3.1 we showed that the connected hulls of the Browder and upper Browder spectra do not in general coincide. However, in Theorem 4.3.2, we pointed out that the (connected hulls of the) Browder and upper Browder spectra of a commutative $COBA$ element coincide whenever T has the Riesz property and each spectral idempotent of an element in the $COBA$ corresponding to any isolated point of the spectrum (but outside of the Fredholm spectrum) of that element is a linear combination of positive null space elements.

Finally, we conclude this dissertation with Chapter 5, where we discuss the following spectral problem: If (A, C) is an OBA and $x \in C$ is such that the spectral radius $r(x)$ of x lies outside the Fredholm spectrum of x, does it follow that $r(x)$ is not an element of the upper Browder spectrum of x? The element satisfying this condition is said to have the *upper Browder spectrum property* (see Definition 5.1.1).

Different types of sufficient conditions for a positive element in a $COBA$ to have the upper Browder spectrum property is studied. Firstly, we consider the finite-dimensional case (Section 5.2), for which (with the help of Wedderburn-Artin theorem) we showed that every finite-dimensional semisimple $COBA$ is isomorphic to a $COBA$ in which all positive elements have the upper Browder spectrum property (see Theorem 5.2.8). This result generalizes [6] (Corollary 5.13) to the $COBA$ setting.

In Section 5.3 we consider arbitrary Banach algebra homomorphisms T with the strong Riesz property and provide sufficient conditions for positive $COBA$ elements to have the upper Browder spectrum property relative to T. In particular, we present Theorem 5.3.4 - a stronger version of Theorem 5.1 in [6] - which states that if T has the strong Riesz property, then every positive element x in the domain of T has the upper Browder spectrum property whenever the spectral idempotent of x corresponding to $r(x)$ is positive. This result (under the condition that the algebra c-cone is closed) has the following three applications (see Corollary 5.3.5):

(1) Any positive element of a $COBA$ whose spectral radius is a simple pole of $(\lambda e - x)^{-1}$ has the upper Browder spectrum property.

(2) All positive elements of commutative semisimple $COBAs$ have the upper Browder spectrum property.

(3) All positive elements of semisimple $COBAs$ with proper and inverse-closed algebra c-cones have the upper Browder spectrum property.

2

Preliminaries

In this chapter we collect some basic definitions, results and notations that will be needed in the rest of the chapters. The results for which proofs are given are either in the interest of completeness or facts that were given in an article without proof.

2.1. Introduction to Banach spaces

We begin with a brief summary on the theory of Banach spaces. Throughout this dissertation all vector spaces will be defined over the complex numbers $\mathbb{C}$.

Definition 2.1.1. *([16], Definition 2.2.1) A norm on a vector space E is a function $\|\cdot\|$: $E \to \mathbb{R}^+ := [0, \infty)$ that satisfies the following properties for all $x, y \in E$ and $\alpha \in \mathbb{C}$:*

(*i*) $\|x\| \geq 0$,

(*ii*) $\|x\| = 0$ *if and only if* $x = 0$,

(*iii*) $\|\alpha x\| = |\alpha|\|x\|$,

(*iv*) $\|x + y\| \leq \|x\| + \|y\|$.

A vector space E together with a norm $\|\cdot\|$ defined on it is called a *normed space*. A complete normed space is referred to as a *Banach space*.

The following are examples of Banach spaces.

Example 2.1.2. *Consider the vector space $C(K)$ of all continuous complex-valued functions on a compact Hausdorff space K. With the norm defined by*

$$\|f\| = \max_{x \in K} |f(x)|,$$

where $f \in C(K)$, $C(K)$ *is a Banach space.*

Example 2.1.3. *Consider the vector space $\ell^\infty = \{(x_n) : \sup_{n \geq 1} |x_n| < \infty\}$ of all bounded sequences of complex numbers. With the norm on ℓ^∞ defined by*

$$\|x\| = \sup_{n\geq 1} |x_n|,$$

where $x = (x_n) \in \ell^\infty$, ℓ^∞ *is a Banach space.*

Example 2.1.4. *Consider the vector space* $\ell^p = \{(x_n) : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$, *where* $p \in [1, \infty)$, *of all p-summable sequences of complex numbers. With the norm on* ℓ^p *defined by*

$$\|x\| = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}},$$

where $x = (x_n) \in \ell^p$, ℓ^p *is a Banach space.*

Example 2.1.5. *Consider the vector space* $B(X)$ *of all bounded linear operators on a Banach space* X. *With the (usual operator) norm defined by*

$$\|T\| = \sup_{\substack{x\in X \\ \|x\|=1}} \|Tx\|,$$

where $T \in B(X)$, $B(X)$ *is a Banach space.*

The next example shows that the direct sum of Banach spaces is also a Banach space.

Example 2.1.6. *Let* X_i *be Banach spaces for each* $i = 1, 2, \ldots, n$, *where* $n \in \mathbb{N}$, *and let* $X = X_1 \oplus \cdots \oplus X_n$ *be the direct sum of* $X_1, X_2, \ldots, X_n$. *With the norm on* X *defined by*

$$\|x\| = \sum_{j=1}^{n} \|x_j\|_{X_j},$$

where $x = (x_1, \ldots, x_n) \in X$ and $||\cdot||_{X_j}$ is the norm on X_j, X *is a Banach space.*

Example 2.1.7. *Consider the quotient space* X/M, *where* M *is a closed subspace of the Banach space* X. *With the norm on* X/M *defined by*

$$\|x + M\| = \inf_{y\in M} \|x - y\|_X,$$

where $x + M \in X/M$ and $||\cdot||_X$ denotes the norm on X, X/M *is a Banach space.*

Next we display some results from a very special class of Banach spaces known as Hilbert spaces. We recall that a Hilbert space is a Banach space, where the norm is induced by an inner product.

Definition 2.1.8. *([16], Definition 3.9.1, Definition 3.10.1) Let* $T : H_1 \to H_2$ *be a bounded linear operator between Hilbert spaces* H_1 *and* H_2. *The Hilbert adjoint operator* T^* *of* T *is the operator* $T^* : H_2 \to H_1$ *defined by* $\langle Tx, y\rangle = \langle x, T^*y\rangle$ *for all* $x \in H_1$ *and* $y \in H_2$. *If* $T = T^*$, *where* $T : H \to H$ *is a bounded linear operator* *on a Hilbert space H, then T is said to be self-adjoint (or Hermitian).*

Definition 2.1.9. *([16], p.470) A bounded linear operator T on a Hilbert space H is said to be positive, written $T \geq 0$, if $\langle Tx, x\rangle \in \mathbb{R}^+$ for all $x \in H$.*

Theorem 2.1.10. *([16], p.470) If S and T are positive operators on a Hilbert space, then $S + T \geq 0$. If, in addition $ST = TS$, then $ST \geq 0$, and hence $\lambda S \geq 0$ for all $\lambda \geq 0$.*

Proposition 2.1.11. *([10], Proposition 2.13) If T is a self-adjoint operator on a Hilbert space H, then $\|T\| = \sup\{|\langle Tx, x\rangle| : \|x\| = 1\}$.*

2.2. Ordered Banach Spaces

In this section we give the definition of a positive cone of a Banach space X and discuss some properties positive cones may have.

Definition 2.2.1. *([23], p.491) A non-empty subset C of a Banach space X is called a positive cone if it satisfies the following:*

(i) $C + C \subseteq C$,

(ii) $\lambda C \subseteq C$ *for all scalars* $\lambda \in \mathbb{R}^+$.

A relation "$\leq$" on any set S is called a *partial order* if it is reflexive and transitive on S.

Every positive cone of a Banach space induces a partial order in the following way: Let X be a Banach space and $C \subseteq X$ a positive cone. We say that

$$y \leq x \text{ if and only if } x - y \in C \text{ for } x, y \in X. \qquad (*)$$

A Banach space X which is ordered by a positive cone C is called an *ordered Banach space* (OBS), denoted by (X, C). In view of $(*)$ it is easy to see that $C = \{x \in X : x \geq 0\}$. The elements of C are therefore referred to as *positive*.

Definition 2.2.2. *Let (X, C) be an OBS. Then C is said to be*

(i) closed if C is a closed (in a topological sense) subset of X;

(ii) proper whenever $C \cap -C = \{0\}$;

(iii) normal if there exists a constant $\alpha > 0$ such that for all $x, y \in X$, $0 \leq x \leq y$ implies that $\|x\| \leq \alpha\|y\|$;

(iv) generating if each element of X is a linear combination (over $\mathbb{C}$) of elements in C.

It is known that every normal positive cone in a Banach space is proper ([23], p.492).

The following are examples of ordered Banach spaces. (It is understood that the norms on these spaces, that were pointed out in Section 2.1, are still valid in the examples that follow.)

Example 2.2.3. *$(C(K), C)$, where $C = \{f \in C(K) : f(x) \in \mathbb{R}^+ \text{ for } x \in K\}$, is an OBS with closed, normal and generating positive cone.*

Example 2.2.4. *(ℓ^∞, C), where $C = \{(x_1, x_2, \ldots) \in \ell^\infty : x_n \in \mathbb{R}^+ \text{ for all } n \in \mathbb{N}\}$, is an OBS with closed, normal and generating positive cone.*

Example 2.2.5. *(ℓ^p, C), where $C = \{(x_n) \in \ell^p : x_n \in \mathbb{R}^+ \text{ for all } n \in \mathbb{N}\}$, is an OBS with closed, normal and generating positve cone.*

Example 2.2.6. *$(M_n(\mathbb{C}), M_n(\mathbb{R}^+))$ is an OBS with closed, normal and generating positive cone.*

Example 2.2.7. *$(B(X), C^{'})$, where (X, C) is an OBS* and

$$C^{'} = \{T \in B(X) : Tx \in C \text{ for all } x \in C\},$$

is an OBS. If C is closed (normal) in X, then $C^{'}$ is closed (normal) in $B(X)$.

The next example, whose proof is rather straightforward, shows that the direct sum of ordered Banach spaces is also an ordered Banach space. This is one of the ways in which new ordered Banach spaces are constructed from existing ones.

Example 2.2.8. *Let (X_i, C_i) be ordered Banach spaces for each $i = 1, \ldots, n$, where $n \in \mathbb{N}$, and let $X = X_1 \oplus \cdots \oplus X_n$ be the direct sum of $X_1, X_2, \ldots, X_n$. Then (X, C), where*

$$C = \{(x_1, x_2, \ldots, x_n) \in X : x_i \in C_i \text{ for all } i \in \{1, \ldots, n\}\},$$

is an OBS. If C_i is closed (normal, generating) in X_i for all $i \in \{1, \ldots, n\}$, then C is closed (normal, generating) in X.

2.3. Banach algebras

This section is a brief introduction of the theory of Banach algebras, which are just Banach spaces endowed with a multiplication satisfying certain properties.

Definition 2.3.1. *([16], p.394) An algebra is a vector space A such that for each ordered pair of elements $x, y \in A$ a unique product $xy \in A$ is defined and has the following properties:*

(i) $(xy)z = x(yz)$,

(ii) $x(y+z) = xy + yz$,

(iii) $(x+y)z = xz + yz$,

(iv) $\alpha(xy) = (\alpha x)y = x(\alpha y)$,

for all $x, y, z \in A$ *and* $\alpha \in \mathbb{C}$.

A is said to be *commutative* if $xy = yx$ for all $x, y \in A$.

We call A an *algebra with unity* if there is a element $e \in A$ such that $ex = xe = x$ for all $x \in A$. If A is an algebra with unity, then $x \in A$ is said to be *invertible* if there exists an element $y \in A$ satisfying $xy = yx = e$. If such y exists, then it is unique, and we refer to it as the *inverse* of x and denote it by x^{-1}. We will write A^{-1} for the set of all invertible elements of an algebra A with unity.

Definition 2.3.2. *([2], p.30) An algebra* A *is said to be a Banach algebra if* A *is a Banach space with a norm* $\|\cdot\|$ *that satisfies* $\|xy\| \leqslant \|x\|\|y\|$ *for all* $x, y \in A$.

By a *unital Banach algebra* we mean a Banach algebra with unity e satisfying $\|e\| = 1$.

The following are examples of unital Banach algebras.

Example 2.3.3. *The Banach space* $C(K)$ *with standard pointwise-defined multiplication of functions and unity the constant function 1 on* K *is a commutative unital Banach algebra.*

Example 2.3.4. *The Banach space* ℓ^∞ *under coordinate-wise multiplication and unity* $e = (1, 1, 1 \ldots)$, *is a commutative unital Banach algebra.*

Example 2.3.5. *The Banach space* $M_n(\mathbb{C})$ *with standard multiplication of matrices and unity the* $n \times n$ *identity matrix is a non-commutative unital Banach algebra.*

Example 2.3.6. *The Banach space* $B(X)$ *with composition of operators as multiplication and unity the identity operator on the Banach space* X *is a non-commutative unital Banach algebra.*

The next example deals with direct sums of Banach algebras.

Example 2.3.7. *The Banach space* $A = A_1 \oplus \cdots \oplus A_n$, *where the* A_i's *are Banach algebras, with multiplication on* A *defined by*

$$xy = (x_1y_1, x_2y_2, \ldots, x_ny_n) \text{ for all } x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in A$$

is a Banach algebra. Moreover, if A_i *is a commutative unital Banach algebra for all* $i \in \{1, \ldots, n\}$, *then* A *is a commutative unital Banach algebra with unity* $e = (e_1, \ldots, e_n)$, where e_i *is the unity of* A_i.

A subset B of a Banach algebra A is called a *subalgebra* of A if it is closed under the addition, multiplication and scalar multiplication defined on A.

Definition 2.3.8. (*[23], p.490*) *Let A and B be algebras with unity. A linear map $T : A \to B$ is called an (algebra) homomorphism if $T(xy) = TxTy$ for all $x, y \in A$ and $Te_A = e_B$.*

By a "*Banach algebra homomorphism*" we mean a homomorphism between two unital Banach algebras.

We recall the following well-known fact about linear mappings that will be useful in Chapter 5.

Proposition 2.3.9. *([16], Theorem 2.6.9(b)) A linear operator with finite-dimensional domain has finite-dimensional range.*

The following result was pointed out in the PhD thesis of R. Benjamin and relies on the facts that the range of a linear operator is a vector space ([16], Theorem 2.6.9(a)) and that every finite-dimensional subspace of a normed space is closed ([16], Theorem 2.4.3). We denote by $\dim A$, the dimension of a vector space A.

Lemma 2.3.10. *([3], Corollary 1.1.7) If $T : A \to B$ is a Banach algebra homomorphism and $\dim A < \infty$, then T has closed range; that is the range $\mathrm{R}(T)$ of T is a closed subspace of B.*

By an *ideal* I of an algebra A we mean a *proper two-sided ideal* of A, that is, $I \subsetneq A$ is a (vector) subspace of A which is closed under multiplication from the left and right by elements of A.

It is easy to verify that, if $0 \neq T : A \to B$ is a Banach algebra homomorphism, then the null space $\mathrm{N}(T)$ of T is an ideal of A. Another example of an ideal is given by the radical of an algebra which we define next.

Definition 2.3.11. (*[2], p.34*) *The radical of a unital Banach algebra A, denoted by $\mathrm{Rad}(A)$, is defined as*

$$\mathrm{Rad}(A) = \{x \in A\colon e - yx \in A^{-1} \text{ for all } y \in A\}.$$

If $\mathrm{Rad}(A) = \{0\}$, then A is said to be *semisimple.* Note that the unital Banach algebras given in Examples **??** to 2.3.6 are all semisimple Banach algebras.

The following example tells us that the quotient of a Banach algebra A by a closed ideal I of A is again a Banach algebra.

Example 2.3.12. *The Banach space A/I, where I is a closed ideal of the Banach algebra A, with quotient multiplication defined by*

$$(x + I)(y + I) = xy + I \text{ for all } x, y \in A$$

is a Banach algebra. Moreover, if A is a commutative unital Banach algebra, then A/I is a commutative unital Banach algebra with unity $e + I$.

It is a well known fact that the set $K(X)$ of all compact operators on a Banach space X is an ideal in $B(X)$.

2.4. Spectral theory in Banach algebras

In this section we formulate several definitions and results relating to the concept of the spectrum of a (unital) Banach algebra element.

Definition 2.4.1. *([2], p.36) Let A be a unital Banach algebra. The spectrum of an element $x \in A$, which we denote by $\sigma(x, A)$, is the set*

$$\sigma(x, A) := \{\lambda \in \mathbb{C} : \lambda e - x \notin A^{-1}\}.$$

Whenever there is no ambiguity we shall drop the A in $\sigma(x, A)$. By $\sigma'(x)$ we denote the set of all non-zero elements of $\sigma(x)$.

If x is an arbitrary element of a unital Banach algebra, then we use the notation iso $\sigma(x)$ (acc $\sigma(x)$) to indicate the set of all *isolated (accumulation) points of the spectrum* of x and write $\rho(x) := \mathbb{C}\backslash\sigma(x)$ for the *resolvent set* of x. By [2] (Theorem 3.2.8), the function $\lambda \to (\lambda e - x)^{-1}$ on $\rho(x)$ (called the *resolvent* on x) is analytic on $\rho(x)$.

It is a well-known fact that the spectrum is a non-empty and compact subset of the complex plane ([2], Theorem 3.2.8).

Examples of the spectrum of a unital Banach algebra element include:

- *If $X \in M_n(\mathbb{C})$, then $\sigma(X) = \{\lambda \in \mathbb{C} : \lambda$ is an eigenvalue of $X\}$.*
- *If $f \in C(K)$, where K is a compact Hausdorff space, then $\sigma(f) = f(K)$.*

The following fact will come in handy when proving one of the main results in Section 5.2.

Lemma 2.4.2. *([2], Exercise 9, p.66) Let A be a unital Banach algebra and $x_1, x_2, \dots, x_n \in A$ be such that $x_i x_j = 0 = x_j x_i$ for $i \neq j$. Then*

$$\sigma(x_1 + \cdots + x_n)\backslash\{0\} = (\sigma(x_1) \cup \cdots \cup \sigma(x_n))\backslash\{0\}.$$

The following result will be useful in the sequel.

Lemma 2.4.3. *Let A_i be a unital Banach algebra for each $i = 1, \dots, n$, where $n \in \mathbb{N}$, and let $A = A_1 \oplus \cdots \oplus A_n$ be the direct sum of $A_1, A_2, \dots, A_n$. Then for $x = (x_1, \dots, x_n) \in A$, $\sigma(x, A) = \bigcup_{i=1}^{n} \sigma(x_i, A_i)$.*

Definition 2.4.4. *([2], p.36) An element x of a unital Banach algebra A is said to be quasinilpotent if $\sigma(x) = \{0\}$.*

We denote by $\mathrm{QN}(A)$ the set of all quasinilpotent elements of a unital Banach algebra A. We further point out that $\mathrm{QN}(A) = \mathrm{Rad}(A)$ whenever A is commutative ([2], Remark 1, p.71).

An element x of a unital Banach algebra A is said to be *almost invertible* if $0 \notin \mathrm{acc}\, \sigma(x)$. We will denote the set of all almost invertible elements of a unital Banach algebra A by A^D. (Take note that $\mathrm{QN}(A) \subseteq A^D$.) We further point out that, in the literature, almost invertible elements are also called *generalized Drazin invertible elements* ([15], Definition 2.3). The latter are elements x for which there exist a unique element, denoted by x^D, satisfying

$$xx^D = x^D x, x^D x x^D = x^D \text{ and } x - xx^D x \in \mathrm{QN}(A).$$

Note that, if x is an invertible element of a unital Banach algebra, then $x^{-1} = x^D$, and in such a case $x^D x = e_A$. But for some non-invertible elements one can still find an x^D such that $x^D x \neq e_A$, and so the inclusion $A^{-1} \subseteq A^D$ always holds. It is also worth noting that $T(A^D) \subseteq B^D$ for any Banach algebra homomorphism $T : A \to B$.

Let A be an algebra (with or without unity). The *quasi-product* of elements x and y of A, denoted by $x \circ y$, is defined by $x \circ y := x + y - xy$. An element $x \in A$ is said to be *quasi-invertible* if there exists $y \in A$ such that $x \circ y = 0 = y \circ x$. Denote by q-A^{-1} the set of quasi-invertible elements of A.

If A is non-unital Banach algebra, then the spectrum of $x \in A$, which we shall denote by $\sigma_1(x, A)$, is given by

$$\sigma_1(x, A) \ := \{0\} \cup \left\{ \lambda \in \mathbb{C} \backslash \{0\} : \frac{1}{\lambda} x \notin q\text{-}A^{-1} \right\}$$

(see [7], p.20). By [7] (Lemma 2, p.20), $\sigma_1(x, A) = \sigma((x, 0), A_1)$ for all $x \in A$, where $A_1 := A \oplus \mathbb{C}$ indicates the *unitization* of a Banach algebra A.

The following result gives a relationship between the spectrum relative to a unital Banach algebra and the spectrum relative to the unitization of the Banach algebra.

Lemma 2.4.5. *([7], Proposition 5, p.16) If A is an algebra with unity and $x \in A$, then x is quasi-invertible if and only if $e - x$ is invertible. Hence, if A is a unital Banach algebra, then*

$$\sigma'(x, A) = \left\{ \lambda \in \mathbb{C} \backslash \{0\} : \frac{1}{\lambda} x \notin q\text{-}A^{-1} \right\} = \sigma'((x, 0), A_1).$$

From Lemma 2.4.5 we have that $\sigma'(x, A) = \sigma_1'(x, A)$ for all elements x of a unital Banach algebra A.

The following result will be used in the proof of Proposition 5.2.5 and states that the non-zero spectrum of an element with finite spectrum relative to a Banach algebra (not necessarily unital) and the non-zero spectrum of this element relative to a "larger" Banach algebra in general coincide. We omit its proof that can be found in [3] (Theorem 1.2.5).

Proposition 2.4.6. *([3], Theorem 1.2.5) Suppose that A is a unital Banach algebra and B is a closed subalgebra of A. If $x \in B$ is such that $\sigma_1(x, B)$ is finite, then $\sigma_1'(x, B) = \sigma'(x, A)$.*

Definition 2.4.7. *([2], p.36) Let A be a unital Banach algebra. The spectral radius of an element $x \in A$, denoted by $r(x, A)$, is defined as*

$$r(x, A) = \sup\{|\lambda| : \lambda \in \sigma(x, A)\}.$$

It suffices to write $r(x)$ if the Banach algebra being discussed is clear from the context.

Take note that, if $T : A \to B$ is a Banach algebra homomorphism, then $\sigma(Tx) \subseteq \sigma(x)$ since the inclusion $T(A^{-1}) \subseteq B^{-1}$ holds, and therefore $r(Tx) \leq r(x)$.

The following representation

$$r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$$

of the spectral radius for a unital Banach algebra element x, will be useful throughout this dissertation. This identity implies that $r(x) \leq \|x\|$.

We now make mention that the spectral radius is subadditive and submultiplicative on commuting elements of a unital Banach algebra.

Proposition 2.4.8. *([2], Corollary 3.2.10) Let A be a unital Banach algebra and $x, y \in A$ be such that $xy = yx$. Then $\sigma(x+y) \subseteq \sigma(x)+\sigma(y)$ and $\sigma(xy) \subseteq \sigma(x)\sigma(y)$. Hence, $r(x + y) \leq r(x) + r(y)$ and $r(xy) \leq r(x)r(y)$.*

An important concept in this dissertation is that of an inessential ideal.

Definition 2.4.9. ([2], *p.106) Let I be an ideal of a unital Banach algebra A. Then I is said to be an inessential ideal if for every $x \in I$ the spectrum $\sigma(x)$ of x has at most* 0 *as an accumulaton point, that is,* $\operatorname{acc} \sigma(x) \subseteq \{0\}$.

It is known that the ideal $K(X)$ of all compact operators on a Banach space X is an inessential ideal of $B(X)$.

Definition 2.4.10. *([12], p.432) A Banach algebra homomorphism $T : A \to B$*

is said to have the Riesz property if $\mathrm{N}(T)$ *is an inessential ideal of* A.

If I is a closed inessential ideal of a unital Banach algebra A, then the quotient map $T : A \to A/I$ defined by $Tx = x + I$ for all $x \in A$ is an example of a Banach algebra homomorphism with the Riesz property.

Next, we give some results on the holomorphic functional calculus that will be used throughout the dissertation.

Let $\Omega \subseteq \mathbb{C}$ be an open set. We denote by $H(\Omega)$ the algebra of all complex-valued functions defined and holomorphic on Ω.

Theorem 2.4.11. *([2], Theorem 3.3.3) Let* A *be a unital Banach algebra and* $x \in A$. *If* Ω *is an open set containing* $\sigma(x)$ *and* Γ *is a smooth contour in* Ω *surrounding* $\sigma(x)$, *then the function* $f \to f(x) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda e - x)^{-1}d\lambda$ *from* $H(\Omega)$ *into* A *has the following properties:*

(i) $(f_1 + f_2)(x) = f_1(x) + f_2(x)$;

(ii) $(f_1 f_2)(x) = f_1(x)f_2(x) = f_2(x)f_1(x)$;

(iii) $1(x) = e$ *and* $I(x) = x$, *where* 1 *and* I *are the unit and identity functions on* $\mathbb{C}$, *respectively;*

(iv) $\sigma(f(x)) = f(\sigma(x))$

for all $f_1, f_2, f \in H(\Omega)$.

In Theorem 2.4.11, property number (iv) is called the *spectral mapping theorem* for holomorphic functions.

The following result, which we state without proof, can be found in the Masters thesis of R. Heymann and will be useful in proving Theorem 2.7.3.

Proposition 2.4.12. *([14], Lemma 2.3.2, the proof of Proposition 2.3.1 together with Proposition 3.3) Let* $T : A \to B$ *be a Banach algebra homomorphism. Then*

$$f(\sigma(Tx)) = \sigma(T(f(x))) \qquad \text{for all } x \in A$$

and every function $f : U \to \mathbb{C}$ *which is holomorphic on a neighbourhood* U *of* $\sigma(x)$.
In addition, if f *is also non-constant on each component of* U, *then* $\mathrm{acc}\ \sigma(f(x)) = f(\mathrm{acc}\ \sigma(x))$.

Theorem 2.4.13. *([2], Theorem 3.3.4) Let* A *be a unital Banach algebra and suppose that* $x \in A$ *has a disconnected spectrum. Take any two disjoint open sets* U_0 *and* U_1 *such that* $\sigma(x) \subseteq U_0 \cup U_1$. *Define* $f \in H(U_0 \cup U_1)$ *by*

$$f(\lambda) = \begin{cases} 0 & \text{if } \lambda \in \mathrm{U}_0 \\ 1 & \text{if } \lambda \in \mathrm{U}_1. \end{cases}$$

Then $p := f(x)$ is an idempotent commuting with x, that is, $p^2 = p$ and $xp = px$. If $\sigma(x) \cap U_0 \neq \emptyset$ and $\sigma(x) \cap U_1 \neq \emptyset$, then p is a non-trivial idempotent and $\sigma(px) = (\sigma(x) \cap U_1) \cup \{0\}$ and $\sigma(x - px) = (\sigma(x) \cap U_0) \cup \{0\}$.

The idempotent p in Theorem 2.4.13 is called the *spectral idempotent of x corresponding to the set $\sigma(x) \cap U_1$.*
When $\sigma(x) \cap U_1 = \{\lambda_0\}$, that is $\lambda_0 \in \operatorname{iso} \sigma(x)$, then p is said to be the *spectral idempotent of x corresponding to λ_0* and is given by

$$\frac{1}{2\pi i} \int_{\Gamma} (\lambda e - x)^{-1} d\lambda,$$

where Γ is a circle centred at λ_0, separating λ_0 from the remaining spectrum of x. The spectral idempotent of x corresponding to $\lambda_0 \in \operatorname{iso} \sigma(x)$ will be denoted by $p(x, \lambda_0)$. Note that $p(x, \lambda_0) = 0$ if and only if $\lambda_0 \notin \sigma(x)$.

The following result will be needed in proving Theorem 2.6.3.

Lemma 2.4.14. *([5], Lemma 2.0.6) Let $T : A \to B$ be a Banach algebra homomorphism, $x \in A$ and $\lambda \in (\operatorname{iso} \sigma(x)) \backslash \sigma(Tx)$. Then $p(x, \lambda) \in \mathrm{N}(T)$.*

Let $R(\lambda, x) = \sum\limits_{n=-\infty}^{\infty} (\lambda - \lambda_0)^n x_n$ be the Laurent expansion of the resolvent $(\lambda e - x)^{-1}$ of x in a deleted neighbourhood of $\lambda_0 \in \operatorname{iso} \sigma(x)$, where x is an element of a unital Banach algebra A. In the above Laurent series expansion, we will always refer to the coefficient x_n ($n \in \mathbb{Z}$) of $(\lambda - \lambda_0)^n$. In particular, we shall refer more to the coefficient x_{-1}, which is known to be equal to the spectral idempotent of x corresponding to $\lambda_0 \in \operatorname{iso} \sigma(x)$.

The following lemma, which will be used to prove Corollary 5.3.3, gives one property that the coefficients in the Laurent expansion of $(\lambda e - x)^{-1}$ have.

Lemma 2.4.15. *([3], Corollary 2.2.5) Let A be a unital Banach algebra, $x \in A$ and $\lambda_0 \in \operatorname{iso} \sigma(x)$. If $k > 1$, then $x_{-k} \in \mathrm{QN}(A)$.*

Proof. Let U_1 and U_0 be disjoint open sets such that U_1 contains $\{\lambda_0\}$ and U_0 contains $\sigma(x) \backslash \{\lambda_0\}$. Choose Γ_1 to be a small circle in U_1 surrounding $\{\lambda_0\}$ and let Γ_2 be a smooth contour in U_0 surrounding $\sigma(x) \backslash \{\lambda_0\}$. Define, for $j > 1$, the function $f_j : U_0 \cup U_1 \to \mathbb{C}$ by

$$f_j(\lambda) = \begin{cases} 0 & \text{if } \lambda \in \mathrm{U}_0 \\ (\lambda - \lambda_0)^{j-1} & \text{if } \lambda \in \mathrm{U}_1. \end{cases}$$

Then $f_j \in H(U_0 \cup U_1)$ and by Theorem 2.4.11

$$\begin{aligned} f_j(x) &= \frac{1}{2\pi i}\int_{\Gamma_1\cup\Gamma_2} f_j(\lambda)(\lambda e - x)^{-1} d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_1} (\lambda-\lambda_0)^{j-1}(\lambda e - x)^{-1} d\lambda. \end{aligned}$$

The coefficients x_k are given by

$$x_k = \frac{1}{2\pi i}\int_{\Gamma_1} \frac{(\lambda e - x)^{-1}}{(\lambda-\lambda_0)^{k+1}} d\lambda.$$

With this, we therefore have that

$$x_{-k} = \frac{1}{2\pi i}\int_{\Gamma_1} (\lambda-\lambda_0)^{k-1}(\lambda e - x)^{-1} d\lambda = f_k(x)$$

for all $k > 1$. Using the spectral mapping theorem we have that

$$\sigma(x_{-k}) = \sigma(f_k(x)) = f_k(\sigma(x)) = \{f_k(\lambda) : \lambda \in \sigma(x)\} = \{0\},$$

that is, $x_{-k} \in \mathrm{QN}(A)$ for all $k > 1$. □

For ease of reference we point out the following result, whose proof can be found in [3] (Corollary 2.2.6).

Proposition 2.4.16. *([3], Corollary 2.2.6) Let A be a unital Banach algebra, $x \in A$ and $\lambda \in \operatorname{iso}\sigma(x)$. Then $\sigma(x \pm x_{-1}) = (\sigma(x)\backslash\{\lambda\}) \cup \{\lambda \pm 1\}$.*

Definition 2.4.17. *([17], p.150) Let I be an ideal of a unital Banach algebra A and $x \in A$. Then $\lambda \in \operatorname{iso}\sigma(x))$ is called a Riesz point of $\sigma(x)$ relative to I if $p(x, \lambda) \in I$.*

Let x be an element of a unital Banach algebra. A point $z \in \sigma(x)$ is called a *pole of order k* of $(\lambda e - x)^{-1}$, where $\lambda \notin \sigma(x)$, if $z \in \operatorname{iso}\sigma(x)$ and k is the smallest natural number such that $(ze - x)^k p(x, z) = 0$ ([21], p.307).

The following result tells us when a Riesz point of the spectrum will also be a pole.

Lemma 2.4.18. *([17], Lemma 2.1) Let A be a semisimple unital Banach algebra, I be an inesssential ideal of A and $x \in A$. Then λ is a Riesz point of $\sigma(x)$ relative to I if and only if λ is a pole of $(\lambda e - x)^{-1}$ and $p(x, \lambda) \in I$.*

We close this section with a brief introduction to one particular class of Banach algebras, called C^*-algebras.

Definition 2.4.19. *([10], p.238) Suppose that A is a Banach algebra. A mapping $x \mapsto x^*$ from A into itself is called an involution on A if it satisfies the following properties for all $x, y \in A$ and $\alpha \in \mathbb{C}$:*

(i) $(x^*)^* = x$

(ii) $(xy)^* = y^* x^*$

(iii) $(\alpha x + y)^* = \bar{\alpha} x^* + y^*$

Definition 2.4.20. *([10], Definition 1.1) A C^*-algebra is a Banach algebra A with an involution such that, for all $x \in A$, $||x^*|| = ||x||$ and $\|x^* x\| = \|x\|^2$.*

We point out two examples of C^*-algebras.

Example 2.4.21. *([10], Example 1.4) The Banach algebra $C(K)$ (as in Example 2.3.3) is a commutative C^*-algebra when equipped with the involution $f \mapsto f^*$, where $f^* : K \to \mathbb{C}$ is defined by $f^*(x) = \overline{f(x)}$ for all $f \in C(K)$ and $x \in K$.*

Example 2.4.22. *([10], Example 1.2) The Banach algebra $B(H)$ (as in Example 2.3.6), where H is a Hilbert space, is a non-commutative C^*-algebra when equipped with the involution $T \mapsto T^*$, where T^* is the Hilbert adjoint of T.*

The following theorem displays an example of a positive cone of a unital C^*-algebra.

Theorem 2.4.23. *([10], Proposition 3.7) Let A be a unital C^*-algebra and $C = \{x \in A\colon x = x^* \text{ and } \sigma(x) \subseteq [0, \infty)\}$. Then C is a closed subset of A and λx, $x + y \in C$ for all $x, y \in C$ and $\lambda \in \mathbb{R}^+$.*

We point out that a C^*-algebra element x satisfying $x = x^*$ is called *hermitian.*

Proposition 2.4.24. *([11], Proposition 12.8(b)) Let A be a unital C^*-algebra and $C = \{x \in A\colon x = x^* \text{ and } \sigma(x) \subseteq [0, \infty)\}$. If $x, y \in C$ and $y - x \in C$, then $\|x\| \leq \|y\|$.*

2.5. Commutatively ordered Banach algebras

In this section, commutatively ordered Banach algebras ($COBAs$) and their associated basic properties are discussed. We start with the following definition.

Definition 2.5.1. *([19], p.4) Let A be a unital Banach algebra. A positive cone C in A is called an algebra c-cone if it satisfies the following properties:*

(i) $xy \in C$ *for all* $x, y \in C$ *such that* $xy = yx$,

(ii) $e \in C$, *where e is the unity of A.*

We call a unital Banach algebra A which is ordered by an algebra c-cone C, that is, (for $x, y \in A$) $y \leq x$ if and only if $x - y \in C$, a *commutatively ordered Banach algebra* ($COBA$) and will denote it by (A, C). If the commutativity-condition $xy = yx$ in (i) is removed, then C is called an *algebra cone.* We remark that a unital Banach algebra which is ordered by an algebra cone C is called an *ordered Banach algebra* (OBA). Every algebra cone is an algebra c-cone. Therefore, every OBA is a $COBA$. It is worth noting that a non-commutative C^*-algebra is not an OBA but is a $COBA$. As a result, the two structures are significantly different.

Next, we draw attention to some basic properties in the context of $COBAs$ that will be applicable to the rest of the dissertation. These are $COBA$ adaptations of the basic properties of $OBAs$ developed by Mouton and Raubenheimer (in [21] and [23]), for which several aspects of the spectral theory in $OBAs$ were generalized to $COBAs$ in the PhD Thesis of K. Muzundu in [22]. In fact, it is this generalization that has motivated us to extend the results of the upper Browder spectrum from $OBAs$ to $COBAs$.

The following gives the definition of normality in the $COBA$ setting.

Definition 2.5.2. *([19], p.4) Let (A, C) be a $COBA$. We say that C is c-normal if there exists a constant $\alpha > 0$ such that for all $x, y \in A$ satisfying $xy = yx$, $0 \leq x \leq y$ implies that $\|x\| \leq \alpha\|y\|$.*

Next is the definition of the monotonicity property of the spectral radius function for $COBAs$.

Definition 2.5.3. *([19], p.5) Let (A, C) be a $COBA$. We say that the spectral radius function in (A, C) is c-monotone if for all $x, y \in A$ satisfying $xy = yx$, we have that $0 \leq x \leq y$ implies that $r(x) \leq r(y)$.*

Proposition 2.5.4. *([22], Proposition 2.1.6) Every c-normal algebra c-cone C in a unital Banach algebra A is proper.*

Proof. Let $x \in C \cap -C$. Then $0 \leq x \leq 0$ and since $x0 = 0x$, we have, by the c-normality of C, that $0 \leq \|x\| \leq \alpha\|0\|$ for some $\alpha > 0$. This implies that $x = 0$, and so $C \cap -C = \{0\}$. Hence C is proper. $\square$

We state and prove the following proposition that will be used to establish Theorem 2.5.6.

Proposition 2.5.5. *([19], Proposition 3.12) Let (A, C) be a $COBA$ and $x, y \in A$ be such that $xy = yx$. If $0 \leq x \leq y$, then $0 \leq x^m \leq y^m$ for all $m \in \mathbb{N}$.*

Proof. We prove this result using mathematical induction. The inequality clearly holds for $m = 1$. Suppose now that the inequality $0 \leq x^m \leq y^m$ is true for some $m > 1$. We need to show that it also holds for $m + 1$. Now, since $xy = yx$ and $0 \leq x \leq y$, we have (by the induction hypothesis) that $0 \leq x^{m+1} = xx^m \leq xy^m \leq yy^m = y^{m+1}$, so that $0 \leq x^{m+1} \leq y^{m+1}$. □

It is well-known that, if C is a normal algebra cone of a unital Banach algebra A, then the spectral radius function in (A, C) is monotone ([23], Theorem 4.1). The following theorem gives a similar result in the $COBA$ setting.

Theorem 2.5.6. *([19], Theorem 4.2) Let (A,C) be a $COBA$ with c-normal algebra c-cone C. Then the spectral radius function in (A, C) is c-monotone.*

Proof. Let $x, y \in A$ be commuting elements such that $0 \leq x \leq y$. By Proposition 2.5.5 we have that $0 \leq x^n \leq y^n$ for all $n \in \mathbb{N}$. Since $x^n y^n = y^n x^n$ and C is c-normal, there exit some $\alpha > 0$ such that $\|x^n\| \leq \alpha \|y^n\|$, so that $r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} \leq \lim_{n\to\infty} \alpha^{\frac{1}{n}} \|y^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|y^n\|^{\frac{1}{n}} = r(y)$. Hence, the spectral radius function in (A, C) is c-monotone. □

H. Raubenheimer and S. Rode proved that if the spectral radius function in an OBA is monotone, then the spectral radius of a positive element in the OBA belongs to the spectrum of that element ([23], Theorem 5.2). We state this result without proof in the $COBA$ setting, a generalization due to Mouton and Muzundu in [19].

Theorem 2.5.7. *([19], Theorem 4.6) Let (A, C) be a $COBA$ with closed algebra c-cone C such that the spectral radius function in (A, C) is c-monotone. If $x \in C$, then $r(x) \in \sigma(x)$.*

If (A, C) is a $COBA$, we say that the algebra c-cone C is *inverse-closed* in A if for every invertible element $x \in C$ we have that $x^{-1} \in C$.

The following result is a $COBA$ counterpart of [8] (Proposition 4.2), a generalization due to Mouton and Muzundu in [22]. We omit its proof, which can be found in [22] (Proposition 3.3.8), as it is the same as that for $OBAs$. This is because every $COBA$ has the property that all powers x^k of a positive element x are positive.

Proposition 2.5.8. *([22], Proposition 3.3.8) Let (A, C) be a $COBA$ with closed and inverse-closed algebra c-cone C. If $x \in C$, then $0 \leq x \leq r(x)e$.*

The following result, which is an immediate consequence of Proposition 2.5.8, is a $COBA$ version of [3] (Corollary 1.8.8) and will be used to prove Corollary 5.3.3. Note that the proof given here is the same as that which Benjamin offered in [3]

(Corollary 1.8.8).

Proposition 2.5.9. *Let (A, C) be a COBA with proper, closed and inverse-closed algebra c-cone C. If $x \in C \cap \mathrm{QN}(A)$, then $x = 0$.*

Proof. Let $x \in C \cap \mathrm{QN}(A)$. From Proposition 2.5.8 we have that $0 \leq x \leq r(x)e = 0$, and hence $x \in C \cap -C$. Because A is proper, it follows that $x = 0$. □

The following proposition is a *COBA* version of [21] (Theorem 3.2) and will be useful in the sequel. The proof is attributed to Mouton and Muzundu ([22], Theorem 3.1.2) and is the same as that in the *OBA* case, since all powers x^j of a positive *COBA* element x are positive.

Proposition 2.5.10. *Let (A, C) be a COBA with closed algebra c-cone C and suppose that $0 \neq x \in C$. If $r(x)$ is a pole of order k of $(\lambda e - x)^{-1}$, so that*

$$(\lambda e - x)^{-1} = \sum_{j=-k}^{\infty} (\lambda - r(x))^j x_j,$$

then there exists $0 \neq u := x_{-k} \in C$ such that $xu = ux = r(x)u$.

Proof. Since $r(x)$ is a pole of order k of $(\lambda e - x)^{-1}$, we have that

$$R(\lambda, x) = \frac{x_{-k}}{(\lambda - r(x))^k} + \frac{x_{-k+1}}{(\lambda - r(x))^{k-1}} + \cdots + x_0 + (\lambda - r(x))x_1 + \cdots$$

in a deleted neighbourhood of $r(x)$. Multiplying both sides of the Laurent series expansion by $(\lambda - r(x))^k$ and then taking the limit as $\lambda \to r(x)^+$, we have that

$$x_{-k} = \lim_{\lambda \to r(x)^+} (\lambda - r(x))^k R(\lambda, x).$$

From the Neumann series $R(\lambda, x) = \sum_{j=0}^{\infty} \frac{x^j}{\lambda^{j+1}}$ for $R(\lambda, x)$ $(\lambda > ||x|| \geq r(x))$ and the fact that all powers x^j are in C, it follows that $R(\lambda, x) \in C$, since C is a closed algebra c-cone and $x \in C$. Hence $x_{-k} \in C$. The result follows by taking $u = x_{-k}$. □

We conclude this section with some examples of *COBAs*.

Example 2.5.11. *The OBS $(\mathbb{C}^n, C)$, where*

$$C = \{(x_1, \ldots, x_n) \in \mathbb{C}^n : x_i \in \mathbb{R}^+ \text{ for all } i \in \{1, \ldots, n\} \text{ and } n \geq 1\},$$

is a COBA with closed, inverse-closed, c-normal and generating algebra c-cone.

Example 2.5.12. *The OBS $(C(K), C)$, where*

$$C = \{f \in C(K) : f(x) \in \mathbb{R}^+ \text{ for all } x \in K\},$$

is a COBA with closed, inverse-closed, c-normal and generating algebra c-cone.

Example 2.5.13. *The OBS* (ℓ^∞, C), *where*

$$C = \{(x_1, x_2, \dots) \in \ell^\infty : x_n \in \mathbb{R}^+ \text{ for all } n \in \mathbb{N}\},$$

is a COBA with closed, inverse-closed, c-normal and generating algebra c-cone.

Proposition 2.5.14. *The OBS* $(M_n(\mathbb{C}), M_n(\mathbb{R}^+))$ *is a COBA with closed and c-normal algebra c-cone.*

We also point out the following examples.

Proposition 2.5.15. *([19], Example 3.3) (B(H),C), where*

$$C = \{T \in B(H) : T \geq 0\}$$

is the set of all positive operators on a Hilbert space H, is a COBA with closed and c-normal algebra c-cone.

Proof. In view of Theorem 2.1.10 we have that C is an algebra c-cone of the Banach algebra $B(H)$, which was discussed in Example 2.3.6. We now show that C is closed, so let (T_n) be a sequence in C that converges to, say, T. Since $T_n \in \mathbb{R}^+$, we have that $\langle T_n x, x\rangle \in \mathbb{R}^+$ for all $n \in \mathbb{N}$ and $x \in H$. Now, by the continuity of the inner product, it follows that $\langle T_n x, x\rangle \to \langle Tx, x\rangle$ as $n \to \infty$, so that $\langle Tx, x\rangle \in \mathbb{R}^+$. We therefore have that $T \in C$, and hence C is closed.
Lastly, let $T_1, T_2 \in B(H)$ be such that $0 \leq T_1 \leq T_2$ and $T_1T_2 = T_2T_1$. Then for every $x \in H$, we have that $\langle T_1 x, x\rangle \leq \langle T_2 x, x\rangle$. From Proposition 2.1.11 it follows that

$$\|T_1\| = \sup\{|\langle T_1 x, x\rangle| : \|x\| = 1\} \leq \sup\{|\langle T_2 x, x\rangle| : \|x\| = 1\} = \|T_2\|,$$

and hence C is c-normal. □

Or more generally:

Example 2.5.16. *([19], Example 3.2)* (A, C), *where A is a* C^**-algebra and*

$$C = \{x \in A : x = x^* \text{ and } \sigma(x) \subseteq [0, \infty)\},$$

is a COBA with closed and c-normal algebra c-cone.

Proof. From Theorem 2.4.23 we have that C is a closed positive cone in A. Further, it is obvious that $e \in C$. We show next that C is closed under multiplication of commuting elements, so let $x, y \in C$ be such that $xy = yx$. Then

$(xy)^* = y^*x^* = yx = xy$ and, in view of Proposition 2.4.8, $\sigma(xy) \subseteq \sigma(x)\sigma(y)$, so that $\sigma(xy) \subseteq [0, \infty)$. This gives $xy \in C$, and hence C is an algebra c-cone of A. Finally, the c-normality of C follows from Proposition 2.4.24. □

It is worth noting that the two examples we have discussed above are examples of $COBAs$ that are not $OBAs$.

It is known that the direct sum of ordered Banach algebras is again an ordered Banach algebra. Below we point out the situation for commutatively ordered Banach algebras, which is easy to prove.

Example 2.5.17. *Let (A_i, C_i) be COBAs for each $i = 1, \ldots, n$, where $n \in \mathbb{N}$, and let $A = A_1 \oplus \cdots \oplus A_n$ be the direct sum of $A_1, A_2, \ldots, A_n$. Then (A, C), where*

$$C = \{(x_1, \ldots, x_n) \in A : x_i \in C_i \text{ for all } i \in \{1, \ldots, n\}\},$$

is a COBA. If C_i is closed (inverse-closed, c-normal, generating) in A_i for all $i \in \{1, \ldots, n\}$, then C is closed (inverse-closed, c-normal, generating) in A.

2.6. Fredholm theory in unital Banach algebras

This section provides a brief review on the basic concepts and results from Fredholm theory in general unital Banach algebras. This theory was introduced by Harte in [12] for a bounded Banach algebra homomorphism and extended by Mouton and Raubenheimer to general Banach algebra homomorphisms in [20].

We start with the following definition.

Definition 2.6.1. *([12], pp.431-432) Let $T : A \to B$ be a Banach algebra homomorphism. Relative to T, an element $x \in A$ is called*

(i) Fredholm if $Tx \in B^{-1}$,

(ii) Browder if there exist commuting elements $y \in A^{-1}$ and $z \in \mathrm{N}(T)$ such that $x = y + z$.

The symbols $\mathcal{F}_T$ and $\mathcal{B}_T$ will denote the sets of Fredholm and Browder elements of A (relative to T), respectively. Clearly, $A^{-1} \subseteq \mathcal{B}_T$ and $A^{-1} \subseteq \mathcal{F}_T$.

The following example, whose proof can be found in [14] (p.8) gives the Fredholm and Browder elements in $C(K)$.

Example 2.6.2. *([12], p.432) Consider the unital Banach algebras of continuous complex-valued functions $A = C(X)$ and $B = C(Y)$ on compact Hausdorff spaces X and Y, respectively. Let $T : A \to B$ be the Banach algebra homomorphism* *defined by*

$$Tf = f \circ \theta \text{ for all } f \in A,$$

where $\theta : Y \to X$ *is a continuous map. Then*

(i) $f \in \mathcal{F}_T$ *if and only if* $\theta(Y) \cap \mathrm{N}(f) = \emptyset$,

(ii) $f \in \mathcal{B}_T$ *if and only if its restriction to* $\theta(Y)$ *has an invertible extension to* X.

Relative to a Banach algebra homomorphism $T : A \to B$, we shall call an element $x \in A$ *almost invertible Fredholm* if it is almost invertible and Fredholm. We will use the notation $A^D \cap \mathcal{F}_T$ for the set of all almost invertible Fredholm elements.

Theorem 2.6.3. *([12], (1.4) and Theorem 1; [20], Corollary 2.5) Let* $T : A \to B$ *be a Banach algebra homomorphism. Then the following inclusions hold:*

$$A^{-1} \subseteq A^D \cap \mathcal{F}_T \subseteq \mathcal{B}_T \subseteq \mathcal{F}_T.$$

In addition, if T *has the Riesz property, then* $A^D \cap \mathcal{F}_T = \mathcal{B}_T$.

Proof. If $x \in A^{-1}$, then $0 \notin \sigma(x)$ by definition, so that $0 \notin \operatorname{acc} \sigma(x)$ since $\sigma(x)$ is closed and $0 \notin \sigma(Tx)$. This gives $x \in A^D \cap \mathcal{F}_T$, proving the first inclusion. For the second inclusion, suppose that $x \in A^D \cap \mathcal{F}_T$ and let $p := p(x, 0)$. If $x \in A^{-1}$, then $x \in \mathcal{B}_T$, and the proof is done. Hence, suppose that $x \notin A^{-1}$. Then $0 \in (\operatorname{iso} \sigma(x)) \backslash \sigma(Tx)$. Now let $y = p + x(e - p)$ and $z = (x - e)p$. Then $x = y + z$ and $yz = zy$ follows from the fact that $xp = px$. Using the holomorphic functional calculus, we show next that $y \in A^{-1}$ and $z \in \mathrm{N}(T)$. So, let U_0 and U_1 be disjoint open sets such that U_0 contains $\sigma(x)\backslash\{0\}$ and U_1 contains $\{0\}$. Also, let Γ_0 be a smooth contour in U_0 surrounding $\sigma(x)\backslash\{0\}$ and Γ_1 be a small circle in U_1 surrounding 0. We define the function $f : U_0 \cup U_1 \to \mathbb{C}$ by

$$f(\lambda) = \begin{cases} 0 & \text{if } \lambda \in U_0, \\ 1 & \text{if } \lambda \in U_1. \end{cases}$$

It then follows from Theorem 2.4.13 that $f(x) = p$. Now let $g : U_0 \cup U_1 \to \mathbb{C}$ be the function defined by $g(\lambda) = f(\lambda) + \lambda(1 - f(\lambda))$ for all $\lambda \in U_0 \cup U_1$. Since f is holomorphic on $U_0 \cup U_1$, g is also holomorphic on $U_0 \cup U_1$. Thus $g(x) \in A$ and

$$g(x) = f(x) + x(e - f(x)) = p + x(e - p) = y,$$

so that

$$\sigma(y) = \sigma(g(x)) = g(\sigma(x)) = \{g(\lambda) : \lambda \in \sigma(x)\} = \{f(\lambda) + \lambda(1 - f(\lambda)) : \lambda \in \sigma(x)\}$$
$$= (\sigma(x)\backslash\{0\}) \cup \{1\}$$

using the spectral mapping theorem. This tells us that $0 \notin \sigma(y)$, and hence $y \in A^{-1}$. By Lemma 2.4.14, $p \in \mathrm{N}(T)$, so that $z = (x - e)p \in \mathrm{N}(T)$, and hence $x = y + z \in \mathcal{B}_T$.

We now prove the last inclusion. Suppose that $x \in \mathcal{B}_T$. Then there exist commuting elements $y \in A^{-1}$ and $z \in \mathrm{N}(T)$ such that $x = y + z$. By the linearity of T and the inclusion $T(A^{-1}) \subseteq B^{-1}$, we have that

$$Tx = T(y + z) = Ty + Tz = Ty \in B^{-1},$$

and hence $x \in \mathcal{F}_T$.

Suppose now that T has the Riesz property. We prove that $\mathcal{B}_T \subseteq A^D \cap \mathcal{F}_T$, so let $x \in \mathcal{B}_T$. Then $x = y + z$ for some commuting elements $y \in A^{-1}$ and $z \in \mathrm{N}(T)$. It suffices to show that $x \in A^D$. Now

$$y^{-1}x = y^{-1}(y + z) = y^{-1}y + y^{-1}z = e + y^{-1}z,$$

with $y^{-1}z \in \mathrm{N}(T)$, and hence $\sigma(y^{-1}x) = 1 + \sigma(y^{-1}z)$ follows from the fact that $\sigma(e) = 1$. Since T has the Riesz property, $\operatorname{acc} \sigma(y^{-1}z) \subseteq \{0\}$, and therefore $\operatorname{acc} \sigma(y^{-1}x) \subseteq \{1\}$, which gives $0 \notin \operatorname{acc} \sigma(y^{-1}x)$.

Also, since the identity $yz = zy$ implies that x commutes with y, we have that y commutes with $y^{-1}x$. Utilizing Proposition 2.4.8 it now follows that

$$\sigma(x) = \sigma(y(y^{-1}x)) \subseteq \sigma(y)\sigma(y^{-1}x).$$

Because $0 \notin \operatorname{acc} \sigma(y)$ and $0 \notin \operatorname{acc} \sigma(y^{-1}x)$, we get that $0 \notin \operatorname{acc} \sigma(x)$, that is $x \in A^D$, and hence the inclusion $\mathcal{B}_T \subseteq A^D \cap \mathcal{F}_T$ follows. □

Remark 2.6.4. *From the proof of Theorem 2.6.3 we have that, if* $x \in A^D \cap \mathcal{F}_T$ *and* $p = p(x, 0)$, *then* $x = y + z$, *where* $y = p + x(e - p) \in A^{-1}$ *and* $z = (x - e)p \in \mathrm{N}(T)$ *commute.*

Remark 2.6.5. *If* $T : A \to B$ *is a Banach algebra homomorphism with the Riesz property, so that* $\mathcal{B}_T = A^D \cap \mathcal{F}_T$, *then (using Proposition 2.4.8) it is not difficult to show that commuting Browder elements are again Browder.*

Next we introduce and discuss the Fredholm, Browder and almost invertible Fredholm spectra of a unital Banach algebra element.

Definition 2.6.6. *([12], pp.433-434) Let $T : A \to B$ be a Banach algebra homomorphism and $x \in A$.*

(i) The Fredholm spectrum of x is the set

$$\{\lambda \in \mathbb{C} : \lambda e - x \notin \mathcal{F}_T\}, \text{ which coincides with } \sigma(Tx).$$

(ii) The Browder spectrum of x is the set

$$\beta_T(x) := \{\lambda \in \mathbb{C} : \lambda e - x \notin \mathcal{B}_T\}.$$

(iii) The almost invertible Fredholm spectrum of x is the set

$$\{\lambda \in \mathbb{C} : \lambda e - x \notin A^D \cap \mathcal{F}_T\}, \text{ which coincides with } \sigma(Tx) \cup \text{ acc } \sigma(x).$$

The following result expresses the Browder spectrum in terms of the (ordinary) spectra of unital Banach algebra elements. The proof of this result is fairly straightforward, and we shall therefore omit it.

Proposition 2.6.7. *([12], p.434, (2.2)) Let $T : A \to B$ be a Banach algebra homomorphism. For any element $x \in A$, we have that*

$$\beta_T(x) = \bigcap_{\substack{y \in \mathrm{N}(T) \\ xy = yx}} \sigma(x + y).$$

Since the Fredholm, almost invertible Fredholm and Browder spectra can be represented in terms of the ordinary spectrum, which is non-empty and compact, these new spectra are all non-empty, closed and bounded subsets of the complex plane.

The following proposition is an immediate consequence of Theorem 2.6.3.

Proposition 2.6.8. *([12], p.434, (2.3)) Let $T : A \to B$ be a Banach algebra homomorphism. For any element $x \in A$, the following inclusions hold:*

$$\sigma(Tx) \subseteq \beta_T(x) \subseteq \sigma(Tx) \cup \text{acc } \sigma(x) \subseteq \sigma(x).$$

In addition, if T has the Riesz property, then $\sigma(Tx) \cup \text{acc } \sigma(x) = \beta_T(x)$.

2.7. A spectral mapping theorem for the Browder spectrum

In this section we present sufficient conditions under which the Browder spectrum satisfies a spectral mapping theorem.

We first recall the following results from complex analysis.

Proposition 2.7.1. *([9], Theorem 3.7, p.78) Let U be a connected open set and $f : U \to \mathbb{C}$ be a holomorphic function on U. Then the following two statements are equivalent.*

(i) f is identically 0;

(ii) $\{z \in U : f(z) = 0\}$ has an accumulation point in U.

Theorem 2.7.2. *([9], Corollary 3.9, p.79) Suppose that f is holomorphic on an open connected set U and that f is not identically zero. Then for each $x \in U$ with $f(x) = 0$, there is a positive integer n and a holomorphic function $g : U \to \mathbb{C}$ such that $g(x) \neq 0$ and $f(z) = (z - x)^n g(z)$ for all $z \in U$.*

The following theorem gives a spectral mapping theorem for the Browder spectrum of a unital Banach algebra element.

Theorem 2.7.3. *([12], Theorem 2) Let $T : A \to B$ be a Banach algebra homomorphism with the Riesz property. Then*

$$\beta_T(f(x)) = f(\beta_T(x))$$

for all $x \in A$ and every function $f : U \to \mathbb{C}$ which is holomorphic on a neighbourhood U of $\sigma(x)$ and non-constant on each component of U.

Proof. We first prove the inclusion $\beta_T(f(x)) \subseteq f(\beta_T(x))$. (We remark that, this inclusion is known to hold for an arbitrary Banach algebra homomorphism, but we give here a proof under the additional assumption that T has the Riesz property.)
Using the spectral mapping theorem and Proposition 2.6.8, we have that $\beta_T(f(x)) \subseteq \sigma(f(x)) = f(\sigma(x))$. Hence, if $t \notin f(\sigma(x))$, then $t \notin \beta_T(f(x))$ and we are done. Therefore, suppose that $t \in f(\sigma(x)) \backslash f(\beta_T(x))$.
We show that $t \notin \beta_T(f(x))$. Define $h : U \to \mathbb{C}$ by $h(z) = t - f(z)$ for all $z \in U$. Then $h \in H(U)$ and h is not identically zero on each component of U. Since $t \in f(\sigma(x))$, h has at least one zero in $\sigma(x)$. From Proposition 2.7.1 we have that $\{z \in U : h(z) = 0\}$ do not have an accumulation point in C, for each component C of U.
Now suppose that h has infinitely many zeros in $\sigma(x) \cap C$. Since $\sigma(x) \cap C$ is bounded, the infinite set

$$K = \{z \in \sigma(x) \cap C : h(z) = 0\}$$

is bounded, and hence contains a convergent sequence from the Bolzano-Weierstrass theorem for complex sequences. Since $\sigma(x) \cap C$ is closed, we conclude that it has an accumulation point in $\sigma(x) \cap C$. But this is a contradiction; hence h can only have a finite number of zeros in $\sigma(x) \cap C$, so that the zeros of h does not have an accumulation point in $\sigma(x)$. Using the Bolzana-Weierstrass theorem again, we get that h has only a finite number of zeros in $\sigma(x)$. Let

$$\{s_1, s_2, \dots, s_n\} = \{z \in \sigma(x) : h(z) = 0\} = \{z \in \sigma(x) : f(z) = t\}.$$

By utilizing Theorem 2.7.2 repeatedly, we can write

$$t - f(z) = h(z) = (s_1 - z)^{m_1}(s_2 - z)^{m_2} \dots (s_n - z)^{m_n} g(z),$$

where $m_1, m_2, \dots, m_n \in \mathbb{N}$ and $g \in H(U)$ satisfies $0 \notin g(\sigma(x))$. Hence by the holomorphic functional calculus, we have that

$$te - f(x) = (s_1 e - x)^{m_1}(s_2 e - x)^{m_2} \dots (s_n e - x)^{m_n} g(x), \tag{2.7.4}$$

where $g(x) \in A^{-1}$. Now since $t \notin f(\beta_T(x))$, we have that $\{s_1, s_2, \dots, s_n\} \cap \beta_T(x) = \emptyset$, and hence, for all $j \in \{1, \dots, n\}$, $s_j e - x \in \mathcal{B}_T$, so that $(s_j e - x)^{m_j} \in \mathcal{B}_T$ in view of Remark 2.6.5. Also note that $g(x) \in A^{-1} \subseteq \mathcal{B}_T$ commutes with each element in the product in (2.7.4). It now follows from the fact that $te - f(x)$ is a product of commuting elements in $\mathcal{B}_T$ and Remark 2.6.5 that $te - f(x) \in \mathcal{B}_T$. This gives $t \notin \beta_T(f(x))$, and hence the inclusion $\beta_T(f(x)) \subseteq f(\beta_T(x))$ is true.
We now show that the inclusion $f(\beta_T(x)) \subseteq \beta_T(f(x))$ also holds. From Theorem 2.6.8 we have that $\operatorname{acc} \sigma(x) \cup \sigma(Tx) = \beta_T(x)$, and hence

$$\begin{aligned} f(\beta_T(x)) &= f(\operatorname{acc} \sigma(x) \cup \sigma(Tx)) \\ &\subseteq f(\operatorname{acc} \sigma(x)) \cup f(\sigma(Tx)) \\ &= \operatorname{acc} \sigma(f(x)) \cup \sigma(T(f(x))) \\ &= \beta_T(f(x)), \end{aligned}$$

where the second last identity follows from Proposition 2.4.12 and the last identity follows from Theorem 2.6.8. This completes the proof. □

2.8. Connected hulls of Fredholm and Browder spectra

Mouton, Mouton and Raubenheimer proved in [18] that, if T is a Banach algebra homomorphism with closed range satisfying the Riesz property, then the connected hulls of the Fredholm and Browder spectra (relative to T) coincide. In this section, we point out a stronger version of their result: the connected hulls of the aforementioned spectra coincide relative to Banach algebra homomorphisms with the strong Riesz property.

We start with the following definition.

Definition 2.8.1. *([13], p.201) Let K be a compact subset of $\mathbb{C}$. The connected hull of K, denoted by ηK, is the union of K with the bounded components of the complement of K.*

Not that, if K is any compact set, then its complement has a unique unbounded component. It is the complement of this unbounded component that gives ηK.

We state the following result that will be useful in the proof of Proposition 5.2.3.

Proposition 2.8.2. *([10], Theorem 5.4(b), p.211) Suppose that B is a closed subalgebra of a unital Banach algebra A such that $e_A \in B$. If $x \in B$, then $\eta\sigma(x, A) = \eta\sigma(x, B)$.*
If, in addition, A is finite-dimensional, then $\sigma(x, A) = \sigma(x, B)$ for all $x \in B$.

Denote by ∂K and $\text{int}(K)$ the topological boundary and interior of a subset K of the complex plane, respectively. We prove the following lemma that is required in the proof of Theorem 2.8.5. This result connects the connected hulls and topological boundaries of compacts sets.

Lemma 2.8.3. *([13], Theorem 1.2) If K_1 and K_2 are compact subsets of $\mathbb{C}$, then $\partial K_1 \subseteq K_2 \subseteq K_1$ implies that $K_1 \subseteq \eta K_2$.*

Proof. Suppose that $\partial K_1 \subseteq K_2 \subseteq K_1$ and let $H := \mathbb{C}\backslash\eta K_2$ be the unbounded component of $\mathbb{C}\backslash K_2$. By definition of the connected hull and the assumption that $\partial K_1 \subseteq K_2$, we have that $H \subseteq \mathbb{C}\backslash K_2 \subseteq \mathbb{C}\backslash\partial K_1$, implying that $H \cap \partial K_1 = 0$. Now, because K_1 is closed and bounded, we have that

$$\begin{aligned}
H = H \cap \mathbb{C} = H \cap (K_1 \cup \mathbb{C}\backslash K_1)& \\
&= (H \cap K_1) \cup (H \cap \mathbb{C}\backslash K_1) \\
&= (H \cap (\text{int}(K_1) \cup \partial K_1)) \cup (H \cap \mathbb{C}\backslash K_1) \\
&= (H \cap \text{int}(K_1)) \cup (H \cap \partial K_1) \cup (H \cap \mathbb{C}\backslash K_1) \\
&= (H \cap \text{int}(K_1)) \cup (H \cap \mathbb{C}\backslash K_1).
\end{aligned}$$

This shows that H is the union of two disjoint open sets, which is a contradiction since H is not made up of separate pieces. Hence, either $H \cap \text{int}(K_1) = \emptyset$ or $H \cap \mathbb{C}\backslash K_1 = \emptyset$ since $(H \cap \text{int}(K_1)) \cap (H \cap \mathbb{C}\backslash K_1) = \emptyset$. Now suppose that $H \cap \mathbb{C}\backslash K_1 = \emptyset$. Then

$$\begin{aligned}
H \cap \mathbb{C}\backslash K_1 = (\mathbb{C}\backslash\eta K_2) \cap (\mathbb{C}\backslash K_1) = \emptyset &\Leftrightarrow \mathbb{C}\backslash(K_1 \cup \eta K_2) = \emptyset \\
&\Leftrightarrow \mathbb{C} = K_1 \cup \eta K_2.
\end{aligned}$$

From the fact that $\mathbb{C}$ is unbounded and $K_1 \cup \eta K_2$ is a bounded subset of $\mathbb{C}$ (both K_1 and ηK_2 are bounded sets); the identity $H \cap \mathbb{C}\backslash K_1 = \emptyset$ is not possible. We therefore conclude that the identity $H \cap \text{int}(K_1) = \emptyset$ holds. To be more specific,

$$H \cap K_1 = (H \cap (\text{int}(K_1) \cup \partial K_1)) = (H \cap \text{int}(K_1)) \cup (H \cap \partial K_1) = \emptyset.$$

Now $H \cap K_1 = \emptyset$ implies that $K_1 \subseteq \mathbb{C}\backslash H$. But since $\mathbb{C}\backslash H = \eta K_2$, we therefore have that $K_1 \subseteq \eta K_2$. □

We recall that a Banach algebra homomorphism T is said to have the Riesz property if $\mathrm{N}(T)$ is an inessential ideal of the domain of T. Next, we give the definition of the strong Riesz property of T.

Definition 2.8.4. *([12], (3.3)) Let $T : A \to B$ be a Banach algebra homomorphism. Then T is said to have the strong Riesz property if for every $x \in A$, $\partial\sigma(x) \subseteq \sigma(Tx) \cup \mathrm{iso}\ \sigma(x)$.*

Evidently, any Banach algebra homomorphism with the strong Riesz property has the Riesz property. Furthermore, it was shown in [18] (Corollary 7.9) that, if a Banach algebra homomorphism T has closed range and the Riesz property, then T has the strong Riesz property.

The next theorem alludes to the fact that the connected hulls of the Browder and Fredholm spectra coincide whenever the relevant Banach algebra homomorphism has the strong Riesz property.

Theorem 2.8.5. *([24], Corollary 2.2) Let $T : A \to B$ be a Banach algebra homomorphism with the strong Riesz property. Then $\eta\sigma(Tx) = \eta\beta_T(x)$ for all $x \in A$.*

Proof. In view of Lemma 2.8.3 we only have to prove the inclusion $\partial\beta_T(x) \subseteq \sigma(Tx)$. Now, since T has the Riesz property, we have from Proposition 2.6.8 that $\beta_T(x) = \sigma(Tx) \cup \mathrm{acc}\ \sigma(x) \subseteq \sigma(x)$. It then follows from this identity that $\mathrm{int}(\sigma(x)) = \mathrm{int}(\beta_T(x))$, and hence

$$\partial\beta_T(x) = \beta_T(x)\backslash\mathrm{int}(\beta_T(x)) \subseteq \sigma(x)\backslash\mathrm{int}(\beta_T(x)) = \sigma(x)\backslash\mathrm{int}(\sigma(x)) = \partial\sigma(x).$$

The strong Riesz property (and hence Riesz property) then gives

$$\begin{aligned}\partial\beta_T(x) &\subseteq \partial\sigma(x) \cap \beta_T(x)\\ &\subseteq (\sigma(Tx) \cup \ \mathrm{iso}\ \sigma(x)) \cap (\sigma(Tx) \cup \ \mathrm{acc}\ \sigma(x))\\ &\subseteq \sigma(Tx);\end{aligned}$$

so that $\beta_T(x) \subseteq \eta\sigma(Tx)$ in view of Lemma 2.8.3. Using Proposition 2.6.8 and the properties of the connected hull, it now follows that

$$\eta\beta_T(x) \subseteq \eta(\eta\sigma(Tx)) = \eta\sigma(Tx) \subseteq \eta\beta_T(x),$$

and hence $\eta\beta_T(x) = \eta\sigma(Tx)$ for all $x \in A$. □

We conclude this section with a sufficient condition for $r(x)$ to be a Riesz point of $\sigma(x)$ relative to $\mathrm{N}(T)$, where T is a Banach algebra homomorphism with the strong Riesz property. This result, whose proof we omit, will be used in the proof of Theorem 5.3.4. We further point out that the proof of Lemma 2.8.6 can be found in [3] (Lemma 5.4.3).

Lemma 2.8.6. *([3], Lemma 5.4.3) Let $T : A \to B$ be a Banach algebra homomorphism with the strong Riesz property and $x \in A$. If $r(x) \in \sigma(x) \backslash \sigma(Tx)$, then $r(x)$ is a Riesz point of $\sigma(x)$ relative to $\mathrm{N}(T)$.*
Moreover, if A is a semisimple unital Banach algebra, then $r(x) \in \sigma(x) \backslash \sigma(Tx)$ if and only if $r(x) > 0$ is a pole of $(\lambda e - x)^{-1}$ and $p(x, r(x)) \in \mathrm{N}(T)$.

3

Upper Browder elements in COBAs

In [5], Benjamin and Mouton introduced and investigated the set $A^{-1}+C\cap \mathrm{N}(T)$, where (A, C) is a general OBA and $T : A \to B$ an arbitrary Banach algebra homomorphism. Their motivation was due to E. Alekhno ([1], Theorem 3) who discovered that

$$B(E)^{-1} + K(E) = B(E)^{-1} + (K \cap K(E)),$$

where K denotes the algebra cone of positive operators on a Banach lattice E. They consequently introduced (upper Weyl and) upper Browder elements in general $OBAs$, which are special kinds of (Weyl and) Browder elements, respectively.

In this chapter, we will present generalizations of some results involving upper Browder elements in $OBAs$ - offered in [4] and [5] - to a $COBA$ setting.

Throughout this chapter, (A, C) will denote an arbitrary $COBA$ and $T : A \to B$ a Banach algebra homomorphism.

3.1. Definition and examples

We begin with the definition of an upper Browder element of A relative to a Banach algebra homomorphism $T : A \to B$.

Definition 3.1.1. *Let (A, C) be a COBA and $T : A \to B$ be a Banach algebra homomorphism. Relative to T, an element $x \in A$ is called upper Browder if there exist commuting elements $y \in A^{-1}$ and $z \in C \cap \mathrm{N}(T)$ such that $x = y + z$.*

We will use the notation $\mathcal{B}_T^+$ for the set of all upper Browder elements of A relative to T.

The next example describes the upper Browder elements in $C(K)$. It should be noted that this example will play an important role as a source of counterexamples in this dissertation.

Example 3.1.2. *([5], Example 3.1.3) Consider the Banach algebra homomorphism $T : A \to B$ induced by composition with the continuous map $\theta : K_2 \to K_1$ defined by $Tf = f \circ \theta$, where $A = C(K_1)$ and $B = C(K_2)$ are the commutatively* *ordered Banach algebras consisting of all continuous complex-valued functions on the compact Hausdorff spaces K_1 and K_2,* respectively. *Then $f \in \mathcal{B}_T^+$ if and only if its restriction to $\theta(K_2)$ has an invertible extension to K_1, say g, satisfying $f \geq g$.*

Proof. Let $C = \{f \in A : f(x) \in \mathbb{R}^+ \text{ for all } x \in K_1\}$ and suppose that $f \in \mathcal{B}_T^+$. Then there exist commuting elements $g \in A^{-1}$ and $h \in C \cap \mathrm{N}(T)$ such that $f = g + h$. Now because $h \in \mathrm{N}(T)$, we have that $h(\theta(K_2)) = \{0\}$, and thus $f|_{\theta(K_2)} = g|_{\theta(K_2)} + h|_{\theta(K_2)} = g|_{\theta(K_2)}$. This shows that, if $f \in \mathcal{B}_T^+$, then $f|_{\theta(K_2)}$ has an invertible extension to K_1, namely g. Furthermore, since $h \in C$, we then have that $f \geq g$.

Conversely, suppose that $f|_{\theta(K_2)}$ has an invertible extension to K_1, say g, satisfying $f \geq g$. Then $f|_{\theta(K_2)} = g|_{\theta(K_2)}$. From this, we therefore have that $T(f - g) = (f - g) \circ \theta = \{0\}$. Hence, $f = g + (f - g)$, where $g \in A^{-1}$ and $f - g \in C \cap \mathrm{N}(T)$, so that $f \in \mathcal{B}_T^+$. □

The following result gives the inclusion properties of the sets of upper Browder, Browder, Fredholm and invertible elements of a *COBA*, a generalization of Proposition 3.1.5 in [3]. This result and its proof carry over to *COBAs* without any modifications since the product of positive elements is not required to be positive.

Proposition 3.1.3. *Let (A, C) be a COBA and $T : A \to B$ be a Banach algebra homomorphism. Then the following inclusions hold in general:*

$$A^{-1} \subseteq \mathcal{B}_T^+ \subseteq \mathcal{B}_T \subseteq \mathcal{F}_T.$$

Proof. We only prove the first inclusion since the other inclusions follow directly. Let $x \in A^{-1}$. Now $x = x + 0$, where $x0 = 0x$. Since T is a linear operator, $T0 = 0$, so that $0 \in C \cap \mathrm{N}(T)$. Hence $x \in \mathcal{B}_T^+$. □

Next, we illustrate with examples that the set of upper Browder elements is generally larger than the set of invertible elements, and strictly contained in the set of Browder elements.

The following example reveals that the equality $A^{-1} = \mathcal{B}_T^+$ is not true in general.

Example 3.1.4. *Consider the COBA $(M_3^u(\mathbb{C}), M_3^u(\mathbb{R}^+))$, where $M_3^u(\mathbb{C})$ is the Banach algebra of all 3×3 upper triangular matrices with complex entries and $T : M_3^u(\mathbb{C}) \to \mathbb{C}$ is the Banach algebra homomorphism defined by*

$$T \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{bmatrix} = x_{11}.$$

Then $(M_3^u(\mathbb{C}))^{-1} \subsetneq \mathcal{B}_T^+$.

Proof. We first note that $\mathrm{N}(T)$ consists of all elements of $M_3^u(\mathbb{C})$ with entry in the first row and first column zero. Consider the element $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in M_3^u(\mathbb{C})$. Then

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \in (M_3^u(\mathbb{C}))^{-1} + (M_3^u(\mathbb{R}^+) \cap \mathrm{N}(T))$$

and the matrices on the right hand side of the identity commute.

Hence, $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{B}_T^+ \backslash (M_3^u(\mathbb{C}))^{-1}$. □

The next example shows that the inclusion $\mathcal{B}_T^+ \subseteq \mathcal{B}_T$ can be strict.

Example 3.1.5. *Consider the COBA* $(C([0,1]), C)$, *where*

$$C = \{f \in C([0,1]) : f(x) \in \mathbb{R}^+ \text{ for all } x \in [0,1]\}.$$

Let $K := [0,1]$ *and* $T : C(K) \to C(K)$ *be the Banach algebra homomorphism induced by composition with the unit function* 1 *on* K. *If* $f \in C(K)$ *is defined by* $f(z) = z$ *for all* $z \in K$, *then* $\mathcal{B}_T^+ \subsetneq \mathcal{B}_T$.

Proof. Since $f|_{1(K)}$ has an extension to K - the constant function 1 on K - which is invertible, we have by Example 2.6.2 that $f \in \mathcal{B}_T$. Suppose that f is also upper Browder relative to T. Then from Example 3.1.2, $f|_{1(K)}$ has an invertible extension to K, say g, satisfying $f \geq g$. Now because g is continuous, $g(1) = 1$ and $g(0) \leq f(0) = 0$, we have from the intermediate value theorem the existence of some $x \in K$ satisfying $g(x) = 0$. This gives a contradiction to the invertibility of g. Hence, $f \notin \mathcal{B}_T^+$, so that $\mathcal{B}_T^+ \subsetneq \mathcal{B}_T$. □

3.2. Algebraic properties of upper Browder elements

In this section, we present some basic algebraic properties of upper Browder elements in *COBAs*.

We start our discussion with a generalization of a result ([1], Lemma 1) by E. Alekhno, whose proof in the Banach algebra setting is due to Benjamin and Mouton.

Proposition 3.2.1. *([5], Lemma 3.2.5) Let A be a unital Banach algebra and I an inessential ideal of A. If $x \in A^{-1}, y \in I$ and $\lambda \in \mathbb{C}$, then there exist $x_1 \in A^{-1}$ and $\mu \in \mathbb{R}^+$ such that $x + \lambda y = x_1 + \mu y$.*

Proof. Let $x \in A^{-1}, y \in I$ and $\lambda \in \mathbb{C}$. If $\lambda \in \mathbb{R}^+$, then the result holds with $x_1 = x$ and $\mu = \lambda$. Hence, suppose that $\lambda \in \mathbb{C}\backslash\{\lambda \in \mathbb{R} : \lambda \geq 0\}$. Seeing that I is an ideal, we have that $x^{-1}y \in I$. Since I is inessential, then acc $\sigma(x^{-1}y) \subseteq \{0\}$. Now suppose that $\frac{1}{\mu - \lambda} \in \sigma(x^{-1}y)$ for all $\mu \in \mathbb{R}^+$. Then $\frac{1}{\frac{1}{n} - \lambda} \in \sigma(x^{-1}y)$ for all $n \in \mathbb{N}$, and hence $-\frac{1}{\lambda} = \lim_{n\to\infty} \frac{1}{\frac{1}{n} - \lambda} \in \sigma(x^{-1}y)$ by the closedness of the spectrum, so that $-\frac{1}{\lambda} \in$ acc $\sigma(x^{-1}y)$. But this contradicts the fact that I is an inessential ideal; hence there exists $\mu \in \mathbb{R}^+$ such that $\frac{1}{\mu - \lambda} \notin \sigma(x^{-1}y)$. For such μ, we have that $(\mu-\lambda)\left[\frac{1}{\mu - \lambda}e - x^{-1}y\right] \in A^{-1}$, that is $e-(\mu-\lambda)x^{-1}y \in A^{-1}$. Consequently,

$$x + \lambda y = x - (\mu - \lambda)y + \mu y = x[e - (\mu - \lambda)x^{-1}y] + \mu y.$$

Let $x_1 = x[e - (\mu - \lambda)x^{-1}y]$. Then the result follows as $x_1 \in A^{-1}$. □

We now point out two immediate consequences of Proposition 3.2.1.

The following corollary will be useful in establishing a number of results in this section. The result can be obtained by applying Proposition 3.2.1 a number of times.

Corollary 3.2.2. *([5], Corollary 3.2.6) Let A be a unital Banach algebra and I an inessential ideal of A. If $x = y + \lambda_1 c_1 + \cdots + \lambda_n c_n$ for some $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}, y \in A^{-1}$ and $c_1, \ldots, c_n \in I$. Then x has the form*

$$x = y_n + \mu_1 c_1 + \mu_2 c_2 + \cdots + \mu_n c_n,$$

where $\mu_1, \ldots, \mu_n \in \mathbb{R}^+$ and $y_n \in A^{-1}$.

Corollary 3.2.3. *([3], Corollary 3.3.1) Let $T : A \to B$ be a Banach algebra homomorphism with the Riesz property. If $x \in A^{-1}, y \in \mathrm{N}(T)$ and $\lambda \in \mathbb{C}$, then there exist $x_1 \in A^{-1}$ and $\mu \in \mathbb{R}^+$ such that $x + \lambda y = x_1 + \mu y$.*

Proof. Since T has the Riesz property, then $\mathrm{N}(T)$ is an inessential ideal of A. It therefore follows from Proposition 3.2.1 that $x + \lambda y = x_1 + \mu y$ for some $x_1 \in A^{-1}$ and $\mu \in \mathbb{R}^+$. □

Corollary 3.2.3 allows us to establish the following fact, which is a generalization of [5] (Corollary 3.2.7) to the *COBA* setting. We remark that this result carries over to the *COBA* setting without any modifications made to the *OBA* result, and that the proof is verbatim to that of [5] (Corollary 3.2.7) since the product of positive elements is not required to be positive.

Lemma 3.2.4. *Let (A, C) be a COBA and $T : A \to B$ be a Banach algebra homomorphism with the Riesz property. If $x \in A^{-1}$ and $y \in C \cap \mathrm{N}(T)$ commute and $\lambda \in \mathbb{C}$, then $x + \lambda y \in \mathcal{B}_T^+$.*

Proof. Let $x \in A^{-1}$ and $y \in C \cap \mathrm{N}(T)$ be commuting elements and $\lambda \in \mathbb{C}$. From Corollary 3.2.3 we have the existence of $x_1 \in A^{-1}$ and $\mu \in \mathbb{R}^+$ such that $x + \lambda y = x_1 + \mu y$. Since x commutes with y, then x_1 also commutes with y. Using the fact that $x_1 \in A^{-1}$ and $\mu y \in C \cap \mathrm{N}(T)$ commute, we have, by Definition 3.1.1 that $x_1 + \mu y \in \mathcal{B}_T^+$. □

Example 3.2 of [4] (p.955) shows that upper Browder elements are not in general stable under perturbation by elements in span $(C \cap \mathrm{N}(T))$, where C is an algebra cone of a unital Banach algebra A and T is a Banach algebra homomorphism with domain A. However, as we point out next, if the *OBA* is commutative - in which case we are also dealing with a *COBA* - then the upper Browder elements remain stable under perturbation by elements in span$(C \cap \mathrm{N}(T))$. The proof we offer follows the lines of [4] (Lemma 3.1).

Proposition 3.2.5. *Let (A, C) be a commutative COBA and $T : A \to B$ be a Banach algebra homomorphism with the Riesz property. Then $\mathcal{B}_T^+ = \mathcal{B}_T^+ +$* span$(C \cap \mathrm{N}(T))$.

Proof. We only show that $\mathcal{B}_T^+ + \text{span}(C \cap \mathrm{N}(T)) \subseteq \mathcal{B}_T^+$ since the reverse inclusion follows trivially from the fact that $0 \in \text{span}(C \cap \mathrm{N}(T))$. Therefore, let $x \in \mathcal{B}_T^+ + \text{span}(C \cap \mathrm{N}(T))$. Then there exist $y \in \mathcal{B}_T^+$ and $z \in \text{span}(C \cap \mathrm{N}(T))$ such that $x = y + z$. Since $y \in \mathcal{B}_T^+$, we can find (commuting) elements $y_1 \in A^{-1}$ and $y_2 \in C \cap \mathrm{N}(T)$ such that $y = y_1 + y_2$. Since $z \in \text{span}(C \cap \mathrm{N}(T))$, there exist $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $z_1, \ldots, z_n \in C \cap \mathrm{N}(T)$ such that $z = \lambda_1 z_1 + \cdots + \lambda_n z_n$. Hence,

$$x = y + z = (y_1 + y_2) + (\lambda_1 z_1 + \cdots + \lambda_n z_n).$$

Applying Corollary 3.2.2, we have that $x = y_n + y_2 + \mu_1 z_1 + \mu_2 z_2 + \cdots + \mu_n z_n$ for some $y_n \in A^{-1}$ and $\mu_1, \ldots, \mu_n \in \mathbb{R}^+$, and thus $x \in \mathcal{B}_T^+$. □

The following example points out that the above result is no longer true if we exclude the commutativity assumption.

Example 3.2.6. *Consider the COBA and Banach algebra homomorphism T as in Example* 3.1.4. *Then* $\mathcal{B}_T^+ + \text{span}(M_3^u(\mathbb{R}^+) \cap \text{N}(T)) \neq \mathcal{B}_T^+$.

Proof. We can write $X = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in M_3^u(\mathbb{C})$ as

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} + \left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] \in \mathcal{B}_T^+ + \text{span}(M_3^u(\mathbb{R}^+) \cap \text{N}(T)).$$

We now show that $X \notin \mathcal{B}_T^+$. Suppose that this is not the case. Then

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 - x_{12} & -x_{13} \\ 0 & 1 - x_{22} & 1 - x_{23} \\ 0 & 0 & -x_{33} \end{pmatrix} + \begin{pmatrix} 0 & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{pmatrix} \in (M_3^u(\mathbb{C}))^{-1} + M_3^u(\mathbb{R}^+) \cap \text{N}(T),$$

where $x_{12}, x_{13}, x_{22}, x_{23} \in \mathbb{R}^+, x_{33} > 0$ and the matrices on the right hand side of the identity commute. Now the commutativity property, however, implies that $x_{23} + x_{33} = 0$ which is impossible as $x_{33} > 0$. Therefore $X \notin \mathcal{B}_T^+$. □

It is known that the set of Browder elements of a unital Banach algebra A relative to any Banach algebra homomorphism on A are closed under non-zero scalar multiplication. In [5], Benjamin and Mouton noted that, upper Browder elements of an *OBA* are not generally closed under non-zero scalar multiplication. It is however proved in [5] (Lemma 3.2.8) that, whenever T has the Riesz property, the set $\mathcal{B}_T^+$ will be closed under non-zero scalar multiplication. As the product of positive elements is not required to be positive, this result and its proof carry over to *COBAs* without any modifications. We include the proof here in the interest of completeness.

Lemma 3.2.7. *Let (A, C) be a COBA and $T : A \to B$ be a Banach algebra homomorphism with the Riesz property. Suppose that $0 \neq \lambda \in \mathbb{C}$, then $\lambda \mathcal{B}_T^+ \subseteq \mathcal{B}_T^+$.*

Proof. Suppose that $0 \neq \lambda \in \mathbb{C}$ and that $x \in \mathcal{B}_T^+$. Then there exist commuting elements $y \in A^{-1}$ and $z \in C \cap \text{N}(T)$ such that $x = y + z$. Seeing that $\lambda \neq 0$, we have that $\lambda y \in A^{-1}$. Utilizing Lemma 3.2.4, it then follows that $\lambda x = \lambda(y + z) = \lambda y + \lambda z \in \mathcal{B}_T^+$. □

We remark that the assumption "T has the Riesz property" in Lemma 3.2.7 cannot in general be dropped. This can be seen in the following example.

Example 3.2.8. *Consider the COBA and Banach algebra homomorphism T as in Example 3.1.5. If $f \in C(K)$ is defined by $f(z) = -z$ for all $z \in K := [0,1]$, then $f \in \mathcal{B}_T^+$ but $-f \notin \mathcal{B}_T^+$.*

Proof. We first note that $T : C(K) \to C(K)$ does not have the Riesz property since $\mathrm{N}(T) = \{h \in C(K) : h(1) = 0\}$ is not an inessential ideal. Since $f|_{1(K)}$ has an extension to K - the constant function -1 on K - which is invertible and satisfies $f \geq -1$, we have that $f \in \mathcal{B}_T^+$. Since $-f$ is the identity function on K, it follows from the proof of Example 3.1.5 that $-f \notin \mathcal{B}_T^+$. $\square$

Example 3.2.8 also reveals that the upper Browder elements are not closed under multiplication. This is confirmed by the following example.

Example 3.2.9. *Again, consider the COBA and Banach algebra homomorphism T as in Example 3.1.5. If $f \in C(K)$ is defined by $f(z) = -z$ for all $z \in K := [0,1]$, then $f \in \mathcal{B}_T^+$ but $f^2 \notin \mathcal{B}_T^+$.*

Proof. We have from Example 3.2.8 that $f \in \mathcal{B}_T^+$. Now using a similar approach as in the proof of Example 3.1.5, we have that $f^2|_{1(K)}$ has no invertible extension, say g, to K satisfying $z^2 = f^2(z) \geq g(z)$ for all $z \in K$. Hence $f^2 \notin \mathcal{B}_T^+$. $\square$

If the OBA in [4] (Corollary 3.3) is commutative - in which case we are also dealing with a commutative $COBA$ - then the product of upper Browder elements is again upper Browder. This result is displayed next and we include its proof - which follows the lines of the proof Benjamin offered in [4] - in the interest of completeness.

Proposition 3.2.10. *Let (A, C) be a commutative COBA and $T : A \to B$ be a Banach algebra homomorphism with the Riesz property. If C is a generating algebra c-cone in A, then $\mathcal{B}_T^+\mathcal{B}_T^+ \subseteq \mathcal{B}_T^+$.*

Proof. Let $x, y \in \mathcal{B}_T^+$. Then $x = x' + x''$ and $y = y' + y''$ for some $x', y' \in A^{-1}$ and $x'', y'' \in C \cap \mathrm{N}(T)$. Now because C is a generating algebra c-cone in A, we have that $x', y' \in \operatorname{span} C$, and hence $x', y' \in \operatorname{span} C \cap A^{-1}$. Thus $x' = \lambda_1 x_1 + \cdots + \lambda_n x_n$ for some $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $x_1, \ldots, x_n \in C$ and $y' = \mu_1 y_1 + \cdots + \mu_m y_m$ for some $\mu_1, \ldots, \mu_m \in \mathbb{C}$ and $y_1, \ldots, y_m \in C$. With this, we therefore have that

$$\begin{aligned} xy &= (x' + x'')(y' + y'') \\ &= x'y' + x''y'' + x'y'' + x''y' \\ &= x'y' + [x''y'' + (\lambda_1 x_1 y'' + \cdots + \lambda_n x_n y'') + (\mu_1 x'' y_1 + \cdots + \mu_m x'' y_m)] \\ &\in A^{-1} + \operatorname{span}(C \cap \mathrm{N}(T)) \subseteq \mathcal{B}_T^+ \end{aligned}$$

by Proposition 3.2.5. Hence $\mathcal{B}_T^+\mathcal{B}_T^+ \subseteq \mathcal{B}_T^+$. $\square$

Next, we give an example to confirm that the commutativity assumption in Proposition 3.2.10 cannot in general be removed.

Example 3.2.11. *([4], Example 3.4) Consider the COBA and Banach algebra homomorphism T as in Example* 3.1.4. *If* $X = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *and* $Y = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, *then X and Y are elements of $\mathcal{B}_T^+$, but neither XY nor YX belongs to $\mathcal{B}_T^+$.*

Proof. Note that X is an invertible matrix in $M_3^u(\mathbb{C})$, and so $X \in \mathcal{B}_T^+$ by Proposition 3.1.3. From Example 3.1.4 we have that $Y \in \mathcal{B}_T^+$ and, by recalling Example 3.2.6, $XY = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \notin \mathcal{B}_T^+$. Now using a similar approach as that in the proof of Example 3.2.6, one can show that $YX = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is not upper Browder in $M_3^u(\mathbb{C})$ relative to T. $\square$

Recall that not all Browder elements of a *COBA* are in general upper Browder - see Example 3.1.5. The following result, which is our own observation, gives conditions under which the sets of Browder and upper Browder elements coincide in a *COBA* setting.

Theorem 3.2.12. *Let (A, C) be a commutative COBA and $T : A \to B$ be a Banach algebra homomorphism with the Riesz property. If the condition*

$$p(x, \lambda) \in \operatorname{span}(C \cap \mathrm{N}(T)) \text{ for all } \lambda \in (\operatorname{iso} \sigma(x))\backslash\sigma(Tx)$$

holds for all $x \in A$, then $\mathcal{B}_T = \mathcal{B}_T^+$.

Proof. For the non trivial inclusion, suppose that $x \in \mathcal{B}_T$. Since T has the Riesz property, we have from Theorem 2.6.3 that $x \in A^D \cap \mathcal{F}_T$. Now if $x \in A^{-1}$, then $x \in \mathcal{B}_T^+$, and we are done. Hence suppose that $x \notin A^{-1}$. Then $0 \in (\operatorname{iso} \sigma(x))\backslash\sigma(Tx)$ and we therefore consider the representation $x = (x - p) + p$, where $p = p(x, 0)$ is the spectral idempotent of x corresponding to 0. We have by the holomorphic functional calculus that $x - p \in A^{-1}$ and by the assumption that $p \in \operatorname{span}(C \cap \mathrm{N}(T))$. From Proposition 3.2.5 it then follows that $x \in \mathcal{B}_T^+$. $\square$

Theorem 3.2.12 immediately gives the following:

Example 3.2.13. *Consider the COBA $(\mathbb{C}^n, C)$, where*

$$C = \{(x_1, \ldots, x_n) \in \mathbb{C}^n : x_i \in \mathbb{R}^+ \text{ for all } i \in \{1, \ldots, n\}\}.$$

Let $T : \mathbb{C}^n \to \mathbb{C}$ be the Banach algebra homomorphism defined by

$$T(x_1, x_2, \ldots, x_n) = x_1.$$

Then $\mathcal{B}_T = \mathcal{B}_T^+$.

Proof. We start by showing that $\mathrm{N}(T) = \mathrm{span}(C \cap \mathrm{N}(T))$. We first note that the inclusion $\mathrm{span}(C \cap \mathrm{N}(T)) \subseteq \mathrm{N}(T)$ is always true because $\mathrm{N}(T)$ is a vector space. It therefore remains to show that $\mathrm{N}(T) \subseteq \mathrm{span}(C \cap \mathrm{N}(T))$, so let $x \in \mathrm{N}(T)$. Then there exist $x_k := y_k + iz_k$, where $y_k, z_k \in \mathbb{R}$ for $2 \leq k \leq n$, such that $x = (0, x_2, \ldots, x_n) = (0, y_2, \ldots, y_n) + i(0, z_2, \ldots, z_n)$. Let $\{y_{k_2}, \ldots, y_{k_t}\} = \{y_2, \ldots, y_n\} \cap \mathbb{R}^+$ and $\{z_{l_2}, \ldots, z_{l_s}\} = \{z_2, \ldots, z_n\} \cap \mathbb{R}^+$ for $s, t \in \mathbb{N}$. Then the elements $y_{k_{t+1}}, \ldots, y_{k_n}, z_{l_{s+1}}, \ldots, z_{l_n}$ are the negative real numbers in $\{y_2, \ldots y_n, z_2, \ldots, z_n\}$ and thus

$$x = \sum_{j=2}^{t} y_{k_j} e_{k_j} + (-1) \sum_{j=t+1}^{n} (-y_{k_j}) e_{k_j} + i \sum_{j=2}^{s} z_{l_j} e_{l_j} + (-i) \sum_{j=s+1}^{n} (-z_{l_j}) e_{l_j},$$

where e_i is the n-tuple with i-th term 1 and all other terms 0. This proves that $x \in \mathrm{span}(C \cap \mathrm{N}(T))$. Therefore $\mathrm{N}(T) = \mathrm{span}(C \cap \mathrm{N}(T))$.
Now let $\lambda \in \sigma(x) \backslash \sigma(Tx)$. Then $p(x, \lambda) \in \mathrm{N}(T)$ by Lemma 2.4.14, and hence $p(x, \lambda) \in \mathrm{span}(C \cap \mathrm{N}(T))$, so that $\mathcal{B}_T = \mathcal{B}_T^+$ follows from Theorem 3.2.12. $\square$

Note that T in Example 3.2.13 has the Riesz property since each element of $\mathrm{N}(T) = \{(0, x_2, \ldots, x_n) \ : x_2, \ldots, x_n \in \mathbb{C}\}$ has finite spectrum.

4

The upper Browder spectrum of a COBA element

In this chapter we present generalizations of some basic properties of the upper Browder spectrum from the OBA setting to the context of $COBAs$. We will, in particular, present sufficient conditions under which the upper Browder spectrum obeys a spectral mapping theorem (Section 4.2) and examine in Section 4.3 the relationship between the connected hulls of the Browder and the upper Browder spectra.

4.1. Definitions and examples

We start our discussion with the definition - to be compared with Definition 4.1.1 in [5] - of the upper Browder spectrum of an element of a $COBA$ (A, C) relative to an arbitrary Banach algebra homomorphism $T : A \to B$.

Definition 4.1.1. *Let* (A, C) *be a COBA and* $T : A \to B$ *a Banach algebra homomorphism. The upper Browder spectrum of an element* $x \in A$ *is the set given by*

$$\beta_T^+(x) = \{\lambda \in \mathbb{C} : \lambda e - x \notin \mathcal{B}_T^+\}.$$

We recall (for Browder elements) that $\lambda e - x$ is Browder if and only if $x - \lambda e$ is Browder. This fact is not true for upper Browder elements, that is the sets

$$\{\lambda \in \mathbb{C} : \lambda e - x \notin \mathcal{B}_T^+\} \text{ and } \{\lambda \in \mathbb{C} : x - \lambda e \notin \mathcal{B}_T^+\}$$

do not coincide in general - see Example 4.1.2. This is because the upper Browder elements of a $COBA$ are not generally closed under non-zero scalar multiplication as seen from Example 3.2.8.

Example 4.1.2. *Consider the COBA and Banach algebra homomorphism* T *as in Example 3.1.5. If* $f \in C(K)$ *is defined by* $f(z) = z$ *for all* $z \in K := [0, 1]$, *then* $\beta_T^+(f) = \{1\}$ *and* $\{\lambda \in \mathbb{C} : f - \lambda e \notin \mathcal{B}_T^+\} = [0, 1]$.

Proof. We already pointed out in Example 3.2.8 that T does not have the Riesz

property. From Example 3.1.2, $\lambda e - f$ is upper Browder relative to T if and only if $(\lambda e - f)_{|1(K)}$ has an invertible extension to K, say g, satisfying $\lambda e - f \geq g$. With this, we have that if $\lambda = 1$, then $(\lambda e - f)_{|1(K)}(1) = 1 - 1 = 0$. This shows that $(\lambda e - f)_{|1(K)}$ is not invertible, and hence does not have an invertible extension to K. Thus $1 \in \beta_T^+(f)$.
If $\lambda \neq 1$, then $(\lambda e - f)_{|1(K)}(1) = \lambda - 1$ and this means that $0 \notin (\lambda e - f)_{|1(K)}(\{1\})$, so that $(\lambda e - f)_{|1(K)}$ is invertible. Now let $g : K \to \mathbb{C}$ be the constant function defined by $g(z) = \lambda - 1$ for all $z \in K$. Then g is an invertible extension of $(\lambda e - f)_{|1(K)}$ to K satisfying $\lambda e - f \geq g$. Hence $\lambda e - f \in \mathcal{B}_T^+$, so that $1 \neq \lambda \notin \beta_T^+(f)$. This gives $\beta_T^+(f) = \{1\}$.
Similarly, $f - \lambda e$ is upper Browder relative to T if and only if $(f - \lambda e)_{|1(K)}$ has an invertible extension to K, say g, satisfying $f - \lambda e \geq g$. Using the intermediate value theorem as in the proof of Example 3.1.5 one can show that, for $\lambda \in [0, 1]$, the function $(f - \lambda e)_{|1(K)}$ has no invertible extension g to K satisfying $f - \lambda e \geq g$. Hence $f - \lambda e$ is not upper Browder. By also using the fact that $A^{-1} \subseteq \mathcal{B}_T^+$, it now follows that

$$[0, 1] \subseteq \{\lambda \in \mathbb{C} : f - \lambda e \notin \mathcal{B}_T^+\} \subseteq \sigma(f) = [0, 1].$$

This evidently shows that $\beta_T^+(f) = \{1\} \neq [0, 1] = \{\lambda \in \mathbb{C} : f - \lambda e \notin \mathcal{B}_T^+\}$. □

Lemma 3.2.7 allows us to establish the following fact which is a generalization of [5] (Proposition 4.1.2).

Corollary 4.1.3. *Let (A, C) be a COBA and $T : A \to B$ be a Banach algebra homomorphism with the Riesz property. If $x \in A$, then*

$$\beta_T^+(x) = \{\lambda \in \mathbb{C} : \lambda e - x \notin \mathcal{B}_T^+\} = \{\lambda \in \mathbb{C} : x - \lambda e \notin \mathcal{B}_T^+\}.$$

Our next result, which is a generalization of [3] (Proposition 4.1.4) shows that the upper Browder spectrum of a *COBA* element can be represented in terms of the ordinary spectrum. Though its proof is verbatim to that of [3] (Proposition 4.1.4), as it does not require that the product of positive elements be positive, we include it in the interest of completeness.

Theorem 4.1.4. *Let (A, C) be a COBA and $T : A \to B$ be a Banach algebra homomorphism. For any element $x \in A$, we have that*

$$\beta_T^+(x) = \bigcap_{\substack{z \in C \cap N(T) \\ xz = zx}} \sigma(x + z).$$

Proof. Suppose that $\lambda \notin \bigcap_{\substack{z \in C \cap N(T) \\ xz = zx}} \sigma(x + z)$. Then there exists an element $w \in$

$C \cap \mathrm{N}(T)$ commuting with x such that $\lambda \notin \sigma(x+w)$, that is $\lambda e - (x+w) \in A^{-1}$. Now let $y := \lambda e - (x+w)$. (We note that w also commutes with y.) From this, we therefore have that $\lambda e - x = y + w$, where $y \in A^{-1}$ and $w \in C \cap \mathrm{N}(T)$ satisfy $yw = wy$. Hence $\lambda e - x \in \mathcal{B}_T^+$, so that $\lambda \notin \beta_T^+(x)$. Hence

$$\beta_T^+(x) \subseteq \bigcap_{\substack{z \in C \cap \mathrm{N}(T) \\ xz = zx}} \sigma(x+z).$$

To prove the converse, we suppose that $\lambda \notin \beta_T^+(x)$. Then $\lambda e - x \in \mathcal{B}_T^+$ by definition, and thus there exist commuting elements $y \in A^{-1}$ and $w \in C \cap \mathrm{N}(T)$ such that $\lambda e - x = y + w$. We note that w commutes with x and that $\lambda e - (x+w) = y \in A^{-1}$. This implies that $\lambda \notin \sigma(x+w)$, and thus $\lambda \notin \bigcap_{\substack{z \in C \cap \mathrm{N}(T) \\ xz = zx}} \sigma(x+z)$. This gives

$$\beta_T^+(x) \supseteq \bigcap_{\substack{z \in C \cap \mathrm{N}(T) \\ xz = zx}} \sigma(x+z).$$

and hence the identity follows. □

The following result follows immediately from Proposition 3.1.3.

Proposition 4.1.5. *Let (A, C) be a COBA and $T : A \to B$ be a Banach algebra homomorphism. For each element $x \in A$, the following inclusions hold:*

$$\sigma(Tx) \subseteq \beta_T(x) \subseteq \beta_T^+(x) \subseteq \sigma(x).$$

Since the upper Browder spectrum can be represented in terms of the ordinary spectrum which is non-empty and compact (cf. Theorem 4.1.4), it follows that the upper Browder spectrum is also non-empty and compact.

We point out in the following examples that the upper Browder spectrum of a *COBA* element is in general strictly contained in the (ordinary) spectrum of that element and larger than the Browder spectrum of the element.

Example 4.1.6. *Consider the COBA and Banach algebra homomorphism T as in Example* 3.1.4. *If* $X = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in M_3^u(\mathbb{C})$, *then* $\beta_T^+(-X) \subsetneq \sigma(-X)$.

Proof. Since $0I - (-X) \in \mathcal{B}_T^+ \backslash (M_3^u(\mathbb{C}))^{-1}$ in view of Example 3.1.4, where I denotes the identity matrix in $M_3^u(\mathbb{C})$, it follows from definition that $0 \in \sigma(-X) \backslash \beta_T^+(-X)$. □

Example 4.1.7. *Consider the COBA and Banach algebra homomorphism T as in Example* 3.1.5. *For $f \in C([0,1])$ defined by $f(z) = z$ for all $z \in [0,1]$, we have that $\beta_T(-f) \subsetneq \beta_T^+(-f)$.*

Proof. Using Example 3.1.5 and a similar reasoning as in the proof of Example 4.1.6, one obtains that $0 \in \beta_T^+(-f)$ but $0 \notin \beta_T(-f)$. □

4.2. Spectral mapping theorem

The aim of this section is to investigate whether the upper Browder spectrum of a *COBA* element satisfies a spectral mapping theorem.

The following result, which is an immediate consequence of Theorem 3.2.12, provides conditions under which the upper Browder spectrum of a *COBA* element obeys the spectral mapping theorem. This result, for a commutative *OBA*, is clear from Theorem 4.1 in [4].

Proposition 4.2.1. *Let (A, C) be a commutative COBA, $T : A \to B$ be a Banach algebra homomorphism with the Riesz property and $x \in A$. If*

$$p(y, \lambda) \in \operatorname{span}(C \cap \mathrm{N}(T)) \text{ for all } \lambda \in (\operatorname{iso} \sigma(y)) \backslash \sigma(Ty)$$

holds for all $y \in A$, then

$$\beta_T^+(f(x)) = f(\beta_T^+(x))$$

for every function $f : U \to \mathbb{C}$ which is holomorphic on an open set U containing $\sigma(x)$ and non-constant on each component of U.

Proof. By Theorem 3.2.12, $\mathcal{B}_T = \mathcal{B}_T^+$. It then follows from Theorem 2.7.3 that $\beta_T^+(f(x)) = f(\beta_T^+(x))$. □

We investigate next the necessity of the assumptions in Proposition 4.2.1. At this stage we do not have an example which illustrates that the commutativity-assumption in Proposition 4.2.1 is essential. We also failed to produce an example that proves that the assumption on the spectral idempotents cannot in general be removed. Nevertheless, we show next that the assumption "T has the Riesz property" in Proposition 4.2.1 is generally essential to conclude the spectral mapping theorem for an element of a commutative *COBA*.

Example 4.2.2. *Consider the commutative COBA and Banach algebra homomorphism T as in Example* 3.1.5. *Let $f \in C(K)$ be defined by $f(z) = z$ for all $z \in K := [0,1]$ and U be any open set containing $\sigma(f) = f(K) = [0,1]$. Then*

$$\beta_T^+(g(f)) \neq g(\beta_T^+(f)),$$

where $g : U \to \mathbb{C}$ defined by $g(\lambda) = -\lambda$ is holomorphic on U and non-constant on each component of U.

Proof. We have from Example 4.1.2 that $\beta_T^+(f) = \{1\}$; hence $g(\beta_T^+(f)) = \{-1\}$. Since $f = 0e - (-f) \notin \mathcal{B}_T^+$ by Example 3.1.5, it follows that $0 \in \beta_T^+(-f)$, and therefore $\beta_T^+(g(f)) \neq g(\beta_T^+(f))$. □

When the condition about the spectral idempotent in Proposition 4.2.1 is removed and the algebra c-cone is generating, then the upper Browder spectrum of a commutative $COBA$ element generally satisfies a one-way spectral mapping theorem. Take note that this result can also be obtained from Theorem 4.8 in [4].

Theorem 4.2.3. *Let (A, C) be a commutative $COBA$ with generating algebra c-cone C, $T : A \to B$ be a Banach algebra homomorphism with the Riesz property and $x \in A$. Then*

$$\beta_T^+(f(x)) \subseteq f(\beta_T^+(x))$$

for every function $f : U \to \mathbb{C}$ which is holomorphic on an open set U containing $\sigma(x)$ and non-constant on each component of U.

Proof. The result can be established by using the same arguments as that in the proof of the inclusion $\beta_T(f(x)) \subseteq f(\beta_T(x))$ in Theorem 2.7.3; just replace the Browder spectrum by the upper Browder spectrum and Remark 2.6.5 and Proposition 2.6.8 by Propositions 3.2.10 and 4.1.5, respectively. □

4.3. Connected hulls

A result by Mouton, Mouton and Raubenheimer ([18], Corollaries 7.6 and 7.8) shows that the connected hulls of the Fredholm, Browder and Weyl spectra relative to a Banach algebra homomorphism T with closed range satisfying the Riesz property coincide. In [5] (Theorem 4.3.5), Benjamin and Mouton proved in an OBA (A, C) that if there is an additional condition that $p(x, \lambda) \in \text{span}(C \cap \text{N}(T))$ for all $\lambda \in (\text{iso}\, \sigma(x)) \backslash \sigma(Tx)$ and $x \in A$, then the connected hulls of the Fredholm, Weyl, Browder and upper Weyl spectra - see [3] ((4.1.6), p.61) - coincide. They further illustrated by example that under the same assumptions, the connected hull of the upper Browder spectrum does not coincide with the connected hulls of any of the aforementioned spectra (see Example 4.3.1 in [5]).

This section therefore presents the conditions (in a $COBA$ setting) under which the connected hulls of the Browder and upper Browder spectra coincide.

We first start by showing that the connected hulls of the Browder and the upper Browder spectra of a *COBA* element do not in general coincide relative to any Banach algebra homomorphism T with closed range satisfying the Riesz property and under the condition that $p(x, \lambda) \in \operatorname{span}(C \cap \mathrm{N}(T))$ for all $\lambda \in$ (iso $\sigma(x)) \backslash \sigma(Tx)$.

Example 4.3.1. *Consider the COBA and Banach algebra homomorphism T as in Example 3.1.4. If $X = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in M_3^u(\mathbb{C})$, then $\eta\beta_T(X) \neq \eta\beta_T^+(X)$.*

Proof. Recall that T has the Riesz property since the spectrum of each element $X \in M_3^u(C)$ - which coincides with the set of all eigenvalues of X - is finite. Also, from Lemma 2.3.10, we have that T has closed range.
We show next that $\mathrm{N}(T) = \operatorname{span}(M_3^u(\mathbb{R}^+) \cap \mathrm{N}(T))$. The inclusion $\operatorname{span}(M_3^u(\mathbb{R}^+) \cap \mathrm{N}(T)) \subseteq \mathrm{N}(T)$ is always true since $\mathrm{N}(T)$ is a vector space. Suppose now that $X \in \mathrm{N}(T)$ and note that $\mathrm{N}(T)$ consists of all elements of $M_3^u(\mathbb{C})$ with entry in the first row and first column zero. Since each element of $\mathrm{N}(T)$ is a matrix with at most 5 non- zero entries, we can force the matrix X, depending on the signs of the real and imaginary parts of the entries in X, to be written as a linear combination of at most 10 elements (where each element is a matrix consisting of only one non- zero entry) in $M_3^u(\mathbb{R}^+) \cap \mathrm{N}(T)$. This shows that $X \in \operatorname{span}(M_3^u(\mathbb{R}^+) \cap \mathrm{N}(T))$, so that $\mathrm{N}(T) = \operatorname{span}(M_3^u(\mathbb{R}^+) \cap \mathrm{N}(T))$.
Now let $\lambda \in \sigma(X) \backslash \sigma(TX)$. Then $p(X, \lambda) \in \mathrm{N}(T)$ by Lemma 2.4.14, and hence the condition $p(X, \lambda) \in \operatorname{span}(M_3^u(\mathbb{R}^+) \cap \mathrm{N}(T))$ for all $\lambda \in \sigma(X) \backslash \sigma(TX)$ holds.
Next we compute $\eta\beta_T(X)$ and $\eta\beta_T^+(X)$. We have by definition that $\lambda \notin \beta_T(X)$ if and only if

$$\lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \lambda - 1 & -1 & 0 \\ 0 & \lambda - 1 & -1 \\ 0 & 0 & \lambda \end{pmatrix} \in \mathcal{B}_T;$$

which happens if and only if $\lambda \neq 1$. Hence

$$\{1\} = \beta_T(X) \subseteq \beta_T^+(X) \subseteq \sigma(X) = \{0, 1\} \tag{4.1.8}$$

by Proposition 4.1.5. Now since $X - 0I \notin \mathcal{B}_T^+$ in view of the proof of Example 3.2.6 and T has the Riesz property, we have from Corollary 4.1.3 that $0 \in \beta_T^+(X)$, and hence $\beta_T^+(X) = \{0, 1\}$ by recalling (4.1.8). This gives $\eta\beta_T(X) = \beta_T(X) = \{1\} \neq \{0, 1\} = \beta_T^+(X) = \eta\beta_T^+(X)$. □

The following result gives conditions under which the (connected hulls of the) Browder and upper Browder spectra coincide.

Theorem 4.3.2. *Let (A, C) be a commutative COBA, $T : A \to B$ be a Banach algebra homomorphism with the Riesz property and $x \in A$. If*

$$p(y, \lambda) \in \operatorname{span}(C \cap \mathrm{N}(T)) \textit{ for all } \lambda \in (\operatorname{iso} \sigma(y)) \backslash \sigma(Ty)$$

holds for all $y \in A$, then $\eta\beta_T(x) = \eta\beta_T^+(x)$.

Proof. The result is an immediate consequence of Theorem 3.2.12. □

5

Upper Browder spectrum property in COBAs

5.1. Background

In [5], Benjamin and Mouton established the following set of inclusions in the context of $OBAs$ (relative to a Banach algebra homomorphism T) :

$$\sigma(Tx) \subseteq \beta_T^+(x) \subseteq \sigma(x),$$

where $\sigma(Tx)$, $\beta_T^+(x)$ and $\sigma(x)$ are the Fredholm, upper Browder and (ordinary) spectra of a Banach algebra element x, respectively.
In [6], they investigated the following natural question arising from the above inclusions: provided that it is known that the spectral radius of x is an element of $\sigma(x)$ but not of $\sigma(Tx)$, is it possible to infer that $r(x) \notin \beta_T^+(x)$? The OBA element satisfying this condition is said to have the *upper Browder spectrum property.*

The above mentioned question was investigated for Banach algebra homomorphisms defined on finite-dimensional $OBAs$ and (more generally) those with closed range having the Riesz property. They discovered, in particular, that all positive elements of a finite-dimensional semisimple OBA have the upper Browder spectrum property.

In this chapter we present generalizations of some of the OBA results Benjamin and Mouton offered in [6] in a $COBA$. We start with the definition - to be compared with [6] (p.581) - of the upper Browder spectrum property in $COBAs$:

Definition 5.1.1. (Upper Browder spectrum property) *Let (A, C) be a COBA and $T : A \to B$ be a Banach algebra homomorphism. Then $x \in C$ is said to have the upper Browder spectrum property (relative to T) if there is the implication $r(x) \in \sigma(x) \backslash \sigma(Tx) \Rightarrow r(x) \notin \beta_T^+(x)$.*

5.2. Finite-Dimensional case

We start by studying the upper Browder spectrum property for elements of a finite dimensional $COBA$. The main result of this section is Theorem 5.2.8, which states that every finite-dimensional semisimple $COBA$ is isomorphic to a $COBA$ in which all positive elements have the upper Browder spectrum property.

Our first result deals with simple Banach algebras - these are Banach algebras which has no ideals besides the zero ideal. An example of a simple Banach algebra which will be useful in establishing key results in this section is given by $M_n(\mathbb{C})$ for $n \geq 1$.

The following lemma shows the conditions under which the Fredholm and the ordinary spectra of an element of a simple unital Banach algebra coincide.

Lemma 5.2.1. *([6], Lemma 5.5) Let A be a simple unital Banach algebra and $T : A \to B$ be a Banach algebra homomorphism with closed range. Then $\eta\sigma(Tx) = \eta\sigma(x)$ for all $x \in A$.*
Hence, if A is finite-dimensional (in which case T automatically has closed range), then $\sigma(Tx) = \beta_T(x) = \sigma(x)$ for all $x \in A$.

Proof. Since A is a simple Banach algebra, the ideal $\mathrm{N}(T)$ coincides with $\{0\}$, and hence T has the Riesz property. The identity $\mathrm{N}(T) = \{0\}$ also implies that $A^{-1} = \mathcal{B}_T$. Also, since T has closed range and the Riesz property, we have from the remark following Definition 2.8.4 that T has the strong Riesz property. This gives $\eta\sigma(Tx) = \eta\beta_T(x) = \eta\sigma(x)$ for all $x \in A$ in view of Theorem 2.8.5.
The last statement follows from the fact that the spectrum of a finite-dimensional Banach algebra element is finite. □

We point out the following result by Benjamin and Mouton whose complete proof can be found in the PhD thesis of Benjamin ([3], Lemma 5.2.3). We will, however, offer the proofs of some consequences of this result in the sequel. By 0_A we mean the zero element of a Banach algebra A.

Lemma 5.2.2. *([6], Lemma 5.6) For $t \in \mathbb{N}$, let $A_1, \ldots, A_t$ be finite-dimensional simple unital Banach algebras and $T : A_1 \oplus \cdots \oplus A_t \to B$ be a Banach algebra homomorphism. For each $j \in \{1, \ldots, t\}$, let $A'_j := \{0_{A_1}\} \oplus \cdots \oplus A_j \oplus \cdots \oplus \{0_{A_t}\}$ and define $T_j : A'_j \to B$ by*

$$T_j(0_{A_1}, \ldots, x_j, \ldots, 0_{A_t}) = T(0_{A_1}, \ldots, x_j, \ldots, 0_{A_t}),$$

where $x_j \in A_j$. Then either $T_j = 0$ or $T_j(A'_j)$ is a unital Banach algebra and the homomorphism $T_j : A'_j \to T_j(A'_j)$ is an isomorphism, i.e. one-to-one and onto.

An immediate consequence of Lemma 5.2.2 in the case where all T_j's are non-zero is given next. This result - whose proof is due to Benjamin and Mouton - extends Lemma 5.2.1 to direct sums of finite-dimensional simple unital Banach algebras.

Proposition 5.2.3. *([6], Proposition 5.7) For $t \in \mathbb{N}$, let $A_1, \dots, A_t$ be finite-dimensional simple unital Banach algebras and $T : A_1 \oplus \cdots \oplus A_t \to B$ be a Banach algebra homomorphism such that all T_j's are non-zero. Then $\sigma(Tx) = \sigma(x)$ for all $x \in A_1 \oplus \cdots \oplus A_t$.*

Proof. Here we assume that $t \geq 2$ since the result for $t = 1$ is clear from Lemma 5.2.1. Since all T_j's are non-zero, we have from Lemma 5.2.2 that $T_j : A'_j \to T_j(A'_j)$ is an isomorphism for all $j \in \{1, \dots, t\}$. By assumption, the only ideal of A_j, for each $j \in \{1, \dots, t\}$, is $\{0_{A_j}\}$, causing $A := A_1 \oplus \cdots \oplus A_t$ to have $2^t - 1$ ideals, each of the form $I_1 \oplus \cdots \oplus I_t$, where $I_j \in \{\{0_{A_j}\}, A_j\}$ for all $j \in \{1, \dots, t\}$. (By the definition of an ideal, A is excluded.)
Since $\mathrm{N}(T)$ is an ideal of A, it will take the form of one of the $2^t - 1$ ideals of A. We now determine $\mathrm{N}(T)$ through a process of elimination.
If $t = 2$, then we note that $\mathrm{N}(T)$ cannot coincide with some A'_j; otherwise we get $T_j = 0$, which contradicts our assumption. We therefore have that $\mathrm{N}(T) = \{0_{A_1}\} \oplus \{0_{A_2}\}$.
Suppose that $t \geq 3$. Then $\mathrm{N}(T)$ - as before - cannot coincide with some A'_j as this would mean that $T_j = 0$, a contradiction. If $\mathrm{N}(T)$ is an ideal with at least two non-zero components and at least one zero component, then some $j \in \{1, \dots, t\}$ can indeed be found satisfying $A'_j \subseteq \mathrm{N}(T)$. This again gives $T_j = 0$, which is not possible. Therefore $\mathrm{N}(T) = \{0_{A_1}\} \oplus \cdots \oplus \{0_{A_t}\}$.
By Lemma 2.3.10 we have that $T(A)$ is a Banach algebra which contains the unit of B. Using Proposition 2.8.2 and the fact that $T : A \to T(A)$ is an isomorphism, it follows that $\sigma(x) = \sigma(Tx, T(A)) = \sigma(Tx, B)$ for all $x \in A$. $\square$

The assumption in Proposition 5.2.3 that "all T_j's must be non-zero" cannot be removed. The following example demonstrates this.

Example 5.2.4. *Consider the finite-dimensional simple unital Banach algebras $M_3(\mathbb{C})$ and $\mathbb{C}$ and the Banach algebra homomorphism $T : M_3(\mathbb{C}) \oplus \mathbb{C} \to \mathbb{C}$ defined by $T(X, \lambda) = \lambda$ for $X \in M_3(\mathbb{C})$. Then $T_1 = 0$ and $\sigma(TA) \neq \sigma(A)$, where $A := \left(\left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right), 1 \right) \in M_3(\mathbb{C}) \oplus \mathbb{C}$.*

Proof. First take note that $T_1 : M_3(\mathbb{C}) \oplus \{0\} \to \mathbb{C}$ defined by $T_1(X, 0) = T(X, 0) = 0$ for $X \in M_3(\mathbb{C})$ is the zero map. Also,
$\sigma(TA) = \sigma \left(T \left(\left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right), 1 \right) \right) = \{1\}$ and, in view of Lemma 2.4.3, $\sigma(A) = \sigma \left(\left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right), M_3(\mathbb{C}) \right) \cup \sigma(1, \mathbb{C}) = \{0, 1, 2\}$, so that $\sigma(TA) \neq \sigma(A)$. $\square$

Our next result is a consequence of Lemma 5.2.2 in the case where some of the T_j's are zero. It stipulates that, if some T_j's are zero, then every element of the direct sum of finite-dimensional simple $COBAs$ has the upper Browder spectrum property - a $COBA$ generalization of [6] (Proposition 5.10). Though the proof of Proposition 5.2.5 is verbatim to that of [6] (Proposition 5.10), as it does not require the product of positive elements to be positive, we include it in the interest of completeness.

Proposition 5.2.5. *For $t \in \mathbb{N}$, let $A_1, A_2, \dots, A_t$ be finite-dimensional simple COBAs with algebra c-cones $C_1, C_2, \dots, C_t$, respectively. Consider the COBA $A = A_1 \oplus \cdots \oplus A_t$ with algebra c-cone $C = C_1 \oplus \cdots \oplus C_t$ and suppose that $T : A \to B$ is a Banach algebra homomorphism with at least one T_j zero. If $x \in A$ satisfies $r(x) \in \sigma(x) \backslash \sigma(Tx)$, then $r(x) \notin \beta_T^+(x)$.*

Proof. Let $x = (x_1, x_2, \dots, x_t) \in A$, where $x_j \in A_j$, be such that $r(x) \in \sigma(x) \backslash \sigma(Tx)$. Also, assume that $T_j = 0$ for all $j \in \{s_1, \dots, s_k\} \subseteq \{1, \dots, t\}$ and $x_j^0 = (0_{A_1}, \dots, x_j, \dots, 0_{A_t})$, where x_j is in the j-th position of the t-tuple. From Lemma 5.2.2 we have that $T_j : A_j' \to T_j(A_j')$ is an isomorphism for all $j \in \{1, \dots, t\} \backslash \{s_1, \dots, s_k\} := D$. Further note that x can be expressed as

$$x = (x_1, \dots, 0_{A_t}) + \cdots + (0_{A_1}, \dots, x_t) = x_1^0 + \cdots + x_t^0$$

with $x_i^0 x_j^0 = 0 = x_j^0 x_i^0$ for all $i \neq j$. Since A is finite, $\sigma(y, A)$ (and hence $\sigma(Ty,\ B)$) is finite for all $y \in A$. By also utilizing Lemma 2.4.2 we have that

$$\begin{aligned} \sigma'(Tx, B) &= \sigma'(T(x_1^0 + \cdots + x_t^0), B) \\ &= \sigma'(Tx_1^0 + \cdots + Tx_t^0, B) \\ &= (\sigma(Tx_1^0,\ B) \cup \cdots \cup \sigma(Tx_t^0, B)) \backslash \{0\} \\ &= \sigma'(Tx_1^0, B) \cup \cdots \cup \sigma'(Tx_t^0, B) \\ &= \sigma'(T_1 x_1^0, B) \cup \cdots \cup \sigma'(T_t x_t^0, B) \\ &= \bigcup_{j \in D} \sigma'(T_j x_j^0, B). \end{aligned} \qquad (\star)$$

Since, in view of Lemma 5.2.2, the range $T_j(A_j')$ of T_j is a closed subalgebra of the Banach algebra B for all $j \in D$, it follows from Proposition 2.4.6 and the remark following Lemma 2.4.5 that $\sigma'(T_j x_j^0, T_j(A_j')) = \sigma'(T_j x_j^0, B)$. Hence, using $(\star)$ and the facts that $T_j : A_j' \to T_j(A_j')$ is an isomorphism and A_j' is isomorphic to A_j, we have that

$$\begin{aligned}
\sigma'(Tx, B) &= \bigcup_{j \in D} \sigma'(T_j x_j^0, B) \\
&= \bigcup_{j \in D} \sigma'(T_j x_j^0, T_j(A_j')) \\
&= \left(\bigcup_{j \in D} \sigma(T_j x_j^0, T_j(A_j')) \right) \backslash \{0\} \\
&= \left(\bigcup_{j \in D} \sigma(x_j^0, A_j') \right) \backslash \{0\} \\
&= \left(\bigcup_{j \in D} \sigma(x_j, A_j) \right) \backslash \{0\}. \qquad (\star\star)
\end{aligned}$$

Let $\lambda \in \mathbb{R}^+$ be such that $\lambda > r(x)(\geq r(x_j))$ and consider the decomposition

$$x - r(x)e_A = (x_1 - r(x)e_{A_1}, \ldots, x_t - r(x)e_{A_t}) = (U_1, \ldots, U_t) + (V_1, \ldots, V_t),$$

where the U_j and V_j are chosen as follows: If $j \in D$, then $r(x) \notin \sigma(x_j)$ in view of $(\star)$, so choose $U_j = x_j - r(x)e_{A_j} \in A_j^{-1}$ and $V_j = 0_{A_j}$. If $j \notin D$, then choose $U_j = x_j - \lambda e_{A_j} \in A_j^{-1}$ and $V_j = (\lambda - r(x))e_{A_j}$. In both cases the U_j's are invertible in A_j, and hence $(U_1, \ldots, U_t) \in A^{-1}$. Next we show that $(V_1, \ldots, V_t)$ is an element of $C \cap \mathrm{N}(T)$. Now, using a similar argument as in the proof of Proposition 5.2.3, one obtains that $\mathrm{N}(T) = I_1 \oplus \cdots \oplus I_t$, where $I_j = A_j$ for $j \notin D$ and $I_j = \{0_{A_j}\}$ for $j \in D$. Since

$$V_j = \begin{cases} (\lambda - r(x))e_{A_j} \in C_j \subseteq A_j & \text{if } j \notin D \\ 0_{A_j} \in \{0_{A_j}\} \subseteq C_j & \text{if } j \in D, \end{cases}$$

it follows that $(V_1, \ldots, V_t) \in C \cap \mathrm{N}(T)$, and thus $x - r(x)e_A \in \mathcal{B}_T^+$. Therefore, $r(x) \notin \beta_T^+(x)$ by Corollary 4.1.3. □

The following result is an immediate application of Proposition 5.2.5:

Corollary 5.2.6. *Let $n_1, n_2, \ldots, n_t \in \mathbb{N}$ for some $t \in \mathbb{N}$, $A_j = M_{n_j}(\mathbb{C})$ and (say) $C_j = M_{n_j}(\mathbb{R}^+)$ for all $j \in \{1, \ldots, t\}$. If $T : A_1 \oplus \cdots \oplus A_t \to B$ is a Banach algebra homomorphism with at least one T_j zero, then the implication*

$$r(x) \in \sigma(x) \backslash \sigma(Tx) \Rightarrow r(x) \notin \beta_T^+(x)$$

holds for all $x \in A_1 \oplus \cdots \oplus A_t$.

Before we give the main result of this section, we formulate the well-known Wedderburn-Artin theorem which states that every finite-dimensional semisimple

(Banach) algebra is algebraically isomorphic to a direct sum of matrix algebras. This theorem emphasizes the fact that all finite-dimensional semisimple (Banach) algebras have a very simple structure over the complex field.

Theorem 5.2.7. *([2], Theorem 2.1.2) Let A be a semisimple finite-dimensional algebra over $\mathbb{C}$. Then there exist $n_1, \ldots, n_k \in \mathbb{N}$ such that A is isomorphic as an algebra to $M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$.*

We are now ready to prove the main result of this thesis. It generalizes [6] (Corollary 5.13) and shows that every finite-dimensional semisimple *COBA* is algebraically isomorphic to a *COBA* in which all positive elements have the upper Browder spectrum property.

Theorem 5.2.8. *Any finite-dimensional semisimple COBA is isomorphic as an algebra to a COBA $(M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_t}(\mathbb{C}), C_1 \oplus \cdots \oplus C_t)$, where each C_i is an algebra c-cone in $M_{n_i}(\mathbb{C})$, with the property that, if $T : M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_t}(\mathbb{C}) \to B$ is a Banach algebra homomorphism and $x \in C_1 \oplus \cdots \oplus C_t$ is such that $r(x) \in \sigma(x) \backslash \sigma(Tx)$, then $r(x) \notin \beta_T^+(x)$.*

Proof. By the Wedderburn-Artin theorem, any finite-dimensional semisimple *COBA* is algebraically isomorphic to $M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_t}(\mathbb{C})$. The result then follows from Corollary 5.2.6. □

5.3. Arbitrary homomorphism with the strong Riesz property

In Section 5.2 we studied the upper Browder spectrum property relative to Banach algebra homomorphisms on finite dimensional *COBAs*. These homomorphisms automatically have closed range and the Riesz property, and hence (by the remark following Definition 2.8.4) the strong Riesz property. In this section we consider arbitrary Banach algebra homomorphism T and provide sufficient conditions for positive *COBA* elements to have the upper Browder spectrum property relative to T.

We start with the generalization of Theorem 5.3.1 in [3], which forms the basis of our results in this section. The proof is verbatim to that of [3] (Theorem 5.3.1) because it does not require the product of positive elements to be positive.

We will denote by $\mathrm{Comm}(x)$ the *commutant* of x, which is the set of all elements which commute with x.

Theorem 5.3.1. *Let (A, C) be a COBA and I be an ideal of A. Suppose that $x \in A$ is such that $r(x)$ is a Riesz point of $\sigma(x)$ relative to I. If $p(x, r(x)) \in C$, then there exists $z := p(x, r(x)) \in C \cap I \cap \mathrm{Comm}(x)$ such that $r(x) \notin \sigma(x + z)$.*

Proof. Let $x \in A$ be such that $r(x)$ is a Riesz point of $\sigma(x)$ relative to an ideal I of A. Then $r(x) \in \operatorname{iso}\sigma(x)$ and $p(x, r(x)) \in I$ by definition, so that $p(x, r(x)) \in C \cap I \cap \mathrm{Comm}(x)$ follows from assumption and the fact that $p(x, r(x))$ commutes with all elements $x \in A$. Now since $p(x, r(x)) = x_{-1}$ in view of the remark following Lemma 2.4.14, it follows from Proposition 2.4.16 that $r(x) \notin \sigma(x + p(x, r(x))) = \{r(x)+1\} \cup (\sigma(x) \backslash \{r(x)\})$. Hence, by taking $z = p(x, r(x))$ we have that $r(x) \notin \sigma(x + z)$. □

The following result, which is a generalization of Corollary 5.3.2(i) in [3] to the *COBA* setting, is an immediate consequence of Theorem 5.3.1.

Proposition 5.3.2. *Let (A, C) be a COBA with closed algebra c-cone C and I be an inessential ideal of A. Suppose that $x \in C$ is such that $r(x)$ is a Riesz point of $\sigma(x)$ relative to I. If $r(x)$ is a simple pole (that is, a pole of order 1) of $(\lambda e - x)^{-1}$, then there exists $z \in C \cap I \cap \mathrm{Comm}(x)$ such that $r(x) \notin \sigma(x + z)$.*

Proof. By Proposition 2.5.10, $p(x, r(x)) = x_{-1}$ is an element of C. The result then follows from Theorem 5.3.1. □

The following result is an immediate consequence of Proposition 5.3.2 and generalizes Corollary 5.3.2 [(ii) and (iii)] in [3].

Corollary 5.3.3. *Let (A, C) be a COBA with closed algebra c-cone and I be an inessential ideal of A. Suppose that $x \in C$ is such that $r(x)$ is a Riesz point of $\sigma(x)$ relative to I. Then, under each of the following assumptions, there exists $z \in C \cap I \cap \mathrm{Comm}(x)$ such that $r(x) \notin \sigma(x + z)$.*

(i) A is commutative and semisimple,

(ii) A is semisimple and C is proper and inverse-closed.

Proof. Let $x \in C$ be such that $r(x)$ is a Riesz point of $\sigma(x)$ relative to I.
(i) Suppose that A is commutative and semisimple. From Lemma 2.4.18 we have that $r(x)$ is a pole of order, say, k of $(\lambda e - x)^{-1}$. Assume, on the other hand, that $r(x)$ is a pole of order $k > 1$ of $(\lambda e - x)^{-1}$. In light of Lemma 2.4.15 and Proposition 2.5.10, $0 \neq x_{-k} \in C \cap \mathrm{QN}(A)$, and therefore, using the fact that A is commutative (see the remark following Definition 2.4.4) and semisimple, $x_{-k} = 0$. This is impossible; hence $r(x)$ is a simple pole of $(\lambda e - x)^{-1}$. The result then follows from Proposition 5.3.2.

(ii) Suppose that A is semisimple with proper and inverse-closed algebra c-cone C. From the semisimpleness of A, as in (i), we have (in view of Lemmas 2.4.15 and 2.4.18 and Proposition 2.5.10) that $0 \neq x_{-k} \in C \cap \mathrm{QN}(A)$ for $k > 1$. By

Proposition 2.5.9, $x_{-k} = 0$, which is a contradiction. Hence $r(x)$ is a simple pole of $(\lambda e - x)^{-1}$. The result then follows from Proposition 5.3.2. $\square$

The next result, which is a stronger version of Theorem 5.1 in [6], gives a condition under which the positive elements of a *COBA* satisfy the upper Browder spectrum property relative to a Banach algebra homomorphism with the strong Riesz property.

Theorem 5.3.4. *Let (A, C) be a COBA, $T : A \to B$ be a Banach algebra homomorphism with the strong Riesz property and $x \in C$ be such that $r(x) \notin \sigma(Tx)$. If $p(x, r(x)) \in C$, then $r(x) \notin \beta_T^+(x)$.*

Proof. If $r(x) \notin \sigma(x)$ (in which case $p(x, r(x)) = 0 \in C$), then we have from Proposition 4.1.5 that $r(x) \notin \beta_T^+(x)$.
Hence, suppose that $r(x) \in \sigma(x) \backslash \sigma(Tx)$. By Lemma 2.8.6, $r(x)$ is a Riesz point of $\sigma(x)$ relative to $\mathrm{N}(T)$, which is an inessential ideal since T has the strong Riesz property. It then follows from Theorem 5.3.1, with $I = \mathrm{N}(T)$, that

$$r(x) \notin \bigcap_{\substack{z \in \mathrm{C} \cap \mathrm{N}(T) \\ xz = zx}} \sigma(x + z),$$

that is $r(x) \notin \beta_T^+(x)$ in view of Theorem 4.1.4. $\square$

In the light of Proposition 5.3.2 (and Corollary 5.3.3), we state the the following result which is an immediate consequence of Theorem 5.3.4.

Corollary 5.3.5. *Let (A, C) be a COBA with closed algebra c-cone C, $T : A \to B$ be a Banach algebra homomorphism with the strong Riesz property and $x \in C$ be such that $r(x) \notin \sigma(Tx)$. Then each of the following conditions ensures that $r(x) \notin \beta_T^+(x)$:*

(i) $r(x)$ is a simple pole of $(\lambda e - x)^{-1}$,

(ii) A is commutative and semisimple,

(iii) A is semisimple and C is proper and inverse-closed.

Proof. In view of the proofs of Proposition 5.3.2 and Corollary 5.3.3, if any of the statements (i) – (iii) hold, then $p(x, r(x)) \in C$. The result then follows from Theorem 5.3.4. $\square$

Discussion

In this dissertation, several results concerning upper Browder elements and the upper Browder spectrum were extended from $OBAs$ to $COBAs$ with minor or no modifications to both the assumptions and the proofs. In Chapter 3 and 4 we refer the reader to Definitions 3.1.1 and 4.1.1 - that gives respectively the definition of an upper Browder element and the upper Browder spectrum of a COBA element, Propositions 3.1.3 and 4.1.5 - that shows respectively the inclusion properties of the invertible, upper Browder, Browder and the Fredholm elements and their spectra, Lemmas 3.2.4 and 3.2.7 - which demonstrates that upper Browder elements are closed under addition of an invertible element and a scaled positive element in the null space of T and under positive scalar multiplication whenever T has the Riesz property, respectively. Finally, Theorem 4.1.4 establishes that the upper Browder spectrum of a COBA element can be represented in terms of the ordinary spectrum.

In Chapter 5, where the upper Browder spectrum property were discussed, all the results and proofs offered in $COBAs$ proceeded directly from the OBA setting without alteration.

In [4], Benjamin proved Propositions 3.2.5, 3.2.10 and 4.2.1 and Theorems 4.2.3 in the context of $OBAs$. In the $COBA$ setting we could not set up these results without the assumption that the $COBA$ is commutative. Since every commutative $COBA$ is an OBA, we were therefore unable to offer meaningful generalizations of the mentioned results in the $COBA$ setting. Hence, these are typically OBA results.
Theorem 3.2.12, which is of our own observation, provides conditions under which the sets of Browder and upper Browder elements of a $COBA$ coincide. From this result one can easily establish Proposition 4.2.1 and Theorem 4.3.2. These results, respectively, give conditions under which the upper Browder spectrum of a $COBA$ element obeys a spectral mapping theorem and the connected hulls of the Browder and upper Browder spectra coincide.

In this study, we noticed a gap in Theorem 5.1 of [6], for which Benjamin offered the following weaker assumptions: if T has closed range satisfying the Riesz property and that the spectral radius function is monotone in the OBA (A, C), then every positive element of a $COBA$ have the upper Browder spectrum property whenever the spectral idempotent of x corresponding to the spectral radius is positive. To bridge this gap, we offered Theorem 5.3.4 - a stronger version -

which states that if T has the strong Riesz property, then every positive element x has the upper spectrum property whenever the spectral idempotent of x corresponding to the spectral radius is positive.

The other gap was that Benjamin and Mouton never presented the conditions under which the connected hulls of the Browder and upper Browder spectra coincide, instead demonstrating that they do not coincide through an example ([3], Example 4.3.3). We were able to close this gap by showing that when the spectral idempotent condition holds for a commutative $COBA$ with T satisfying the Riesz property, the connected hulls of the aforementioned spectra coincide (see Theorem 4.3.2). Because a commutative $COBA$ is also a commutative OBA, the OBA result follows.

Conclusion

This study has revealed that a number of results established on the upper Browder spectrum of an OBA element readily extend to $COBAs$ with minor or no modifications to both the assumptions and the proofs. There are, however, some results which could not be generalized meaningfully from the OBA setting to the context of $COBAs$ - see, for example, Propositions 3.2.5 and 3.2.10 which stipulate respectively, for a commutative COBA with T satisfying the Riesz property, that upper Browder elements remain stable under perturbation by a positive element in the null space of T, and that the product of upper Browder elements is again upper Browder.
It therefore seems from the foregoing that the $COBA$ setting is in fact the appropriate setting for developing Fredholm theory in Banach algebras ordered by positive cones.

In this dissertation we did not manage to answer the following questions:

(i) *Is there an example of a non-commutative* $COBA$ (A, C) *and a Banach algebra homomorphism (defined on* A*) with the Riesz property satisfying the spectral idempotent condition in Proposition 4.2.1 such that the equality in the formulation of the mentioned result does not hold?*

(ii) *Is there an example of a commutative* $COBA$ (A, C) *and a Banach algebra homomorphism (defined on* A*) with the Riesz property not satisfying the spectral idempotent condition in Proposition 4.2.1 such that the equality in the formulation of the mentioned result does not hold?*

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